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RESEARCH ARTICLE

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Du Fort–Frankel finite difference scheme for Burgers equation

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Abstract In this paper we apply the Du Fort–Frankel finite difference scheme on Burgers equation and solve three test problems. We calculate the numerical solutions using Mathematica 7.0 for different values of viscosity. We have considered smallest value of viscosity as 10^{-4} and observe that the numerical solutions are in good agreement with the exact solution.

Mathematics Subject Classification 65N06

الملخص

في هذه الورقة نطبّق مخطط الفروقات المحدودة لدي فورت - فرانكل (Du Fort-Frankel) على معادلة برجرز ونحل ثلاث مسائل اختبارية. نحسب الحلول العددية باستخدام Mathematica 7.0 لعدة قيم مختلفة من اللزوجة. أصغر قيمة للزوجة اعتبرناها هي 10^{-4}، ولاحظنا أن الحلول العددية تتفق بشكل جيد مع الحلول المضبوطة.

1 Introduction

The Burgers equation which is going to be examined is

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial x} - \frac{\nu_d}{2}\frac{\partial^2 w}{\partial x^2} = 0, \qquad (x,t) \in (0,1) \times (0,T] \tag{1.1}$$

with the initial condition

$$w(x,0) = f(x), \quad 0 \le x \le 1, \tag{1.2}$$

and the boundary conditions

$$w(0,t) = g_1(t), \quad w(1,t) = g_2(t), \quad 0 \le t \le T, \tag{1.3}$$

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where $\nu_d > 0$ is the coefficient of viscous diffusion, and $f(x)$, $g_1(t)$ and $g_2(t)$ are the sufficiently smooth given functions in space and time domain.

Burgers equation (1.1) can model several physical phenomenon such as traffic flow, shock waves, turbulence problems, cosmology, seismology and continuous stochastic processes. It can also be used to test the various numerical algorithm. Due to its wide range of applicability, researchers [3,5,7] were attracted to it and studied properties of its solution using various numerical techniques.

In [3] the Group-Explicit method is introduced which is semi explicit, unconditionally stable and is of order $O\left(\Delta t + (\Delta x)^2 + \frac{\Delta t}{\Delta x}\right)$ with consistency condition $\frac{\Delta t}{\Delta x} \to 0$ as $\Delta t, \Delta x \to 0$. In [5] using the Hopf–Cole transformation the Burgers equation is reduced into linear heat equation and a standard explicit finite difference approximation is derived. Then assuming that this explicit finite difference scheme has product solution, they derived exact explicit finite difference solution which converges to the Fourier solution as mesh size tends to zero. In [7] Douglas finite difference scheme is considered which is unconditionally stable. In [6] a numerical method is proposed to approximate the solution of the nonlinear Burgers equation which is based on collocation of modified cubic B-splines over finite elements. They computed the numerical solutions to the Burgers equation without transforming the equation and without using the linearization. In a recent review article [2] different techniques for the solution of nonlinear Burgers equation are presented.

In this paper we consider Du Fort–Frankel [8, p. 102] finite difference scheme which is unconditionally stable and has local truncation error $O((\Delta x)^2 + (\Delta t)^2 + (\Delta t/\Delta x)^2)$ which tends to zero as $(\Delta x, \Delta t) \to (0, 0)$ provided $(\Delta t/\Delta x) \to 0$. Du Fort–Frankel method is explicit, so matrix inversions are not required for computations and it is therefore simpler to program and cheaper to solve (on a per time-step basis). We compare the absolute errors for our results to the absolute errors of Douglas finite difference scheme [7] and present this comparison using graphs. It can be observed that if we can have little control over the mesh sizes then the results are promising even for very small coefficient of viscosity ($\nu_d = 10^{-4}$).

