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Dehui Yuan · Xiaoling Liu

G-semipreinvexity and its applications

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Abstract In this paper, we introduce some new kinds of generalized convexity, which include (semistrict) G-semipreinvexity and (semistrict) G-semipreincavity. Moreover, we establish the relations with common generalized convexity, present properties of (semistrictly) G-semipreinvex and (semistrictly) G-semipreincave functions, and also give characterizations of the classes of G-semipreinvex and G-semipreincave functions. Moreover, we deal with programming involving G-semipreinvex functions. Our results extend the existing ones in the literature.

Mathematics Subject Classification 90C26 · 90C46

الملخص

في هذه الورقة، نقدّم أنواعاً جديدة من التحدّب المعمّم والتي تشمل G-semipreinvexity وَ G-semipreincavity شبه الفعلية. نثبت أيضاً علاقات مع التحدب المعمَّم العادي، ونعرض خصائص هذه الدوال الجديدة كما نعطي تمييزات لها. بالاضافة إلى ذلك، نتعامل مع برمجة تحتوي على دوال لها خاصية G-semipreinvexity. تمدد نتائجنا بعض النتائج الموجودة في دراسات سابقة.

1 Introduction

It is well known that convexity has been playing a key role in mathematical programming, engineering and optimization theory. The research on characterizations and generalizations of convexity is one of the most important aspects in mathematical programming and optimization theory. There have been many attempts to weaken the convexity assumptions to treat many practical problems. Therefore, many concepts of generalized convex functions have been introduced and applied to mathematical programming problems in the literature [1,2,10,12–21,23,24,26–34]. One of these concepts, invexity, was introduced by Hanson in [17]. Hanson [17] proved that invexity has a common property that Karush–Kuhn–Tucker conditions are sufficient for global optimality of nonlinear programming under the invexity assumptions. Ben-Israel and Mond [15] introduced the concept of preinvex functions which is a special case of invexity.

On the other hand, Avriel [11] introduced the definition of r-convex functions which is another generalization of convex functions, and he discussed some characterizations and its relations with other generalizations of convex functions in the literature. In [3], Antczak introduced the concept of a class of r-preinvex functions, which is a generalization of r-convex functions and preinvex functions, and obtained some optimality results under appropriate r-preinvexity conditions for constrained optimization problems. Further, he introduced the concept of $V - r$-invexity for differentiable multiobjective programming problems in [4].

D. Yuan (✉) · X. Liu
Department of Mathematics, Hanshan Normal University, Chaozhou 521041, Guangdong, China
E-mail: dhyuan@hstc.edu.cn

Recently, Antczak [5] extended further invexity to G-invexity for scalar differentiable functions. In the natural way, Antczak's definition of G-invexity was also extended to the case of differentiable vector-valued functions in [6]. With vector G-invexity, Antczak [7] proved new duality results for nonlinear differentiable multiobjective programming problems. To deal with programming which is not necessarily differential, Antczak [8] introduced the concept of G-preinvexity, which unifies the concepts of nondifferentiable convexity, preinvexity and r-preinvexity. Furthermore, relations between different preinvexity concepts introduced in the literatures were also discussed in [9,22].

Based on the semiinvex set concept, Yang and Chen [35] introduced a wider class of nonconvex functions, called semipreinvex functions, which includes the preinvex functions and arc-connected convex functions and preserves some nice properties that convex functions have. Noor proved that many results in mathematical programming involving convex functions actually hold for semipreinvex functions in [25]. Yang et al. [36] further discussed some basic properties of semipreinvex functions. In 2011, Zhao et al. [37] proposed the concept of r-semipreinvex functions and obtained some important characterizations and optimality results in nonlinear programming. Note that the concept r-semipreinvexity unifies the concepts of r-preinvexity and semipreinvexity.

Motivated by [8,9,35,37], we present some new concepts of generalized convexity, which include (semistrict) G-semipreinvexity and (semistrict) G-semipreincavity, in this paper. We have managed to deal with their relations with some common generalized convexity. The rest of the paper is organized as follows. In Sect. 2, we present new concepts of generalized convexity, and we discuss their relations with common generalized convexity introduced in the literature. In Sect. 3, we present properties of (semistrictly) G-semipreinvex functions and (semistrictly) G-semipreincave functions. In Sect. 4, we give characterizations of the class of G-semipreinvex and G-semipreincave functions. In Sect. 5, we deal with the programming involving G-semipreinvex functions. Section 6 gives some conclusions.

2 Definitions and preliminaries

In this section, we provide some definitions and some results which we will use throughout the paper. The following definition is taken from [35].

Definition 2.1 Let $X \subset \mathbb{R}^n$, $\eta : X \times X \times [0,1] \to \mathbb{R}^n$. The set X is said to be semiinvex at $u \in X$ with respect to η if for all $x \in X$, $\lambda \in [0,1]$ such that

$$u + \lambda\eta(x, u, \lambda) \in X.$$

X is said to be a semiinvex set with respect to η if X is semiinvex at each $u \in X$. If $\eta(x, u, \lambda)$ is independent with respect to the third argument λ, then semiinvex set with respect to η is invex one as defined in literatures.

Definition 2.2 Let X be a nonempty semiinvex subset of $\mathbb{R}^n$. A real-valued function $f : X \to \mathbb{R}$ is said to be G-semipreinvex at u on X with respect to η if there exist a continuous real-valued function $G : I_f(X) \to \mathbb{R}$ such that G is strictly increasing on its domain, and a vector-valued function $\eta : X \times X \times [0,1] \to \mathbb{R}^n$ such that for all $x \in X$

$$f(u+\lambda\eta(x, u, \lambda)) \leq G^{-1}(\lambda G(f(x)) + (1-\lambda)G(f(u))), \lambda \in (0,1) \tag{1}$$

with $\lim_{\lambda\to 0^+} \lambda\eta(x, u, \lambda) = 0$, where $I_f(X) = \{f(x) : x \in X\}$. If inequality (2) holds for any $u \in X$, then f is G-semipreinvex on X with respect to η; f is said to be strictly G-semipreinvex on X with respect to η if strict inequality (2) holds for all $x, u \in X$ such that $x \neq u$; f is said to be semistrictly G-semipreinvex on X with respect to η if strict inequality (2) holds for all $x, u \in X$ such that $f(x) \neq f(u)$.

