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Strong convergence to fixed points of non-Lipschitzian mappings

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Abstract In this paper, we first show the strong convergence of the modified Moudafi iteration process when E is a real uniformly convex Banach space, S is AQT self-mapping and T is ANI self-mapping satisfying Condition (**B**). Next, we show the strong convergence of the modified Mann iteration process when T is ANI self-mapping satisfying Condition (**A**), which generalizes the result due to Kim (J. Nonlinear Convex Anal. 13(3):449–457, 2012). Finally, we show the strong convergence of the Schu iteration process when T is ANI self-mapping satisfying Condition (**A**), which generalizes the result due to Rhoades (J. Math. Anal. Appl. 183:118–120, 1994).

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الملخص

في هذه الورقة، نثبت أولاً التقارب القوي لعملية تكرار مودافي المعدَّلة عندما يكون E فضاءَ باناخ حقيقياً محدباً بانتظام، وَ S راسمَ AQT ذاتياً، وَ T راسمَ ANI ذاتياً يحقق شرط (**B**). بعد ذلك، نثبت التقارب القوي لعملية تكرار مان المعدَّلة عندما يكون T راسمَ ANI ذاتياً يحقق شرط (**A**)، حيث يعمم ذلك النتيجة المنسوبة لِكِمْ [1]. في الختام، نثبت التقارب القوي لعملية تكرار شو عندما يكون T راسمَ ANI ذاتياً يحقق شرط (**A**)، حيث يعمم ذلك النتيجة المنسوبة لِرودِسْ [2].

1 Introduction

Let C be a nonempty closed convex subset of a real Banach space E and let T be a mapping of C into itself. Then, T is said to be *asymptotically nonexpansive* [3] if there exists a sequence $\{k_n\}$, $k_n \geq 1$, with $\lim_{n\to\infty} k_n = 1$, such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$

for all $x, y \in C$ and $n \geq 1$. In particular, if $k_n = 1$ for all $n \geq 1$, T is said to be *nonexpansive*. We denote by $F(T)$ the set of all fixed points of T, i.e., $F(T) = \{x \in C : Tx = x\}$. T is said to be *asymptotically nonexpansive in the intermediate sense* (in brief, ANI) [1] provided T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0.$$

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T is said to be *asymptotically quasi-nonexpansive type* (in brief, AQT) ([6], cf., [7]) provided

$$\limsup_{n\to\infty} \sup_{x\in C, w\in F(T)} (\|T^n x - w\| - \|x - w\|) \le 0.$$

For a mapping T of C into itself, we consider the following iteration scheme: $x_1 \in C$,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n \tag{1.1}$$

for all $n \ge 1$, where $\{\alpha_n\}$ is a real sequence in [0, 1]. Such an iteration scheme was introduced by Schu [11] (cf. Mann [8]). For a mapping T of C into itself, we consider the following iteration scheme: $x_1 \in C$,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n[\beta_n x_n + (1-\beta_n)T^n x_n] \tag{1.2}$$

for all $n \ge 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in [0, 1]. If $\beta_n = 0$ for all $n \ge 1$, then (1.2) reduces to an iteration scheme (1.1). For two mappings S, T of C into itself, we consider the following modified Moudafi iteration scheme (cf. Moudafi [9]): $x_1 \in C$,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n[\beta_n S^n x_n + (1-\beta_n)T^n x_n] \tag{1.3}$$

for all $n \ge 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in [0, 1]. If $S = I$, then (1.3) reduces to an iteration scheme (1.2).

Recently, Kim [5] proved the following result. Let E be a real uniformly convex Banach space and C be a nonempty closed convex subset of E, and let T be a nonexpansive mapping of C into itself satisfying Condition (**A**) with $F(T) \neq \emptyset$. Suppose that for any x_1 in C, the sequence $\{x_n\}$ is defined by (1.2) such that $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$ and $\sum_{n=1}^{\infty} \beta_n < \infty$. Then, $\{x_n\}$ converges strongly to some fixed point of T, which generalized the result due to Senter-Dotson [12].

