SURVEY PAPER

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Advancements in fixed point theory in modular function spaces

Abstract The purpose of this paper is to give an outline of the recent results in fixed point theory for asymptotic pointwise contractive and nonexpansive mappings, and semigroups of such mappings, defined on some subsets of modular function spaces. Modular function spaces are natural generalizations of both function and sequence variants of many important, from applications perspective, spaces such as Lebesgue, Orlicz, Musielak–Orlicz, Lorentz, Orlicz–Lorentz, Calderon–Lozanovskii spaces and many others. In the context of the fixed point theory, we will discuss foundations of the geometry of modular function spaces, and other important techniques like extensions of the Opial property and normal structure to modular spaces. We will present a series of existence theorems of fixed points for nonlinear mappings, and of common fixed points for semigroups of mappings. We will also discuss the iterative algorithms for the construction of the fixed points of the asymptotic pointwise nonexpansive mappings and the convergence of such algorithms.

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الملخص

الغرض من هذه الورقة هو تقديم الخطوط العريضة للنتائج الأخيرة لنظرية النقطة الثابتة المتعلقة بالتقارب النقطي لراسمات انكماشية وغير تمددية وأشباه زمر هذه الراسمات المعرفة على مجموعات جزئية من فضاءات دالية ترددية. هذه الفضاءات الدالية الترددية هي تعميمات طبيعية لكل من الدوال والمتتاليات المتغيرة من الجانب التطبيقي ذات الأهمية في فضاءات مثل فضاءات لوبيغ، وأورليشز، وموزيلاك ـ أورليشز، ولورانتز، وأوليشز ـ لورانز، وكالديرون ـ لوزانوفسكي وغيرها كثير. وفي موضوع نظرية النقطة الثابتة، سوف نناقش أسس هندسة الفضاءات الدالية الترددية وغيرها من التقنيات الهامة مثل تمديد خواص أوبيال (Opial) والبنية العمودية للفضاءات الدالية الترددية. وفي هذا الاطار سوف نقدم سلسلة من مبرهنات الوجود لنقاط ثابتة لراسمات غير خطية، ونقاط ثابتة مشتركة لشبه زمرة الراسمات. وسوف نناقش أيضا خوارزميات تكرارية لإنشاء النقاط الثابتة للتقارب النقطي لراسمات غير تمددية وتقارب هذه الخوارزميات.

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1 Introduction

The purpose of this paper is to give an outline of the fixed point theory for mappings defined on some subsets of modular function spaces which are natural generalizations of both function and sequence variants of many important, from applications perspective, spaces such as Lebesgue, Orlicz, Musielak–Orlicz, Lorentz, Orlicz–Lorentz, Calderon–Lozanovskii spaces, and many others.

The importance for applications of modular function spaces consists in the richness of structure of modular function spaces, that—besides being Banach spaces (or F-spaces in a more general settings)—are equipped with modular equivalents of norm or metric notions, and also are equipped with almost everywhere convergence

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and convergence in submeasure. In many cases, particularly in applications to integral operators, approximation and fixed point results, modular type conditions are much more natural and modular type assumptions can be more easily verified than their metric or norm counterparts. There are also important results that can be proved only using the apparatus of modular function spaces. Khamsi et al. [36] gave an example of a mapping which is ρ-nonexpansive but it is not norm-nonexpansive. They demonstrated that for a mapping T to be norm-nonexpansive in a modular function space L_ρ, a stronger than ρ-nonexpansiveness assumption is needed: $\rho(\lambda(T(x) - T(x)) \leq \rho(\lambda(x - y))$ for any $\lambda \geq 0$. From this perspective, the fixed point theory in modular function spaces should be considered as complementary to the fixed point theory in normed spaces and in metric spaces.

The theory of contractions and nonexpansive mappings defined on convex subsets of Banach spaces has been well developed since the 1960s (see e.g. [10,15,20,21,23,40]), and generalized to metric spaces (see e.g. [4,22,33]), and modular function spaces (see e.g. [31,36,37]). The corresponding fixed point results were then extended to larger classes of mappings like asymptotic mappings [32,41], pointwise contractions [39] and asymptotic pointwise contractions and nonexpansive mappings [26,34,42,43].

The proof of the principal fixed point existence result—Theorem 4.9—is of the existential nature and does not describe any algorithm for constructing a fixed point of an asymptotic pointwise ρ-nonexpansive mapping. It is well known that the fixed point construction iteration processes for generalized nonexpansive mappings have been successfully used to develop efficient and powerful numerical methods for solving various non-linear equations and variational problems, often of great importance for applications in various areas of pure and applied science. The author proved convergence to fixed points of some iterative algorithms applied to asymptotic pointwise nonexpansive mappings in Banach spaces [47]. The convergence of similar algorithms in modular function spaces was demonstrated in [13]. Existence of common fixed points of semigroups of pointwise Lipschitzian mappings in Banach spaces has been proved in [48]. Recently the weak and strong convergence of such processes to common fixed points of semigroups of mappings in Banach spaces was demonstrated by Kozlowski and Sims [50]. We would like to emphasize that all convergence theorems presented in this paper define constructive algorithms that can be actually implemented. When dealing with specific applications of these theorems, one should take into consideration how additional properties of the mappings, sets and modulars involved, can influence the actual implementation of the algorithms defined in this paper.

The existence of common fixed points for families of contractions and nonexpansive mappings in Banach spaces have been investigated since the early 1960s, see e.g. DeMarr [14], Browder [10], Belluce and Kirk [6,7], Lim [52], Bruck [11]. The asymptotic approach for finding common fixed points of semigroups of Lipschitzian (but not pointwise Lipschitzian) mappings has been also investigated for some time, see e.g. Tan and Xu [64]. It is worthwhile mentioning the recent studies on the special case, when the parameter set for the semigroup is equal to $\{0, 1, 2, 3, \ldots\}$ and $T_n = T^n$, the nth iterate of an asymptotic pointwise nonexpansive mapping, i.e., such a $T : C \rightarrow C$ that there exists a sequence of functions $\alpha_n : C \rightarrow [0, \infty)$ with $\|T^n(x) - T^n(y)\| \leq \alpha_n(x)\|x - y\|$. Kirk and Xu [43] proved the existence of fixed points for asymptotic pointwise contractions and asymptotic pointwise nonexpansive mappings in Banach spaces, while Hussain and Khamsi [26] extended this result to metric spaces, and Khamsi and Kozlowski [34,35] to modular function spaces. In the context of modular function spaces with Δ_2-property, Khamsi [30] discussed the existence of nonlinear semigroups in Musielak–Orlicz spaces and considered some applications to differential equations. The general existence of common fixed points for semigroups of mappings acting in modular function spaces was proved by Kozlowski [49].

The paper is organized as follows:

(a) Section 2 provides necessary preliminary material and establishes the terminology and key notation conventions.
(b) Section 3 gives a brief exposition of the theory of the modular function space geometry and associated notions.
(c) Section 4 presents the fixed point existence theorems for asymptotic pointwise contractive and nonexpansive mappings acting in modular function spaces.
(d) Section 5 discusses convergence to fixed points for generalized Mann and Ishikawa iterative processes.
(e) Section 6 explores existence of common fixed points of semigroups of mappings acting in modular function spaces.

2 Modular function spaces

Let Ω be a nonempty set and Σ be a nontrivial σ-algebra of subsets of Ω. Let $\mathcal{P}$ be a δ-ring of subsets of Ω, such that $E \cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and $A \in \Sigma$. Let us assume that there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $\Omega = \bigcup K_n$. By $\mathcal{E}$ we denote the linear space of all simple functions with supports from $\mathcal{P}$. By $\mathcal{M}_\infty$ we will denote the space of all extended measurable functions, i.e., all functions $f : \Omega \to [-\infty, \infty]$ such that there exists a sequence $\{g_n\} \subset \mathcal{E}$, $|g_n| \leq |f|$ and $g_n(\omega) \to f(\omega)$ for all $\omega \in \Omega$. By 1_A we denote the characteristic function of the set A.