Remark 2.3 If $G(a) = a$ for $a \in I_f$, then G-semipreinvex is semipreinvex as defined [35]. An analogous terminology holds in the case of (semistrictly) G-semipreincave functions with respect to η, for which the monotonicity of G should be changed to decreasing.

Remark 2.4 Every G-preinvex function with respect to η as defined in [5,8] is G-semipreinvex with respect to η; every semipreinvex function with respect to η as defined in [35,36] is G-semipreinvex with respect to η, where $G(a) = a$, $a \in \mathbb{R}$. The converse results are, in general, not true. For example, see Example 2.6.

Remark 2.5 Every semistrictly G-preinvex function with respect to η as defined in [22] is semistrictly G-semipreinvex with respect to η. The converse result is, in general, not true. See Example 2.6, too.

Example 2.6 Let X be a subset in $\mathbb{R}^2$ defined as follows.

$$X =: \{x = (x_1, x_2) | 0 < x_2 < x_1^2, 0 < x_1 < 2\} \cup \{(0, 0)\}.$$

Denote $x = (x_1, x_2)$, $u = (u_1, u_2) \in X$. Define

$$\eta(x, u, \lambda) = \begin{cases} \left(x_1, \frac{1}{2}\lambda x_2\right), & u = (0, 0) \\ x - u, & u \neq (0, 0) \end{cases}$$

and

$$f(x) = \ln(x_1 + x_2 + 2), x = (x_1, x_2) \in X,$$
$$G(a) = e^a, a \in \mathbb{R}.$$

Then, it is easy to check that f is both a semistrictly G-semipreinvex function and a G-semipreinvex function on X with respect to η. However, f is not a G-preinvex function on X with respect to η and f is also not a semistrictly G-preinvex function on X with respect to η, because X is not invex but semiinvex.

From Definition 2.2, the inverse of function G must exist. It implies the function G must be strictly monotonous. Thus, we can assume that G is a strictly monotonous function on its domain D_G. Now we give a useful lemma.

Lemma 2.7 *Let $f : X \to \mathbb{R}$. Then*

(i) *f is (semistrictly) G-semipreinvex on X with respect to η if and only if $G(f)$ is (semistrictly) semipreinvex on X with respect to η;*

(ii) *f is (semistrictly) G-semipreincave on X with respect to η if and only if $G(f)$ is (semistrictly) semipreincave on X with respect to η; or f is (semistrictly) G-semipreincave on X with respect to η if and only if $-G(f)$ is (semistrictly) semipreinvex on X with respect to η.*

Proof (i) We only prove the G-semipreinvexity case (the proof of the semistrict G-semipreinvexity case is analogous). By the monotonicity of G, we know that the inequality (2) is equivalent with

$$G(f(u + \lambda\eta(x, u, \lambda))) \leq \lambda G(f(x)) + (1 - \lambda)G(f(u)), \ \lambda \in (0, 1).$$

Therefore, by Definition 2.2, f is G-semipreinvex on X with respect to η if and only if $G(f)$ is semipreinvex on X with respect to η.

Similar to part (i), we can prove (ii). This completes the proof. □

Now, we discuss the relationships between r-semipreinvexity and G-semipreinvexity. For convenience, we present the definition of r-semipreinvex function, given in [37, Definition 2.7]

Definition 2.8 Let X be a nonempty semiinvex subset of $\mathbb{R}^n$. A real-valued function $f : X \to \mathbb{R}$ is said to be r-semipreinvex on X with respect to η if, for all $x, u \in X$, $\lambda \in (0, 1)$,

$$f(u+\lambda\eta(x, u, \lambda)) \leq \begin{cases} \log\left(\lambda e^{rf(x)} + (1 - \lambda)e^{rf(u)}\right)^{\frac{1}{r}}, & r \neq 0 \\ \lambda f(x) + (1 - \lambda) f(u), & r = 0 \end{cases} \tag{2}$$

with $\lim_{\lambda\to 0^+} \lambda\eta(x, u, \lambda) = 0$. The term of r-semipreincave is defined in a similar way with the sense of the inequality reversed. f is said to be be strictly r-semipreinvex with respect to η, if the inequality (2) is strict, for all $x, y \in X$, $x \neq y$, $\lambda \in (0, 1)$. f is said to be be (strictly) semipreinvex with respect to η, if it is (strictly) r-semipreinvex with respect to η for $r = 0$.

Theorem 2.9 *Let function $f : X \to \mathbb{R}$ be r-semipreinvex on X with respect to η. Define*

$$G(a) = \begin{cases} e^{ra}, & r \neq 0 \\ a, & r = 0 \end{cases}.$$

Then f is G-semipreinvex when $r \geq 0$ and G-semipreincave when $r < 0$.

Proof It is easy to check that G is increasing when $r \geqslant 0$ and decreasing when $r < 0$. From Definition 2.2 and Remark 2.3, we obtain the required results. □

Example 2.10 Let $X = (-5, 5)$. Define

$$f_1(x) = \log(x^2 + 1),\ f_2(x) = \log(5 - |x|),\ x \in X.$$

Moreover, define

$$G_1(x) = e^x,\ G_2(x) = -e^x,\ x \in R \quad \text{and} \quad \eta(x, y; \lambda) = x - y.$$

Then f_1 is G_1-semipreinvex and f_2 is G_2-semipreincave on X with respect to η. Take $\eta_1(x, y; \lambda)$ as $\eta(x, y; \lambda)$ defined in [37, Example 2.4] and $\eta_2(x, y; \lambda)$ as $\eta(x, y; \lambda)$ defined in [37, Example 2.5]. Then f_1 and f_2 are G_1-semipreinvex on X with respect to $\eta_1(x, y; \lambda)$ and $\eta_2(x, y; \lambda)$, respectively.

