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# On strongly $\varphi_h$-convex functions in inner product spaces

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**Abstract** In this paper, we introduce the notion of strongly $\varphi_h$-convex functions with respect to $c > 0$ and present some properties and representation of such functions. We obtain a characterization of inner product spaces involving the notion of strongly $\varphi_h$-convex functions. Finally, a version of Hermite–Hadamard-type inequalities for strongly $\varphi_h$-convex functions is established.

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**الملخص**

نقدم في هذه الورقة مفهوم الدوال المحدبة-$\varphi_h$ بقوة بالنسبة لـ $c > 0$ ونعرض بعض خصائص وتمثيلات هذه الدوال. نحصل على تمييزات لفضاءات ضرب داخلي تتضمن مفهوم الدوال المحدبة-$\varphi_h$ بقوة. أخيراً، نثبت نسخة من نوع متراجحات هيرميت – هدمارد للدوال المحدبة-$\varphi_h$ بقوة.

## 1 Introduction

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are very important in the literature (see, e.g., [7], [13, p. 137]). These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval $I$ of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

The inequality (1) has evoked the interest of many mathematicians. Especially in the last three decades, numerous generalizations, variants and extensions of this inequality have been obtained (e.g., [2,3,6–10,13,19,20,24], and the references cited therein).

Let $I$ be an interval in $\mathbb{R}$ and $h : (0, 1) \to (0, \infty)$ be a given function. A function $f : I \to (0, \infty)$ is said to be $h$-convex if

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y) \tag{2}$$

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for all $x, y \in I$ and $t \in (0, 1)$ [21]. This notion unifies and generalizes the known classes of functions, $s$-convex functions, Gudunova–Levin functions and $P$-functions, which are obtained by putting in (2), $h(t) = t$, $h(t) = t^s$, $h(t) = \frac{1}{t}$, and $h(t) = 1$, respectively. Many properties of them can be found, for instance, in [1,8,9,16–18,21,23].

Let us consider a function $\varphi : [a, b] \to [a, b]$ where $[a, b] \subset \mathbb{R}$. Youness have defined the $\varphi$-convex functions in [22]:

**Definition 1.1** A function $f : [a, b] \to \mathbb{R}$ is said to be $\varphi$-convex on $[a, b]$ if for every two points $x \in [a, b]$, $y \in [a, b]$ and $t \in [0, 1]$, the following inequality holds:

$$f(t\varphi(x) + (1-t)\varphi(y)) \leq tf(\varphi(x)) + (1-t)f(\varphi(y)).$$

Obviously, if the function $\varphi$ is the identity, then the classical convexity is obtained from the previous definition. Many properties of the $\varphi$-convex functions can be found, for instance, in [4,5,15,16,22].

Recall also that a function $f : I \to \mathbb{R}$ is called strongly convex with modulus $c > 0$, if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - ct(1-t)(x-y)^2$$

for all $x, y \in I$ and $t \in (0, 1)$. Strongly convex functions have been introduced by Polyak in [14] and they play an important role in optimization theory and mathematical economics. Various properties and applications of them can be found in the literature (see [1,11,12,14], and the references cited therein).

In this paper, we introduce the notion of strongly $\varphi_h$-convex functions defined in normed spaces and present some properties of them. In particular, we obtain a representation of strongly $\varphi_h$-convex functions in inner product spaces and, using the methods of [1,12,15], we give a characterization of inner product spaces, among normed spaces, which involves the notion of strongly $\varphi_h$-convex function. Finally, a version of Hermite–Hadamard-type inequalities for strongly $\varphi_h$-convex functions is presented. This result generalizes the Hermite–Hadamard-type inequalities obtained by Sarikaya in [15] for strongly $\varphi$-convex functions, and for $c = 0$, coincides with the Hermite–Hadamard inequalities for $\varphi_h$-convex functions proved by Sarikaya in [16].