2 Exact solution

Hopf [4] and Cole [1] suggested that (1.1) can be reduced to the linear heat equation

$$\phi_t = \frac{\nu_d}{2}\phi_{xx} \tag{2.1}$$

by the non-linear transformation $\psi = -\nu_d(\log\phi)$ and $w = \psi_x$. The Fourier series solution to the linearized heat equation (2.1) is

$$\phi(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \exp\left(-\frac{\nu_d n^2\pi^2 t}{2}\right)\cos n\pi x \tag{2.2}$$

with Fourier coefficients as

$$A_0 = \int_0^1 \exp\left(-\frac{1}{\nu_d}\int_0^x w_0(\xi)\mathrm{d}\xi\right)\mathrm{d}x$$

and

$$A_n = 2\int_0^1 \exp\left(-\frac{1}{\nu_d}\int_0^x w_0(\xi)\mathrm{d}\xi\right)\cos(n\pi x)\mathrm{d}x$$

where $w_0(\xi) = w(\xi, 0)$. Using the Hopf–Cole transformation we have the exact solution

$$w(x,t) = \pi\nu_d \frac{\sum_{n=1}^{\infty} A_n \exp\left(-\frac{\nu_d n^2\pi^2 t}{2}\right) n \sin n\pi x}{A_0 + \sum_{n=1}^{\infty} A_n \exp\left(-\frac{\nu_d n^2\pi^2 t}{2}\right)\cos n\pi x}. \tag{2.3}$$

3 Description of the method

The solution domain is discretized into uniform mesh. We divide the interval [0, 1] into N equal sub-intervals and divide the interval $[0, T]$ into M equal sub-intervals.

Let $h = 1/N$ be the mess width in space and $x_i = ih$ for $i = 1, 2, \ldots, N$. Let $k = T/M$ be the mess width in time and $t_j = jk$ for $k = 0, 1, \ldots, M$.

Du Fort–Frankel discretization [8, p. 102] to linearized heat equation (2.1) is given by

$$(1+2r)\phi_{i,j+1} = (1-2r)\phi_{i,j-1} + 2r(\phi_{i-1,j} + \phi_{i+1,j}) \tag{3.1}$$

where $r = \dfrac{\nu_d k}{2h^2}$ is the discrete approximation to $\phi(x_i, t_j)$ at the point (i, j). The approximate solution of Burgers equation (1.1) in terms of approximate solution of heat equation (2.1) using Hopf–Cole transformation is given by $w_{i,j}(x,t) = \frac{-\nu_d \phi_x}{\phi}\Big|_{i,j} = \frac{-\nu_d}{2}\left(\frac{\phi_{i+1,j} - \phi_{i-1,j}}{h\phi_{i,j}}\right)$. It is stable for all values of r and has the truncation error of $O\left(k^2 + h^2 + \left(\frac{k}{h}\right)^2\right)$ which will tend to zero as $(h,k) \to (0,0)$ provided $\frac{k}{h} \to 0$. Initial data are given on one-line only; the first row ($j = 1$) of values must be calculated by another method. Here we use Schmidt process $\phi_{i,1} = \frac{r\nu_d}{2}(\phi_{i+1,0} + \phi_{i-1,0}) + (1 - r\nu_d)\phi_{i,0}$ to obtain the values at first row ($j = 1$).

4 Numerical results and discussion

In this section we demonstrate the accuracy of the present method by solving three test problems and compare it with the exact solution at different nodal points.The computed results are displayed in Tables 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16 and Figs. 1, 2, 3, 4, 5, 6 at different nodal points for different values of viscosity.

4.1 Problem 1

Consider Equation (1.1) with boundary conditions and initial condition as

$$w(0,t) = w(1,t) = 0, \;\; t > 0, \tag{4.1}$$

$$w(x,0) = \sin \pi x. \tag{4.2}$$

The exact solution of the Burgers equation (1.1) is (2.3) with given Fourier coefficients

$$A_0 = \int_0^1 \exp\left(\frac{-1}{\pi \nu_d}(1 - \cos \pi x)\right) \mathrm{d}x, \quad A_n = 2\int_0^1 \exp\left(\frac{-1}{\pi \nu_d}(1 - \cos \pi x)\cos n\pi x\right) \mathrm{d}x.$$

In Table 2 we have compared our computed numerical solutions with the exact solution for $N = 10, k = 0.001$ and $\nu_d = 2$. In Table 3 we compare the numerical solutions with the exact solutions for different values of ν_d (0.5, 0.125, 0.03125) and N (20, 40, 80). In Tables 5, 8 and 11 we have displayed the numerical and analytical solutions for very small ν_d values 10^{-2}, 10^{-3} and 10^{-4}. In Table 14 we compare our results with results of Kutluay et al. [5] for $\nu_d = 0.02$, $h = 0.0125$ and $k = 0.0001$. This comparison shows that Du Fort–Frankel is giving good results.