Remark 2.11 From the above example, we observe that one can obtain certain preinvexity through selecting the real function G or the vector function η. Thus, we have more freedom when considering the generalized convexity of a function.

3 Properties of G-semipreinvex functions

In this section, we present properties of (semistrictly) G-semipreinvex functions and (semistrictly) G-semipreincave functions.

Theorem 3.1 *Let X be a nonempty semiinvex set in $\mathbb{R}^n$ with respect to $\eta : X \times X \times [0, 1] \to \mathbb{R}^n$, $f : X \to \mathbb{R}$ be a (semisrictly) G_1-semipreinvex function on X with respect to η, and $G_2 : I_{G_1(f)}(X) \to \mathbb{R}$ be both a convex function and an increasing function. Then f is (semistrictly) $G_2(G_1)$-semipreinvex on X with respect to the same η.*

Proof Here, we only prove the case that f is semistrictly G_1-semipreinvex on X with respect to η (the proof of the case that f is G_1-semipreinvex is analogous). By Lemma 2.7 (ii), $G_1(f)$ is semistrictly semipreinvex on X with respect to the same η. Therefore, for any $x, u \in X$, $f(x) \neq f(u)$, the inequality

$$G_1(f(u + \lambda\eta(x, u, \lambda))) < \lambda G_1(f(x)) + (1 - \lambda)G_1(f(u)),\ \lambda \in (0, 1)$$

holds. Note the convexity and monotonicity of G_2, we have

$$\begin{aligned} G_2(G_1(f(u + \lambda\eta(x, u, \lambda))) &< G_2\left(\lambda G_1(f(x)) + (1 - \lambda)G_1(f(u))\right) \\ &\leq \lambda G_2(G_1(f(x))) + (1 - \lambda)G_2(G_1(f(u))). \end{aligned}$$

Hence, $G_2(G_1(f))$ is semistrictly semipreinvex on X with respect to η. Again, by Lemma 2.7 (ii), f is semistrictly $G_2(G_1)$-semipreinvex on X with respect to η. This completes the proof. □

Theorem 3.2 *Let X be a nonempty semiinvex set in $\mathbb{R}^n$ with respect to $\eta : X \times X \times [0, 1] \to \mathbb{R}^n$, $f_i : X \to \mathbb{R}$ be G-semipreinvex on X with respect to the same η, G, $i \in \mathbb{N}$, where $\mathbb{N}$ is a finite or infinite index set. Then function $h(x) = \sup_{i\in\mathbb{N}} f_i(x)$ is G-semipreinvex on X with respect to the same η, G.*

Proof If f_i is G-semipreinvex on X with respect to the same η, G, $i \in \mathbb{N}$. Then, by Lemma 2.7 (i), $G(f_i)$ is semipreinvex on X with respect to the same η, $i \in \mathbb{N}$. Therefore, for any $x, u \in X$, the inequality

$$G(f_i(u + \lambda\eta(x, u, \lambda))) \leq \lambda G(f_i(x)) + (1 - \lambda)G(f_i(u)),\ \lambda \in (0, 1)$$

holds for $i \in \mathbb{N}$. Define $h^*(x) = \sup_{i\in\mathbb{N}} G(f_i(x))$. Then,

$$h^*(x) = \sup_{i\in\mathbb{N}} G(f_i(x)) = G(\sup_{i\in\mathbb{N}} f_i(x)) = G(h(x)).$$

Therefore, we have

$$G(h(u + \lambda\eta(x, u, \lambda))) \leq \lambda G(h(x)) + (1 - \lambda)G(h(u)).$$

Hence, $G(h)$ is semipreinvex on X with respect to η. Again, by Lemma 2.7 (i), h is G-semipreinvex on X with respect to η, G. This completes the proof. □

We point out that semistrict G-semipreinvexity does not possess an analogous property, see the following example.

Example 3.3 Let $X_1 = [-6, -2] \subset \mathbb{R}$, $X_1 = [-1, 6] \subset \mathbb{R}$, and $X = X_1 \cup X_2$. Define

$$f_1(x) = \begin{cases} 1, & x = 0 \\ 0, & x \in X\backslash\{0\} \end{cases}, \qquad f_2(x) = \begin{cases} 1, & x = 1 \\ 0, & x \in X\backslash\{1\} \end{cases}$$

and define

$$G(a) = a, a \in \mathbb{R},$$
$$\eta(x, y, \lambda) = \begin{cases} x - y, & x, y \in X_2 \\ x - y, & x, y \in X_1 \\ 7 - y, & x \in X_2, y \in X_1 \\ -y, & x \in X_1, y \in X_2\backslash\{0\} \\ \frac{1}{6}x, & x \in X_1, y = 0. \end{cases}$$

It is obvious that both f_1 and f_2 are semistrictly G-semipreinvex on X. Moreover, it can be verified that

$$h(x) = \sup\{f_i(x), i = 1, 2\} = \begin{cases} 1, & x = 0 \text{ or } x = 1 \\ 0, & x \in X\backslash\{0, 1\}. \end{cases}$$

Now take $x = -1$, $y = 1$, $\lambda = \frac{1}{2}$, then

$$G(h(x)) = G(h(-1)) = 0 < 1 = G(h(1)) = G(h(y)).$$

However,

$$G(h(y + \lambda\eta(x, y, \lambda))) = h(0) = 1 > \frac{1}{2} = \frac{1}{2}G(h(-1)) + \frac{1}{2}G(h(1)).$$

Hence, h is not semistrictly G-semipreinvex on X.