On the other hand, Rhoades [10] proved the following result. Let E be a real uniformly convex Banach space and C be a nonempty bounded closed convex subset of E, and let $T : C \to C$ be a completely continuous asymptotically nonexpansive mapping with $\{k_n\}$ satisfying $k_n \ge 1$, $\sum_{n=1}^{\infty}(k_n^r - 1) < \infty$, $r = \max\{2, p\}$. Then, for any $x_1 \in C$, the sequence $\{x_n\}$ defined by (1.1), where $\{\alpha_n\}$ satisfies $a \le \alpha_n \le 1 - a$ for all $n \ge 1$ and some $a > 0$ converge strongly to some fixed point of T, which extended the result of Schu [11] to uniformly convex Banach spaces.

In this paper, we first prove that the iteration $\{x_n\}$ defined by (1.3) converges strongly to a common fixed point of S and T, when E is a real uniformly convex Banach space, $S : C \to C$ is AQT and $T : C \to C$ is ANI satisfying Condition (**B**). Next, we prove that if $T : C \to C$ is ANI satisfying Condition (**A**), the iteration $\{x_n\}$ defined by (1.2) converges strongly to some fixed point of T, which generalizes the result due to Kim [5]. Finally, we prove that if $T : C \to C$ is ANI satisfying Condition (**A**), the iteration $\{x_n\}$ defined by (1.1) converges strongly to some fixed point of T, which generalizes the result due to Rhoades [10].

2 Preliminaries

Throughout this paper we denote by E a real Banach space. A Banach space E is said to be *uniformly convex* if the modulus of convexity $\delta_E = \delta_E(\epsilon)$, $0 \le \epsilon \le 2$, of E defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : x, y \in E, \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \epsilon\right\}$$

satisfies the inequality $\delta_E(\epsilon) > 0$ for every $\epsilon \in (0, 2]$. When $\{x_n\}$ is a sequence in E, then $x_n \to x$ will denote strong convergence of the sequence $\{x_n\}$ to x.

Condition 2.1 [12] *A mapping $T : C \to C$ with $F(T) \neq \emptyset$ is said to satisfy Condition* (**A**) *if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that*

$$\|x - Tx\| \ge f(d(x, F(T)))$$

for all $x \in C$, where $d(x, F(T)) = \inf_{z\in F(T)} \|x - z\|$.

Condition 2.2 [2] *Two mappings $S, T : C \to C$ with $\mathbf{F} = F(S) \bigcap F(T) \neq \emptyset$, where C is a subset of E, are said to satisfy Condition* (**B**) *if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that either $\|x - Sx\| \ge f(d(x, \mathbf{F}))$ or $\|x - Tx\| \ge f(d(x, \mathbf{F}))$ for all $x \in C$, where $d(x, \mathbf{F}) = \inf_{z\in \mathbf{F}} \|x - z\|$.*

3 Strong convergence theorems

We first begin with the following lemma.

Lemma 3.1 [14] *Let* $\{a_n\}$ *and* $\{b_n\}$ *be sequences of nonnegative real numbers such that* $\sum_{n=1}^{\infty} b_n < \infty$ *and*

$$a_{n+1} \leq a_n + b_n$$

for all $n \geq 1$. *Then,* $\lim_{n\to\infty} a_n$ *exists.*

Lemma 3.2 [4] *Let* E *be a uniformly convex Banach space. Let* $x, y \in E$. *If* $\|x\| \leq 1$, $\|y\| \leq 1$, *and* $\|x - y\| \geq \epsilon > 0$, *then* $\|\lambda x + (1-\lambda)y\| \leq 1 - 2\lambda(1-\lambda)\delta(\epsilon)$ *for* λ *with* $0 \leq \lambda \leq 1$.