## 2 Main result

In what follows, $(X, \|.\|)$ denotes a real normed space, $D$ stands for a convex subset of $X$, $\varphi : D \to D$ is a given function and $c$ is a positive constant. Let $h : (0, 1) \to (0, \infty)$ be a given function. We say that a function $f : D \to (0, \infty)$ is strongly $\varphi_h$-convex with modulus $c$ if

$$f(t\varphi(x) + (1-t)\varphi(y)) \leq h(t)f(\varphi(x)) + h(1-t)f(\varphi(y)) - ct(1-t)\|\varphi(x) - \varphi(y)\|^2 \quad (3)$$

for all $x, y \in D$ and $t \in (0, 1)$. In particular, if $f$ satisfies (3) with $h(t) = t$, $h(t) = t^s$ $(s \in (0, 1))$, $h(t) = \frac{1}{t}$, and $h(t) = 1$, then $f$ is said to be strongly $\varphi$-convex, strongly $\varphi_s$-convex, strongly $\varphi$-Gudunova–Levin function and strongly $\varphi$-$P$-function, respectively. The notion of $\varphi_h$-convex function corresponds to the case $c = 0$. We start with the following lemma which gives some relationships between strongly $\varphi_h$-convex functions and $\varphi_h$-convex functions in the case where $X$ is a real inner product space (that is, the norm $\|.\|$ is induced by an inner product: $\|.\| :=< x|x >$).

*Remark 2.1* Let $h : (0, 1) \to (0, \infty)$ be a given function such that $h(t) \geq t$ for all $t \in (0, 1)$. If $f$ is strongly $\varphi$-convex on $I$, then for $x, y \in I$ and $t \in (0, 1)$,

$$\begin{aligned} f(t\varphi(x) + (1-t)\varphi(y)) &\leq tf(\varphi(x)) + (1-t)f(\varphi(y)) - ct(1-t)\|\varphi(x) - \varphi(y)\|^2 \\ &\leq h(t)f(\varphi(x)) + h(1-t)f(\varphi(y)) - ct(1-t)\|\varphi(x) - \varphi(y)\|^2, \end{aligned}$$

i.e., $f : I \to [0, \infty)$ is strongly $\varphi_h$-convex.

**Lemma 2.2** *Let $h_1, h_2 : (0, 1) \to (0, \infty)$ be given functions such that $h_2(t) \leq h_1(t)$ for all $t \in (0, 1)$. If $f$ is strongly $\varphi_{h_2}$-convex on $I$, then for $x, y \in I$, $f$ is strongly $\varphi_{h_1}$-convex on $I$.*

*Proof* Since $f$ is strongly $\varphi_{h_2}$-convex on $I$, thus for $x, y \in I$ and $t \in (0, 1)$, we have

$$\begin{aligned} f(t\varphi(x) + (1-t)\varphi(y)) &\le h_2(t)f(\varphi(x)) + h_2(1-t)f(\varphi(y)) - ct(1-t)\|\varphi(x) - \varphi(y)\|^2 \\ &\le h_1(t)f(\varphi(x)) + h_1(1-t)f(\varphi(y)) - ct(1-t)\|\varphi(x) - \varphi(y)\|^2. \end{aligned}$$

□

**Lemma 2.3** *Let $h : (0, 1) \to (0, \infty)$ be a given function. If $f, g : I \to [0, \infty)$ are strongly $\varphi_h$-convex functions on $I$ and $\alpha > 0$, then for all $t \in (0, 1)$ $f + g$ and $\alpha f$ are strongly $\varphi_h$-convex on $I$.*

*Proof* By definition of strongly $\varphi_h$-convexity, the proof is obvious. □

**Lemma 2.4** *Let $(X, \|.\|)$ be a real inner product space, $D$ be a convex subset of $X$ and $c$ be a positive constant and $\varphi : D \to D$. Assume that $h : (0, 1) \to (0, \infty)$ is a given function.*

(i) *If $h(t) \le t,\ t \in (0, 1)$ and a function $f : D \to (0, \infty)$ is strongly $\varphi_h$-convex with modulus $c$, then the function $g = f - c\|.\|^2$ is $\varphi_h$-convex.*
(ii) *If $h(t) \le t,\ t \in (0, 1)$ and the function $g = f - c\|.\|^2$ is $\varphi_h$-convex, then the function $f : D \to (0, \infty)$ is strongly $\varphi$-convex with modulus $c$.*
(iii) *If $h(t) \ge t,\ t \in (0, 1)$ and a function $f : D \to (0, \infty)$ is strongly $\varphi_h$-convex with modulus $c$, then the function $g = f - c\|.\|^2$ is $\varphi_h$-convex.*