Definition 2.1 Let $\rho : \mathcal{M}_\infty \to [0, \infty]$ be a nontrivial, convex and even function. We say that ρ is a regular convex function pseudomodular if:

  (i) $\rho(0) = 0$;
 (ii) ρ is monotone, i.e., $|f(\omega)| \leq |g(\omega)|$ for all $\omega \in \Omega$ implies $\rho(f) \leq \rho(g)$, where $f, g \in \mathcal{M}_\infty$;
(iii) ρ is orthogonally subadditive, i.e., $\rho(f 1_{A \cup B}) \leq \rho(f 1_A) + \rho(f 1_B)$ for any $A, B \in \Sigma$ such that $A \cap B \neq \emptyset$, $f \in \mathcal{M}_\infty$;
 (iv) ρ has the Fatou property, i.e., $|f_n(\omega)| \uparrow |f(\omega)|$ for all $\omega \in \Omega$ implies $\rho(f_n) \uparrow \rho(f)$, where $f \in \mathcal{M}_\infty$;
  (v) ρ is order continuous in $\mathcal{E}$, i.e., $g_n \in \mathcal{E}$ and $|g_n(\omega)| \downarrow 0$ implies $\rho(g_n) \downarrow 0$.

Similarly as in the case of measure spaces, we say that a set $A \in \Sigma$ is ρ-null if $\rho(g 1_A) = 0$ for every $g \in \mathcal{E}$. We say that a property holds ρ-almost everywhere if the exceptional set is ρ-null. As usual we identify any pair of measurable sets whose symmetric difference is ρ-null as well as any pair of measurable functions differing only on a ρ-null set. With this in mind we define

$$\mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho) = \{f \in \mathcal{M}_\infty; |f(\omega)| < \infty \; \rho - a.e\}, \tag{2.1}$$

where each $f \in \mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho)$ is actually an equivalence class of functions equal ρ-a.e. rather than an individual function. Where no confusion exists we will write $\mathcal{M}$ instead of $\mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho)$.

Definition 2.2 Let ρ be a regular convex function pseudomodular.

(1)   We say that ρ is a regular convex function semimodular if $\rho(\alpha f) = 0$ for every $\alpha > 0$ implies $f = 0$ $\rho - a.e.$;
(2)   We say that ρ is a regular convex function modular if $\rho(f) = 0$ implies $f = 0$ $\rho - a.e.$;

The class of all nonzero regular convex function modulars defined on Ω will be denoted by $\Re$.

Let us denote $\rho(f, E) = \rho(f 1_E)$ for $f \in \mathcal{M}$, $E \in \Sigma$. It is easy to prove that $\rho(f, E)$ is a function pseudomodular in the sense of Def.2.1.1 in [46] (more precisely, it is a function pseudomodular with the Fatou property). Therefore, we can use all results of the standard theory of modular function spaces as per the framework defined by Kozlowski [44–46], see also Musielak [57] for the basics of the general modular theory.

Remark 2.3 We limit ourselves to convex function modulars in this paper. However, omitting convexity in Definition 2.1 or replacing it by s-convexity would lead to the definition of nonconvex or s-convex regular function pseudomodulars, semimodulars and modulars as in [46].

Definition 2.4 [44–46] Let ρ be a convex function modular.

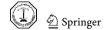(a)   A modular function space is the vector space $L_\rho(\Omega, \Sigma)$, or briefly L_ρ, defined by

$$L_\rho = \{f \in \mathcal{M}; \rho(\lambda f) \to 0 \; as \; \lambda \to 0\}.$$

(b)   The following formula defines a norm in L_ρ (frequently called Luxemburg norm):

$$\|f\|_\rho = \inf\{\alpha > 0; \rho(f/\alpha) \leq 1\}.$$

In this way, Orlicz space is an example of modular function space where the function modular ρ is defined by

$$\rho(f) = \int_{\mathbb{R}} \phi(|f(t)|) dt, \tag{2.2}$$

and Musielak–Orlicz space by

$$\rho(f) = \int_{\mathbb{R}} \phi(t, |f(t)|)dt, \tag{2.3}$$

provided ϕ satisfies necessary conditions, see [46,51,57].

In the following theorem, we recall some of the properties of modular spaces that will be used later on in this paper.

Theorem 2.5 [44–46] *Let $\rho \in \Re$.*

(1) $(L_\rho, \|f\|_\rho)$ *is complete and the norm $\| \cdot \|_\rho$ is monotone w.r.t. the natural order in $\mathcal{M}$.*

(2) $\|f_n\|_\rho \to 0$ *if and only if $\rho(\alpha f_n) \to 0$ for every $\alpha > 0$.*

(3) *If $\rho(\alpha f_n) \to 0$ for an $\alpha > 0$, then there exists a subsequence $\{g_n\}$ of $\{f_n\}$ such that $g_n \to 0$ $\rho - a.e.$*

(4) *If $\{f_n\}$ converges uniformly to f on a set $E \in \mathcal{P}$, then $\rho(\alpha(f_n - f), E) \to 0$ for every $\alpha > 0$.*

(5) *Let $f_n \to f$ $\rho - a.e.$ There exists a nondecreasing sequence of sets $H_k \in \mathcal{P}$ such that $H_k \uparrow \Omega$ and $\{f_n\}$ converges uniformly to f on every H_k (Egoroff Theorem).*

(6) $\rho(f) \leq \liminf \rho(f_n)$ *whenever $f_n \to f$ $\rho - a.e.$ (Note: this property is equivalent to the Fatou Property).*

(7) *Defining $L_\rho^0 = \{f \in L_\rho;\ \rho(f, \cdot) \text{ is order continuous}\}$ and $E_\rho = \{f \in L_\rho;\ \lambda f \in L_\rho^0 \text{ for every } \lambda > 0\}$ we have:*

    (a) $L_\rho \supset L_\rho^0 \supset E_\rho$,

    (b) E_ρ *has the Lebesgue property, i.e., $\rho(\alpha f, D_k) \to 0$ for $\alpha > 0$, $f \in E_\rho$ and $D_k \downarrow \emptyset$.*

    (c) E_ρ *is the closure of $\mathcal{E}$ (in the sense of $\| \cdot \|_\rho$).*

The following definition plays an important role in the theory of modular function spaces.

Definition 2.6 Let $\rho \in \Re$. We say that ρ has the Δ_2-property if

$$\sup_n \rho(2f_n, D_k) \to 0$$

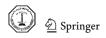whenever $D_k \downarrow \emptyset$ and $\sup_n \rho(f_n, D_k) \to 0$.

Theorem 2.7 *Let $\rho \in \Re$. The following conditions are equivalent:*

(a) ρ *has Δ_2,*

(b) L_ρ^0 *is a linear subspace of L_ρ,*

(c) $L_\rho = L_\rho^0 = E_\rho$,

(d) *if $\rho(f_n) \to 0$, then $\rho(2f_n) \to 0$,*

(e) *if $\rho(\alpha f_n) \to 0$ for an $\alpha > 0$, then $\|f_n\|_\rho \to 0$, i.e., the modular convergence is equivalent to the norm convergence.*

We will also use another type of convergence which is situated between norm and modular convergence. It is defined, among other important terms, in the following definition.

Definition 2.8 Let $\rho \in \Re$.

(a) We say that $\{f_n\}$ is ρ-convergent to f and write $f_n \to f$ (ρ) if and only if $\rho(f_n - f) \to 0$.

(b) A sequence $\{f_n\}$ where $f_n \in L_\rho$ is called ρ-Cauchy if $\rho(f_n - f_m) \to 0$ as $n, m \to \infty$.

(c) A set $B \subset L_\rho$ is called ρ-closed if for any sequence of $f_n \in B$, the convergence $f_n \to f$ (ρ) implies that f belongs to B.

(d) A set $B \subset L_\rho$ is called ρ-bounded if its ρ-diameter $\delta_\rho(B) = \sup\{\rho(f - g); f \in B, g \in B\}$ is finite.

(e) A set $B \subset L_\rho$ is called strongly ρ-bounded if there exists $\beta > 1$ such that $M_\beta(B) = \sup\{\rho(\beta(f-g)); f \in B, g \in B\} < \infty$.

(f) A set $B \subset L_\rho$ is called ρ-compact if for any $\{f_n\}$ in C, there exists a subsequence $\{f_{n_k}\}$ and an $f \in C$ such that $\rho(f_{n_k} - f) \to 0$.

(g) A set $C \subset L_\rho$ is called ρ-a.e. closed if for any $\{f_n\}$ in C which ρ-a.e. converges to some f, then we must have $f \in C$.