4.2 Problem 2

As a second example consider (1.1) with the boundary condition (4.1) and initial condition $w(x,0) = 4x(1-x)$, $0 < x < 1$. The exact solution (2.3) can be obtained in the similar fashion as in Problem 4.1 with the Fourier coefficients as follows:

$$A_0 = \int_0^1 \exp\left(\frac{-2x^2}{3\nu_d}(3 - 2x)\right) \mathrm{d}x, \quad A_n = 2\int_0^1 \exp\left(\frac{-2x^2}{3\nu_d}(3 - 2x)\right) \cos(n\pi x)\mathrm{d}x.$$

Table 1 Comparison of numerical solutions with the exact solutions at different space points of Problems 4.1, 4.2, 4.3 at $T = 0.1$, for $\nu_d = 2$, $h = 0.1$ and $k = 0.001$

x	Problem 4.1		Problem 4.2		Problem 4.3	
	Computed solution	Exact solution	Computed solution	Exact solution	Computed solution	Exact solution
0.1	0.109517	0.10954	0.112918	0.11289	0.306694	0.307354
0.2	0.209758	0.20979	0.216311	0.21625	0.596984	0.598069
0.3	0.291865	0.2919	0.301055	0.30097	0.852678	0.853758
0.4	0.34791	0.34792	0.358998	0.35886	1.05241	1.05295
0.5	0.371591	0.37158	0.383589	0.38342	1.17139	1.1709
0.6	0.359088	0.35905	0.370858	0.37066	1.18336	1.18163
0.7	0.309965	0.30991	0.320261	0.32007	1.06648	1.0638
0.8	0.227876	0.22782	0.235537	0.23537	0.813219	0.810417
0.9	0.120722	0.12069	0.12481	0.12472	0.441575	0.439768

Table 2 Comparison of numerical solutions with the exact solutions at different space points at $T = 0.1$ and $k = 0.001$ for different values of ν_d for Problem 4.1

x	$h = 0.05$, $\nu_d = 0.5$		$h = 0.025$, $\nu_d = 0.125$		$h = 0.0125$, $\nu_d = 0.03125$	
	Computed solution	Exact solution	Computed solution	Exact solution	Computed solution	Exact solution
0.1	0.201986	0.20241	0.228169	0.228675	0.234829	0.235166
0.2	0.392435	0.393201	0.445803	0.446428	0.460554	0.459645
0.3	0.559111	0.560073	0.641386	0.641476	0.666655	0.661882
0.4	0.68847	0.689456	0.801371	0.800237	0.839693	0.828473
0.5	0.765389	0.76625	0.909044	0.906278	0.961708	0.942984
0.6	0.773827	0.774471	0.943621	0.939401	1.00881	0.984667
0.7	0.699498	0.699912	0.880778	0.876009	0.951415	0.928516
0.8	0.535755	0.535983	0.698132	0.694143	0.760721	0.766497
0.9	0.292094	0.292192	0.391534	0.389365	0.428993	0.284701

Table 3 Comparison of numerical solutions with the exact solutions at different space points at $T = 0.1$ and $k = 0.001$ for different values of ν_d for Problem 4.2

x	$h = 0.05$, $\nu_d = 0.5$		$h = 0.025$, $\nu_d = 0.125$		$h = 0.0125$, $d = 0.03125$	
	Computed solution	Exact solution	Computed solution	Exact solution	Computed solution	Exact solution
0.1	0.211009	0.211315	0.248593	0.24903	0.263585	0.263835
0.2	0.408366	0.408941	0.477793	0.478258	0.501686	0.500182
0.3	0.578747	0.579501	0.673289	0.673046	0.704549	0.698449
0.4	0.709105	0.709887	0.824944	0.823281	0.86484	0.851921
0.5	0.786093	0.786732	0.92286	0.919457	0.972233	0.952114
0.6	0.795418	0.795784	0.954344	0.949488	1.01244	0.98746
0.7	0.722576	0.722638	0.901529	0.89614	0.966159	0.941272
0.8	0.557692	0.557546	0.738341	0.733728	0.807204	0.783701
0.9	0.306233	0.306075	0.43355	0.430905	0.497518	0.548686