*Proof* (i) Assume that $f$ is strongly $\varphi_h$-convex with modulus $c$. Using properties of the inner product and assumption $h(t) \le t,\ t \in (0, 1)$, we obtain

$$\begin{aligned} &g(t\varphi(x) + (1-t)\varphi(y)) \\ &\quad = f(t\varphi(x) + (1-t)\varphi(y)) - c\|t\varphi(x) + (1-t)\varphi(y)\|^2 \\ &\quad \le h(t)f(\varphi(x)) + h(1-t)f(\varphi(y)) - ct(1-t)\|\varphi(x) - \varphi(y)\|^2 - c\|t\varphi(x) + (1-t)\varphi(y)\|^2 \\ &\quad \le h(t)f(\varphi(x)) + h(1-t)f(\varphi(y)) - c\left(t(1-t)\left[\|\varphi(x)\|^2 - 2 < \varphi(x)|\varphi(y) > + \|\varphi(y)\|^2\right]\right. \\ &\qquad \left. + \left[t^2\|\varphi(x)\|^2 + 2t(1-t) < \varphi(x)|\varphi(y) > + (1-t)^2\|\varphi(y)\|^2\right]\right) \\ &\quad = h(t)f(\varphi(x)) + h(1-t)f(\varphi(y)) - ct\|\varphi(x)\|^2 - c(1-t)\|\varphi(y)\|^2 \\ &\quad \le h(t)f(\varphi(x)) + h(1-t)f(\varphi(y)) - ch(t)\|\varphi(x)\|^2 - ch(1-t)\|\varphi(y)\|^2 \\ &\quad = h(t)g(\varphi(x)) + h(1-t)g(\varphi(y)) \end{aligned}$$

which gives that $g$ is a $\varphi_h$-convex function.

(ii) Since $g$ is a $\varphi_h$-convex function, and using the assumption $h(t) \le t,\ t \in (0, 1)$, we get

$$\begin{aligned} f(t\varphi(x) + (1-t)\varphi(y)) &= g(t\varphi(x) + (1-t)\varphi(y)) + c\|t\varphi(x) + (1-t)\varphi(y)\|^2 \\ &\le h(t)g(\varphi(x)) + h(1-t)g(\varphi(y)) \\ &\quad + c\left(t^2\|\varphi(x)\|^2 + 2t(1-t) < \varphi(x)|\varphi(y) > + (1-t)^2\|\varphi(y)\|^2\right) \\ &\le t\left[g(\varphi(x)) + c\|\varphi(x)\|^2\right] + (1-t)\left[g(\varphi(y)) + c\|\varphi(y)\|^2\right] \\ &\quad - ct(1-t)\left[\|\varphi(x)\|^2 - 2 < \varphi(x)|\varphi(y) > + \|\varphi(y)\|^2\right] \\ &= tf(\varphi(x)) + (1-t)f(\varphi(y)) - ct(1-t)\|\varphi(x) - \varphi(y)\|^2 \end{aligned}$$

which shows that $f$ is strongly $\varphi$-convex with modulus $c$.

(iii) In a similar way, we can prove it. This completes the proof. □

The following example shows that the assumption that $X$ is an inner product space is essential in the above lemma.

*Example.* Let $X = \mathbb{R}^2$ and $h(t) = t,\ t \in (0, 1)$. Let us consider a function $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$, defined by $\varphi(x) = x$ for every $x \in \mathbb{R}^2$ and $\|x\| = \max\{|x_1|, |x_2|\}$ for $x = (x_1, x_2)$. Take $f = \|.\|^2$. Then $g = f - \|.\|^2$ is $\varphi_h$-convex being the zero function. However, $f$ is not strongly $\varphi_h$-convex with modulus 1. Indeed, for $x = (1, 0)$ and $y = (0, 1)$, we have

$$f\left(\frac{x+y}{2}\right) = \frac{1}{2} \ge \frac{3}{4} = \frac{f(x) + f(y)}{2} - \frac{1}{4}\|x - y\|^2$$

which contradicts (3).