But we have the following result:

Theorem 3.4 *Let X be a nonempty semiinvex set in $\mathbb{R}^n$ with respect to $\eta : X \times X \times [0, 1] \to \mathbb{R}^n$, $f_i : X \to \mathbb{R}$ be both G-semipreinvex and semistrictly G-semipreinvex on X with respect to the same η, G, $i \in \mathbb{N}$, where $\mathbb{N}$ is a finite or infinite index set. Define function $h(x) := \sup_{i\in\mathbb{N}} f_i(x)$, for every $x \in X$. Assume that for every $x \in X$, there exists an $i_0 := i(x) \in \mathbb{N}$, such that $h(x) = f_{i_0}(x)$. Then function $h(x)$ is both G-semipreinvex and semistrictly G-semipreinvex on X with respect to the same η, G.*

Proof By Theorem 3.2, we know that h is G-semipreinvex on X with respect to η. It suffices to show that h is a semistrictly G-semipreinvex on X with respect to η. Assume that h is not a semistrictly G-semipreinvex on X. Then, there exist x, y,$h(x) \neq h(y)$ such that

$$G(h(y + \lambda\eta(x, y, \lambda))) \geq \lambda G(h(x)) + (1 - \lambda)G(h(y)), \forall\lambda \in (0, 1).$$

By the G-semipreinvexity of h, we have

$$G(h(y + \lambda\eta(x, y, \lambda))) \leq \lambda G(h(x)) + (1 - \lambda)G(h(y)).$$

Hence

$$G(h(y + \lambda\eta(x, y, \lambda))) = \lambda G(h(x)) + (1 - \lambda)G(h(y)). \tag{3}$$

Denote $z = y + \lambda\eta(x, y, \lambda))$. From the assumptions of the theorem, there exist $i(z) := i_0$, $i(x) := i_1$ and $i(y) := i_2$, satisfying

$$h(z) = h_{i_0}(z), h(x) = h_{i_1}(x), h(y) = h_{i_2}(y).$$

Then, (3) implies that

$$G(f_{i_0}(z)) = \lambda G(f_{i_1}(x)) + (1 - \lambda)G(f_{i_2}(y)). \tag{4}$$

(i) If $f_{i_0}(x) \neq f_{i_0}(y)$, by the semistrict G-semipreinvexity of f_{i_0}, we have

$$G(f_{i_0}(z)) < \lambda G(f_{i_0}(x)) + (1-\lambda)G(f_{i_0}(y)). \tag{5}$$

From $f_{i_0}(x) \leq f_{i_1}(x)$, $f_{i_0}(y) \leq f_{i_2}(y)$ and (5), we have

$$G(f_{i_0}(z)) < \lambda G(f_{i_1}(x)) + (1-\lambda)G(f_{i_2}(y)),$$

which contradicts (3).

(ii) If $f_{i_0}(x) = f_{i_0}(y)$, by the G-semipreinvexity of f_{i_0}, we have

$$G(f_{i_0}(z)) \leq \lambda G(f_{i_0}(x)) + (1-\lambda)G(f_{i_0}(y)). \tag{6}$$

Since $h(x) \neq h(y)$, at least one of the inequalities $f_{i_0}(x) \leq f_{i_1}(x) = h(x)$ and $f_{i_0}(y) \leq f_{i_2}(y) = h(y)$ has to be a strict inequality. From (6), we obtain

$$G(h(z)) = G(f_{i_0}(z)) < \lambda G(h(x)) + (1-\lambda)G(h(y)),$$

which contradicts (3). This completes the proof. □

Similar to Theorems 3.1, 3.2 and 3.4, respectively, we have the following three Theorems 3.5, 3.6 and 3.7 for (semistrictly) G-semipreincave functions.

Theorem 3.5 *Let X be a nonempty semiinvex set in $\mathbb{R}^n$ with respect to $\eta : X \times X \times [0,1] \to \mathbb{R}^n$, $f : X \to \mathbb{R}$ be (semistrictly) G_1-semipreincave on X with respect to η, and $G_2 : I_{G_1(f)}(X) \to \mathbb{R}$ be both concave and decreasing. Then f is (semistrictly) $G_2(G_1)$-semipreincave on X with respect to the same η.*

Theorem 3.6 *Let X be a nonempty semiinvex set in $\mathbb{R}^n$ with respect to $\eta : X \times X \times [0,1] \to \mathbb{R}^n$, $f_i : X \to \mathbb{R}$ be G-semipreincave on X with respect to the same η and G for each $i \in \mathbb{N}$, where $\mathbb{N}$ is a finite or infinite index set. Define*

$$h(x) := \inf_{i \in \mathbb{N}} f_i(x), \ x \in X.$$

Then h is G-semipreincave on X with respect to the same η and G.

Theorem 3.7 *Let X be a nonempty semiinvex set in $\mathbb{R}^n$ with respect to $\eta : X \times X \times [0,1] \to \mathbb{R}^n$, $f_i : X \to \mathbb{R}$ be both G-semipreincave and semistrictly G-semipreincave on X with respect to the same η and G for each $i \in \mathbb{N}$, where $\mathbb{N}$ is a finite or infinite index set. Define*

$$h(x) := \inf_{i \in \mathbb{N}} f_i(x), \ x \in X.$$

Assume that for every $x \in X$, there exists an $i_0 := i(x) \in \mathbb{N}$, such that $h(x) = f_{i_0}(x)$. Then h is both G-semipreincave and semistrictly G-semipreincave on X with respect to the same η, G.

4 Characterizations of *G*-semipreinvex functions

In this section, we give some characterizations of the class of G-semipreinvex and G-semipreincave functions. Firstly, we have the following Theorems 4.1 and 4.2 for G-semipreinvex functions.