Table 4 Comparison of numerical solutions with the exact solutions at different space points at $T = 0.1$ and $k = 0.001$ for different values of ν_d for Problem 4.3

x	$h = 0.05$, $\nu_d = 0.5$		$h = 0.025$, $\nu_d = 0.125$		$h = 0.0125$, $\nu_d = 0.03125$	
	Computed solution	Exact solution	Computed solution	Exact solution	Computed solution	Exact solution
0.1	0.39334	0.394264	0.395957	0.396695	0.396928	Can
0.2	0.785588	0.787021	0.793275	0.793091	0.800338	not
0.3	1.17496	1.17609	1.19301	1.18862	1.21617	be
0.4	1.55788	1.55755	1.59555	1.58211	1.64929	computed
0.5	1.92636	1.92321	2.00002	1.97127	2.10304	using
0.6	2.26047	2.2532	2.4034	2.35182	2.5787	Mathematica
0.7	2.50327	2.49142	2.79839	2.71544	3.07465	
0.8	2.48185	2.4675	3.16762	3.04407	3.58445	
0.9	1.76276	1.75224	3.42563	3.25473	4.09173	

Table 5 Comparison of the numerical solutions with the exact solutions at different space points of Problem 4.1 at $T = 10$ for $\nu_d = 0.01$ and $k = 0.01$

x	Numerical solutions				Exact solution
	$N = 10$	$N = 20$	$N = 40$	$N = 80$	
0.1	0.00969751	0.00969206	0.00969306	0.0096873	0.00965798
0.2	0.0194008	0.0193849	0.0193867	0.0193746	0.0193482
0.3	0.0291023	0.0290791	0.0290812	0.0290618	0.0290787
0.4	0.0388169	0.0387738	0.0387761	0.0387478	0.0388117
0.5	0.0485068	0.0484601	0.0484638	0.0484249	0.0484579
0.6	0.0581065	0.058076	0.0580905	0.0580402	0.0579005
0.7	0.0670385	0.0672139	0.0672847	0.067229	0.0669137
0.8	0.0722915	0.0732623	0.0735637	0.0735353	0.073627
0.9	0.0592958	0.0618627	0.0626532	0.0627219	0.0641923

Table 6 Comparison of the numerical solutions with the exact solutions at different space points of Problem 4.2 at $T = 10$ for $\nu_d = 0.01$ and $k = 0.01$

x	Numerical solutions				Exact solution
	$N = 10$	$N = 20$	$N = 40$	$N = 80$	
0.1	0.0097647	0.00974103	0.00974118	0.0097353	0.00973453
0.2	0.0195351	0.0194827	0.0194828	0.0194723	0.019469
0.3	0.0293001	0.0292253	0.0292254	0.0292125	0.0292031
0.4	0.0390783	0.0389682	0.0389681	0.0389565	0.0389358
0.5	0.0488257	0.0487025	0.0487038	0.048699	0.0486598
0.6	0.0584871	0.0583674	0.05838	0.0583909	0.058324
0.7	0.067481	0.0675619	0.0676319	0.06768	0.0675733
0.8	0.0728243	0.0736923	0.0739968	0.0741447	0.0739831
0.9	0.0598502	0.0623514	0.0631584	0.0634842	0.0632856

Table 7 Comparison of the numerical solutions with the exact solutions at different space points of Problem 4.3 at $T = 10$ for $\nu_d = 0.01$ and $k = 0.01$

x	Numerical solutions				Exact solution
	$N = 10$	$N = 20$	$N = 40$	$N = 80$	
0.1	0.00993853	0.00986103	0.00985735	0.00985101	0.00984981
0.2	0.0198835	0.0197225	0.0197152	0.0197038	0.0197005
0.3	0.0298153	0.0295845	0.029574	0.0295602	0.0295514
0.4	0.039764	0.0394462	0.0394332	0.0394209	0.0394
0.5	0.0496677	0.0492993	0.0492864	0.0492814	0.0492395
0.6	0.0595009	0.0590865	0.0590844	0.0590962	0.0590263
0.7	0.068669	0.0684217	0.0684784	0.0685293	0.0684243
0.8	0.0742707	0.0747558	0.0750563	0.0752115	0.0750494
0.9	0.0613733	0.0635649	0.0644039	0.0647492	0.064521