The assumption that $X$ is an inner product space in Lemma 2.4 is essential. Moreover, it appears that the fact that for every $\varphi_h$-convex function $g : X \to \mathbb{R}$ the function $f = g + c\,\|.\|^2$ is strongly $\varphi_h$-convex characterizes inner product spaces among normed spaces. Similar characterizations of inner product spaces by strongly convex, strongly $h$-convex and strongly $\varphi$-convex functions are presented in [1,12,15], respectively.

**Theorem 2.5** *Let $(X, \|.\|)$ be a real normed space, $D$ be a convex subset of $X$ and $\varphi : D \to D$. Assume that $h : (0, 1) \to (0, \infty)$ and $h\left(\frac{1}{2}\right) = \frac{1}{2}$. Then the following conditions are equivalent.*

- *(i)* *$(X, \|.\|)$ is a real inner product.*
- *(ii)* *For every $c > 0$, $h(t) \geq t,\ t \in (0, 1)$, and for every $\varphi_h$-convex function $g : D \to (0, \infty)$ defined on $D$, the function $f = g + c\,\|.\|^2$ is strongly $\varphi_h$-convex with modulus $c$.*
- *(iii)* *$\|.\|^2 : X \to (0, \infty)$ is strongly $\varphi_h$-convex with modulus 1.*

*Proof* The implication (i) $\Rightarrow$ (ii) follows by Lemma 2.4. To see that (ii) $\Rightarrow$ (iii) take $g = 0$. Clearly, $g$ is $\varphi_h$-convex function, whence $f = c\,\|.\|^2$ is strongly $\varphi_h$-convex with modulus $c$. Consequently, $\|.\|^2$ is strongly $\varphi_h$-convex with modulus 1. Finally, to prove iii) $\Rightarrow$ i) observe that by the strongly $\varphi_h$-convexity of $\|.\|^2$ and assumption $h\left(\frac{1}{2}\right) = \frac{1}{2}$, we obtain

$$\left\|\frac{\varphi(x)+\varphi(y)}{2}\right\|^2 \leq \frac{\|\varphi(x)\|^2}{2} + \frac{\|\varphi(y)\|^2}{2} - \frac{1}{4}\,\|\varphi(x)+\varphi(y)\|^2$$

and hence

$$\|\varphi(x)+\varphi(y)\|^2 \leq 2\,\|\varphi(x)\|^2 + 2\,\|\varphi(y)\|^2 \tag{4}$$

for all $x, y \in X$. Now, putting $u = \varphi(x) + \varphi(y)$ and $v = \varphi(x) - \varphi(y)$ in (4), we have

$$2\,\|u\|^2 + 2\,\|v\|^2 \leq \|u+v\|^2 + \|u-v\|^2 \tag{5}$$

for all $u, v \in X$.

Conditions (4) and (5) mean that the norm $\|.\|^2$ satisfies the parallelogram law, which implies, by the classical Jordan–Von Neumann theorem, that $(X, \|.\|)$ is an inner product space. This completes the proof. □

Now, we give new Hermite–Hadamard-type inequalities for strongly $\varphi_h$-convex functions with modulus $c$ as follows:

**Theorem 2.6** *Let $h : (0, 1) \to (0, \infty)$ be a given function. If a function $f : I \to (0, \infty)$ is Lebesgue integrable and strongly $\varphi_h$-convex with modulus $c > 0$ for the continuous function $\varphi : [a, b] \to [a, b]$, then*

$$\begin{aligned}
&\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) + \frac{c}{24h\left(\frac{1}{2}\right)}(\varphi(a)-\varphi(b))^2 \\
&\leq \frac{1}{\varphi(b)-\varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x)\mathrm{d}x \\
&\leq [f(\varphi(a)) + f(\varphi(b))] \int_0^1 h(t)\mathrm{d}t - \frac{c}{6}(\varphi(a)-\varphi(b))^2.
\end{aligned} \tag{6}$$

