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Unit graphs of rings of polynomials and power series

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Abstract Let R be a commutative ring. The unit graph of R, denoted by $G(R)$, is a graph with all elements of R as vertices and two distinct vertices $x, y \in R$ are adjacent if and only if $x + y \in U(R)$ where $U(R)$ denotes the set of all units of R. In this paper, we examine the preservation of the connectedness, diameter, girth, and some other properties, such as chromatic index, clique number and planarity of the unit graph $G(R)$ under extensions to polynomial and power series rings.

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الملخص

لتكن R حلقة إبدالية. بيان العكس لِـ R، والذي يرمز له بِـ $G(R)$، هو البيان الذي تكون رؤوسه جميع عناصر R، ويكون فيه رأسان مختلفان $x, y \in R$ متجاورين إذا كان وفقط إذا كان $x + y \in U(R)$، حيث ترمز $U(R)$ لمجموعة العناصر القابلة للعكس في R. في هذه الورقة، نفحص الحفاظ على الترابط، والقطر، والنطاق، وبعض الخصائص الأخرى مثل المؤشر اللَّوْني، وعدد الجماعة، والاستواء لبيان العكس $G(R)$ تحت تأثير التمديدات لحلقات كثيرات الحدود وحلقات سلاسل القوى.

1 Introduction

The investigation of graphs related to various algebraic structures is a very large and growing area of research. In particular, Cayley graphs have attracted serious attention in the literature, since they have many useful applications, see [7,8,10,12,14,18] for examples of recent results and further references. Several other classes of graphs associated with algebraic structures have been also actively investigated. For example, see [1,5,9,11,13].

Let R be a commutative ring with non-zero identity, and $U(R)$ and $Z(R)$ be the sets of all unit elements and zero-divisors of R, respectively. In addition, suppose that $\mathbb{N}$ (resp., $\mathbb{N}_0$) is the set of positive (resp., non-negative) integers. The concept of the unit graph of R was first introduced by Grimaldi [5]. His work was based on the ring $\mathbb{Z}_n$, where n is a positive integer and $\mathbb{Z}_n$ is the ring of integers modulo n. He defined a graph $G(\mathbb{Z}_n)$ such that its vertices are all elements of $\mathbb{Z}_n$ and two distinct vertices x and y are adjacent if and only if $x + y$ is a unit of $\mathbb{Z}_n$. Recently, Ashrafi et al. [1], generalized $G(\mathbb{Z}_n)$ to $G(R)$, the unit graph of R, where R is an arbitrary associative ring with non-zero identity.

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In the Sects. 2 and 3, we examine the preservation and lack thereof of the connectedness, diameter and girth of the unit graph $G(R)$ under extensions to rings of polynomials and power series.

In Sect. 4, we introduce some special subgraphs of $G(R[x])$, denoted by $G_n(R[x])$, where n is a non-negative integer. In fact, the vertices of $G_n(R[x])$ are all of the polynomials with degree n in $R[x]$. In this way, one can exploit the properties of these subgraphs to investigate some properties of $G(R[x])$. We also compute the chromatic index of $G(R[x])$ and $G_n(R[x])$, for a non-negative integer n. For instance, we prove the following theorem. Recall that a graph G is said to be class 1 if the chromatic index of G equals its maximum vertex the valency, where valency of a vertex is the number of edges incident to it.

Theorem 1.1 *Let R be a finite ring. Then*

(i) *if R is reduced, then $G(R[x])$ is class 1;*
(ii) *for each $n \in \mathbb{N}_0$, $G_n(R[x])$ is class 1.*

In Sect. 5, we compare the clique numbers of the unit graphs of R, $R[x]$ and $R[[x]]$. Recall that the clique number of a graph G is the number of vertices of the largest complete subgraph in G. The following theorem consists of some of our main results, in this context.

Theorem 1.2 (i) $\mathrm{clique}(G(R[x])) = \begin{cases} \infty \text{ if there is } a \in R \text{ such that } 2a \in U(R) \text{ and } R \text{ is not reduced,} \\ \mathrm{clique}(G(R)) \text{ else.} \end{cases}$

(ii) $\mathrm{clique}(G(R[[x]])) = \begin{cases} \infty \text{ if there is an element } a \in R \text{ such that } 2a \in U(R), \\ \mathrm{clique}(G(R)) \text{ else.} \end{cases}$

In the last section, among other things, we investigate the planarity of $G(R[x])$ and $G(R[[x]])$. In fact, we show that $G(R[[x]])$ is never planar. In addition, $G(R[x])$ is not planar whenever R is a non-reduced ring. Moreover, in the case that R is a finite ring, we provide some circumstances under which $G(R[x])$ is planar.

In this paper, we also extend or give some new versions of Propositions 2.4 and 4.6, Lemma 2.7, Theorem 4.3 and Corollary 4.4 in [1].

Throughout the paper, R is a commutative ring with non-zero identity. We also denote Jacobson radical and nilradical of R by $\mathrm{J}(R)$ and $\mathrm{Nil}(R)$, respectively. Let G be a graph. Then $V(G)$ and $E(G)$ denote the set of vertices and edges of G, respectively. The valency of a vertex a, denoted by $V(a)$, is the number of edges of G incident to a. For every non-negative integer r, G is called r-regular if the valency of each vertex is equal to r. The distance between two distinct vertices a and b, denoted by $\mathrm{d}_G(a, b)$ or briefly $d(a, b)$, is the length of the shortest path connecting a and b, if such a path exists; otherwise, we set $\mathrm{d}_G(a, b) := \infty$. In addition, for two distinct vertices a and b in G, the notation $a \sim b$ means that a and b are adjacent. A graph G is said to be connected if there exists a path between any two distinct vertices, and it is complete if every two distinct vertices are adjacent. Let $\chi(G)$ denote the chromatic number of the graph G, that is the minimal number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors. For a positive integer r, an r-partite graph is one whose vertex set can be partitioned into r subsets so that no edge has both ends in any one subset. A complete r-partite graph is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite graph (2-partite graph) with part sizes m and n is denoted by $K_{m,n}$.

2 Connectedness of unit graphs of rings of polynomials and power series

In this section, we are going to study connectedness of unit graphs of the rings $R[x]$, $R[[x]]$ and some of their relations with the unit graph of R. To this end, firstly, we begin this section by the definition of the unit graph of a ring R and some elementary remarks around the rings of polynomials and power series which may be valuable in turn. These can be immediately gained from elementary notes about polynomials and power series.

Definition 2.1 (*See* [1, Definitions and Remarks 2.1]) The unit graph of R, denoted by $G(R)$, is a graph whose vertices are all of the elements of R and distinct two vertices a and b in $G(R)$ are adjacent if and only if $a + b \in U(R)$. If we omit the word "distinct" in the definition, we obtain the closed unit graph denoted $\overline{G}(R)$; this graph may have loops.

Remarks 2.2 (1) Two elements $f(x) = \sum a_i x^i$ and $g(x) = \sum b_i x^i$ are adjacent in $G(R[x])$ if and only if $a_0 + b_0 \in U(R)$ and for each $i \in \mathbb{N}$, $a_i + b_i$ is a nilpotent element of R.

(2) Two elements $f(x)=\sum a_ix^i$ and $g(x)=\sum b_ix^i$ are adjacent in $G(R[[x]])$ if and only if $a_0+b_0\in U(R)$.
(3) Every element in $U(R)$ is adjacent to every element in $\mathrm{Nil}(R)$. In addition, it is clear that for each two distinct nilpotent elements a and b in R, we have $d(a,b)=2$ in $G(R)$.
(4) Since $G(R)$ is a subgraph of the comaximal graph which is introduced in [11], in the light of [13, Theorem 2.3], if R is finite, $\chi(G(R))\le t+\ell$ when t is the number of maximal ideals of R and ℓ is the number of units of R.
(5) $U(R)$ is a dominating set for $G(R)$ if and only if $U(R[[x]])$ is a dominating set for $G(R[[x]])$, where a dominating set of a graph G is a subset of the vertex set, say S, such that every vertex not in S is adjacent to a vertex in S.
(6) If $J(R[x])=\{0\}$ (or equivalently R is reduced), then $U(R)=U(R[x])$. Hence, $f(x)=\sum_{i=1}^m a_ix^i$ is adjacent to $g(x)=\sum_{i=1}^n b_ix^i$ in $G(R[x])$ if and only if $m=n$, $a_0+b_0\in U(R)$ and for all $i=1,\dots,n$, $a_i=-b_i$.
(7) For every edge $a_0\sim b_0$ in $G(R)$ and for all $\sum_{i=1}^\infty a_ix^i$ and $\sum_{i=1}^\infty b_ix^i$ in $R[[x]]$, $\sum_{i=0}^\infty a_ix^i$ and $\sum_{i=0}^\infty b_ix^i$ are adjacent in $G(R[[x]])$ and so $G(R[[x]])$ has a complete bipartite subgraph of infinite size.
(8) For all elements $f(x)=\sum a_ix^i$ and $g(x)=\sum b_ix^i$ in $U(R[x])$, $f(x)$ is adjacent to $g(x)$ if and only if $a_0+b_0\in U(R)$.
(9) It is clear that $G(R)$ is an induced subgraph of $G(R[x])$ and $G(R[x])$ is a subgraph of $G(R[[x]])$.
(10) If $2\notin U(R)$, then $\overline{G(R)}=G(R)$.
(11) If $Z(R)\trianglelefteq R$, then the induced subgraph of $G(R)$ with vertices in $Z(R)$ is a totally disconnected graph.
(12) Since for all a in R, $a+x$ is not adjacent to $-a$, the unit graphs $G(R[x])$ and $G(R[[x]])$ are never complete.