Table 8 Comparison of the numerical solutions with the exact solutions at different space points of Problem 4.1 at $T = 100$ for $\nu_d = 0.001$ and $k = 0.1$

x	Numerical solutions				Exact solution
	$N = 10$	$N = 20$	$N = 40$	$N = 80$	
0.1	0.0010008	0.00100288	0.00100595	0.00099697	0.000993238
0.2	0.00200233	0.00200579	0.00201193	0.00199413	0.00199047
0.3	0.00300231	0.00300874	0.00301794	0.00299165	0.00299277
0.4	0.00400428	0.00401165	0.00402393	0.00398964	0.00399563
0.5	0.0050013	0.00501376	0.00502925	0.0049877	0.00498799
0.6	0.00599212	0.00600971	0.00602928	0.00598162	0.00595643
0.7	0.00691669	0.00696259	0.00699038	0.00693937	0.00688528
0.8	0.00748855	0.00762217	0.00767449	0.00762928	0.00762049
0.9	0.00620309	0.00651922	0.0066196	0.00660351	0.00678091

Table 9 Comparison of the numerical solutions with the exact solutions at different space points of Problem 4.2 at $T = 100$ for $\nu_d = 0.001$ and $k = 0.1$

x	Numerical solutions				Exact solution
	$N = 10$	$N = 20$	$N = 40$	$N = 80$	
0.1	0.0010008	0.00100359	0.00101213	0.000998562	0.000997441
0.2	0.00200233	0.0020072	0.00202429	0.00199731	0.00199488
0.3	0.00300231	0.00301087	0.00303651	0.0029964	0.00299231
0.4	0.00400428	0.0040145	0.00404871	0.00399596	0.00398964
0.5	0.0050013	0.00501733	0.0050603	0.00499556	0.00498627
0.6	0.00599212	0.00601404	0.00606661	0.00599101	0.00597782
0.7	0.0069167	0.00696778	0.00703393	0.00695031	0.006932
0.8	0.00748855	0.00762849	0.00772315	0.00764188	0.0076171
0.9	0.00620309	0.00652623	0.0066636	0.00661617	0.00658827

Table 10 Comparison of the numerical solutions with the exact solutions at different space points of Problem 4.3 at $T = 100$ for $\nu_d = 0.001$ and $k = 0.1$

x	Numerical solutions				Exact solution
	$N = 10$	$N = 20$	$N = 40$	$N = 80$	
0.1	0.0010008	0.00100406	0.0010302	0.00101229	0.000998426
0.2	0.00200233	0.00200815	0.00206047	0.00202471	0.00199695
0.3	0.00300231	0.00301229	0.00309088	0.0030374	0.00299551
0.4	0.00400428	0.0040164	0.00412141	0.0040504	0.00399383
0.5	0.0050013	0.00501973	0.00515148	0.00506327	0.0049913
0.6	0.00599212	0.00601695	0.00617643	0.00607173	0.00598399
0.7	0.0069167	0.00697126	0.00716217	0.00704349	0.00694017
0.8	0.00748855	0.00763274	0.00786613	0.00774459	0.00762715
0.9	0.00620309	0.00653094	0.00679141	0.00670742	0.00659685

Table 11 Comparison of the numerical solutions with the exact solutions at different space points of Problem 4.1 at $T = 1000$ for $\nu_d = 0.0001$ and $k = 0.1$

x	Numerical solutions				Exact solution
	$N = 10$	$N = 20$	$N = 40$	$N = 80$	
0.1	0.000100009	0.000100013	0.000100061	0.000100566	0.000099602
0.2	0.000200018	0.000200026	0.000200121	0.000201132	0.000199612
0.3	0.00030002	0.000300036	0.00030018	0.000301697	0.00030014
0.4	0.000399992	0.000400033	0.00040023	0.000402254	0.000400728
0.5	0.000499781	0.000499937	0.000500206	0.000502743	0.000500245
0.6	0.000598536	0.000599211	0.00059965	0.000602721	0.000597326
0.7	0.000691161	0.000694172	0.000695243	0.000698944	0.000690475
0.8	0.000747855	0.000759866	0.000763432	0.000768091	0.000764653
0.9	0.000619565	0.000649842	0.000658978	0.000664544	0.000681876