Theorem 4.1 *Let X be a nonempty semiinvex set in $\mathbb{R}^n$ with respect to $\eta : X \times X \times [0,1] \to \mathbb{R}^n$. Assume that G is continuous on its domain D_G and D_G is an open subset of $\mathbb{R}$. Then $f : X \to \mathbb{R}$ is G-semipreinvex on X with respect to η if and only if, for all $x, u \in X$, $\lambda \in [0,1]$, $s, t \in D_G$, one has*

$$f(x) < s \quad \text{and} \quad f(u) < t \Longrightarrow G(f(u + \lambda\eta(x,u,\lambda))) < \lambda G(s) + (1-\lambda)G(t). \tag{7}$$

Proof Let f be G-semipreinvex on X with respect to η, and $f(x) < s$, $f(u) < t$, $0 < \lambda < 1$. From Definition 2.2, one obtains that

$$f(u+\lambda\eta(x,u,\lambda)) \le G^{-1}(\lambda G(f(x)) + (1-\lambda)G(f(u))).$$

This follows that

$$G(f(u+\lambda\eta(x,u,\lambda))) < \lambda G(s) + (1-\lambda)G(t).$$

Conversely, let $x, u \in X, \lambda \in [0,1]$. Note that G is continuous on D_G, D_G is an open interval and $I_f \subset D_G$. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$G(r) < G(f(x)) + \varepsilon, \quad \text{for all } |r - f(x)| < \delta$$

and

$$G(r) < G(f(u)) + \varepsilon, \quad \text{for all } |r - f(u)| < \delta.$$

Define $s = f(x) + \frac{\delta}{2}, t = f(u) + \frac{\delta}{2}$. From (7), we obtain

$$\begin{aligned} G(f(u+\lambda\eta(x,u,\lambda))) &< \lambda G\left(f(x) + \frac{\delta}{2}\right) + (1-\lambda)G\left(f(u) + \frac{\delta}{2}\right) \\ &< \lambda G(f(x)) + (1-\lambda)G(f(u)) + 2\varepsilon. \end{aligned}$$

Let $\varepsilon \to 0$, it follows that

$$G(f(u+\lambda\eta(x,u,\lambda))) \le \lambda G(f(x)) + (1-\lambda)G(f(u)).$$

From Lemma 2.7 (i), we deduce that f is G-semipreinvex on X with respect to η. □

Theorem 4.2 *Let X be a nonempty semiinvex set in $\mathbb{R}^n$ with respect to $\eta : X \times X \times [0,1] \to \mathbb{R}^n$. Assume that G is continuous on its domain D_G and D_G is an open subset of $\mathbb{R}$. Then $f : X \to \mathbb{R}$ is G-semipreinvex on X with respect to η if and only if the set*

$$F_G(f) = \{(x,s) : x \in X, s \in \mathbb{R}, G(f(x)) < s\} \tag{8}$$

is semiinvex set with respect to $\bar{\eta} : F_G(f) \times F_G(f) \times [0,1] \to \mathbb{R}^{n+1}$ defined by

$$\bar{\eta}((u,t),(x,s),\lambda) := (\eta(x,u,\lambda), s-t), \lambda \in [0,1]$$

for all $(x,s), (u,t) \in F_G(f)$.

Proof Let $(x,s) \in F_G(f)$ and $(u,t) \in F_G(f)$, i.e.,

$$G(f(x)) < s \quad \text{and} \quad G(f(u)) < t.$$

We obtain from the G-semipreinvexity of f that

$$\begin{aligned} G(f(u+\lambda\eta(x,u,\lambda))) &\le \lambda G(f(x)) + (1-\lambda)G(f(u)) \\ &< \lambda s + (1-\lambda)t = t + \lambda(s-t), \ \forall \lambda \in [0,1]. \end{aligned}$$

Thus,

$$(u+\lambda\eta(x,u,\lambda), t+\lambda(s-t)) \in F_G(f), \ \forall \lambda \in [0,1].$$

That is,

$$(u,t) + \lambda(\eta(x,u,\lambda),(s-t)) \in F_G(f), \ \forall \lambda \in [0,1].$$

Hence $F_G(f)$ is a semiinvex set with respect to $\bar{\eta}$:

$$\bar{\eta}((u,t),(x,s),\lambda) = (\eta(x,u,\lambda), s-t), \lambda \in [0,1].$$

Conversely, assume that $F_G(f)$ is a semiinvex set with respect to $\bar{\eta}$. Let $x, u \in X$ and $s, t \in \mathbb{R}$ such that

$$G(f(x)) < s \quad \text{and} \quad G(f(u)) < t.$$

Then,

$$(x, s) \in F_G(f) \quad \text{and} \quad (u, t) \in F_G(f).$$

Note that $F_G(f)$ is a semiinvex set with respect to $\bar{\eta}$. It implies that

$$(u, t) + \lambda\bar{\eta}((x, s), (u, t), \lambda) \in F_G(f), \lambda \in [0, 1].$$

That is,

$$(u, t) + \lambda(\eta(x, u, \lambda), s - t) \in F_G(f), \lambda \in [0, 1]$$

or

$$(u + \lambda\eta(x, u, \lambda), t + \lambda(s - t)) \in F_G(f), \lambda \in [0, 1].$$

Thus,

$$G(f(u + \lambda\eta(x, u, \lambda))) < \lambda s + (1 - \lambda)t, \lambda \in [0, 1].$$

Taking $s = G(f(x)) + \varepsilon$ and $t = G(f(u)) + \varepsilon$ for $\varepsilon > 0$ sufficiently small in above inequality, we obtain

$$G(f(u + \lambda\eta(x, u, \lambda))) < \lambda G(f(x)) + (1 - \lambda)G(f(u)) + \varepsilon, \lambda \in [0, 1].$$

Let $\varepsilon \to 0$, it follows that

$$G(f(u + \lambda\eta(x, u, \lambda))) \leq \lambda G(f(x)) + (1 - \lambda)G(f(u)), \lambda \in [0, 1].$$

From Lemma 2.7 (i), f is G-semipreinvex on X with respect to η. □

Similarly, we can establish Theorems 4.3 and 4.4 for G-semipreincave functions. Therefore, we simply state them here.