Lemma 3.3 *Let* E *be a uniformly convex Banach space. Let* C *be a nonempty closed convex subset of* E *and let* $S, T : C \to C$ *be AQT with* $\mathbf{F} = F(S) \cap F(T) \neq \emptyset$. *For* $z \in \mathbf{F}$, *put*

$$c_n = \sup_{x\in C}(\|S^n x - z\| - \|x - z\|) \vee \sup_{x\in C}(\|T^n x - z\| - \|x - z\|) \vee 0$$

for all $n \geq 1$. *Suppose that* $\sum_{n=1}^{\infty} c_n < \infty$ *and the sequence* $\{x_n\}$ *is defined by* (1.3). *Then,* $\lim_{n\to\infty} \|x_n - z\|$ *exists.*

Proof Since

$$\begin{aligned}
\|x_{n+1} - z\| &= \|(1-\alpha_n)x_n + \alpha_n[\beta_n S^n x_n + (1-\beta_n)T^n x_n] - z\| \\
&\leq (1-\alpha_n)\|x_n - z\| + \alpha_n\|[\beta_n S^n x_n + (1-\beta_n)T^n x_n] - z\| \\
&\leq (1-\alpha_n)\|x_n - z\| + \alpha_n\{\beta_n\|S^n x_n - z\| + (1-\beta_n)\|T^n x_n - z\|\} \\
&\leq (1-\alpha_n)\|x_n - z\| + \alpha_n\{\beta_n\|x_n - z\| + \beta_n c_n + (1-\beta_n)\|x_n - z\| + (1-\beta_n)c_n\} \\
&= (1-\alpha_n)\|x_n - z\| + \alpha_n\{\|x_n - z\| + c_n\} \\
&\leq \|x_n - z\| + c_n
\end{aligned}$$

for all $n \geq 1$. By Lemma 3.1, we see that $\lim_{n\to\infty} \|x_n - z\|$ exists. □

Theorem 3.4 *Let* E *be a uniformly convex Banach space. Let* C *be a nonempty closed convex subset of* E *and let* $S : C \to C$ *be AQT and* $T : C \to C$ *be ANI with* $\mathbf{F} = F(S) \cap F(T) \neq \emptyset$. *Put*

$$c_n = \sup_{x\in C, w\in\mathbf{F}}(\|S^n x - w\| - \|x - w\|) \vee \sup_{x,y\in C}(\|T^n x - T^n y\| - \|x - y\|) \vee 0$$

for all $n \geq 1$. *Suppose that* $\sum_{n=1}^{\infty} c_n < \infty$ *and the sequence* $\{x_n\}$ *is defined by* (1.3) *such that* $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$ *and* $\sum_{n=1}^{\infty} \beta_n < \infty$. *Then,* $\liminf_{n\to\infty} \|x_n - Tx_n\| = 0$.

Proof For any $z \in \mathbf{F}$, by Lemma 3.3, $\lim_{n\to\infty} \|x_n - z\| (\equiv c)$ exists. If $c = 0$, then the conclusion is obvious. So, we assume $c > 0$. Put $y_n = \beta_n S^n x_n + (1-\beta_n)T^n x_n$. Since

$$\begin{aligned}
\|y_n - z\| &= \|\beta_n S^n x_n + (1-\beta_n)T^n x_n - z\| \\
&\leq \beta_n\|S^n x_n - z\| + (1-\beta_n)\|T^n x_n - z\| \\
&\leq \beta_n\{\|x_n - z\| + c_n\} + (1-\beta_n)\{\|x_n - z\| + c_n\} \\
&= \|x_n - z\| + c_n,
\end{aligned}$$

Using Lemma 3.2 and Takahashi [13], we obtain

$$\begin{aligned}
\|x_{n+1} - z\| &= \|(1-\alpha_n)x_n + \alpha_n y_n - z\| \\
&= \|(1-\alpha_n)(x_n - z) + \alpha_n(y_n - z)\| \\
&\leq \Big(\|x_n - z\| + c_n\Big)\left[1 - 2\alpha_n(1-\alpha_n)\delta_E\left(\frac{\|x_n - y_n\|}{\|x_n - z\| + c_n}\right)\right].
\end{aligned}$$