The following theorem, which shows $G(R[x])$ is always disconnected, is a generalization of Proposition 4.6 in [1].

Theorem 2.3 (See [1, Proposition 4.6]) *The unit graph $G(R[x])$ is always disconnected.*

Proof We show that there is not any path between the polynomials x and x^2. To this end, suppose, in contrary, that there is a path such as

$$x\sim f_1(x)\sim\cdots\sim f_n(x)\sim x^2$$

in $G(R[x])$, where for each $i=1,\dots,n$, $f_i(x)=\sum a_{i,j}x^j$. Since $x+f_1(x)$ is unit in $R[x]$, $1+a_{1,1}$ must be nilpotent and so $a_{1,1}$ is a unit. Hence, by similar arguments one can show that for all $i=1,\dots,n$, $a_{i,1}$ is unit. Now, since $f_n(x)+x^2$ is a unit element in $R[x]$, $a_{n,1}$ must be a nilpotent, which is a contradiction. □

For the next results of this section, we need to recall some definitions in the context of generating elements of a ring additively by its units, which is initiated and studied in 1953–1954 by Wolfson [16] and Zelinsky [17], independently. There exist several papers devoted to this context, (e.g., [4,6,15], etc.).

Definition 2.4 Let R be a ring and $k\in\mathbb{N}$. An element $r\in R$ is said to be k-good if we may write $r=u_1+\cdots+u_k$ for some unit elements $u_1,\dots,u_k$ in R. The ring R is said to be k-good if all elements of R are k-good. The unit sum number of R, which is denoted by $u(R)$ is defined as follows.

- $u(R)=\min\{k\mid R \text{ is } k-\text{good}\}$ if R is k-good for some $k\ge 1$;
- $u(R)=w$ if R is not k-good for every $k\in\mathbb{N}$, but every element of R is k-good for some k, (that is, when at least $U(R)$ generates R additively);
- $u(R)=\infty$ otherwise, (that is, when $U(R)$ does not generate R additively).

If $u(R)=k$ for some $k\in\mathbb{N}$ or $u(R)=w$, we say that $u(R)$ is *finite* and we write $u(R)\le w$.
The following lemma is needed in the sequel.

Lemma 2.5 *Let k be a positive integer and $f(x)=\sum_{i=0}^n a_ix^i$ an element of $R[x]$ such that a_0 is a k-good element in R. Then the following statements hold:*

(i) *$d(f(x),0)\le k$ in $G(R[[x]])$;*
(ii) *$d(f(x),\sum_{i=1}^n a_ix^i)\le k$ in $G(R[x])$ if k is even;*

(iii) *if k is an odd number, for all nilpotent elements $t_1, \dots, t_n$ of R, we have $d(f(x), \sum_{i=1}^{n}(t_i - a_i)x^i) \le k$ in $G(R[x])$.*

Proof Let $a_0 = u_1 + \cdots + u_k$, where $u_1, \dots, u_k$ are unit elements in R. Then, for every $1 \le i \le k$, set $b_i := (-1)^i \sum_{j=1}^{k-i} u_j$ (note that $b_k = 0$). Considering the path $f(x) \sim b_1 \sim \cdots \sim b_k$ from $f(x)$ to zero in $G(R[[x]])$ proves (i).

Now, let $t_1, \dots, t_n$ be nilpotent elements of R. Then for all even numbers i with $1 \le i \le k$, set $g_i(x) := b_i + \sum_{i=1}^{n} a_i x^i$ and for all odd numbers i with $1 \le i \le k$, set $g_i(x) := b_i + \sum_{i=1}^{n}(t_i - a_i)x^i$. Now, $f(x) \sim g_1(x) \sim \cdots \sim g_k(x)$ is a path from $f(x)$ to $g_k(x)$ in $G(R[x])$. This proves (ii) and (iii) simultaneously. □

The following result, which is one of our main results of this section, is a slight generalization of Theorem 4.3 in [1].

Theorem 2.6 *The following conditions are equivalent.*

(1) *$u(R) \le w$;*
(2) *$d(f(x), 0)$ is finite and $d(f(x), 0) \le u(R)$ in $G(R[[x]])$ for all $f(x)$ in $R[[x]]$;*
(3) *$d(a, 0)$ is finite and $d(a, 0) \le u(R)$ in $G(R)$ for all a in R;*
(4) *$G(R)$ is connected;*
(5) *$G(R[[x]])$ is connected.*

Proof ($1 \Rightarrow 2$) follows from Lemma 2.5.

($2 \Rightarrow 3$) Let $a \in R$. In view of (2), there is a positive integer k such that $d(a, 0) = k$ in $G(R[x])$ and $k \le u(R)$. Let $a \sim f_1(x) \sim \cdots \sim f_{k-1}(x) \sim 0$ be a path in $G(R[[x]])$, where for all $1 \le i \le k-1$, $f_i(x) = \sum a_{i,j}x^j$. Then it is clear that $a \sim a_{1,0} \sim \cdots \sim a_{k-1,0} \sim 0$ induces a path from a to zero in $G(R)$. Hence, $d(a, 0)$ is finite and $d(a, 0) \le u(R)$ in $G(R)$ as desired.

($3 \Rightarrow 4$) is clear.

($4 \Rightarrow 5$) Let $G(R)$ be connected and $f(x) = \sum a_i x^i$ and $g(x) = \sum b_i x^i$ be two elements in $R[[x]]$. At first, suppose that $a_0 = b_0$. Since $G(R)$ is connected a_0 is connected to some element such as c. Therefore, $f(x) \sim c \sim g(x)$ is a path in $G(R[[x]])$. In addition, if $a_0 \ne b_0$, since $G(R)$ is connected, there is a path, say $a_0 \sim c_1 \sim \cdots \sim c_n \sim b_0$, in $G(R)$. Therefore, $f(x) \sim c_1 \sim \cdots \sim c_n \sim g(x)$ is a path from $f(x)$ to $g(x)$ in $G(R[[x]])$.

($5 \Rightarrow 1$) Let a be an arbitrary element of R. Then since $G(R[[x]])$ is connected, $d(0, a) = k$ in $G(R[[x]])$ for some positive integer k. Suppose that $a_0 = 0, a_k = a$ and $a_0 \sim (a_1 + \sum b_{1,j}x^j) \sim \cdots \sim (a_{k-1} + \sum b_{k-1,j}x^j) \sim a_k$ is a path from zero to a in $G(R[[x]])$. Hence, there are unit elements $u_1, \dots, u_k$ of R such that $a_i + a_{i+1} = u_{i+1}$ for all $0 \le i \le k-1$. Therefore, we have $a = \sum_{i=1}^{k}(-1)^{k-i}u_i$. This implies that every element of R is k-good, for some positive integer k of R. □

The following corollary is a slight generalization of Corollary 4.4 in [1].

Corollary 2.7 *Let S be a subset of R in which every element is k-good for some positive integer k. If S is a dominating set for $G(R)$ or $G(R[[x]])$, then $G(R)$ and $G(R[[x]])$ are connected.*

Proof Let a be an arbitrary element of R not in S. Then there is a k-good element s of R such that $a+s \in U(R)$, where k is a positive integer. Therefore, a is a $(k+1)$-good element. Hence, $u(R) \le w$. Now, the result follows from Theorem 2.6. □

Recall that a unit-regular ring R, which is a special type of Von Neumann regular ring, is a ring in which for each element $x \in R$ there exists a unit element $u \in R$ such that $xux = x$.

Corollary 2.8 *Let R be a unit-regular ring in which 2 is unit. Then $G(R[[x]])$ is connected.*

Proof If R is a unit-regular ring, then each $x \in R$ can be written as $x = eu$, where e is an idempotent and u is a unit. We may write $e = (1+e) - 1$. Now, since 2 is a unit in R, $(1+e)$ is a unit with $(1+e)^{-1} = 1 - 1/2e$. This gives that e is the sum of two units and hence x is the sum of two units. This implies that R is 2-good. Therefore, for each $x \in R$, there are units u and v such that $x = u + v$. This yields that $U(R)$ is a dominating set for $G(R)$. Now the result follows from Corollary 2.7. □

In the following proposition, we exploit some properties of the unit graph $G(R[x])$ to find some upper bound for $u(R)$.