(h) A set $C \subset L_\rho$ is called ρ-a.e. compact if for any $\{f_n\}$ in C, there exists a subsequence $\{f_{n_k}\}$ which ρ-a.e. converges to some $f \in C$.

(i) Let $f \in L_\rho$ and $C \subset L_\rho$. The ρ-distance between f and C is defined as

$$d_\rho(f, C) = inf\{\rho(f - g); g \in C\}.$$

Let us note that ρ-convergence does not necessarily imply ρ-Cauchy condition. Also, $f_n \to f$ does not imply in general $\lambda f_n \to \lambda f$, $\lambda > 1$. Using Theorem 2.5 it is not difficult to prove the following

Proposition 2.9 *Let* $\rho \in \Re$.

  (i) L_ρ *is ρ-complete,*
 (ii) ρ-balls $B_\rho(x, r) = \{y \in L_\rho; \rho(x - y) \leq r\}$ *are ρ-closed and ρ-a.e. closed.*

Let us compare different types of compactness introduced in Definition 2.8.

Proposition 2.10 *Let* $\rho \in \Re$. *The following relationships hold for sets* $C \subset L_\rho$:

  (i) *If C is ρ-compact, then C is ρ-a.e. compact.*
 (ii) *If C is $\|.\|_\rho$-compact, then C is ρ-compact.*
(iii) *If ρ satisfies Δ_2, then $\|.\|_\rho$-compactness and ρ-compactness are equivalent in L_ρ.*

Proof (i) follows from Theorem 2.5 part (3).
 (ii) follows from Theorem 2.5 part (2).
(iii) follows from (ii) and from Theorem 2.7 part (e).

$\square$

3 Geometrical properties of modular function spaces

Let us start with the introduction of modular definitions of pointwise contractions, asymptotic pointwise mappings and associated notions [34,35].

Definition 3.1 Let $\rho \in \Re$ and let $C \subset L_\rho$ be nonempty and ρ-closed. A mapping $T : C \to C$ is called a pointwise contraction if there exists $\alpha : C \to [0, 1)$ such that
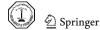
$$\rho(T(f) - T(g)) \leq \alpha(f)\rho(f - g) \quad \text{for any} \ \ f, g \in C, n \geq 1.$$

Definition 3.2 Let $\rho \in \Re$ and let $C \subset L_\rho$ be nonempty and ρ-closed. A mapping $T : C \to C$ is called an asymptotic pointwise mapping if there exists a sequence of mappings $\alpha_n : C \to [0, \infty)$ such that

$$\rho(T^n(f) - T^n(g)) \leq \alpha_n(f)\rho(f - g) \quad \text{for any} \ f, g \in L_\rho.$$

  (i) If $\alpha_n(f) = 1$ for every $f \in L_\rho$ and every $n \in \mathbb{N}$, then T is called ρ-nonexpansive or shortly nonexpansive.
 (ii) If $\{\alpha_n\}$ converges pointwise to $\alpha : C \to [0, 1)$, then T is called asymptotic pointwise contraction.
(iii) If $\lim \sup_{n \to \infty} \alpha_n(f) \leq 1$ for any $f \in L_\rho$, then T is called asymptotic pointwise nonexpansive.
(iv) If $\lim \sup_{n \to \infty} \alpha_n \leq 1$ for any $f \in L_\rho$, i.e., α_n is constant for every n, then T is called asymptotically nonexpansive.
 (v) If $\lim \sup_{n \to \infty} \alpha_n(f) \leq k$ for any $f \in L_\rho$ with $0 < k < 1$, then T is called strongly asymptotic pointwise contraction.

Questions are sometimes asked whether the theory of modular function spaces provides general methods for the consideration of fixed point properties, similarly as this is the case in the Banach space setting. We believe that recent results, see e.g. [3,13,34,35], provide further evidence for the existence of such a general theory. Indeed, the most common approach in the Banach space fixed point theory for generalized nonexpansive mappings is to assume the uniform convexity of the norm which implies the reflexivity, and—via the Milman Theorem—guarantees the weak compactness of the closed bounded sets. As we will see, the notion of a uniform convexity of function modulars in conjunction with the property (R) being the modular equivalence of the Banach space reflexivity [34,35,37], equips us with the powerful tools for proving the fixed point property in modular function spaces. Let us recall that the property (R) represents the most important, from the fixed point theory viewpoint, geometric characterization of reflexive spaces: every nonincreasing sequence of nonempty,

convex, bounded sets has a nonempty intersection. The property (R) also aligns well to the metric equivalents of reflexivity defined by the notions of compact convexity structures [26]. This idea has been further developed in [3] to introduce notions of admissible sets and related modular versions of normal and compact convexity structures. All of this provides a set of powerful techniques for proving existence of common fixed points for commutative families of mappings acting in modular function spaces, and for investigating the topological properties of the set of common fixed points.

Let us start with the discussion of the modular equivalents of uniform convexity of ρ. As demonstrated below, one concept of uniform convexity in normed spaces generates several different types of uniform convexity in modular function spaces. This is due primarily to the fact that in general modulars are not homogeneous.

Definition 3.3 Let $\rho \in \mathfrak{R}$. We define the following uniform convexity type properties of the function modular ρ:

(i) Let $r > 0, \varepsilon > 0$. Define

$$D_1(r, \varepsilon) = \{(f, g); f, g \in L_\rho, \rho(f) \leq r, \rho(g) \leq r, \rho(f - g) \geq \varepsilon r\}.$$

Let

$$\delta_1(r, \varepsilon) = \inf\left\{1 - \frac{1}{r}\rho\left(\frac{f+g}{2}\right); (f, g) \in D_1(r, \varepsilon)\right\}, \quad \text{if } D_1(r, \varepsilon) \neq \emptyset,$$

and $\delta_1(r, \varepsilon) = 1$ if $D_1(r, \varepsilon) = \emptyset$. We say that ρ satisfies $(UC1)$ if for every $r > 0, \varepsilon > 0, \delta_1(r, \varepsilon) > 0$. Note, that for every $r > 0, D_1(r, \varepsilon) \neq \emptyset$, for $\varepsilon > 0$ small enough.

(ii) We say that ρ satisfies $(UUC1)$ if for every $s \geq 0, \varepsilon > 0$ there exists

$$\eta_1(s, \varepsilon) > 0$$

depending on s and ε such that

$$\delta_1(r, \varepsilon) > \eta_1(s, \varepsilon) > 0 \quad \text{for } r > s.$$

(iii) Let $r > 0, \varepsilon > 0$. Define

$$D_2(r, \varepsilon) = \left\{(f, g); f, g \in L_\rho, \rho(f) \leq r, \rho(g) \leq r, \rho\left(\frac{f-g}{2}\right) \geq \varepsilon r\right\}.$$

Let

$$\delta_2(r, \varepsilon) = \inf\left\{1 - \frac{1}{r}\rho\left(\frac{f+g}{2}\right); (f, g) \in D_2(r, \varepsilon)\right\}, \quad \text{if } D_2(r, \varepsilon) \neq \emptyset,$$

and $\delta_2(r, \varepsilon) = 1$ if $D_2(r, \varepsilon) = \emptyset$. We say that ρ satisfies $(UC2)$ if for every $r > 0, \varepsilon > 0, \delta_2(r, \varepsilon) > 0$. Note, that for every $r > 0, D_2(r, \varepsilon) \neq \emptyset$, for $\varepsilon > 0$ small enough.

(iv) We say that ρ satisfies $(UUC2)$ if for every $s \geq 0, \varepsilon > 0$ there exists

$$\eta_2(s, \varepsilon) > 0$$

depending on s and ε such that

$$\delta_2(r, \varepsilon) > \eta_2(s, \varepsilon) > 0 \quad \text{for } r > s.$$

(v) We say that ρ is strictly convex, (SC), if for every $f, g \in L_\rho$ such that $\rho(f) = \rho(g)$ and

$$\rho\left(\frac{f+g}{2}\right) = \frac{\rho(f) + \rho(g)}{2}$$

there holds $f = g$.

Remark 3.4 (i) Let us observe that for $i = 1, 2, \ \delta_i(r, 0) = 0$, and $\delta_i(r, \varepsilon)$ is an increasing function of ε for every fixed r.