Table 12 Comparison of the numerical solutions with the exact solutions at different space points of Problem 4.2 at $T = 1000$ for $\nu_d = 0.0001$ and $k = 0.1$

x	Numerical solutions				Exact solution
	$N = 10$	$N = 20$	$N = 40$	$N = 80$	
0.1	0.000100009	0.000100013	0.000100061	0.000100681	9.99748E-05
0.2	0.000200018	0.000200026	0.000200121	0.000201363	0.000199949
0.3	0.00030002	0.000300036	0.000300181	0.000302044	0.000299923
0.4	0.000399992	0.000400033	0.00040023	0.000402717	0.000399887
0.5	0.000499781	0.000499937	0.000500206	0.000503322	0.000499783
0.6	0.000598536	0.000599211	0.00059965	0.000603416	0.000599177
0.7	0.000691161	0.000694172	0.000695244	0.000699754	0.000694872
0.8	0.000747855	0.000759866	0.000763433	0.000768994	0.000763799
0.9	0.000619565	0.000649842	0.000658979	0.000665357	0.000661322

Table 13 Comparison of the numerical solutions with the exact solutions at different space points of Problem 4.3 at $T = 1000$ for $\nu_d = 0.0001$ and $k = 0.1$

x	Numerical solutions				Exact solution
	$N = 10$	$N = 20$	$N = 40$	$N = 80$	
0.1	0.000100009	0.000100013	0.000100061	0.00010081	0.000099979
0.2	0.000200018	0.000200026	0.000200121	0.00020162	0.000199969
0.3	0.00030002	0.000300036	0.000300181	0.000302429	0.000299961
0.4	0.000399992	0.000400033	0.00040023	0.000403231	0.00039993
0.5	0.000499781	0.000499937	0.000500206	0.000503966	0.000499814
0.6	0.000598536	0.000599211	0.00059965	0.00060419	0.000599225
0.7	0.000691161	0.000694172	0.000695244	0.000700655	0.000695008
0.8	0.000747855	0.000759866	0.000763433	0.000769998	0.000763953
0.9	0.000619565	0.000649842	0.000658979	0.000666261	0.000661156

Table 14 Comparison of the numerical solutions with the exact solutions at different times of Problem 4.1 for $\nu_d = 0.02$, $h = 0.0125$ and $k = 0.0001$

x	T	Numerical solutions		
		Kutluay et al. [5]	Du Fort–Frankel	Exact solution
0.25	0.4	0.34244	0.342253	0.34191
	0.6	0.26905	0.269002	0.26896
	0.8	0.22145	0.221457	0.22148
	1	0.18813	0.188159	0.18819
	3	0.07509	0.0751073	0.07511
0.5	0.4	0.67152	0.668399	0.66071
	0.6	0.53406	0.532218	0.52942
	0.8	0.44143	0.440339	0.43914
	1	0.37568	0.374999	0.37442
	3	0.1502	0.150182	0.15018
0.75	0.4	0.94675	0.939743	0.91026
	0.6	0.78474	0.778807	0.76724
	0.8	0.65659	0.652555	0.6474
	1	0.56135	0.558643	0.55605
	3	0.22502	0.224845	0.22481

Table 15 Comparison of the numerical solutions with the exact solutions at different times of Problem 4.2 for $\nu_d = 2$, $h = 0.025$ and $k = 0.0001$

x	T	Numerical solutions		
		Kutluay et al. [5]	Du Fort–Frankel	Exact solution
0.25	0.01	0.65915	0.660037	0.66006
	0.05	0.42582	0.426343	0.42629
	0.1	0.26121	0.261555	0.26148
	0.15	0.16132	0.161552	0.16148
	0.25	0.06103	0.0611377	0.06109
0.5	0.01	0.9189	0.919707	0.91972
	0.05	0.62745	0.628165	0.62808
	0.1	0.38304	0.383542	0.38342
	1.15	0.23382	0.234169	0.23406
	0.25	0.08715	0.0873058	0.08723
0.75	0.01	0.68304	0.683706	0.68364
	0.05	0.46481	0.465329	0.46525
	0.1	0.28129	0.281668	0.28157
	0.15	0.16957	0.169826	0.16974
	0.25	0.06223	0.0623431	0.06229