Hence, we obtain

$$2\alpha_n(1-\alpha_n)\Big(\|x_n - z\| + c_n\Big)\delta_E\left(\frac{\|x_n - y_n\|}{\|x_n - z\| + c_n}\right) \leq \|x_n - z\| - \|x_{n+1} - z\| + c_n.$$

Since δ_E is strictly increasing, continuous and by $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$, we obtain

$$\liminf_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.1}$$

By Lemma 3.3, $\{x_n\}$ is bounded and thus

$$\begin{aligned} \|y_n - T^n x_n\| &= \|\beta_n S^n x_n + (1-\beta_n)T^n x_n - T^n x_n\| \\ &= \beta_n \|S^n x_n - T^n x_n\| \\ &\le \beta_n M, \end{aligned}$$

where $M = \sup_{n\ge 1} \|T^n x_n - S^n x_n\| < \infty$. Since $\sum_{n=1}^{\infty} \beta_n < \infty$, we obtain

$$\lim_{n\to\infty} \|y_n - T^n x_n\| = 0. \tag{3.2}$$

Since

$$\|x_n - T^n x_n\| \le \|x_n - y_n\| + \|y_n - T^n x_n\|,$$

by (3.1) and (3.2), we obtain

$$\liminf_{n\to\infty} \|x_n - T^n x_n\| = 0. \tag{3.3}$$

Since

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|(1-\alpha_n)x_n + \alpha_n y_n - x_n\| \\ &= \alpha_n \|y_n - x_n\| \\ &\le \|y_n - T^n x_n\| + \|T^n x_n - x_n\|, \end{aligned}$$

by (3.2) and (3.3), we obtain

$$\liminf_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.4}$$

Since

$$\begin{aligned} \|y_{n+1} - z\| &= \|\beta_{n+1} S^{n+1} x_{n+1} + (1-\beta_{n+1})T^{n+1} x_{n+1} - z\| \\ &\le \beta_{n+1}\|S^{n+1} x_{n+1} - z\| + (1-\beta_{n+1})\|T^{n+1} x_{n+1} - z\| \\ &\le \beta_{n+1}\{\|x_{n+1} - z\| + c_{n+1}\} + (1-\beta_{n+1})\{\|x_{n+1} - z\| + c_{n+1}\} \\ &= \|x_{n+1} - z\| + c_{n+1}, \end{aligned}$$

by Lemma 3.2 and Takahashi [13], we obtain

$$\begin{aligned} \|x_{n+2} - z\| &= \|(1-\alpha_{n+1})x_{n+1} + \alpha_{n+1} y_{n+1} - z\| \\ &= \|(1-\alpha_{n+1})(x_{n+1} - z) + \alpha_{n+1}(y_{n+1} - z)\| \\ &\le \Big(\|x_{n+1} - z\| + c_{n+1}\Big)\left[1 - 2\alpha_{n+1}(1-\alpha_{n+1})\delta_E\left(\frac{\|x_{n+1} - y_{n+1}\|}{\|x_{n+1} - z\| + c_{n+1}}\right)\right]. \end{aligned}$$

As in the same method as above, we obtain

$$\liminf_{n\to\infty} \|x_{n+1} - y_{n+1}\| = 0. \tag{3.5}$$

Since $\{x_n\}$ is bounded and thus

$$\begin{aligned} \|y_{n+1} - T^{n+1} x_{n+1}\| &= \|\beta_{n+1} S^{n+1} x_{n+1} + (1-\beta_{n+1})T^{n+1} x_{n+1} - T^{n+1} x_{n+1}\| \\ &= \beta_{n+1}\|S^{n+1} x_{n+1} - T^{n+1} x_{n+1}\| \\ &\le \beta_{n+1} M', \end{aligned}$$

where $M' = \sup_{n\geq 1} \|T^{n+1}x_{n+1} - S^{n+1}x_{n+1}\| < \infty$. Since $\sum_{n=1}^{\infty} \beta_n < \infty$, we obtain

$$\lim_{n\to\infty} \|y_{n+1} - T^{n+1}x_{n+1}\| = 0. \tag{3.6}$$

Thus

$$\|x_{n+1} - T^{n+1}x_{n+1}\| \leq \|x_{n+1} - y_{n+1}\| + \|y_{n+1} - T^{n+1}x_{n+1}\|.$$