Proposition 2.9 *Assume that k is a positive integer and $a_1, \dots, a_n$ are some k-good elements of R. Let S be a subset of $R[x]$ composed from some polynomials with coefficients in $\{a_1, \dots, a_n\} \cup \{0\}$, which is a dominating set for $G(R[x])$. Then $u(R) \le k$ and so $G(R)$ and $G(R[[x]])$ are connected.*

Proof Let t be a positive integer and a be an arbitrary element of R. Since S is a dominating set for $G(R[x])$, there is a polynomial $f(x) = \sum b_i x^i$ such that $ax^t + f(x) \in U(R[x])$ and for each $i \in \mathbb{N}_0$, $b_i \in \{a_1, \dots, a_n\} \cup \{0\}$. Therefore, there is a nilpotent element r of R such that $a + b_t = r$. Now, if $b_t = 0$, then $a = (r-1) + 1$ and so a is a 2-good element of R. Otherwise, since b_t is a k-good element of R, there are unit elements $u_1, \dots, u_k$ in R such that $a = (r - u_1) + (-u_2) + \cdots + (-u_k)$ and so a is a k-good element of R. Therefore, $u(R) \le k$ as required. Furthermore, Theorem 2.6 implies that $G(R)$ and $G(R[[x]])$ are connected. □

3 Diameter and girth in $G(R)$, $G(R[x])$ and $G(R[[x]])$

Suppose that G is a graph with vertex set V. Then recall that the diameter of G, which is denoted by $\mathrm{diam}(G)$, is defined as follows.

$$\mathrm{diam}(G) := \sup\{d(a,b) \mid a, b \in V\}$$

When R is a ring and $a, b \in R$, we use the notation $d(a,b)$ or precisely $d_R(a,b)$ instead of $d_{G(R)}(a,b)$. Note that for a finite graph G, it is easy to see that $\mathrm{diam}(G) = \infty$ if and only if G is disconnected.

Also recall that the girth of a graph G, denoted by $\mathrm{gr}(G)$, is the length of the shortest cycle in G if G has a cycle; otherwise, $\mathrm{gr}(G) = \infty$. Moreover, for an arbitrary commutative ring R, we use the notations $\mathrm{diam}(R)$ and $\mathrm{gr}(R)$ instead of $\mathrm{diam}(G(R))$ and $\mathrm{gr}(G(R))$, respectively.

In the following lemma, we compare the distance between two vertices in $G(R)$, $G(R[x])$ and $G(R[[x]])$.

Lemma 3.1 (i) *For all $a, b \in R$, $d(a,b)$ in $G(R)$, $G(R[x])$ and $G(R[[x]])$ has the same value.*
(ii) *For all $f(x), g(x) \in R[x]$, $d_{R[[x]]}(f(x), g(x)) \le d_{R[x]}(f(x), g(x))$.*

Proof (i) Denote $d(a,b)$ in $G(R)$, $G(R[x])$ and $G(R[[x]])$, respectively by m, n and t. It is clear that $t \le n \le m$. It is enough to prove the result when t is finite. To this end, let $a \sim f_1(x) \sim \cdots \sim f_{t-1}(x) \sim b$ be a path from a to b in $G(R[[x]])$, where $f_i(x) = \sum a_{i,j} x^j$ for every $i = 1, \dots, t-1$. Hence, $a \sim a_{1,0} \sim \cdots \sim a_{t-1,0} \sim b$ induces a path in $G(R)$ with length smaller than $t+1$. Now, if $t < m$, then $a \sim a_{1,0} \sim \cdots \sim a_{t-1,0} \sim b$ induces a path in $G(R)$ with length smaller than m which is a contradiction. Therefore, we must have $m = t$.

(ii) follows from Remarks 2.2(9). □

Remarks 3.2 Note that in part (ii) of the above lemma, the inequality may be strict. For instance, as we mentioned in the proof of Theorem 2.3, x is not connected to x^2 in $G(R[x])$ at all, but $d(x, x^2) = 2$ in $G(R[[x]])$, because $x \sim 1 \sim x^2$ is a path from x to x^2 in $G(R[[x]])$.

The following result has an important role for the remainder of this section.

Lemma 3.3 *Let $f(x) = \sum a_i x^i$ and $g(x) = \sum b_i x^i$ be two distinct elements of $R[[x]]$. Then*

(i) *if $a_0 = b_0$, then*

$$d_{R[[x]]}(f(x), g(x)) = \begin{cases} 1 & 2a_0 \in U(R), \\ 2 & 2a_0 \notin U(R); \end{cases}$$

and

(ii) *if $a_0 \ne b_0$, then $d_{R[[x]]}(f(x), g(x)) = d_R(a_0, b_0)$.*

Proof (i) If $2a_0 \in U(R)$, it is clear that $f(x)$ is adjacent to $g(x)$. Otherwise, we have the path $f(x) \sim (1 - a_0) \sim g(x)$ in $G(R[[x]])$.

(ii) Let $a_0 \ne b_0$. Then one can easily check that a_0 is connected to b_0 if and only if $f(x)$ is connected to $g(x)$. Now, suppose that $d_{R[[x]]}(f(x), g(x)) = m$ and $d_R(a_0, b_0) = n$. Let $a_0 \sim c_1 \sim \cdots \sim c_{n-1} \sim b_0$ be one of the shortest paths from a_0 to b_0 in $G(R)$. Then $f(x) \sim c_1 \sim \cdots \sim c_{n-1} \sim g(x)$ is a path from $f(x)$ to $g(x)$. Therefore, $m \le n$. Conversely, Let $c_{0,0} := a_0$, $c_{m,0} := b_0$ and $f(x) \sim h_1(x) \sim \cdots \sim h_{m-1}(x) \sim g(x)$ be one of the shortest paths in $G(R[[x]])$ from $f(x)$ to $g(x)$ in $G(R[[x]])$, when $h_i(x) = \sum c_{i,j}x^j$ for all $i = 1, \ldots, m-1$. Then $c_{0,0} \sim c_{1,0} \sim \cdots \sim c_{m-1,0} \sim c_{m,0}$ is a path from a_0 to b_0 of length m, because, if $c_{i,0} = c_{j,0}$ for some distinct integers $i, j \in \{0, 1, \ldots, m\}$, then $f(x) \sim h_1(x) \sim \cdots \sim h_i(x) \sim h_{j+1}(x) \sim \cdots \sim h_{m-1}(x) \sim g(x)$ is a path from $f(x)$ to $g(x)$ with length less than m which is a contradiction with $d_{R[[x]]}(f(x), g(x)) = m$. Hence, $n \leq m$. Therefore, $d_{R[[x]]}(f(x), g(x)) = d_R(a_0, b_0)$ as required. □

The following corollary is an immediate consequence of Lemma 3.1 and Remarks 2.2(9).

Corollary 3.4 *For a commutative ring R with non-zero identity,*

(i) *we have the inequality*

$$\mathrm{diam}(R) \leq \mathrm{diam}(R[[x]]);$$

and

(ii) *if $G(R)$ has a cycle, then we have the following inequalities of natural numbers.*

$$\mathrm{gr}(R[[x]]) \leq \mathrm{gr}(R[x]) \leq \mathrm{gr}(R).$$

In the following result, we characterize the diameter of $G(R[[x]])$ in terms of the diameter of $G(R)$.

Theorem 3.5 (i) *Let R be a division ring. Then $R[[x]]$ is connected and* $\mathrm{diam}(R[[x]]) = 2$.
(ii) *If $G(R)$ is complete, then* $\mathrm{diam}(R[[x]]) = 2$.
(iii) *If $G(R)$ is not complete, then* $\mathrm{diam}(R) = \mathrm{diam}(R[[x]])$.

Proof (i) Let $f(x) = \sum a_i x^i$ and $g(x) = \sum b_i x^i$ be two elements of $R[[x]]$. Since every non-zero element of R is unit, if $a_0 = 0$ or $b_0 = 0$ and not both, then $f(x)$ is adjacent to $g(x)$. In addition, if a_0 and b_0 are nonzero, $f(x) \sim 0 \sim g(x)$ is a path from $f(x)$ to $g(x)$. Moreover, if $a_0 = b_0 = 0$, then $f(x) \sim 1 \sim g(x)$ is a shortest path from $f(x)$ to $g(x)$. Therefore, $\mathrm{diam}(R[[x]]) = 2$ as required.

(ii) immediately follows from Theorem 3.4 in [1] and (i).

(iii) Suppose that $G(R)$ is not complete and n, m are two natural numbers such that $\mathrm{diam}(R) = n$ and $\mathrm{diam}(R[[x]]) = d(f(x), g(x)) = m$, where $f(x) = \sum a_i x^i$ and $g(x) = \sum b_i x^i$ are two distinct elements of $R[[x]]$ (note that, in the light of Theorem 2.6, one may assume that $G(R)$ and so $G(R[[x]])$ are connected). If $a_0 = b_0$, by part (12) of Remarks 2.2 and Lemma 3.3(i), $d(f(x), g(x)) = 2$ and so the result follows from Corollary 3.4(i) in this case. Now, let $a_0 \neq b_0$. Then Lemma 3.3(ii) insures that $d(a_0, b_0) = m$. Therefore, we should have $n \geq m$. Now, Corollary 3.4(i) completes the proof. □

Corollary 3.6 *Let R be a division ring such that* $\mathrm{char}(R) \neq 2$. *Then*

$$\mathrm{diam}(R) = \mathrm{diam}(R[[x]]) = 2.$$

Proof The result follows from [1, Theorem 3.4], Corollary 3.4(i) and Theorem 3.5(i). □

We end this section by the following proposition which investigates the girths of $G(R[x])$ and $G(R[[x]])$.