Table 16 Comparison of the numerical solutions with the exact solutions at different times of Problem 4.2 for $\nu_d = 0.02$, $h = 0.0125$ and $k = 0.001$

x	T	Numerical solutions		
		Kutluay et al. [5]	Du Fort–Frankel	Exact solution
0.25	0.4	0.36296	0.363255	0.36226
	0.6	0.28217	0.282427	0.28204
	0.8	0.23043	0.230651	0.23045
	1	0.19463	0.194812	0.19469
	3	0.07611	0.0761492	0.07613
0.5	0.4	0.69591	0.693785	0.68368
	0.6	0.55351	0.552281	0.54832
	0.8	0.45625	0.455572	0.45371
	1	0.38705	0.386677	0.38568
	3	0.1522	0.152233	0.15218
0.75	0.4	0.95925	0.95431	0.9205
	0.6	0.80197	0.797371	0.78299
	0.8	0.67267	0.66948	0.66272
	1	0.57501	0.572888	0.56932
	3	0.22796	0.227869	0.22774

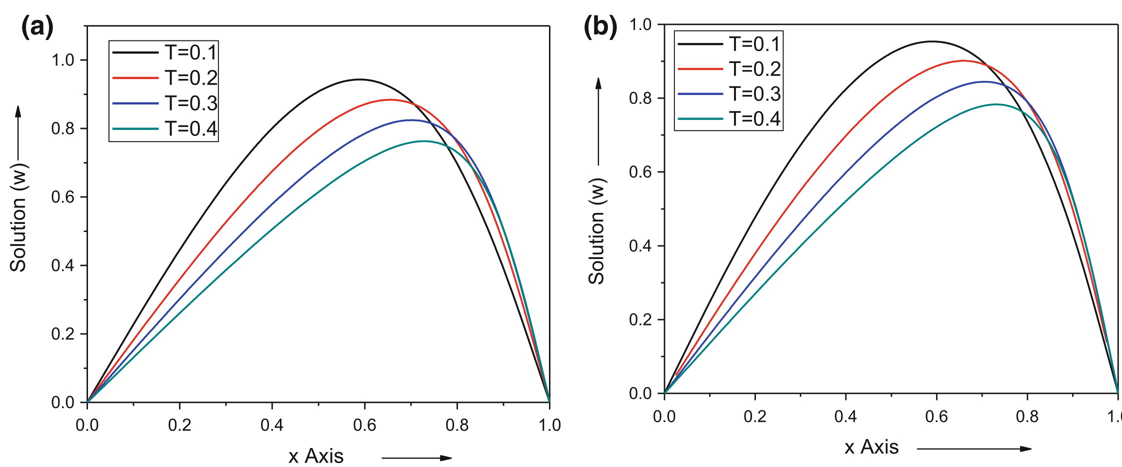

Fig. 1 Numerical solution at different times for $N = 40$, $\nu_d = 0.125$ and $k = 0.001$ for **a** Problem 4.1 and **b** Problem 4.2

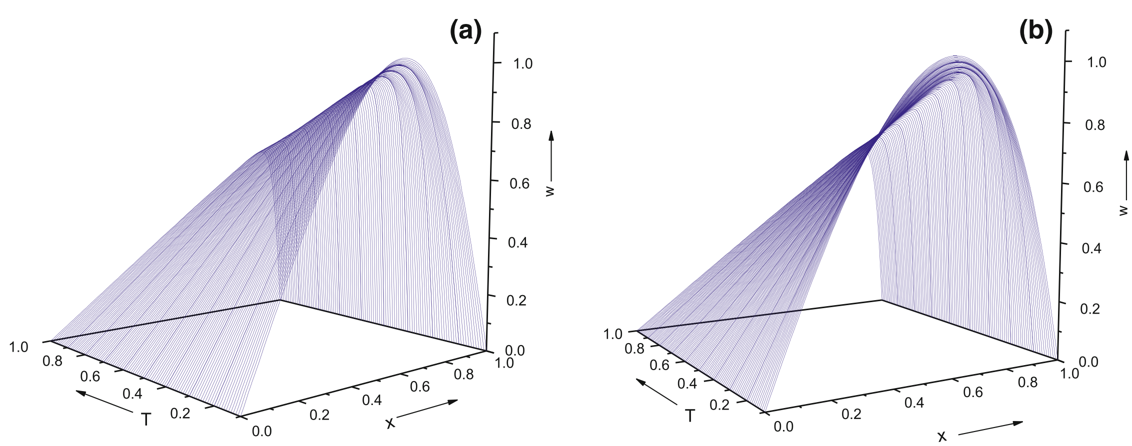

Fig. 2 Numerical solutions of Problem 4.1 at different times for $N = 80$, $\nu_d = 0.03125$ and $k = 0.001$ for **a** Problem 4.1 and **b** Problem 4.2