*Proof* From the strong $\varphi_h$-convexity of $f$, we have

$$\begin{aligned}
f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) &= f\left(\frac{t\varphi(a)+(1-t)\varphi(b)}{2} + \frac{(1-t)\varphi(a)+t\varphi(b)}{2}\right) \\
&\leq h\left(\frac{1}{2}\right)[f\left(t\varphi(a)+(1-t)\varphi(b)\right) + f((1-t)\varphi(a)+t\varphi(b))] \\
&\quad - \frac{c}{4}(1-2t)^2(\varphi(a)-\varphi(b))^2.
\end{aligned}$$

Integrating the above inequality over the interval (0, 1), we obtain

$$f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)+\frac{c}{12}(\varphi(a)-\varphi(b))^2$$
$$\leq h\left(\frac{1}{2}\right)\left[\int_0^1 f(t\varphi(a)+(1-t)\varphi(b))\,\mathrm{d}t+\int_0^1 f((1-t)\varphi(a)+t\varphi(b))\,\mathrm{d}t\right].$$

In the first integral, we substitute $x=t\varphi(a)+(1-t)\varphi(b)$. Meanwhile, in the second integral, we also use the substitution $x=(1-t)\varphi(a)+t\varphi(b)$, we obtain

$$f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)+\frac{c}{12}(\varphi(a)-\varphi(b))^2\leq\frac{2h(\frac{1}{2})}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(x)\mathrm{d}x.$$

To prove the second inequality, we start from the strong $\varphi_h$-convexity of $f$ meaning that for every $t\in(0,1)$, one has

$$f(t\varphi(a)+(1-t)\varphi(b))\leq h(t)f(\varphi(a))+h(1-t)f(\varphi(b))-ct(1-t)(\varphi(a)-\varphi(b))^2.$$

Integrating the above inequality over the interval (0, 1), we get

$$\int_0^1 f(t\varphi(a)+(1-t)\varphi(b))\mathrm{d}t\leq[f(\varphi(a))+f(\varphi(b))]\int_0^1 h(t)\mathrm{d}t-c(\varphi(a)-\varphi(b))^2\int_0^1 t(1-t)\mathrm{d}t.$$

The previous substitution in the first side of this inequality leads to

$$\frac{1}{(\varphi(a)-\varphi(b))}\int_{\varphi(b)}^{\varphi(a)} f(x)\,\mathrm{d}x\leq[f(\varphi(a))+f(\varphi(b))]\int_0^1 h(t)\mathrm{d}t-\frac{c}{6}(\varphi(a)-\varphi(b))^2$$

which gives the second inequality of (6). This completes the proof. □

*Remark 2.7* If $h(t)=t,\ t\in(0,1)$, then the inequalities (6) coincide with the Hermite–Hadamard type inequalities for strongly $\varphi$-convex functions proved by Sarikaya in [15].

**Corollary 2.8** *Under the assumptions of Theorem 2.6 with* $h(t)=t^s\ (s\in(0,1)),\ t\in(0,1)$*, we have*

$$2^{s-1}f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)+\frac{c2^s}{24}(\varphi(a)-\varphi(b))^2$$
$$\leq\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(x)\mathrm{d}x$$
$$\leq\frac{f(\varphi(a))+f(\varphi(b))}{s+1}-\frac{c}{6}(\varphi(a)-\varphi(b))^2.$$

These inequalities are associated Hermite–Hadamard type inequalities for strongly $\varphi_s$-convex functions.

**Corollary 2.9** *Under the assumptions of Theorem 2.6 with* $h(t)=\frac{1}{t},\ t\in(0,1)$*, we have*

$$\frac{1}{4}f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)+\frac{c}{48}(\varphi(a)-\varphi(b))^2\leq\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(x)\mathrm{d}x\ \ (\leq\infty).$$

This inequality is associated Hermite–Hadamard type inequalities for strongly $\varphi$-Godunova–Levin functions.