(ii)   Note that

$$\delta_1(r, \varepsilon) = inf\{\delta^{'}(r, h); \ h \in L_\rho, \ \rho(h) \geq r\varepsilon\}, \tag{3.1}$$

$$\delta_2(r, \varepsilon) = inf\left\{\delta^{'}(r, h); \ h \in L_\rho, \ \rho\left(\frac{h}{2}\right) \geq r\varepsilon\right\}, \tag{3.2}$$

where

$$\delta^{'}(r, h) = inf\left\{1 - \frac{1}{r}\rho\left(f + \frac{h}{2}\right); \ f \in L_\rho, \rho(f) \leq r, \rho(f + h) \leq r\right\}. \tag{3.3}$$

Proposition 3.5 [35] *The following conditions characterize relationship between the above defined notions:*

(1) $(UUCi)$ *implies* (UCi) *for* $i = 1, 2.$;
(2) $\delta_1(r, \varepsilon) \leq \delta_2(r, \varepsilon)$;
(3) $(UC1)$ *implies* $(UC2)$;
(4) $(UC2)$ *implies* (SC);
(5) $(UUC1)$ *implies* $(UUC2)$;
(6)   *If* $\rho \in \Re$ *satisfies* Δ_2, *then* $(UUC1)$ *and* $(UUC2)$ *are equivalent;*
(7)   *If* ρ *is homogeneous (e.g. is a norm), then all conditions* $(UC1)$, $(UC2)$, $(UUC1)$, $(UUC2)$ *are equivalent and* $\delta_1(r, 2\varepsilon) = \delta_1(1, 2\varepsilon) = \delta_2(1, \varepsilon) = \delta_2(r, \varepsilon)$.

Remark 3.6 Observe that, denoting $\rho_\alpha(u) = \alpha\rho(u)$, and the corresponding moduli of convexity by $\delta_{\rho_\alpha, i}$, where $i = 1, 2$, we have

$$\delta_{\rho_\alpha, i}(r, \varepsilon) = \delta_{\rho, i}\left(\frac{r}{\alpha}, \varepsilon\right), \tag{3.4}$$

or

$$\delta_{\rho, i}(r, \varepsilon) = \delta_{\rho_\alpha, i}(r\alpha, \varepsilon). \tag{3.5}$$

Hence, ρ is (UCx), where (UCx) is any of the conditions from Definition 3.3, if and only if there exists $\alpha > 0$ such that ρ_α is (UCx). In particular, taking $\alpha = \frac{1}{r}$, it is enough to prove any of the conditions defining (UCx) with $r = 1$.

Remark 3.7 Note that the uniform convexity of ρ defined in [37] coincides with our $(UC2)$. In the same paper, the authors proved that in Orlicz spaces over a finite, atomless measure space, both conditions $(UC2)$ and $(UUC2)$ are equivalent.

Remark 3.8 It is known that for a wide class of modular function spaces with the Δ_2 property, the uniform convexity of the Luxemburg norm is equivalent to $(UC1)$. For example, in Orlicz spaces this result can be traced to early papers by Luxemburg [53], Milnes [55], Akimovic [2], and Kaminska [28]. It is also known that, under suitable assumptions, $(UC2)$ in Orlicz spaces is equivalent to the very convexity of the Orlicz function [37,63] and that the uniform convexity of the Orlicz function implies $(UC1)$ [28]. Typical examples of Orlicz functions that do not satisfy the Δ_2 condition but are uniformly convex (and hence very convex) are: $\varphi_1(t) = e^{|t|} - |t| - 1$ and $\varphi_2(t) = e^{t^2} - 1$, [51,55]. See also [25] for the discussion of some geometrical properties of Calderon–Lozanovskii and Orlicz–Lorentz spaces.

The notion of bounded away sequences of real numbers will be used extensively throughout this section.

Definition 3.9 A sequence $\{t_n\} \subset (0, 1)$ is called bounded away from 0 if there exists $0 < a < 1$ such that $t_n \geq a$ for every $n \in \mathbb{N}$. Similarly, $\{t_n\} \subset (0, 1)$ is called bounded away from 1 if there exists $0 < b < 1$ such that $t_n \leq b$ for every $n \in \mathbb{N}$.

The following lemma provides a modular equivalent of a well-known norm property in uniformly convex Banach spaces, see e.g. [62]. It introduces a useful technique which is used extensively for investigating convergence to fixed points in the $(UUC1)$ modular function spaces. It was introduced in [35] for the case $t_n = \frac{1}{2}$ and extended to more general case in [13].

Lemma 3.10 *Let* $\rho \in \Re$ *be* $(UUC1)$ *and let* $\{t_n\} \subset (0, 1)$ *be bounded away from* 0 *and* 1. *If there exists* $R > 0$ *such that*

$$\limsup_{n \to \infty} \rho(f_n) \leq R, \ \limsup_{n \to \infty} \rho(g_n) \leq R, \tag{3.6}$$

$$\lim_{n \to \infty} \rho(t_n f_n + (1 - t_n)g_n) = R, \tag{3.7}$$

then

$$\lim_{n \to \infty} \rho(f_n - g_n) = 0.$$

In the next theorem, we investigate relationship between the uniform convexity of function modulars and the Unique Best Approximant property (for other results on best approximation in modular function spaces, see e.g. [38]). This result, Theorem 3.11 below, is used in the proofs of Theorem 3.12 and Theorem 3.14 to establish relationship between the modular uniform convexity and the property (R) which is a modular equivalent of the Milman–Pettis theorem stating that uniform convexity of a Banach space implies its reflexivity.

Theorem 3.11 [35] *Assume* $\rho \in \Re$ *is* $(UUC2)$. *Let* $C \subset L_\rho$ *be nonempty, convex, and* ρ-*closed. Let* $f \in L_\rho$ *be such that* $d = d_\rho(f, C) < \infty$. *There exists then a unique best* ρ-*approximant of* f *in* C, *i.e., a unique* $g_0 \in C$ *such that*

$$\rho(f - g_0) = d_\rho(f, C).$$

The uniqueness part follows immediately from the Strict Convexity (SC) of ρ (see Proposition 3.5, Part 4). The existence can be proved using the properties of the modulus of convexity and the completeness of the space L_ρ.

The Unique Best Approximant property of the $(UUC2)$ modular function spaces is used for establishing the property (R). As elaborated previously, this is parallel to the well known fact that uniformly convex Banach spaces are reflexive. The countable version of this theorem was proved in [35]. In this paper, we provide a more general version following [3]. The property (R) will be essential for the proof of several fixed point theorems in modular function spaces.

Theorem 3.12 [3,35] *Assume* $\rho \in \Re$ *is* $(UUC2)$. *Let* $\{C_\alpha\}_{\alpha \in \Gamma}$ *be a nonincreasing family of nonempty, convex,* ρ-*closed subsets of* L_ρ, *where* $(\Gamma, \prec)$ *is upward directed. Assume that there exists* $f \in L_\rho$ *such that* $\sup_{\alpha \in \Gamma} d_\rho(f, C_\alpha) < \infty$. *Then,* $\bigcap_{\alpha \in \Gamma} C_\alpha \neq \emptyset$.

Following [37], let us formally define the property (R).

Definition 3.13 We say that L_ρ has property (R) if and only if every nonincreasing sequence $\{C_n\}$ of nonempty, ρ-bounded, ρ-closed, convex subsets of L_ρ has nonempty intersection.

As an immediate consequence of Theorem 3.12 we get the following result.

Theorem 3.14 [35] *Let* $\rho \in \Re$ *be* $(UUC2)$. *Then* L_ρ *has property* (R).

We will establish now a modular version of the parallelogram inequality for uniformly convex modular function spaces. The parallelogram property plays a critical role in the proof of the main fixed point theorem. See the papers of Xu [65] and Beg [5] for the norm and metric versions, respectively.

Lemma 3.15 *For each* $0 < s < r$ *and* $\varepsilon > 0$ *set*

$$\Psi(r, s, \varepsilon) = \inf \left\{ \frac{1}{2}\rho^2(f) + \frac{1}{2}\rho^2(g) - \rho^2 \left(\frac{f + g}{2} \right) \right\}, \tag{3.8}$$

where the infimum is taken over all $f, g \in L_\rho$ *such that* $\rho(f) \leq r$, $\rho(g) \leq r$, $\max(\rho(f), \rho(g)) \geq s$, *and* $\rho(f - g) \geq r\varepsilon$. *If* $\rho \in \Re$ *is* $(UUC1)$, *then* $\Psi(r, s, \varepsilon) > 0$ *for any* $0 < s < r$ *and* $\varepsilon > 0$. *Moreover, for a fixed* $r, s > 0$, *we have*

 (i) $\Psi(r, s, 0) = 0$
 (ii) $\Psi(r, s, \varepsilon)$ *is a nondecreasing function of* ε;

(iii)   *if* $\lim_{n\to\infty} \Psi(r, s, t_n) = 0$, *then* $\lim_{n\to\infty} t_n = 0$.