Proposition 3.7 *For a commutative ring R with non-zero identity,*

(i) $\mathrm{gr}(R[[x]]) \leq 4$ *and if $2 \in U(R)$, then* $\mathrm{gr}(R[[x]]) = 3$;
(ii) *if R is not reduced, then* $\mathrm{gr}(R[x]) \leq 4$ *and if also $2 \in U(R)$, then* $\mathrm{gr}(R[x]) = 3$;
(iii) *if R is reduced such that $2 \notin U(R)$ and $G(R)$ has a cycle, then* $\mathrm{gr}(R) = \mathrm{gr}(R[x]) = \mathrm{gr}(R[[x]])$.

Proof (i) $1 \sim x \sim (1+x) \sim x^2 \sim 1$ is a cycle in $G(R[[x]])$. Therefore, $\mathrm{gr}(R[[x]]) \leq 4$. If $2 \in U(R)$, then $1 \sim x \sim (1+x) \sim 1$ is a cycle in $G(R[[x]])$. Therefore, $\mathrm{gr}(R[[x]]) = 3$.

(ii) Since R is not reduced, there exists a non-zero nilpotent element c in R. Now, $1 \sim cx \sim (1-cx) \sim cx^2 \sim 1$ is a cycle in $R[x]$ and so $\mathrm{gr}(R[x]) \leq 4$. If also $2 \in U(R)$, then $1 \sim (1+cx) \sim (1+cx^2) \sim 1$ is a cycle in $R[x]$ and so $\mathrm{gr}(R[x]) = 3$. (iii) Since R is reduced and $2 \notin U(R)$, by means of Remarks 2.2(6), it is clear that every cycle in $G(R[x])$ (or similarly $G(R[[x]])$) induces a cycle with equal or smaller length, in $G(R)$. Therefore, $\mathrm{gr}(R) \leq \mathrm{gr}(R[x])$ and $\mathrm{gr}(R) \leq \mathrm{gr}(R[[x]])$. Now, the result follows from Corollary 3.4(ii). □

4 Chromatic index of $G(R[x])$ and some of its special subgraphs

In this section we study some special subgraphs of $G(R[x])$ to investigate some properties of $G(R[x])$ better. To this end, we introduce the following special subgraphs of $G(R[x])$.

Definitions and notations 4.1 Let G be a graph, a be a vertex of G and $S \subseteq V(G)$. Then the S-neighborhood of a in G, which is denoted by $N_S(a)$, is the set of all vertices in S which are adjacent to a in G. In addition, the number of elements of $N_S(a)$ is called the S-valency of a and is denoted by $V_S(a)$. Note that when $S = V(G)$, our definition coincides with well known definitions of neighborhood and valency of a vertex.

Let n be a non-negative integer. We use the notion $G_n(R[x])$ for the induced subgraph of $G(R[x])$ whose vertex set consists of all polynomials in $R[x]$ of degree n. We accept with the contraction that the degree of zero polynomial is zero. Therefore, by this contraction, we have $G_0(R[x]) = G(R)$.

In the next proposition, for each $a \in R$ and $n \in \mathbb{N}$, we are going to find the number of neighbors of a with degree n in the unit graph $G(R[x])$.

Proposition 4.2 *Let $a \in R$, $n \in \mathbb{N}$, $|U(R)| = \alpha$ and $|\mathrm{Nil}(R)| = \beta$ where α, β are two positive integers. Then*

(i) *if $\beta = 1$ (i. e., R is reduced), then $V_{G_n(R[x])}(a) = 0$;*
(ii) *if $\beta > 1$, then $V_{G_n(R[x])}(a) = \alpha\beta^{n-1}(\beta - 1)$.*

Proof (i) Since R is reduced and $n \in \mathbb{N}$, the result immediately follows from Remarks 2.2(6).

(ii) Let $n \in \mathbb{N}$, $\beta > 1$ and $f(x) = \sum_{i=0}^{n} a_i x^i \in N_{G_n(R[x])}(a)$. Then $a + f(x) \in U(R[x])$. Therefore, we should have $a + a_0 \in U(R)$, $a_1, \ldots, a_{n-1}$ are nilpotent elements of R and a_n is a non-zero nilpotent element of R. Therefore, there are α, β and $\beta - 1$ possibilities for a_0, each of $a_1, \ldots, a_{n-1}$ and a_n, respectively. Therefore, $V_{G_n(R[x])}(a) = \alpha\beta^{n-1}(\beta - 1)$. □

In the following proposition we investigate the valency of $f(x) = \sum_{i=0}^{n} a_i x^i$ in $G_n(R[x])$, when n is a natural number. Note that the case $n = 0$ is calculated by Ashrafi et al. in Proposition 2.4 in [1].

Proposition 4.3 *Assume that $|U(R)| = \alpha$, $|\mathrm{Nil}(R)| = \beta$, where α, β are two positive integers. Let $n \in \mathbb{N}$ and $f(x) = \sum_{i=0}^{n} a_i x^i$. Then*

(i) *if R is reduced, then $V_{G_n(R[x])}(f(x)) = \alpha$ and so $G_n(R[x])$ and also $G(R[x])$ are α-regular graphs;*
(ii) *if $a_n \notin \mathrm{Nil}(R)$, then $V_{G_n(R[x])}(f(x)) = \alpha\beta^n$;*
(iii) *if $2a_0 \in U(R)$ and $a_1, \ldots, a_n \in \mathrm{Nil}(R)$, then*

$$V_{G_n(R[x])}(f(x)) = \alpha\beta^{n-1}(\beta - 1),$$

otherwise $V_{G_n(R[x])}(f(x)) = \alpha\beta^{n-1}(\beta - 1) - 1$.

Proof (i) Let $g(x) = \sum_{i=0}^{n} b_i x^i \in N_{G_n(R[x])}(f(x))$. Then if R is reduced, then we should have $a_0 + b_0 \in U(R)$ and $b_i = -a_i$, for each $1 \leq i \leq n$. In addition, since $a_n \neq 0$, if we have $a_n = -a_n$, we should have $2 \notin U(R)$ and so $2a_0 \notin U(R)$. Hence, $f(x) + f(x) \notin U(R[x])$. This implies that there are α possibilities for b_0 and so for $g(x)$. Therefore, $G_n(R[x])$ is α-regular in this case. In addition, in view of Remarks 2.2(6), we know that two polynomials with different degrees can't be adjacent. Hence, $G(R[x])$ is also α-regular.

(ii) and (iii) In general case, if $g(x) = \sum_{i=0}^{n} b_i x^i \in N_{G_n(R[x])}(f(x))$, then $b_0 + a_0 \in U(R)$, for each $i = 1, \ldots, n-1$ there is a nilpotent element $t_i \in \mathrm{Nil}(R)$ such that $b_i = t_i - a_i$ and there is $t_n \in \mathrm{Nil}(R) \setminus \{a_n\}$ such that $b_n = t_n - a_n$. Therefore, for b_0 there are α possibilities and for each of $b_1, \ldots, b_{n-1}$, there are β possibilities. In addition, for b_n, there are β possibilities if $a_n \notin \mathrm{Nil}(R)$ and $\beta - 1$ possibilities if $a_n \in \mathrm{Nil}(R)$. Moreover, if $2a_0 \in U(R)$ and for all $i = 1, \ldots, n$, $2a_i \in \mathrm{Nil}(R)$, then one of the mentioned possibilities, which is enumerated, is the case that $g(x) = f(x)$. So we must decrease the gained valency 1 unit. These completes our proof. □

Recall that the chromatic index of a graph G, which is denoted by $\chi'(G)$, is the smallest number of colors such that one can associate colors to edges of G so that every pair of distinct edges meeting at a common vertex are assigned two different colors. There is a strong result for characterizing the chromatic index of a graph gained by Vizing. Vizing's theorem says that if G is a graph whose maximum vertex valency is Δ, then $\Delta \leq \chi'(G) \leq \Delta + 1$ (see [2, p. 93]). This result divides the graphs into two classes according to their chromatic index; graphs with chromatic index Δ are called class 1, and graphs with chromatic index $\Delta + 1$

are called class 2. Ashrafi et al. in [1], showed that all unit graphs of finite rings are class 1. Now, since $R[x]$ is an infinite ring, we characterize the class of some finite subgraphs of it.

In the following result, for each $n \in \mathbb{N}_0$, we prove that $G_n(R[x])$ is class 1. In addition, if R is reduced, then we show that $G(R[x])$ is always class 1.