By (3.5) and (3.6), we get

$$\liminf_{n\to\infty} \|x_{n+1} - T^{n+1}x_{n+1}\| = 0. \tag{3.7}$$

Since

$$\begin{aligned}
&\|x_n - Tx_n\| \\
&\quad \leq \|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - T^{n+1}x_n\| + \|T^{n+1}x_n - Tx_n\| \\
&\quad \leq 2\|x_n - x_{n+1}\| + c_{n+1} + \|x_{n+1} - T^{n+1}x_{n+1}\| + \|T^{n+1}x_n - Tx_n\|
\end{aligned}$$

and by the uniform continuity of T, (3.3), (3.4) and (3.7), we have $\liminf_{n\to\infty} \|x_n - Tx_n\| = 0$. □

Theorem 3.5 *Let E be a uniformly convex Banach space. Let C be a nonempty closed convex subset of E and let $S : C \to C$ be AQT and $T : C \to C$ be ANI satisfying Condition* (**B**) *with $\mathbf{F} = F(S) \cap F(T) \neq \emptyset$. Put*

$$c_n = \sup_{x\in C, w\in \mathbf{F}} (\|S^n x - w\| - \|x - w\|) \vee \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0$$

for all $n \geq 1$. Suppose that $\sum_{n=1}^{\infty} c_n < \infty$ and the sequence $\{x_n\}$ is defined by (1.3) such that $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$ and $\sum_{n=1}^{\infty} \beta_n < \infty$. Then, $\{x_n\}$ converges strongly to a common fixed point of S and T.

Proof For any $z \in \mathbf{F}$, as in the proof of Lemma 3.3, we obtain

$$\|x_{n+1} - z\| \leq \|x_n - z\| + c_n. \tag{3.8}$$

Taking the infimum over all $z \in \mathbf{F}$ on both sides and by Lemma 3.1, we see that $\lim_{n\to\infty} d(x_n, \mathbf{F}) (\equiv r)$ exists. We first claim that $\lim_{n\to\infty} d(x_n, \mathbf{F}) = 0$. In fact, assume that $r = \lim_{n\to\infty} d(x_n, \mathbf{F}) > 0$. Then, we can choose $n_0 \in N$ such that $0 < \frac{r}{2} < d(x_n, \mathbf{F})$ for all $n \geq n_0$. Using Condition (**B**), Theorem 3.4 and taking lim inf on both sides, we obtain

$$0 < f\left(\frac{r}{2}\right) \leq f(d(x_n, \mathbf{F})) \leq \|Tx_n - x_n\| \to 0$$

as $n \to \infty$. This is a contradiction. So, we obtain $r = 0$. Next, we claim that $\{x_n\}$ is a Cauchy sequence. Let $\epsilon > 0$ be given. Since $\lim_{n\to\infty} d(x_n, \mathbf{F}) = 0$ and $\sum_{n=1}^{\infty} c_n < \infty$, there exists $n_0 \in N$ such that for all $n \geq n_0$, we obtain

$$d(x_n, \mathbf{F}) < \frac{\epsilon}{4} \quad \text{and} \quad \sum_{i=n_0}^{\infty} c_i < \frac{\epsilon}{4}. \tag{3.9}$$

Let $n, m \geq n_0$ and $p \in \mathbf{F}$. Then, by (3.8), we obtain

$$\begin{aligned}
\|x_n - x_m\| &\leq \|x_n - p\| + \|x_m - p\| \\
&\leq \|x_{n_0} - p\| + \sum_{i=n_0}^{n-1} c_i + \|x_{n_0} - p\| + \sum_{i=n_0}^{m-1} c_i \\
&\leq 2\left[\|x_{n_0} - p\| + \sum_{i=n_0}^{\infty} c_i\right].
\end{aligned}$$