**Corollary 2.10** *Under the assumptions of Theorem 2.6 with* $h(t) = 1,\ t \in (0, 1)$*, we have*

$$\frac{1}{2} f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) + \frac{c}{24}(\varphi(a)-\varphi(b))^2 \le \frac{1}{\varphi(b)-\varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x)\mathrm{d}x \le f(\varphi(a)) + f(\varphi(b)) - \frac{c}{6}(\varphi(a)-\varphi(b))^2.$$

These inequalities are associated Hermite–Hadamard type inequalities for strongly $\varphi$-$P$-convex functions.

**Theorem 2.11** *Let* $h : (0, 1) \to (0, \infty)$ *be a given function. If* $f : I \to (0, \infty)$ *is Lebesgue integrable and strongly* $\varphi_h$*-convex with modulus* $c > 0$ *for the continuous function* $\varphi : [a, b] \to [a, b]$*, then*

$$\begin{aligned} &\frac{1}{\varphi(b)-\varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x)\, f(a+b-x)\, \mathrm{d}x \\ &\le \left[f^2(\varphi(a)) + f^2(\varphi(b))\right] \int_0^1 h(t)h(1-t)\mathrm{d}t + 2f(\varphi(a))f(\varphi(b)) \int_0^1 h^2(t)\mathrm{d}t \\ &-2c\,(\varphi(a)-\varphi(b))^2\,[f(\varphi(a)) + f(\varphi(b))] \int_0^1 t(1-t)h(t)\mathrm{d}t + \frac{c^2}{30}(\varphi(a)-\varphi(b))^4. \end{aligned} \tag{7}$$

*Proof* Since $f$ is strongly $\varphi_h$-convex with respect to $c > 0$, we have that for all $t \in (0, 1)$

$$f(t\varphi(a) + (1-t)\varphi(b)) \le h(t)f(\varphi(a)) + h(1-t)f(\varphi(b)) - ct(1-t)(\varphi(a)-\varphi(b))^2 \tag{8}$$

and

$$f((1-t)\varphi(a) + t\varphi(b)) \le h(1-t)f(\varphi(a)) + h(t)f(\varphi(b)) - ct(1-t)(\varphi(a)-\varphi(b))^2. \tag{9}$$

Multiplying both sides of (8) by (9), it follows that

$$\begin{aligned} &f(t\varphi(a) + (1-t)\varphi(b))f((1-t)\varphi(a) + t\varphi(b)) \\ &\le h(t)h(1-t)\left[f^2(\varphi(a)) + f^2(\varphi(b))\right] + \left(h^2(t) + h^2(1-t)\right) f(\varphi(a))f(\varphi(b)) \\ &\quad -ct(1-t)(\varphi(a)-\varphi(b))^2\,[f(\varphi(a)) + f(\varphi(b))]\,[h(t) + h(1-t)] \\ &\quad +c^2t^2(1-t)^2(\varphi(a)-\varphi(b))^4. \end{aligned} \tag{10}$$

Integrating the inequality (10) with respect to $t$ over $(0, 1)$, we obtain

$$\begin{aligned} &\int_0^1 f(t\varphi(a) + (1-t)\varphi(b))f((1-t)\varphi(a) + t\varphi(b))\mathrm{d}t \\ &\le \left[f^2(\varphi(a)) + f^2(\varphi(b))\right] \int_0^1 h(t)h(1-t)\mathrm{d}t + 2f(\varphi(a))f(\varphi(b)) \int_0^1 h^2(t)\mathrm{d}t \\ &-2c\,(\varphi(a)-\varphi(b))^2\,[f(\varphi(a)) + f(\varphi(b))] \int_0^1 t(1-t)h(t)\mathrm{d}t \\ &+\frac{c^2}{30}(\varphi(a)-\varphi(b))^4. \end{aligned}$$

If we change the variable $x := t\varphi(a) + (1-t)\varphi(b),\ t \in (0, 1)$, we get the required inequality in (7). This proves the theorem. □