Theorem 4.4 *Let R be a finite ring. Then*

(i) *if R is reduced, then $G(R[x])$ is class 1;*
(ii) *for each $n \in \mathbb{N}_0$, $G_n(R[x])$ is class 1.*

Proof (i) Let $f(x) = \sum a_i x^i$. Then since R is reduced, if $g(x) = \sum b_i x^i$ and $h(x) = \sum c_i x^i$ are two different neighbors of $f(x)$, in the light of Remarks 2.2(6), $b_0 \neq c_0$. In this regard, if we color the edge $f(x) \sim g(x)$ in $G(R[x])$ with color $a_0 + b_0$, then

$$C = \left\{a_0 + b_0 \mid f(x) = \sum a_i x^i \text{ and } g(x) = \sum b_i x^i \text{ are adjacent in } G(R[x])\right\}$$

is a set of colors for edge coloring of $G(R[x])$. Therefore, $\chi'(G(R[x])) \leq |C| \leq |U(R)|$. Using part (i) of Proposition 4.3 in conjunction with Vizing's theorem insures that $|U(R)| \leq \chi'(G(R[x]))$. Therefore, we have $\chi'(G(R[x])) = |U(R)|$ which implies that $G(R[x])$ is class 1.

(ii) One can check that

$$C = \left\{(a_0 + b_0, \ldots, a_n + b_n) \mid f(x) = \sum_{i=0}^{n} a_i x^i \text{ and } g(x) = \sum_{i=0}^{n} b_i x^i \text{ are adjacent in } G_n(R[x])\right\}$$

is a set of colors for edge coloring of $G_n(R[x])$. Now, if $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{i=0}^{n} b_i x^i$ are adjacent, then $a_0 + b_0 \in U(R)$ and $a_i + b_i \in \mathrm{Nil}(R)$ for all i with $1 \leq i \leq n$. Hence,

$$C \subseteq U(R) \times \underbrace{\mathrm{Nil}(R) \times \cdots \times \mathrm{Nil}(R)}_{n\text{-times}}.$$

Therefore, $\chi'(G_n(R[x])) \leq |U(R)||\mathrm{Nil}(R)|^n$. Proposition 4.3 shows that $\Delta = |U(R)||\mathrm{Nil}(R)|^n$. Therefore, we have $\chi'(G_n(R[x])) \leq \Delta$. Now the result follows from Vizing's theorem. □

5 Clique number

Recall that a graph with n vertices in which each pair of distinct vertices is joined by an edge is called a complete graph, and denoted by K_n. A clique of a graph is a complete subgraph of it and a coclique of a graph is a set of pairwise nonadjacent vertices. Let G be a graph, the number of vertices of one of the largest cliques of G is called clique number of G and is denoted by clique(G). Let R be a ring, throughout this section, we use the notation clique(R) instead of clique$(G(R))$. In this section we will compare clique(R), clique$(R[x])$ and clique$(R[[x]])$. We begin this section by the following proposition, which presents a necessary condition for characterizing the clique number of $G(R)$.

Proposition 5.1 *Let $2 \in U(R)$ and n be a natural number with $n \geq 3$ such that* clique$(R) = n$. *Then there exist elements $a_1, \ldots, a_n$ in R such that for all $1 \leq i \leq n$ and all $3 \leq k \leq n$, a_i is k-good.*

Proof We show that if $n \geq 3$ and K_n is a complete subgraph of $G(R)$ with vertices $a_1, \ldots, a_n$, then for all $3 \leq k \leq n$ and $1 \leq i \leq n$, a_i is k-good. To this end, we use induction on n. Let $n = 3$ and K_3 be a complete subgraph of $G(R)$ with vertices a, b and c. Then there are unit elements u, v and w of R such that $a + b = u$, $b + c = v$ and $c + a = w$. Therefore, preliminary calculations and the fact that $2 \in U(R)$ implies that $a = u/2 + w/2 - v/2$, $b = u/2 + v/2 - w/2$ and $c = w/2 + v/2 - u/2$. This means that a, b and c are 3-good elements. Now suppose, inductively, that the result has been proven for each complete subgraph of $G(R)$ with the number of vertices less than n, and let K_n be a complete subgraph of $G(R)$ with the vertices $a_1, \ldots, a_n$. Since every $n - 1$ elements of $\{a_1, \ldots, a_n\}$ construct a complete subgraph of $G(R)$, by inductive hypothesis, it is enough to show that for all $1 \leq i \leq n$, a_i is an n-good element of $G(R)$. Let i and j be two distinct integers with $1 \leq i, j \leq n$. Then since a_j is an $(n-1)$-good element and a_i is adjacent to a_j, a_i is an n-good element of R as required. □

The following two lemmas are some completer versions of Lemma 2.7 in [1].

Lemma 5.2 *If $r \in R$, then*

(a) *for all $r, s \in R$, $r + J(R)$ and $s + J(R)$ are adjacent in $G(R/J(R))$ if and only if every element in $r + J(R)$ is adjacent to every element in $s + J(R)$;*
(b) *$r + J(R)$ is a clique in $G(R)$ if and only if $2r$ is invertible;*
(c) *$r + J(R)$ is a coclique in $G(R)$ if and only if $2r$ is not invertible.*

Proof (a) ($\Rightarrow$) follows from [1, Lemma 2.7(a)].

($\Leftarrow$) Let $j, j' \in J(R)$. Then by our assumption $(r + j) + (s + j') \in U(R)$. Therefore, there is a unit element u in R such that $r + s - u = -(j + j') \in J(R)$. Hence, $r + s - u + J(R) = J(R)$, which implies that $(r + J(R)) + (s + J(R)) \in U(R/J(R))$ as required.

(b) ($\Rightarrow$) Let $j, j' \in J(R)$. Then there is a unit element u in R such that $2r + j + j' = u$. Therefore, $2ru^{-1} = 1 - u^{-1}(j + j')$. Now, since $u^{-1}(j + j') \in J(R)$, $2ru^{-1}$ and so $2r$ is invertible. Note that if $1 - u^{-1}(j + j')$ is not unit, it should be contained in some maximal ideal $\mathfrak{m}$ of R. Since $u^{-1}(j + j') \in \mathfrak{m}$, we should have $1 \in \mathfrak{m}$ which is contradiction.

($\Leftarrow$) follows from [1, Lemma 2.7(c)].

(c) immediately follows from (b) and [1, Lemma 2.7(b)]. □

Lemma 5.3 *Assume that R is not reduced and $r \in R$.*

(a) *The following statements are equivalent:*
   1) *$r + J(R)$ is a coclique in $G(R)$;*
   2) *$r + \mathrm{Nil}(R)$ is a coclique in $G(R)$;*
   3) *$2r \notin U(R)$.*
(b) *The following statements are equivalent:*
   1) *$r + J(R)$ is a clique in $G(R)$;*
   2) *$r + \mathrm{Nil}(R)$ is a clique in $G(R)$;*
   3) *$2r \in U(R)$.*

Proof (a) ($1 \Rightarrow 2$) follows from the fact that $r + \mathrm{Nil}(R) \subseteq r + J(R)$.
($2 \Rightarrow 3$) Since $r + \mathrm{Nil}(R)$ is a coclique of $G(R)$, for every two nilpotent elements s_1 and s_2 of R, there isn't any unit element u of R such that $(r + s_1) + (r + s_2) = u$. Hence, $2r$ can't be a unit.
($3 \Rightarrow 1$) follows from Lemma 5.2(c).

(b) ($1 \Rightarrow 2$) follows from the fact that $r + \mathrm{Nil}(R) \subseteq r + J(R)$.
($2 \Rightarrow 3$) Since $r + \mathrm{Nil}(R)$ is a clique of $G(R)$, for every two nilpotent elements s_1 and s_2 of R, there is a unit element u of R such that $(r + s_1) + (r + s_2) = u$. Hence, $2r = u - s_1 - s_2$ which is a unit.
($3 \Rightarrow 1$) follows from Lemma 5.2(b). □

Note that in view of Lemma 5.3(a), $\mathrm{Nil}(R)$ and $J(R)$ are some coclique of $G(R)$. In the following theorem we investigate the clique number of $G(R[[x]])$.

Theorem 5.4 (i) *If $\mathrm{clique}(R) \neq \mathrm{clique}(R[[x]])$, then there is an element $a \in R$ such that $2a$ is invertible.*
(ii) *If $\mathrm{clique}(R)$ is finite, then $\mathrm{clique}(R[[x]]) = \infty$ if and only if there exists an element $a \in R$ such that $2a$ is invertible.*
(iii) $\mathrm{clique}(R[[x]]) = \begin{cases} \infty & \text{if there is an element } a \in R \text{ such that } 2a \in U(R), \\ \mathrm{clique}(R) & \text{else.} \end{cases}$

Proof (i) It is clear that $\mathrm{clique}(R) \leq \mathrm{clique}(R[[x]])$. Therefore, suppose that

$$\mathrm{clique}(R) < n \leq \mathrm{clique}(R[[x]]).$$

Then there is a complete subgraph with vertices $\{f_1, \ldots, f_n\}$ in $G(R[[x]])$, where for each $1 \leq i \leq n$ we have $f_i = \sum a_{i,j} x^j$. Therefore, the vertices $a_{1,0}, \ldots, a_{n,0}$ induce a complete subgraph in $G(R)$. Now, since $\mathrm{clique}(R) < n$, there are distinct integers i, i' with $1 \leq i, i' \leq n$ such that $a_{i,0} = a_{i',0}$ and $a_{i,0} + a_{i',0} \in U(R)$. Setting $a = a_{i,0}$ implies the result.