Taking the infimum over all $p \in \mathbf{F}$ on both sides and by (3.9), we obtain

$$\begin{aligned}\|x_n - x_m\| &\leq 2\left[d(x_{n_0}, \mathbf{F}) + \sum_{i=n_0}^{\infty} c_i\right] \\ &< 2\left(\frac{\epsilon}{4} + \frac{\epsilon}{4}\right) = \epsilon\end{aligned}$$

for all $n, m \geq n_0$. This implies that $\{x_n\}$ is a Cauchy sequence. Let $\lim_{n\to\infty} x_n = q$. Then $d(q, \mathbf{F}) = 0$. Since $\mathbf{F}$ is closed, we obtain $q \in \mathbf{F}$. Hence, $\{x_n\}$ converges strongly to a common fixed point of S and T. □

As a direct consequence, taking $S = I$ in Theorem 3.4, we have the following result.

Theorem 3.6 *Let E be a uniformly convex Banach space. Let C be a nonempty closed convex subset of E and let $T : C \to C$ be ANI with $F(T) \neq \emptyset$. Put*

$$c_n = \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0$$

for all $n \geq 1$. Suppose that $\sum_{n=1}^{\infty} c_n < \infty$ and the sequence $\{x_n\}$ is defined by (1.2) such that $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$ and $\sum_{n=1}^{\infty} \beta_n < \infty$. Then, $\liminf_{n\to\infty} \|x_n - Tx_n\| = 0$.

As a direct consequence, taking $S = I$ in Theorem 3.5, we have the following result which carries over Theorem 5 of Kim [5] to ANI.

Theorem 3.7 *Let E be a uniformly convex Banach space. Let C be a nonempty closed convex subset of E and let $T : C \to C$ be ANI satisfying Condition (**A**) with $F(T) \neq \emptyset$. Put*

$$c_n = \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0$$

for all $n \geq 1$. Suppose that $\sum_{n=1}^{\infty} c_n < \infty$ and the sequence $\{x_n\}$ is defined by (1.2) such that $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$ and $\sum_{n=1}^{\infty} \beta_n < \infty$. Then, $\{x_n\}$ converges strongly to some fixed point of T.

As a direct consequence, taking $\beta_n = 0$ for all $n \geq 1$ in Theorem 3.7, we have the following result which carries over Theorem 2 of Rhoades [10] to ANI under much less restriction on the iterative parameter $\{\alpha_n\}$.

Theorem 3.8 *Let E be a uniformly convex Banach space. Let C be a nonempty closed convex subset of E and let $T : C \to C$ be ANI satisfying Condition (**A**) with $F(T) \neq \emptyset$. Put*

$$c_n = \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0$$

for all $n \geq 1$. Suppose that $\sum_{n=1}^{\infty} c_n < \infty$ and the sequence $\{x_n\}$ is defined by (1.1) such that $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$. Then, $\{x_n\}$ converges strongly to some fixed point of T.

Remark 3.9 If $T : C \to C$ is completely continuous, then it is demicompact and, if T is continuous and demicompact, it satisfies Condition (**A**); see Senter and Dotson [12].

Remark 3.10 If $\{\alpha_n\}$ is bounded away from both 0 and 1, i.e., $a \leq \alpha_n \leq b$ for all $n \geq 1$ and some $a, b \in (0, 1)$, then $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$ holds. However, the converse is not true.

We give an example of an ANI which is not a Lipschitz function.