Let us introduce a notion of a ρ-type, a powerful technical tool which will be used in the proofs of our fixed point results.

Definition 3.16 Let $C \subset L_\rho$ be convex and ρ-bounded. A function $\tau : C \to [0, \infty]$ is called a ρ-type (or shortly a type) if there exists a sequence $\{x_k\}$ of elements of C such that for any $x \in C$ there holds

$$\tau(x) = \limsup_{k\to\infty} \rho(x_k - x).$$

Note that τ is convex provided ρ is convex.

Definition 3.17 Let τ be a ρ-type defined on C. A sequence $\{z_n\} \subset C$ is called a minimizing sequence for τ if $\lim_{n\to\infty} \tau(z_n) = \inf\{\tau(x) : x \in C\}$.

The following lemma establishes a crucial minimizing sequence property of uniformly convex modular function spaces. It will be used in conjunction with the parallelogram property in the proof of the main fixed point result in modular function spaces—Theorem 4.9.

Lemma 3.18 *Assume that $\rho \in \Re$ is $(UUC1)$. Let C be a ρ-closed ρ-bounded convex nonempty subset. Let τ be a ρ-type defined on C. Then any minimizing sequence of τ is ρ-convergent. Its limit is independent of the minimizing sequence.*

Let us introduce modular notions of the Chebyshev radius and of the Chebyshev center which play an important role in the fixed point theory in modular function spaces and will also allow us to introduce a concept of a ρ-normal structure.

Definition 3.19 [36] Let $\rho \in \Re$ and $C \subset L_\rho$ be nonempty.

(a)   The quantity $r_\rho(f, C) = \sup\{\rho(f - g) : g \in C\}$ is called the ρ-Chebyshev radius of C with respect to f.
(b)   The ρ-Chebyshev radius of C is defined by $R_\rho(C) = \inf\{r_\rho(f, C) : f \in C\}$.
(c)   The ρ-Chebyshev center of C is defined as the set

$$\mathcal{C}_\rho(C) = \{f \in C : r_\rho(f, C) = R_\rho(C)\}$$

Note that $R_\rho(C) \le r_\rho(f, C) \le \delta_\rho(C)$ for all $f \in C$, and observe that there is no reason, in general, for $\mathcal{C}_\rho(C)$ to be nonempty.

Definition 3.20 [3] Let $\rho \in \Re$ and $C \subset L_\rho$ be nonempty and ρ-bounded. We say that $A \subset L_\rho$ is an admissible subset of C if

$$A = \bigcap_{i\in I} B_\rho(b_i, r_i) \cap C,$$

where $b_i \in C$, $r_i \ge 0$ and I is an arbitrary index set. By $\mathcal{A}(C)$ we denote the family of all admissible subsets of C.

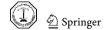Observe that if C is ρ-bounded, then $C \in \mathcal{A}(C)$.

The concept of a normal structure was introduced by Brodskii and Milman [9] for the case of linear normed spaces. It was frequently used to prove existence theorems in fixed point theory. The modular version of normal structure was initially introduced in [36].

Definition 3.21 [3,36] Let $\rho \in \Re$ and $C \subset L_\rho$ be nonempty.

(1)   We say that $\mathcal{A}(C)$ is ρ-normal if for any nonempty $A \in \mathcal{A}(C)$, which has more than one point, we have $R_\rho(C) < \delta_\rho(C)$.
(2)   We say that $\mathcal{A}(C)$ is compact if for any family $\{A_\alpha\}_{\alpha\in\Gamma} \subset \mathcal{A}(C)$ there holds

$$\bigcap_{\alpha\in\Gamma} A_\alpha \ne \emptyset,$$

provided that $\bigcap_{\alpha\in F} A_\alpha \ne \emptyset$ for any finite subset F of Γ.

We summarize relationships between the above notions in the following remark.

Remark 3.22 (i)   It follows directly from the definition that if $\mathcal{A}(L_\rho)$ is compact, then L_ρ has property R.
  (ii)   Let $\rho \in \Re$ be $(UUC2)$ and let $C \subset L_\rho$ be nonempty, convex, ρ-closed, and ρ-bounded. It follows from Theorem 3.12 that $\mathcal{A}(C)$ is compact.
 (iii)   It is easy to prove that under the assumptions of remark (ii), $\mathcal{A}(C)$ is ρ-normal [3].

Let us recall the definition of the Opial property and the Strong Opial property in modular function spaces [31, 34].

Definition 3.23 We say that L_ρ satisfies the ρ-a.e. Opial property if for every $\{f_n\} \in L_\rho$ which is ρ-a.e. convergent to 0 such that there exists $\beta > 1$ for which

$$\sup_n\{\rho(\beta f_n)\} < \infty, \tag{3.9}$$

the following inequality holds for any $g \in E_\rho$ not equal to 0

$$\liminf_{n\to\infty} \rho(f_n) \leq \liminf_{n\to\infty} \rho(f_n + g). \tag{3.10}$$

Definition 3.24 We say that L_ρ satisfies the ρ-a.e. Strong Opial property if for every $\{f_n\} \in L_\rho$ which is ρ-a.e. convergent to 0 such that there exists $\beta > 1$ for which

$$\sup_n\{\rho(\beta f_n)\} < \infty, \tag{3.11}$$

the following equality holds for any $g \in E_\rho$

$$\liminf_{n\to\infty} \rho(f_n + g) = \liminf_{n\to\infty} \rho(f_n) + \rho(g). \tag{3.12}$$

Remark 3.25 Note that the ρ-a.e. Strong Opial property implies ρ-a.e. Opial property [31].

Remark 3.26 In addition, note that, in virtue of Theorem 2.1 in [31], every convex, orthogonally additive function modular ρ has the ρ-a.e. Strong Opial property. Let us recall that ρ is called orthogonally additive if $\rho(f, A \cup B) = \rho(f, A) + \rho(f, B)$ whenever $A \cap B = \emptyset$. Therefore, all Orlicz and Musielak–Orlicz spaces must have the Strong Opial property.

Note that the Opial property in the norm sense does not necessarily hold for several classical Banach function spaces. For instance the norm Opial property does not hold for L^p spaces for $1 \leq p \neq 2$ while the modular Strong Opial property holds in L^p for all $p \geq 1$.

A typical method of proof for the fixed point theorems is to construct a fixed point by finding an element on which a specific type function attains its minimum. To be able to proceed with this method, one has to know that such an element indeed exists. In modular function spaces, the ρ-types are not in general lower semicontinuous in any strong or weak sense and therefore one needs additional assumptions to ensure that ρ-types attain their minima. It turns out that for ρ-a.e. compact sets C the Strong Opial property can be such a convenient additional assumption.

Theorem 3.27 [49] *Let $\rho \in \Re$. Assume that L_ρ has the ρ-a.e. Strong Opial property. Let $C \subset E_\rho$ be a nonempty, strongly ρ-bounded and ρ-a.e. compact convex set. Then any ρ-type defined in C attains its minimum in C.*

4 Existence of fixed points for mappings in modular function spaces

Fixed point theorems has been used extensively in the theory of integral equations and integral inequalities. Since 1930s, many prominent mathematicians like Orlicz and Birnbaum recognized that using the methods of L^p-spaces alone created many complications and in some cases did not allow to solve some non-power type integral equations, see [8]. Hence introduction and then intensive application of the theory of Orlicz spaces and Musielak–Orlicz spaces for solving such problems, see e.g. [51, 57]. These attempts, however, used the norm structures implied by the modulars and hence met several difficulties; some of them discussed towards the end of the previous chapter. As mentioned there, there are examples of ρ-nonexpansive mappings that are

not nonexpansive with respect to the norm associated with this modular. Consider for instance the following situation: let $\Omega = (0, \infty)$, let Σ be the σ-algebra of all Lebesgue measurable subsets of $(0, \infty)$. Define a function modular by

$$\rho(f) = \frac{1}{e^2} \int_0^\infty |f(t)|^{t+1} dm(t). \tag{4.1}$$

Let B be a set of all measurable functions $f : (0, \infty) \to \mathbb{R}$ such that $0 \leq f(t) \leq \frac{1}{2}$. It can be shown [36] that the operator $T : B \to B$ is ρ-nonexpansive but it is not nonexpansive with respect to the norm associated with ρ. In the same paper (see also [46]) it is shown that in some prominent cases such as Urysohn operators of certain types, given an operator T one can construct a function modular ρ_T in such a way that T is ρ_T-nonexpansive.