Theorem 4.3 *Let X be a nonempty semiinvex set in $\mathbb{R}^n$ with respect to $\eta : X \times X \times [0, 1] \to \mathbb{R}^n$. Assume that G is continuous on its domain D_G and D_G is an open subset of $\mathbb{R}$. Then $f : X \to \mathbb{R}$ is G-semipreincave on X with respect to η if and only if, for all $x, u \in X$, $\lambda \in [0, 1]$, $s, t \in D_G$,*

$$f(x) < s \quad \text{and} \quad f(u) < t \Longrightarrow G(f(u + \lambda\eta(x, u, \lambda))) > \lambda G(s) + (1 - \lambda)G(t).$$

Theorem 4.4 *Let X be a nonempty semiinvex set in $\mathbb{R}^n$ with respect to $\eta : X \times X \times [0, 1] \to \mathbb{R}^n$. Assume that G is continuous on its domain D_G and D_G is an open subset of $\mathbb{R}$. Then $f : X \to \mathbb{R}$ is G-semipreinvex on X with respect to η if and only if the set*

$$F_G(f) = \{(x, s) : x \in X, s \in \mathbb{R}, G(f(x)) > s\}$$

is semiinvex set with respect to $\bar{\eta} : F_G(f) \times F_G(f) \times [0, 1] \to \mathbb{R}^{n+1}$ defined by

$$\bar{\eta}((u, t), (x, s), \lambda) := (\eta(x, u, \lambda), s - t), \lambda \in [0, 1]$$

for all $(x, s), (u, t) \in F_G(f)$.

5 Programming with G-semipreinvexity

In this section, we present some optimality properties of G-semipreinvex and semistrictly G-semipreinvex functions. Moreover, we deal with programming involving G-semipreinvex functions.

Theorem 5.1 *Let X be a nonempty semiinvex set in $\mathbb{R}^n$ with respect to $\eta : X \times X \times [0, 1] \to \mathbb{R}^n$, and $f : X \to \mathbb{R}$ be G-semipreinvex or G-semipreincave on X with respect to η. If $\bar{x} \in X$ is a local minimum to the problem of minimizing $f(x)$ subject to $x \in X$, then $\bar{x}$ is a global one.*

Proof Case i: f is G-semipreinvex on X with respect to η. Then, by Lemma 2.7 (i), $G(f)$ is semipreinvex on X with respect to η. Since G is increasing on its domain $I_f(x)$, then $\bar{x} \in X$ is a local minimum to the problem of minimizing $f(x)$ subject to $x \in X$ if and only if $\bar{x} \in X$ is a local minimum to the problem of minimizing $G(f)(x)$ subject to $x \in X$. Therefore, by Theorem 2 in [35], $\bar{x} \in X$ is a global one to the problem of minimizing $G(f)(x)$ subject to $x \in X$. Hence $\bar{x} \in X$ is a global one to the problem of minimizing $f(x)$ subject to $x \in X$.

Case ii: f is G-semipreincave on X with respect to η. Then, by Lemma 2.7 (iii), $G(f)$ is semipreincave on X with respect to η. Thus, $-G(f)$ is semipreinvex on X with respect to η. Since G is decreasing on its domain $I_f(x)$, then $\bar{x} \in X$ is a local minimum to the problem of minimizing $f(x)$ subject to $x \in X$ if and only if $\bar{x} \in X$ is a local maximum to the problem of maximizing $G(f)(x)$ subject to $x \in X$, or if and only if $\bar{x} \in X$ is a local minimum to the problem of minimizing $-G(f)(x)$ subject to $x \in X$. Similar to the proof of Case i, we obtain the required result. □

Similar to the proof of Theorem 5.1, we can establish the following Theorem 5.2. Therefore, we simply state it here.

Theorem 5.2 *Let X be a nonempty semiinvex set in $\mathbb{R}^n$ with respect to $\eta : X \times X \times [0, 1] \to \mathbb{R}^n$, and $f : X \to \mathbb{R}$ be semistrictly G-semipreinvex (G-semipreincave) on X with respect to η. If $\bar{x} \in X$ is a local minimum to the problem of minimizing $f(x)$ subject to $x \in X$, then $\bar{x}$ is a global one.*

From Example 2.6, Theorems 5.1 and 5.2, we can conclude that these new generalized convex functions constitutes an important class of generalized convex functions in mathematical programming.

Next, we consider the nonlinear programming with inequality constraint.

$$(P) \quad \min f(x) :$$
$$s.t.\ g_i(x) \leq 0,\ i = 1, 2, \ldots, k, x \in X$$

where X is a nonempty subset of $\mathbb{R}^n$, and f and $g_i (i = 1, \ldots, k)$ denote real-valued function on X. Denote the set of all feasible solutions for (P) by

$$E = \{x : g_i(x) \leq 0, i = 1, 2, \ldots, k, x \in X\}$$

Theorem 5.3 *Suppose that g_i is G_i-semipreinvex with respect to η on X for $i = 1, 2, \ldots, k$. Then the set of all feasible solutions E for (P) is semiinvex with respect to the same η.*

Proof Consider $x, y \in E$. Then

$$g_i(x) \leq 0, g_i(y) \leq 0, i = 1, \ldots, k.$$

This, together with the G_i-semipreinvexity of g_i $(i = 1, \ldots, k)$ on X and $E \subset X$, follows that

$$G_i(g_i(y + \lambda\eta(x, y, \lambda))) \leq \lambda G_i(g_i(x)) + (1 - \lambda)G_i(g_i(y)) = G_i(0)$$

holds for all $\lambda \in [0, 1]$, $i = 1, \ldots, k$. Thus, $y + \lambda\eta(x, y, \lambda) \in E$. This shows that the set E is semiinvex with respect to η. □

Theorem 5.4 *Suppose that g_i is G_i-semipreincave with respect to η on X for $i = 1, 2, \ldots, k$. Then the set of all feasible solutions E for (P) is semiinvex with respect to the same η.*