Example 3.11 Let $E = \mathbb{R}$ and $C = [-\pi, \pi]$ and let $|h| < 1$. Let $T : C \to C$ be defined by

$$Tx = hx \cos nx$$

for each $x \in C$ and for all $n \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of all positive integers. Clearly $F(T) = \{0\}$. Since

$$\begin{aligned}&T(x) = hx \cos nx, \\ &T^2 x = T(Tx) = h(hx \cos nx) \cos n(hx \cos nx) = h^2 x \cos nx \cos nhx \cos n(\cos nx) \ldots,\end{aligned}$$

we obtain $\{T^n x\} \to 0$ uniformly on C. Thus

$$\limsup_{n\to\infty}\{\|T^n x - T^n y\| - \|x - y\| \vee 0\} = 0$$

for all $x, y \in C$. Hence T is ANI. But it is not a Lipschitz function. In fact, suppose that there exists $h > 0$ such that $|Tx - Ty| \le h|x - y|$ for all $x, y \in C$. If we take $x = \frac{\pi}{2n}$ and $y = \frac{\pi}{n}$, then

$$|Tx - Ty| = \left| h\frac{\pi}{2n}\cos n\frac{\pi}{2n} - h\frac{\pi}{n}\cos n\frac{\pi}{n} \right| = \frac{h\pi}{n},$$

whereas,

$$h|x - y| = h\left|\frac{\pi}{2n} - \frac{\pi}{n}\right| = \frac{h\pi}{2n}.$$

We also give an example of two mappings $S, T : C \to C$ which satisfy all assumptions of S, T in Theorem 3.5, i.e., S is AQT and T is ANI satisfying Condition (**B**) with $\mathbf{F} = F(S) \cap F(T) \neq \emptyset$. But S, T are not Lipschitzian.

Example 3.12 Let $E = \mathbb{R}$ and $C = [0, 4]$. Define $S, T : C \to C$ by

$$Sx = \begin{cases} 2, & x \in [0, 2], \\ \sqrt{8 - 2x}, & x \in [2, 4]. \end{cases}$$

and

$$Tx = \begin{cases} 2, & x \in [0, 2], \\ \frac{1}{\sqrt{3}}\sqrt{16 - x^2}, & x \in [2, 4]. \end{cases}$$

Note that $S^n x = 2$, $T^n x = 2$ for all $x \in C$ and $n \ge 2$ and $\mathbf{F} = F(S) \cap F(T) = \{2\}$. Clearly, S is AQT on C, T is both uniformly continuous and ANI on C. We first show that S satisfies Condition (**B**). In fact, if $x \in [0, 2]$, then $|x - 2| = |x - Sx|$. Similarly, if $x \in [2, 4]$, then

$$|x - 2| = x - 2 \le x - \sqrt{8 - 2x} = |x - Sx|.$$

Next, we show that T satisfies Condition (**B**). In fact, if $x \in [0, 2]$, then $|x - 2| = |x - Tx|$. Similarly, if $x \in [2, 4]$, then

$$|x - 2| = x - 2 \le x - \frac{1}{\sqrt{3}}\sqrt{16 - x^2} = |x - Tx|.$$

So, we get either $d(x, \mathbf{F}) = |x - 2| \le |x - Sx|$ or $d(x, \mathbf{F}) = |x - 2| \le |x - Tx|$ for all $x \in C$. But S, T are not Lipschitzian. We first show that S is not Lipschitzian. Indeed, suppose not, i.e., there exists $h > 0$ such that

$$|Sx - Sy| \le h|x - y|$$

for all $x, y \in C$. If we take $x = 4 - \frac{2}{(h+1)^2} > 2$ and $y = 4$, then

$$\sqrt{8 - 2x} \le h(4 - x) \Leftrightarrow \frac{2}{h^2} \le 4 - x = \frac{2}{(h + 1)^2} \Leftrightarrow h + 1 \le h.$$

This is a contradiction. Next, we show that T is not Lipschitzian. Indeed, suppose not, i.e., there exists $h > 0$ such that

$$|Tx - Ty| \le h|x - y|$$

for all $x, y \in C$. If we take $x = 4 - \frac{1}{3(h+1)^2} > 2$ and $y = 4$, then

$$\frac{1}{\sqrt{3}}\sqrt{16 - x^2} \le h(4 - x) \Leftrightarrow \frac{1}{3h^2} \le \frac{4 - x}{4 + x} = \frac{1}{24h^2 + 48h + 23}.$$

This is a contradiction.

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