**Theorem 2.12** *Let $h:(0,1)\to(0,\infty)$ be a given function. If $f,g:I\to(0,\infty)$ is Lebesgue integrable and strongly $\varphi_h$-convex with modulus $c>0$ for the continuous function $\varphi:[a,b]\to[a,b]$, then*

$$\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(x)\mathrm{d}x \le M(a,b)\int_0^1 h^2(t)\mathrm{d}t + N(a,b)\int_0^1 h(t)h(1-t)\mathrm{d}t$$
$$-c\left(\varphi(a)-\varphi(b)\right)^2 S(a,b)\int_0^1 t\left(1-t\right)h(t)\mathrm{d}t + \frac{c^2}{30}\left(\varphi(a)-\varphi(b)\right)^4 \quad (11)$$

*where*

$$M(a,b) = f(\varphi(a))g(\varphi(a)) + f(\varphi(b))g(\varphi(b))$$
$$N(a,b) = f(\varphi(a))g(\varphi(b)) + f(\varphi(b))g(\varphi(a))$$
$$S(a,b) = f\left(\varphi(a)\right) + f\left(\varphi(b)\right) + g\left(\varphi(a)\right) + g\left(\varphi(b)\right).$$

*Proof* Since $f,g:I\to(0,\infty)$ is strongly $\varphi_h$-convex with modulus $c>0$, we have

$$f\left(t\varphi(a)+(1-t)\varphi(b)\right) \le h(t)f\left(\varphi(a)\right) + h(1-t)f\left(\varphi(b)\right) - ct\left(1-t\right)\left(\varphi(a)-\varphi(b)\right)^2 \quad (12)$$
$$g\left(t\varphi(a)+(1-t)\varphi(b)\right) \le h(t)g\left(\varphi(a)\right) + h(1-t)g\left(\varphi(b)\right) - ct\left(1-t\right)\left(\varphi(a)-\varphi(b)\right)^2. \quad (13)$$

Multiplying both sides of (12) by (13), it follows that

$$\begin{aligned} & f\left(t\varphi(a)+(1-t)\varphi(b)\right) g\left(t\varphi(a)+(1-t)\varphi(b)\right) \\ & \le h^2(t)f\left(\varphi(a)\right)g\left(\varphi(a)\right) + h^2(1-t)f(\varphi(b))f(\varphi(b)) \\ & \quad + h(t)h(1-t)\left[f(\varphi(a))g(\varphi(b)) + f(\varphi(b))g(\varphi(a))\right] \\ & \quad - ct\left(1-t\right)h(t)\left(\varphi(a)-\varphi(b)\right)^2\left[f\left(\varphi(a)\right)+g\left(\varphi(a)\right)\right] \\ & \quad - ct\left(1-t\right)h(1-t)\left(\varphi(a)-\varphi(b)\right)^2\left[f\left(\varphi(b)\right)+g\left(\varphi(b)\right)\right] \\ & \quad + c^2t^2\left(1-t\right)^2\left(\varphi(a)-\varphi(b)\right)^4. \end{aligned}$$

Integrating the above inequality over the interval (0, 1), we get

$$\begin{aligned} & \int_0^1 f\left(t\varphi(a)+(1-t)\varphi(b)\right) g\left(t\varphi(a)+(1-t)\varphi(b)\right)\mathrm{d}t \\ & \le \left[f\left(\varphi(a)\right)g\left(\varphi(a)\right) + f(\varphi(b))f(\varphi(b))\right]\int_0^1 h^2(t)\mathrm{d}t \\ & \quad + \left[f(\varphi(a))g(\varphi(b)) + f(\varphi(b))g(\varphi(a))\right]\int_0^1 h(t)h(1-t)\mathrm{d}t \\ & \quad - c\left(\varphi(a)-\varphi(b)\right)^2\left[f\left(\varphi(a)\right)+g\left(\varphi(a)\right)+f\left(\varphi(b)\right)+g\left(\varphi(b)\right)\right]\int_0^1 t\left(1-t\right)h(t)\mathrm{d}t \\ & \quad + c^2\left(\varphi(a)-\varphi(b)\right)^4\int_0^1 t^2\left(1-t\right)^2\mathrm{d}t. \end{aligned}$$

In the first integral, we substitute $x=t\varphi(a)+(1-t)\varphi(b)$ and simple integrals calculated, we obtain the required inequality in (11). □

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