(ii) The "only if" implication immediately follows from (i). Therefore, assume that a is an element of R such that $2a \in U(R)$. Hence, for all $n, m \in \mathbb{N}_0$, $a + x^n$ is adjacent to $a + x^m$ in $G(R[[x]])$. This ensures that $\mathrm{clique}(R[[x]]) = \infty$.

(iii) immediately follows from (i), (ii) and the fact that $\mathrm{clique}(R) \leq \mathrm{clique}(R[[x]])$. □

In the sequel, we are interested in comparing the clique numbers of $G(R)$, $G(R[x])$ and $G_n(R[x])$, when $n \in \mathbb{N}_0$ (note that $G_0(R[x]) = G(R)$).

Theorem 5.5 *Let R be a reduced ring. Then we have*

(i) $\text{clique}(R[x]) = \max\{\text{clique}(G_n(R[x])) \mid n \in \mathbb{N}_0\}$,
(ii) $\text{clique}(R[x]) = \text{clique}(R)$*, and*
(iii) *for all $n \in \mathbb{N}$,* $\text{clique}(G_n(R[x])) = 2$ *or*

$$\text{clique}(G_n(R[x])) = \text{clique}(G_0(R[x])) = \text{clique}(R).$$

Proof (i) Since R is reduced, part (6) of Remarks 2.2 implies the result.

(ii) It is clear that $\text{clique}(R) \leq \text{clique}(R[x])$. Therefore, suppose, in contrary, that there is an integer n with $\text{clique}(R) < n \leq \text{clique}(R[x])$. Then there is a complete subgraph with vertices $f_1, \ldots, f_n$ in $R[x]$, where for each $1 \leq i \leq n$ we have $f_i = \sum a_{i,j}x^j$. In view of part (6) of Remarks 2.2, for every distinct integers i, i' with $1 \leq i, i' \leq n$, we have $\deg f_i = m$, $a_{i,0} + a_{i',0} \in U(R)$ and $a_{i,m} = -a_{i',m}$ (note that since $\text{clique}(R) < n$, we have $m \in \mathbb{N}$). Therefore, $\{a_{1,0}, \ldots, a_{n,0}\}$ induces a complete subgraph in $G(R)$. Now, since $\text{clique}(R) < n$, there are distinct integers i, i' with $1 \leq i, i' \leq n$ such that $a_{i,0} = a_{i',0}$ and $a_{i,0} + a_{i',0} \in U(R)$. Hence $2a_{i,0} \in U(R)$, which implies that $2 \in U(R)$. Now, if $n \geq 3$, for all integers $1 \leq i \leq n$ we have $a_{i,m} = -a_{i,m}$. This yields that 2 is a zero divisor, which is a contradiction. Therefore, $n \leq 2$. On the other hand, we know that $\text{clique}(R) \geq 2$, because zero is adjacent to 1. Hence, we have

$$2 \geq n > \text{clique}(R) \geq 2,$$

which is impossible. This completes the proof.

(iii) Let $n \in \mathbb{N}$. In the light of (i) and (ii), we have $\text{clique}(G_n(R[x])) \leq \text{clique}(R)$. In this regard, if $\text{clique}(G_n(R[x])) = \infty$, there isn't any thing to prove. Therefore, on the contrary, assume that $\text{clique}(G_n(R[x])) = t < k \leq \text{clique}(R)$. Let $f_1, \ldots, f_t$ be the vertices of a maximal complete subgraph of $G_n(R[x])$ such that $f_i = \sum_{j=0}^{n} a_{i,j}x^j$. In view of Remarks 2.2(6), for all $1 \leq i, i' \leq t$ with $i \neq i'$ and all $1 \leq j \leq n$, $a_{i,j} = -a_{i',j}$. Therefore, if $t \geq 3$, for all $1 \leq i, i' \leq t$ with $i \neq i'$ and all $1 \leq j \leq n$, we should have $a_{i,j} = a_{i',j}$ and $2a_{i,j} = 0$. On the other hand, let $b_1, \ldots, b_k$ be the vertices of a complete subgraph of $G(R)$. Then $\{b_1 + a_{1,n}x^n, \ldots, b_k + a_{1,n}x^n\}$ makes a complete subgraph in $G_n(R[x])$. This yields that $\text{clique}(G_n(R[x])) \geq k$, which is a contradiction. In addition, if $t \leq 2$, since x^n is adjacent to $1 - x^n$ in $G_n(R[x])$, we have $\text{clique}(G_n(R[x])) \geq 2$, which implies that $\text{clique}(G_n(R[x])) = 2$ as desired. □

In the following result, which is one of our main results of this section, for an arbitrary ring R, we compare the clique number of $G(R[x])$ with the clique number of $G(R)$.

Theorem 5.6 (i) *If* $\text{clique}(R) \neq \text{clique}(R[x])$*, then there is an element $a \in R$ such that $2a$ is invertible and R is not reduced.*
(ii) *If* $\text{clique}(R)$ *is finite, then* $\text{clique}(R[x]) = \infty$ *if and only if there exists an element $a \in R$ such that $2a$ is invertible and R is not reduced.*
(iii) $\text{clique}(R[x]) = \begin{cases} \infty & \text{there is } a \in R \text{ such that } 2a \in U(R) \text{ and } R \text{ is not reduced,} \\ \text{clique}(R) & \text{else.} \end{cases}$

Proof (i) Similar to proof of part (i) of Theorem 5.4, there is an element $a \in R$ such that $2a$ is invertible. In addition, in view of part (ii) of Theorem 5.5, R is not reduced.

(ii) the "only if" implication immediately follows from (i). To prove the "if" part, assume that c is a non-zero nilpotent element of R and a is an element of R such that $2a \in U(R)$. Therefore, for all $n, m \in \mathbb{N}$, $a + cx^n$ is adjacent to $a + cx^m$. Hence, $\text{clique}(R[x]) = \infty$ as required.

(iii) immediately follows from (i), (ii) and the fact that $\text{clique}(R) \leq \text{clique}(R[x])$. □

Proposition 5.7 *Let 2 be a zero-divisor. Then for all $n \in \mathbb{N}$ we have*

$$\text{clique}(R) = \text{clique}(G_n(R[x])) = \text{clique}(R[x]) = \text{clique}(R[[x]]).$$

Proof If $\text{clique}(R) \neq \text{clique}(R[x])$ or $\text{clique}(R) \neq \text{clique}(R[[x]])$, in view of part (i) of Theorems 5.4 and 5.6, we have that 2 is invertible which contradicts our assumption. Therefore,

$$\text{clique}(R) = \text{clique}(R[x]) = \text{clique}(R[[x]]).$$

It is enough to prove that clique(R) = clique$(G_n(R[x]))$, for all $n \in \mathbb{N}$. To this end, suppose that $a_1, \ldots, a_t$ be the vertices of a complete subgraph of $G(R)$. Since 2 is a zero divisor, there is a non-zero element $a \in R$ such that $2a = 0$. Now, for all $n \in \mathbb{N}$, $\{a_1 + ax^n, \ldots, a_t + ax^n\}$ makes a complete subgraph in $G_n(R[x])$. Therefore, clique$(R) \leq$ clique$(G_n(R[x]))$.

Conversely, suppose that $f_1, \ldots, f_t$ be the vertices of a complete subgraph in $G_n(R[x])$ such that for each $1 \leq i \leq t$, $f_i = \sum_{j=0}^{n} a_{i,j} x^j$. Then for all distinct integers i, i' with $1 \leq i, i' \leq t$, we should have $a_{i,0} \neq a_{i',0}$, because if $a_{i,0} = a_{i',0}$ for some distinct integers i, i' with $1 \leq i, i' \leq t$, since f_i is adjacent to f'_i, we should have $2a_{i,0} \in U(R)$ and so 2 is invertible, which is impossible. Hence, $\{a_{1,0}, \ldots, a_{t,0}\}$ makes a complete subgraph in $G(R)$. Therefore, clique$(G_n(R[x])) \leq$ clique(R). Therefore, clique(R) = clique$(G_n(R[x]))$ as desired. □