For ρ-contractions the most natural way seems to be to obtain a version of Banach contraction principle somehow mimicking Banach's original proof. Indeed, the pioneering result from [36] attempts to do this. We quote this result using the notation from the current paper. As in many early results, the authors assumed the Δ_2 property.

Theorem 4.1 [36] *Let $\rho \in \Re$ satisfy Δ_2. Let $C \subset L_\rho$ be nonempty, ρ-closed and ρ-bounded. Let $T : C \to C$ be a ρ-contraction. Then T has a unique fixed point $x_0 \in C$. Moreover the orbit $\{T^n(x)\}$ converges to x_0 for any $x \in C$.*

The authors generalized the above result to a non-Δ_2 case but assumed in addition that C is ρ-a.e. compact and that $C - C \subset L_\rho^0$, see Theorem 2.9 in [36]. Recent results obtained by Khamsi and Kozlowski extended Theorem 4.1 to the case of pointwise ρ-contractions and asymptotic pointwise ρ-contractions replacing Δ_2 by generally less restrictive condition of uniform continuity of ρ.

Definition 4.2 We will say that the function modular ρ is uniformly continuous if for every $\varepsilon > 0$ and $L > 0$ there exists $\delta > 0$ such that

$$|\rho(g) - \rho(h + g)| \leq \varepsilon \quad \text{if } \rho(h) \leq \delta \quad \text{and } \rho(g) \leq L. \tag{4.2}$$

Theorem 4.3 [34] *Let us assume that $\rho \in \Re$ is uniformly continuous and has property (R). Let $C \subset L_\rho$ be nonempty, convex, ρ-closed and ρ-bounded. Let $T : C \to C$ be a pointwise ρ-contraction. Then T has a unique fixed point $x_0 \in C$. Moreover the orbit $\{T^n(x)\}$ converges to x_0 for any $x \in C$.*

Proof The proof uses the technique of the ρ-Chebyshev radius. In particular, it is based on a result showing that if ρ is uniformly continuous, then the infimum of a sequence of ρ-Chebyshev radii

$$r(x) = \inf_{n \geq 0} r_\rho(x, K_n) = \inf_{n \geq 0} \sup\{\rho(x - y); y \in K_n\} \tag{4.3}$$

is ρ-lower semicontinuous in K_∞, where K_∞ is a nonempty intersection of a nonincreasing sequence of nonempty, convex, ρ-closed subsets of C. See Lemma 3.1 and Theorem 3.1 in [34] for details. □

Theorem 4.4 [34] *Let us assume that $\rho \in \Re$ is uniformly continuous and has property (R). Let $C \subset L_\rho$ be nonempty, convex, ρ-closed, and ρ-bounded. Let $T : C \to C$ be an asymptotic pointwise ρ-contraction. Then T has a unique fixed point $x_0 \in C$. Moreover, the orbit $\{T^n(x)\}$ converges to x_0 for any $x \in C$.*

Proof The proof utilizes the technique of ρ-types and is based on the fact that if ρ is uniformly continuous then any ρ-type is ρ-lower semicontinuous. See Lemma 4.1 and Theorem 4.1 in [34] for details. □

Remark 4.5 Let us mention that uniform continuity holds for a large class of function modulars. For instance, it can be proved that in Orlicz spaces over a finite atomless measure [63] or in sequence Orlicz spaces [28] the uniform continuity of the Orlicz modular is equivalent to the Δ_2-type condition.

In order to deal with the modulars not necessarily being uniformly continuous, the following results have been proven. Observe how they generalize Theorem 4.1 to the pointwise and asymptotic pointwise case. The proofs again use the similar techniques to these applied for the uniformly continuous case.

Theorem 4.6 [34] *Let $\rho \in \Re$. Assume that L_ρ has the ρ-a.e. Strong Opial property. Let $C \subset E_\rho$ be a nonempty, ρ-a.e. compact convex set such that there exists $\beta > 1$ such that $\delta_\rho(\beta C) = \sup\{\rho(\beta(x - y)); x, y \in C\} < \infty$. Then any $T : C \to C$ pointwise ρ-contraction has a unique fixed point $x_0 \in C$. Moreover, the orbit $\{T^n(x)\}$ converges to x_0, for any $x \in C$.*

Theorem 4.7 [34] *Let $\rho \in \Re$. Assume that L_ρ has the ρ-a.e. Strong Opial property. Let $C \subset E_\rho$ be a nonempty, ρ-a.e. compact convex set such that there exists $\beta > 1$ such that $\delta_\rho(\beta C) = \sup\{\rho(\beta(x - y)); x, y \in C\} < \infty$. Then any $T : C \to C$ asymptotic pointwise ρ-contraction has a unique fixed point $x_0 \in C$. Moreover the orbit $\{T^n(x)\}$ converges to x_0, for any $x \in C$.*

Similarly as in the Banach space setting, the fixed point existence theorems for the ρ-nonexpansive mappings (and for their pointwise and pointwise asymptotic generalizations) were much harder to obtain. Early attempts assumed several growth control conditions and assumed some absolute continuity type behavior of ρ-convergent sequences of functions (see e.g. Theorem 2.13 in [36]). A more advanced results started showing up in the early 2000s. Let us quote an interesting result by Dominguez–Benavides, Khamsi and Samadi, Theorem 4.2 in [18]. Note however that this theorem still assumes Δ_2 and ρ-a.e. compactness and hence is still far from the elegance of the Browder/Gohde/Kirk classic fixed point result for Banach spaces.

Theorem 4.8 [18] *Let $\rho \in \Re$ satisfy Δ_2 and C be a ρ-closed, ρ-bounded, convex and ρ-a.e. compact subset of L_ρ. Then any $T : C \to C$ asymptotically nonexpansive has a fixed point.*

It was only after a proper modular function space geometry was established by Khamsi and Kozlowski [35] that it was possible to prove an elegant modular version of the Browder/Gohde/Kirk fixed point theorem. Below we present this main result of the modular function space fixed point theory. We also outline its proof. The reader is referred to [35] for further details.

Theorem 4.9 [35] *Assume $\rho \in \Re$ is $(UUC1)$. Let C be a ρ-closed ρ-bounded convex nonempty subset of L_ρ. Then any $T : C \to C$ pointwise asymptotically nonexpansive mapping has a fixed point. Moreover, the set of all fixed points $F(T)$ is convex and ρ-closed.*
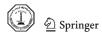
Observe that the statement of the above theorem is completely parallel to that of the Browder/Gohde/Kirk classic fixed point theorem but formulated purely in terms of function modulars without any reference to norms. Also, note that Theorem 4.9 extends outside nonexpansiveness and assumes merely asymptotic pointwise ρ-nonexpansiveness of the mapping T. Therefore, Theorem 4.9 can be actually understood as the modular equivalent of the theorem by Kirk and Xu [43].

The working of our theory can be summarized as follows:

(1)   The Uniform Convexity Property implies The Unique Best Approximant Property (Theorem 3.11).
(2)   The Uniform Convexity Property via The Unique Best Approximant Property implies The Property (R) (Theorem 3.14).
(3)   The Uniform Convexity Property implies The Parallelogram Property (Lemma 3.15).
(4)   The Parallelogram Property implies The Minimizing Sequence Property for type functions when the minimum is strictly positive (Lemma 3.18).
(5)   The Property (R) implies The Minimizing Sequence Property for type functions when the minimum is equal to zero (Lemma 3.18).
(6)   The Minimizing Sequence Property for type functions implies the Fixed Point Property for asymptotic pointwise nonexpansive mappings (Theorem 4.9); the modular limit of a minimizing sequence for a type function defined by an orbit is an obvious candidate for a fixed point. This is indeed the case.