Proof Consider $x, y \in E$. Then

$$g_i(x) \leq 0, g_i(y) \leq 0, i = 1, \ldots, k.$$

Note that G_i is increasing for $i = 1, \ldots, k$, we obtain that

$$G_i(g_i(x)) \geq G_i(0), G_i(g_i(y)) \geq G_i(0), i = 1, \ldots, k.$$

This, together with the G_i-semipreincavity of g_i $(i = 1, \ldots, k)$ on X and $E \subset X$, follows that

$$G_i(g_i(y + \lambda\eta(x, y, \lambda))) \geq \lambda G_i(g_i(x)) + (1 - \lambda)G_i(g_i(y)) = G_i(0)$$

holds for all $\lambda \in [0, 1]$, $i = 1, \ldots, k$. Thus, $y + \lambda\eta(x, y, \lambda) \in E$. This shows that the set E is semiinvex with respect to η. □

Theorem 5.5 *Let f and $g_i (i = 1, 2, \ldots, k)$ are G_i-semipreinvex with respect to η on X, respectively. Moreover, $\bar{x}$ be a local minimum in (P). Then $\bar{x}$ is a global minimum in (P).*

Proof The result can be obtained from Theorems 5.1 and 5.3. □

Theorem 5.6 *Let f and $g_i (i = 1, 2, \ldots, k)$ are G_i-semipreincave with respect to η on X, respectively. Moreover, $\bar{x}$ be a local minimum in (P). Then $\bar{x}$ is a global minimum in (P).*

Proof The result can be obtained from Theorems 5.1 and 5.4. □

Theorem 5.7 *Let $\bar{x}$ be a global minimum in (P) and $\eta : X \times X \times [0, 1] \to R^n$ be a vector-valued function with $\eta(x, y, \lambda) \neq 0$, for all $x, y \in E, x \neq y$. If f is strictly G_f-semipreinvex on X with respect to η and $g_i (i = 1, 2, \ldots, k)$ is G_i-semipreinvex with respect to η on X, then $\bar{x}$ is the unique optimal solution in (P).*

Proof By Theorem 5.3, we know that the set of feasible solutions E is semiinvex set with respect to η. By contradiction, let $\hat{x} \neq \bar{x}$ be an optimal solution in (P), then $\hat{x} \in E$ and

$$f(\hat{x}) = f(\bar{x}). \tag{9}$$

Since E is semiinvex with respect to η, we have

$$\bar{x} + \lambda\eta(\hat{x}, \bar{x}, \lambda) \in E$$

From the strictly G_f semipreincavity of f with respect to η on X and (9), it follows that

$$G_f(f(\bar{x} + \lambda\eta(\hat{x}, \bar{x}, \lambda))) < \lambda G_f(f(\bar{x})) + (1 - \lambda)G_f(f(\hat{x})) = G_f(f(\bar{x}))$$

holds for all $\lambda \in [0, 1]$. Then it means that $\bar{x}$ is not a global solution to the Problem (P). This is a contradiction to the assumptions. □

Theorem 5.8 *Let $\bar{x}$ be a global minimum in (P) and $\eta : X \times X \times [0, 1] \to R^n$ be a vector-valued function with $\eta(x, y, \lambda) \neq 0$, for all $x, y \in E, x \neq y$. If f is strictly G_f-semipreincave on X with respect to η and $g_i (i = 1, 2, \ldots, k)$ is G_i-semipreincave with respect to η on X, then $\bar{x}$ is the unique optimal solution in (P).*

Proof By Theorem 5.4, we know that the set of feasible solutions E is semiinvex set with respect to η. On the contrary, let $\hat{x} \neq \bar{x}$ be an optimal solution in (P), then $\hat{x} \in E$ and

$$f(\hat{x}) = f(\bar{x}). \tag{10}$$

Since G is decreasing on its domain $I_f(x)$, then both $\bar{x}$ and $\hat{x}$ are optimal to the problem of minimizing $f(x)$ subject to $x \in E$. Thus, they are optimal to the problem of maximizing $G(f)(x)$ subject to $x \in E$, or they are optimal to the problem of minimizing $-G(f)(x)$ subject to $x \in E$.

Again, since E is semiinvex with respect to η, we have

$$\bar{x} + \lambda\eta(\hat{x}, \bar{x}, \lambda) \in E$$

From the strict G_f-semipreinvexity of f with respect to η on X and (10), it follows that

$$G_f(f(\bar{x} + \lambda\eta(\hat{x}, \bar{x}, \lambda))) > \lambda G_f(f(\bar{x})) + (1 - \lambda)G_f(f(\hat{x})) = G_f(f(\bar{x})), \lambda \in [0, 1],$$

or

$$-G_f(f(\bar{x} + \lambda\eta(\hat{x}, \bar{x}, \lambda))) < \lambda(-G_f(f(\bar{x}))) + (1 - \lambda)(-G_f(f(\hat{x}))) = -G_f(f(\bar{x}))$$

for $\lambda \in [0, 1]$. It means that $\bar{x}$ is not a global minimum to the problem of minimizing $-G(f)(x)$ subject to $x \in E$ or $\bar{x}$ is not a global minimum in (P). This is a contradiction to the assumption that $\bar{x}$ is a global minimum to the problem of minimizing $f(x)$ subject to $x \in E$.

6 Conclusions

In this paper, we have introduced some new kinds of generalized convexity, which include (semistrict) G-semipreinvexity and (semistrict) G-semipreincavity. From Example 2.6, Theorems 5.1 and 5.2, we can conclude that these new generalized convex functions constitute an important class of generalized convex functions in mathematical programming. Moreover, we have established the relationships among these new generalized convexity defined in this paper and the common generalized convexity introduced in the literature. Basing on these relationships and using the well-known results pertaining to the common generalized convex functions, we have obtained results about these new generalized convexity.