Theorem 5.8 *If R is not a reduced ring, then for all $n \in \mathbb{N}_0$ we have*

$$\text{clique}(G_n(R[x])) = \text{clique}(R).$$

Proof Suppose that $n \in \mathbb{N}, a_1, \ldots, a_t$ are the vertices of a complete subgraph of $G(R)$ and s is a non-zero nilpotent element of R. Then $\{a_1 + sx^n, \ldots, a_t + sx^n\}$ makes a complete subgraph in $G_n(R[x])$. Therefore, clique$(G_n(R[x])) \geq$ clique(R). If clique$(G_n(R[x])) = 2$, since clique$(R) \geq 2$, the result holds in this case. Therefore, assume that

$$\text{clique}(G_n(R[x])) \geq k \geq \text{clique}(R),$$

where k is an integer greater than 2. Now, suppose that $\{f_1, \ldots, f_k\}$ is a complete subgraph in $G_n(R[x])$ such that for each $1 \leq i \leq k$, $f_i = \sum_{j=0}^{n} a_{i,j} x^j$. Then for every integers i and j with $1 \leq i \leq k$ and $1 \leq j \leq n$, if $a_{i,j} \neq 0$, then $k \geq 0$ implies that there are elements b and c in R such that $a_{i,j} + b, b + c, c + a_{i,j} \in \mathrm{Nil}(R)$. Therefore, there are nilpotent elements r_1, r_2 and r_3 in R such that $a_{i,j} + b = r_1$, $b + c = r_2$ and $c + a_{i,j} = r_3$. Therefore, $2a_{i,j} = r_1 + r_3 - r_2 \in \mathrm{Nil}(R)$, which implies that $2 \in Z(R)$ or $a_{i,j} \in \mathrm{Nil}(R)$. Now, if $2 \in Z(R)$, then Proposition 5.7 completes the proof. Assume that $2 \notin Z(R)$ and for all integers i and j with $1 \leq i \leq k$ and $1 \leq j \leq n, a_{i,j} \in \mathrm{Nil}(R)$. Then since for each distinct integers $1 \leq i, j \leq k$ we have $f_i \neq f_j$, for every $1 \leq i \leq k$, there is an integer $t_i \in \mathbb{N}_0$ and elements $b_{i,1}, \ldots, b_{i,t_i} \in \{a_{i,1}, \ldots, a_{i,n}\}$ such that $\{a_{1,0} + \sum_{j=1}^{t_1} b_{1,j}, \ldots, a_{k,0} + \sum_{j=1}^{t_k} b_{k,j}\}$ is a complete subgraph in $G(R)$. Therefore, in this situation we have clique$(G_n(R[x]))$ = clique(R) as desired. □

The following corollary immediately follows from Theorems 5.5(iii) and 5.8.

Corollary 5.9 *For an arbitrary commutative ring R with non-zero identity, we have* clique$(G_n(R[x])) = 2$ *or* clique$(G_n(R[x]))$ = clique(R), *for all $n \in \mathbb{N}$.*

Let G_1 and G_2 be two vertex-disjoint graphs. We recall that the union of the graphs G_1 and G_2, which is denoted by $G_1 \cup G_2$, is a graph with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. Now, we present the following lemma.

Lemma 5.10 *Let R be a reduced ring such that*

(i) char$(R) = 2$, *or*
(ii) char$(R) \neq 2$ *and $G(R)$ doesn't contain any cycle of odd length.*

Then for all $n \in \mathbb{N}$, the unit graph $G_n(R[x])$ is the union of isomorphic copies of $G_0(R[x])$, and so $G(R[x])$ is the union of isomorphic copies of $G(R)$.

Proof Suppose that $n \in \mathbb{N}$ and $f(x)$ and $g(x)$ are two distinct vertices of $G_n(R[x])$ with constants a and b, respectively. Then, since R is reduced, we have that $f(x)$ is adjacent to $g(x)$ in $G_n(R[x])$ if and only if a is adjacent to b in $G(R)$ and $f(x) - a = -(g(x) - b)$. In addition, assume that $h(x)$ is a polynomial in $G_n(R[x])$ with zero constant. Then, if char$(R) = 2$, it is easy to see that for every two distinct elements c and d in R, the vertices $c + h(x)$ and $d + h(x)$ are adjacent in $G_n(R[x])$ if and only if the vertices c and d are adjacent in $G(R)$. Therefore, we have that the unit graph $G_n(R[x])$ is the union of isomorphic copies of $G_0(R[x])$. In addition, by part (6) of Remarks 2.2, we have that $G(R[x])$ is the union of isomorphic copies of $G(R)$, in this case. Now, in the case that char$(R) \neq 2$ and $G(R)$ doesn't contain any cycle of odd length, assume that $a_1 \sim \cdots \sim a_n$ is a path in $G(R)$. Then for each polynomial $h(x)$ in $G_n(R[x])$ with zero constant, we have the paths $(a_1 + h(x)) \sim (a_2 - h(x)) \sim \cdots \sim (a_{n-1} + (-1)^n h(x)) \sim (a_n + (-1)^{n+1} h(x))$ and $(a_1 - h(x)) \sim (a_2 + h(x)) \sim \cdots \sim (a_{n-1} + (-1)^{n-1} h(x)) \sim (a_n + (-1)^n h(x))$ in $G_n(R[x])$. Therefore, the result holds true. □

We recall that a cycle graph is a graph which consists of a single cycle and the number of edges in a cycle is called its length.

Corollary 5.11 *If R is a finite ring and $G(R)$ is connected, then we have that $G(R[x])$ is the union of cycle graphs if and only if $R \cong \mathbb{Z}_6$.*

Proof Firstly, we show that R is a reduced ring. To this end, assume on the contrary that R is not reduced and choose a non-zero element a in $\mathrm{Nil}(R)$. Assume that $1 \sim f_1(x) \sim \cdots \sim f_n(x) \sim 1$ is a cycle in $G(R[x])$ and set $t = \max\{\deg(f_i(x)) \mid i = 1, \ldots, n\}$. Now, considering the cycle $1 \sim ax^t \sim (1 + ax^t) \sim ax^{t+1} \sim 1$ implies that $G(R[x])$ is not the union of cycle graphs which is a contradiction. Hence, R is a reduced ring. Now, in view of Remarks 2.2(6), $G(R)$ is also the union of cycle graphs. Connectedness of $G(R)$ insures that it must be a cycle graph. Therefore, since R is a reduced ring, by [1, Theorem 3.2], one can conclude that $R \cong \mathbb{Z}_6$. Conversely, if $R \cong \mathbb{Z}_6$, then by Lemma 5.10, the result holds true. □

We end this section by the following proposition.

Proposition 5.12 *Let R be a finite field.*

(i) *If* $\mathrm{char}(R) = 2$, *then $G(R[x])$ is the union of complete graphs and*

$$\mathrm{clique}(G(R[x])) = |R| = \chi(G(R[x])),$$

where $|R|$ is the number of elements in R.

(ii) *If* $\mathrm{char}(R) \neq 2$, *then $G(R[x])\backslash G(R)$ is the union of bipartite graphs. In this case $G(R[x])\backslash G(R)$ is $(m-1)$-regular, where $|R| = m$ and*

$$\chi(G(R[x])\backslash G(R)) = 2.$$

Proof (i) Since R is a field with $\mathrm{char}(R) = 2$, by [1, Theorem 3.4], $G(R)$ is a complete graph. Now, by Lemma 5.10, we have that $G(R[x])$ is the union of complete graphs which all of them are isomorphic to $G(R)$. Therefore, $\mathrm{clique}(G(R[x])) = |R| = \chi(G(R[x]))$.

(ii) Assume that a is a non-zero element in R. Since R is a field and $\mathrm{char}(R) \neq 2$, we have that $a \neq -a$. Note that in $G(R)$, the vertex a is adjacent to all vertices except $-a$. Now, suppose that $f(x)$ is a non-zero polynomial in $R[x]$ with zero constant. Then, by Remarks 2.2(6), $a + f(x)$ (resp., $a - f(x)$) is adjacent to $b - f(x)$ (resp., $b + f(x)$) fr all $b \in R \setminus \{-a\}$ and so the valencies of the vertices $a + f(x)$ and $a - f(x)$ are equal to $m - 1$. In addition, for every non-zero polynomial $f(x)$ in $R[x]$, we have a bipartite subgraph with parts $\{r + f(x) \mid r \in R\}$ and $\{r - f(x) \mid r \in R\}$. Hence, it is easy to see that $G(R[x])\backslash G(R)$ is the union of bipartite graphs and also $G(R[x])\backslash G(R)$ is $(m-1)$-regular. Clearly in this case, $\chi(G(R[x])\backslash G(R)) = 2$. □

6 Planarity of $G(R[x])$ and $G(R[[x]])$

In this section, we investigate the planarity of the unit graphs $G(R[x])$ and $G(R[[x]])$ and some other properties of $G(R[x])$. Firstly, we recall that a graph is said to be planar if it can be drawn in the plane so that its edges intersect only at their ends. A subdivision of a graph G is a graph obtained from G by replacing edges with pairwise internally disjoint paths. A remarkable simple characterization of the planar graphs was given by Kuratowski in 1930. Kuratowski's Theorem says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$, cf. [2, p. 153].

In the next proposition, we show that $G(R[[x]])$ is never planar and also $G(R[x])$ is not planar, whenever R is a non-reduced ring.

Proposition 6.1 (i) *The graph $G(R[[x]])$ is not planar.*
(ii) *If R is a non-reduced ring, then $G(R[x])$ is not planar.*

Proof (i) Since all of the vertices of the set $\{1, 1 + x, 1 + x^2\}$ are adjacent to all of the vertices of the set $\{0, x, x^2\}$ in $G(R[[x]])$, $K_{3,3}$ is a subgraph of $G(R[[x]])$. Therefore, by Kuratowski's Theorem, $G(R[[x]])$ is not planar.