5 Convergence of fixed points iterative algorithms in modular function spaces

Assume $\rho \in \Re$ is $(UUC1)$. Let C be a ρ-closed ρ-bounded convex nonempty subset of L_ρ. Let $T : C \to C$ be a pointwise asymptotically nonexpansive mapping. According to Theorem 4.9, the mapping T has a fixed point. The proof of this important theorem is of the existential nature and does not describe any algorithm for constructing a fixed point of an asymptotic pointwise ρ-nonexpansive mapping. This chapter aims at filling this gap.

Denoting $a_n(x) = \max(\alpha_n(x), 1)$, we note that without loss of generality we can assume that T is asymptotically pointwise nonexpansive if

$$\rho(T^n(x) - T^n(y)) \le a_n(x)\rho(x - y) \quad \text{for all } x, y \in C, \ n \in \mathbb{N}, \tag{5.1}$$

$$\lim_{n \to \infty} a_n(x) = 1, \ a_n(x) \ge 1 \quad \text{for all } x \in C, \ \text{and } n \in \mathbb{N}. \tag{5.2}$$

Define $b_n(x) = a_n(x) - 1$. In view of (5.2), we have

$$\lim_{n \to \infty} b_n(x) = 0. \tag{5.3}$$

The above notation will be consistently used throughout this paper.

By $\mathcal{T}(C)$ we will denote the class of all asymptotic pointwise nonexpansive mappings $T : C \to C$.

In this section, we will impose some restrictions on the behavior of a_n and b_n. This type of assumptions is typical for controlling the convergence of iterative processes for asymptotically nonexpansive mappings, see e.g. [47].

Definition 5.1 Define $\mathcal{T}_r(C)$ as a class of all $T \in \mathcal{T}(C)$ such that

$$\sum_{n=1}^{\infty} b_n(x) < \infty \ \text{ for every } x \in C, \tag{5.4}$$

$$a_n \text{ is a bounded function for every } n \ge 1. \tag{5.5}$$

The following modular version of the Demiclosedness Principle will be used in the proof of our convergence Theorem 5.6. Our proof the Demiclosedness Principle uses the parallelogram inequality valid in the modular spaces with the $(UUC1)$ property (see Lemma 4.2 in [35]).

Theorem 5.2 *Demiclosedness Principle. Let $\rho \in \Re$. Assume that*

(1) ρ *is* $(UCC1)$,
(2) ρ *has Strong Opial Property,*
(3) ρ *has Δ_2 property and is uniformly continuous.*

Let $C \subset L_\rho$ be a nonempty, convex, strongly ρ-bounded and ρ-closed, and let $T \in \mathcal{T}_r(C)$. Let $\{x_n\} \subset C$, and $x \in C$. If $x_n \to x \ \rho - a.e.$ and $\rho(T(x_n) - x_n) \to 0$, then $x \in F(T)$.

Following the original paper by Mann [54], let us start with the definition of the generalized Mann iteration process.

Definition 5.3 Let $T \in \mathcal{T}_r(C)$ and let $\{n_k\}$ be an increasing sequence of natural numbers. Let $\{t_k\} \subset (0, 1)$ be bounded away from 0 and 1. The generalized Mann iteration process generated by the mapping T, the sequence $\{t_k\}$, and the sequence $\{n_k\}$, denoted by $gM(T, \{t_k\}, \{n_k\})$ is defined by the following iterative formula:

$$x_{k+1} = t_k T^{n_k}(x_k) + (1 - t_k)x_k, \quad \text{where } x_1 \in C \text{ is chosen arbitrarily.} \tag{5.6}$$
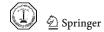
Definition 5.4 We say that a generalized Mann iteration process $gM(T, \{t_k\}, \{n_k\})$ is well defined if

$$\limsup_{k \to \infty} a_{n_k}(x_k) = 1. \tag{5.7}$$

Remark 5.5 Observe that by the definition of asymptotic pointwise nonexpansiveness, $\lim_{k \to \infty} a_k(x) = 1$ for every $x \in C$. Hence we can always select a subsequence $\{a_{n_k}\}$ such that (5.7) holds. In other words, by a suitable choice of $\{n_k\}$ we can always make $gM(T, \{t_k\}, \{n_k\})$ well defined.

Theorem 5.6 [13] *Let $\rho \in \Re$. Assume that*

(1) ρ *is* $(UCC1)$,
(2) ρ *has Strong Opial Property,*
(3) ρ *has Δ_2 property and is uniformly continuous.*

Let $C \subset L_\rho$ be a nonempty, $\rho - a.e.$ compact, convex, strongly ρ-bounded and ρ-closed, and let $T \in \mathcal{T}_r(C)$. Assume that a sequence $\{t_k\} \subset (0, 1)$ is bounded away from 0 and 1. Let $\{n_k\} \subset \mathbb{N}$ and $gM(T, \{t_k\}, \{n_k\})$ be a well-defined generalized Mann iteration process. Assume, in addition, that the set of indices $\mathcal{J} = \{j; n_{j+1} = 1 + n_j\}$ is quasi-periodic. Then there exists $x \in F(T)$ such that $x_n \to x$ ρ-a.e.

Remark 5.7 It is easy to see that we can always construct a sequence $\{n_k\}$ with the quasi-periodic properties specified in the assumptions of Theorem 5.6. When constructing concrete implementations of this algorithm, the difficulty will be to ensure that the constructed sequence $\{n_k\}$ is not "too sparse" in the sense that the generalized Mann process $gM(T, \{t_k\}, \{n_k\})$ remains well defined. The similar, quasi-periodic type assumptions are common in the asymptotic fixed point theory, see e.g. [12,47,50].

The two-step Ishikawa iteration process is a generalization of the one-step Mann process. The Ishikawa iteration process [27], provides more flexibility in defining the algorithm parameters which is important from the numerical implementation perspective.

Definition 5.8 Let $T \in \mathcal{T}_r(C)$ and let $\{n_k\}$ be an increasing sequence of natural numbers. Let $\{t_k\} \subset (0, 1)$ be bounded away from 0 and 1, and $\{s_k\} \subset (0, 1)$ be bounded away from 1. The generalized Ishikawa iteration process generated by the mapping T, the sequences $\{t_k\}$, $\{s_k\}$, and the sequence $\{n_k\}$, denoted by $gI(T, \{t_k\}, \{s_k\}, \{n_k\})$ is defined by the following iterative formula:

$$x_{k+1} = t_k T^{n_k}(s_k T^{n_k}(x_k) + (1 - s_k)x_k) + (1 - t_k)x_k, \quad \text{where } x_1 \in C \text{ is chosen arbitrarily.} \quad (5.8)$$

Definition 5.9 We say that a generalized Ishikawa iteration process $gI(T, \{t_k\}, \{s_k\}, \{n_k\})$ is well defined if

$$\limsup_{k \to \infty} a_{n_k}(x_k) = 1. \quad (5.9)$$

Remark 5.10 Observe that, by the definition of asymptotic pointwise nonexpansiveness, $\lim_{k \to \infty} a_k(x) = 1$ for every $x \in C$. Hence we can always select a subsequence $\{a_{n_k}\}$ such that (5.9) holds. In other words, by a suitable choice of $\{n_k\}$ we can always make $gI(T, \{t_k\}, \{s_k\}, \{n_k\})$ well defined.

Theorem 5.11 [13] *Let $\rho \in \Re$. Assume that*

(1)   *ρ is $(UCC1)$,*
(2)   *ρ has Strong Opial Property,*
(3)   *ρ has Δ_2 property and is uniformly continuous.*

Let $C \subset L_\rho$ be a nonempty, $\rho - a.e.$ compact, convex, strongly ρ-bounded and ρ-closed, and let $T \in \mathcal{T}_r(C)$. Let $T \in \mathcal{T}_r(C)$. Let $\{t_k\} \subset (0, 1)$ be bounded away from 0 and 1, and $\{s_k\} \subset (0, 1)$ be bounded away from 1. Let $\{n_k\}$ be such that the generalized Ishikawa process $gI(T, \{t_k\}, \{s_k\}, \{n_k\})$ is well defined. If, in addition, the set $\mathcal{J} = \{j; n_{j+1} = 1 + n_j\}$ is quasi-periodic, then $\{x_k\}$ generated by $gI(T, \{t_k\}, \{s_k\}, \{n_k\})$ converges ρ-a.e. to a fixed point $x \in F(T)$.