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References

1. Antczak, T.: Lipschitz r-invex functions and nonsmooth programming. Numer. Funct. Anal. Optim. **23**, 265–283 (2002)
2. Antczak, T.: Generalized (p, r)-invexity in mathematical programming. Numer. Funct. Anal. Optim. **24**, 437–453 (2003)
3. Antczak, T.: r-Preinvexity and r-invexity in mathematical programming. Comput. Math. Appl. **50**, 551–566 (2005)
4. Antczak, T.: Optimality and duality for nonsmooth multiobjective programming problems with $V - r$-invexity. J. Global Optim. **45**, 319–334 (2009)
5. Antczak, T.: New optimality conditions and duality results of G-type in differentiable mathematical programming. Nonlinear Anal. **66**, 1617–1632 (2007)
6. Antczak, T.: On G-invex multiobjective programming. Part I. Optimality. J. Global Optim. **43**(1), 97–109 (2009)
7. Antczak, T.: On G-invex multiobjective programming. Part II. Duality. J. Global Optim. **43**(1), 111–140 (2009)
8. Antczak, T.: G-pre-invex functions in mathematical programming. J. Comput. Appl. Math. **217**(1), 212–226 (2008)
9. Antczak, T.: Relationships between pre-invex concepts. Nonlinear Anal. **60**(2), 349–367 (2005)
10. Antczak, T.: Nonsmooth minimax programming under locally Lipschitz (ϕ, ρ)-invexity. Appl. Math. Comput. **217**(23), 9606–9624 (2011)
11. Avriel, M.: r-Convex functions. Math. Program. **2**, 309–323 (1972)
12. Bector, C.R.; Chandra, S.; Gupta, S.; Suneja, S.K.: Univex sets, functions and univex nonlinear programming, Generalized Convexity. Lecture Notes in Economics and mathematical Systems, Proc., pp. 3–18, Pecz, Hungary (1992)
13. Bector, C.R.; Singh, C.: B-vex functions. J. Optim. Theory Appl. **71(2)**, 237–253 (1991)
14. Bector, C.R.; Suneja, S.; Lalitha, C.S.: Generalized B-vex functions and generalized B-vex programming. J. Optim. Theory Appl. **76**(3), 561–576 (1993)
15. Ben-Israel, A.; Mond, B.: What is invexity? Bull. Aust.Math. Soc. B. **28**, 1–9 (1986)
16. Gulati, T.: Duality in nondifferentiable multiobjective fractional programming problem with generalized invexity. J. Appl. Math. Comput. **35**(1), 103–118 (2011)
17. Hanson, M.A.: On sufficiency of the Kuhn-Tucker conditions. J. Math. Anal. Appl. **80**, 545–550 (1981)
18. Ivanov, V.I.: Second-order invex functions in nonlinear programming. Optimization **61**(5), 489–503 (2012)
19. Jayswal, A.; Kumar, D.; Kumar, R.: Second order duality for nondifferentiable multiobjective programming problem involving (F, α, ϱ, d)-V-type I functions. Optim. Lett. **4**(2), 211–226 (2010)
20. Jayswal, A.; Kumar, R.; Kumar, D.: Multiobjective fractional programming problems involving $(p, r) - \rho - (\eta, \theta)$-invex function. J. Appl. Math. Comput. **39**(1), 35–51 (2012)
21. Long, X.J.; Peng, J.W.: Semi-B-preinvex functions. J. Optim. Theory Appl. **131**, 301–305 (2006)
22. Luo, H.Z.; Wu, H.X.: On the relationships between G-preinvex functions and semistrictly G-preinvex functions. J. Comput. Appl. Math. **222**(2), 372–380 (2008)
23. Mohan, S.R.; Neogy, S.K: On invex sets and preinvex functions. J. Math. Anal. Appl. **189**, 901–908 (1995)
24. Mandal, P.; Nahak, C.: Symmetric duality with $(p, r) - \rho - (\eta, \theta)$-invexity. Appl. Math. Comput. **217**, 8141C8148 (2011)
25. Noor, M.: Nonconvex functions and variational inequalities. J. Optim. Theory Appl. **87**(3), 615–630 (1995)
26. Peng, J.W.; Zhu, D.L.: Semistrictly B-preinvex functions. Acta Math. Sci. A. **26**(2), 200–206 (2006)
27. Pini, R.: Invexity and generalized convexity. Optimization **22**(4), 513–525 (1991)
28. Pini, R.; Singh, C.: (ϕ_1, ϕ_2) convexity. Optimization **40**(2), 103–120 (1997)
29. Pini, R.; Singh, C.: (ϕ_1, ϕ_2) optimality and duality under differentiability. Optimization **41**(2), 101–116 (1997)
30. Suneja, S.K.; Singh, C.; Bector, C.R.: Generalization of preinvex and B-vex functions. J. Optim. Theory Appl. **76**, 577–587 (1993)
31. Stefanescu, M.V.; Stefanescu, A.: On semi-infinite minmax programming with generalized invexity. Optimization **61**(11), 1307–1319 (2012)
32. Weir, T.; Mond, B.: Pre-invex functions in multiple objective optimization. J. Math. Anal. Appl. **136**, 29–38 (1988)
33. Yang, X.M.; Li, D.: Semistrictly preinvex functions. J. Math. Anal. Appl. **258**(1), 287–308 (2001)
34. Yang, X.M.; Yang, X.Q.; Teo, K.L.: Explicitly B-preinvex functions. J. Comput. Appl. Math. **146**(1), 25–36 (2002)
35. Yang, X.Q.; Chen, G.Y.: A Class of nonconuex functions and prevariational inequalities. J. Math. Anal. Appl. **169**, 359–373 (1992)

36. Yang, X.M.; Yang, X.Q.; Teo, K.L.: On properties of semipreinvex functions. Bull. Aust. Math. Soc. **68**, 449–459 (2003)
37. Zhao, K.; Liu, X.; Chen Z.: A class of r-semipreinvex functions and optimality in nonlinear programming. J. Global Optim. **49**(1), 37–47 (2011)