(ii) Assume that a is a non-zero nilpotent element in R. Then all of the vertices of the set $\{1, 1+ax, 1+ax^2\}$ are adjacent to all of the vertices of the set $\{0, ax, ax^2\}$ and so $K_{3,3}$ is a subgraph of $G(R[x])$. Therefore, $G(R[x])$ is not planar in this case. □

Now, for a finite ring R, we investigate the planarity of $G(R[x])$. In the following result, we use the notation $\mathbb{F}_4$ for a field with four elements.

Theorem 6.2 *Suppose that R is a finite ring. Then $G(R[x])$ is planar if and only if R is isomorphic to the ring $\underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{\ell\text{-times}} \times S$, where $\ell \geq 0$ and $S \cong \mathbb{Z}_2$, $S \cong \mathbb{Z}_3$ or $S \cong \mathbb{F}_4$.*

Proof Assume that R is isomorphic to the ring $\underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{\ell\text{-times}} \times S$, where $\ell \geq 0$ and $S \cong \mathbb{Z}_2$, $S \cong \mathbb{Z}_3$ or $S \cong \mathbb{F}_4$. We are supposed to show that $G(R[x])$ is planar. Firstly, suppose that $R \cong \underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{\ell\text{-times}} \times S$, where $\ell \geq 0$ and $S \cong \mathbb{Z}_2$, then the valency of all vertices of the unit graph $G(R)$ is one. In other words, $G(R)$ is the union of complete graphs K_2 and so it is planar. Now, since R is reduced and $\text{char}(R) = 2$, in view of Lemma 5.10, $G(R[x])$ is the union of isomorphic copies of $G(R)$. Therefore, the planarity of $G(R)$ implies that $G(R[x])$ is planar. Let $R \cong \underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{\ell\text{-times}} \times S$, where $\ell \geq 0$ and $S \cong \mathbb{Z}_3$. Then, since $\text{char}(R) \neq 2$ and $G(R)$ doesn't contain any cycle of odd length, by Lemma 5.10, we have that $G(R[x])$ is the union of isomorphic copies of $G(R)$. Hence, it is easy to see that $G(R[x])$ is planar. Finally, assume that $R \cong \underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{\ell\text{-times}} \times S$, where $\ell \geq 0$ and $S \cong \mathbb{F}_4$. Clearly, by [1, Theorem 5.14], $G(R)$ is planar. Now, since R is reduced and $\text{char}(R) = 2$, in view of Lemma 5.10, $G(R[x])$ is the union of isomorphic copies of $G(R)$. Therefore $G(R[x])$ is planar. This completes the proof of this part. Conversely, suppose that $G(R[x])$ is planar. Since $G(R)$ is a subgraph of $G(R[x])$, $G(R)$ is also planar. Therefore, by [1, Theorem 5.14], R is isomorphic to one of the following rings:
$\mathbb{Z}_5, \mathbb{Z}_3 \times \mathbb{Z}_3, \underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{\ell\text{-times}} \times S$, where $\ell \geq 0$ and $S \cong \mathbb{Z}_2, S \cong \mathbb{Z}_3, S \cong \mathbb{Z}_4, S \cong \mathbb{F}_4$ or $S \cong \left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z}_2\right\}$. If $R \cong \underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{\ell\text{-times}} \times S$, where $\ell \geq 0$ and $S \cong \mathbb{Z}_4$ or $S \cong \left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z}_2\right\}$, then R is not reduced and so by Proposition 6.1(ii), $G(R[x])$ is not planar. If $R \cong \mathbb{Z}_5$, then the vertices of the sets $\{x, 1+4x, 1+x, 2+x, 3+x\}$ and $\{-x, 1-x, 2-x\}$ form a subgraph of the unit graph $G(R[x])$, which is isomorphic to a subdivision of $K_{3,3}$, and so it is not planar. If $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, then $G(R[x])$ contains a subdivision of $K_{3,3}$, which consists of the vertices $\{(0,0)+(1,1)x, (1,0)+(1,1)x, (0,1)+(1,1)x\}$ and $\{(1,1)-(1,1)x, (2,1)-(1,1)x, (1,2)-(1,1)x\}$ such that the paths $((1,0)+(1,1)x) \sim ((0,2)-(1,1)x) \sim ((2,0)+(1,1)x) \sim ((2,1)-(1,1)x)$ and $((0,1)+(1,1)x) \sim ((1,0)-(1,1)x) \sim ((0,2)+(1,1)x) \sim ((1,2)-(1,1)x)$ connect the vertices $(1,0)+(1,1)x, (2,1)-(1,1)x$ and $(0,1)+(1,1)x, (1,2)-(1,1)x$, respectively. Hence, $G(R[x])$ is not planar in this case. Now, assume that R is isomorphic to the ring $\underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{\ell\text{-times}} \times S$, where $\ell \geq 0$ and $S \cong \mathbb{Z}_2, S \cong \mathbb{Z}_3$ or $S \cong \mathbb{F}_4$. Then by an argument similar to that above, we conclude that $G(R[x])$ is planar. □

We end this paper by the following result which provides some conditions under which $G(R[x])$ is not planar.

Proposition 6.3 *Let R be a ring such that*

(i) $\text{char}(R) \neq 2$, *or*
(ii) *R is not reduced.*

Then, if there exist distinct elements a_1, a_2 and a_3 of R with $a_i + a_j \in U(R)$, for all $1 \leq i, j \leq 3$, then $G(R[x])$ is not planar.

Proof Suppose that $\text{char}(R) \neq 2$. Then all of the vertices of the set $\{a_1 + x, a_2 + x, a_3 + x\}$ are adjacent to all of the vertices of the set $\{a_1 - x, a_2 - x, a_3 - x\}$ and so $K_{3,3}$ is a subgraph of $G(R[x])$. Therefore, $G(R[x])$ is not planar. Now, in the case that R is not reduced, assume that b is a non-zero nilpotent element of R. Then, all of the vertices of the set $\{a_1, a_2, a_3\}$ are adjacent to all of the vertices of the set $\{a_1 + bx, a_2 + bx, a_3 + bx\}$. Hence, we obtain a subgraph isomorphic to $K_{3,3}$ in the structure of $G(R[x])$. So $G(R[x])$ is not planar. □

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References

1. Ashrafi, N.; Maimani, H.R.; Pournaki, M.R.; Yassemi, S.: Unit graphs associated with rings. Commun. Algebra **38**, 2851–2871 (2010)
2. Bondy, J.A.; Murty, U.S.R.: Graph Theory with Applications. American Elsevier Publishing Co., Inc., New York (1976)
3. Ehrlich, G.: Unit-regular rings. Portugal. Math. **27**, 209–212 (1968)
4. Goldsmith, B.; Pabst, S.; Scott, A.: Unit sum numbers of rings and modules. Q. J. Math. Oxford Ser. (2) **49**(195), 331–344 (1998)
5. Grimaldi, R.P.: Graphs from rings. Proceedings of the twentieth southeastern conference on combinatorics, graph theory, and computing (Boca Raton, FL, 1989), Congr. Numer. **71**, 95–103 (1990)
6. Henriksen, M.: Two classes of rings generated by their units. J. Algebra **31**, 182–193 (1974)
7. Kelarev, A.V.: Graph Algebras and Automata. Marcel Dekker, New York (2003)
8. Kelarev, A.V.; Ryan, J.; Yearwood, J.: Cayley graphs as classifiers for data mining: the influence of asymmetries. Discrete Math. **309**, 5360–5369 (2009)
9. Kumar, N.M.; Sharma, P.K.: A Graphical Representation of Rings via Automorphism Groups (arXiv:1003.0174)
10. Li, C.H.; Praeger, C.E.: On the isomorphism problem for finite Cayley graphs of bounded valency. Eur. J. Combin. **20**, 279–292 (1999)
11. Maimani, H.R.; Salimi, M.; Sattari, A.; Yassemi, S.: Comaximal graph of commutative rings. J. Algebra **319**(4), 1801–1808 (2008)
12. Praeger, C.E.: Finite transitive permutation groups and finite vertex-transitive graphs. In: Graph Symmetry: Algebraic Methods and Applications. Kluwer, Dordrecht, pp. 277–318 (1997)
13. Sharma, P.K.; Bhatwadekar, S.M.: A note on graphical representation of rings. J. Algebra **176**(1), 124–127 (1995)
14. Thomson, A.; Zhou, S.: Gossiping and routing in undirected triple-loop networks. Networks **55**, 341–349 (2010)
15. Vamos, P.: 2-good rings. Q. J. Math. **56**(3), 417–430 (2005)
16. Wolfson, K.G.: An ideal-theoretic characterization of the ring of all linear transformations. Am. J. Math. **75**, 358–386 (1953)
17. Zelinsky, D.: Every linear transformation is a sum of nonsingular ones. Proc. Am. Math. Soc. **5**, 627–630 (1954)
18. Zhou, S.: A class of arc-transitive Cayley graphs as models for interconnection networks. SIAM J. Discrete Math. **23**, 694–714 (2009)