It is interesting that, provided C is ρ-compact, both generalized Mann and Ishikawa processes converge strongly to a fixed point of T even without assuming the Opial property.

Theorem 5.12 [13] *Let $\rho \in \Re$ satisfy conditions $(UUC1)$ and Δ_2. Let $C \subset L_\rho$ be a ρ-compact, ρ-bounded and convex set, and let $T \in \mathcal{T}_r(C)$. Let $\{t_k\} \subset (0, 1)$ be bounded away from 0 and 1, and $\{s_k\} \subset (0, 1)$ be bounded away from 1. Let $\{n_k\}$ be such that the generalized Mann process $gM(T, \{t_k\}, \{n_k\})$ (resp., Ishikawa process $gI(T, \{t_k\}, \{s_k\}, \{n_k\})$) is well defined. Then there exists a fixed point $x \in F(T)$ such that then $\{x_k\}$ generated by $gM(T, \{t_k\}, \{n_k\})$ (resp., $gI(T, \{t_k\}, \{s_k\}, \{n_k\})$) converges strongly to a fixed point of T, that is*

$$\lim_{k \to \infty} \rho(x_k - x) = 0. \quad (5.10)$$

6 Semigroups of mappings in modular function spaces

Let us recall that a family $\{T_t\}_{t\geq 0}$ of mappings forms a semigroup if $T_0(x) = x$, $T_{s+t} = T_s(T_t(x))$. Such a situation is quite typical in mathematics and applications. For instance, in the theory of dynamical systems, the modular function space L_ρ would define the state space and the mapping $(t, x) \to T_t(x)$ would represent the evolution function of a dynamical system. The question about the existence of common fixed points, and about the structure of the set of common fixed points, can be interpreted as a question whether there exist points that are fixed during the state space transformation T_t at any given point of time t, and if yes—what the structure of a set of such points may look like. In the setting of this paper, the state space may be an infinite dimensional. Therefore, it is natural to apply these result to not only to deterministic dynamical systems but also to stochastic dynamical systems.

Let us start with the modular definitions of Lipschitzian—in the modular sense—mappings, and of associated definitions of semigroups of nonlinear mappings.

Definition 6.1 Let $\rho \in \Re$ and let $C \subset L_\rho$ be nonempty and ρ-closed. A mapping $T : C \to C$ is called a ρ-Lipschitzian if there exists a constant $0 < L$ such that

$$\rho(T(f) - T(g)) \leq L\rho(f - g) \quad \text{for any } f, g \in L_\rho.$$

Definition 6.2 A one-parameter family $\mathcal{F} = \{T_t; t \geq 0\}$ of mappings from C into itself is said to be a ρ-Lipschitzian (resp., ρ-nonexpansive) semigroup on C if $\mathcal{F}$ satisfies the following conditions:

(i) $T_0(x) = x$ for $x \in C$;
(ii) $T_{t+s}(x) = T_t(T_s(x))$ for $x \in C$ and $t, s \geq 0$;
(iii) for each $t \geq 0$, T_t is ρ-Lipschitzian (resp., ρ-nonexpansive).

Definition 6.3 A one-parameter family $\mathcal{F} = \{T_t; t \geq 0\}$ of mappings from C into itself is said to be a ρ-contractive semigroup on C if $\mathcal{F}$ satisfies the following conditions:

(i) $T_0(x) = x$ for $x \in C$;
(ii) $T_{t+s}(x) = T_t(T_s(x))$ for $x \in C$ and $t, s \geq 0$;
(iii) for each $t \geq 0$, T_t is a ρ-contraction with a constant $0 < L_t < 1$ such that $\limsup_{t\to\infty} L_t < 1$.

The following two theorems demonstrate the existence of common fixed points for contractive and nonexpansive semigroups, respectively.

Theorem 6.4 [49] *Let $\rho \in \Re$. Assume that L_ρ has the ρ-a.e. Strong Opial property. Let $C \subset E_\rho$ be a nonempty, ρ-a.e. compact convex subset such that $\delta_\rho(\beta C) = \sup\{\rho(\beta(x - y)); x, y \in C\} < \infty$, for some $\beta > 1$. Let $\mathcal{F}$ be a ρ-contractive semigroup on C. Then $\mathcal{F}$ has a unique common fixed point $z \in C$ and for each $u \in C$, $\rho(T_t(u) - z) \to 0$ as $t \to \infty$.*

Theorem 6.5 [49] *Assume $\rho \in \Re$ is $(UUC1)$. Let C be a ρ-closed ρ-bounded convex nonempty subset. Let $\mathcal{F}$ be a nonexpansive semigroup on C. Then the set $F(\mathcal{F})$ of common fixed points is nonempty, ρ-closed and convex.*

Recently, Al-Mezel et al. [3] proved a partial generalization of the second theorem.

Theorem 6.6 [3] *Assume $\rho \in \Re$ is $(UUC1)$. Let C be a ρ-closed ρ-bounded convex nonempty subset of L_ρ. Then any family $\mathcal{F} = \{T_i : i \in I\}$ of commutative ρ-nonexpansive mappings defined on C has a common fixed point. Moreover, the set of all common fixed points denoted $F(\mathcal{F})$ is a one-local retract of C.*

Let us recall that a nonempty subset D of C is said to be a one-local retract of C if for every family $\{B_i : i \in I\}$ of ρ-balls centered in D such that $C \cap \left(\bigcap_{i\in I} B_i\right) \neq \emptyset$, there holds $D \cap \left(\bigcap_{i\in I} B_i\right) \neq \emptyset$. Theorem 6.6 is a corollary from even more general result:

Theorem 6.7 [3] *Let $\rho \in \Re$ and let C be a ρ-closed ρ-bounded convex nonempty subset of L_ρ. Assume that $\mathcal{A}(\mathcal{C})$ is compact and ρ-normal. Then any family $\mathcal{F} = \{T_i : i \in I\}$ of commutative ρ-nonexpansive mappings defined on C has a common fixed point. Moreover, the set of all common fixed points denoted $F(\mathcal{F})$ is a one-local retract of C.*

One can ask a legitimate question about existence of natural examples of semigroups of nonlinear mappings in modular function spaces and their applications. We will present examples addressing these issues. Khamsi [30] considered the following initial value problem.

Theorem 6.8 [30] *Let ρ be a convex Musielak–Orlicz function modular, and $C \subset L_\rho$ be ρ-closed, ρ-bounded and convex. Let $T : C \to C$ be ρ-nonexpansive and norm-continuous, and let $f \in C$ and $A > 0$ be fixed. Consider the following initial value problem:*

$$\begin{cases} u(0) = f \\ u'(t) + (I - T)u(t) = 0, \end{cases} \tag{6.1}$$

where the unknown function $u : [0, A] \to L_\rho$. Assume, in addition, that ρ satisfies the Δ_2 condition. Then there exists a solution u_f to (6.1)*, $u_f(t) \in C$ for every $t \in [0, A]$ and the solution $u_f(t)$ can be obtain as the ρ-limit of $\{u_n(t)\}$ where u_n are defined by the following recurrent sequence:*

$$\begin{cases} u_0(t) = f \\ u_{n+1} = e^{-t}f + \int_0^t e^{s-t}T(u_n(s))ds. \end{cases} \tag{6.2}$$

Let us define

$$S_t(f) = u_f. \tag{6.3}$$

It can be proved that $\{S_t\}$ forms a ρ-nonexpansive semigroup of nonlinear mappings in the sense of Definition 6.2. Hence, if in addition ρ is $(UCC1)$, it follows from Theorem 6.5 that the set of common fixed points for $\{S_t\}$ is nonempty. To interpret this fact, observe that if f_0 is such a common fixed point and we place the initial value of our system (6.1) at f_0, then this point becomes a stationary point of the system, i.e., the constant function $u_{f_0}(t) = f_0$ for every t is the solution of (6.1).

These results can be extended to systems where T is a ρ-Lipschitz operator [1], and applied to the perturbed integral equations in modular function spaces [24].

There exists an extensive literature on the question of representation of some types of semigroups of non-linear mapings acting in Banach spaces, see e.g. [19,29,58–60]. It would be interesting to consider similar representation questions in modular function spaces.

Similarly, it would be interesting to discuss the modular ergodic theory for non-linear semigroups defined in modular function spaces. For the Banach space results of this type, see e.g. [56,61,64].

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