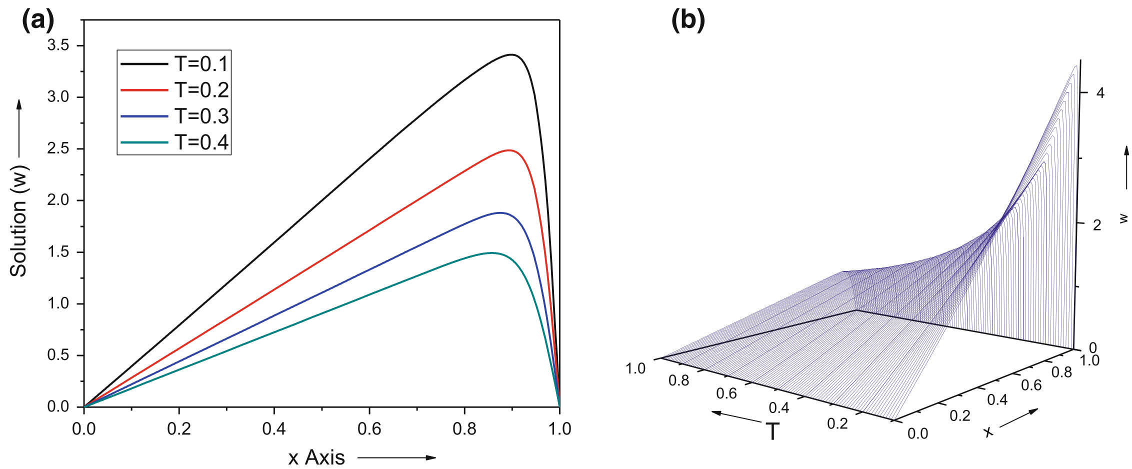

Fig. 3 Numerical solutions of Problem 4.3 at different times for **a** $N = 40$, $\nu_d = 0.125$ and $k = 0.001$, **b** $N = 80$, $\nu_d = 0.03125$ and $k = 0.001$

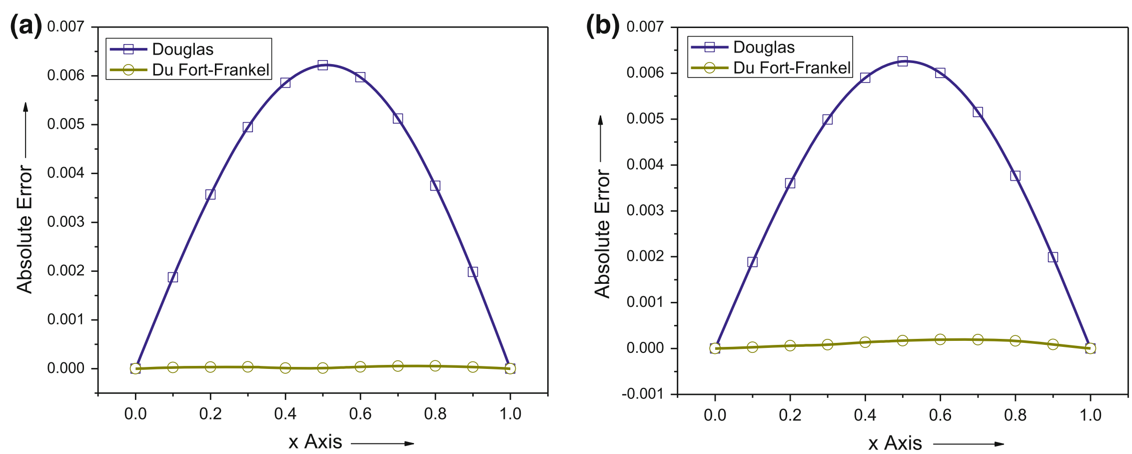

Fig. 4 Comparison of the absolute error for $N = 10$, $T = 0.1$, $\nu_d = 2$ and $k = 0.001$ for **a** Problem 4.1 and **b** Problem 4.2

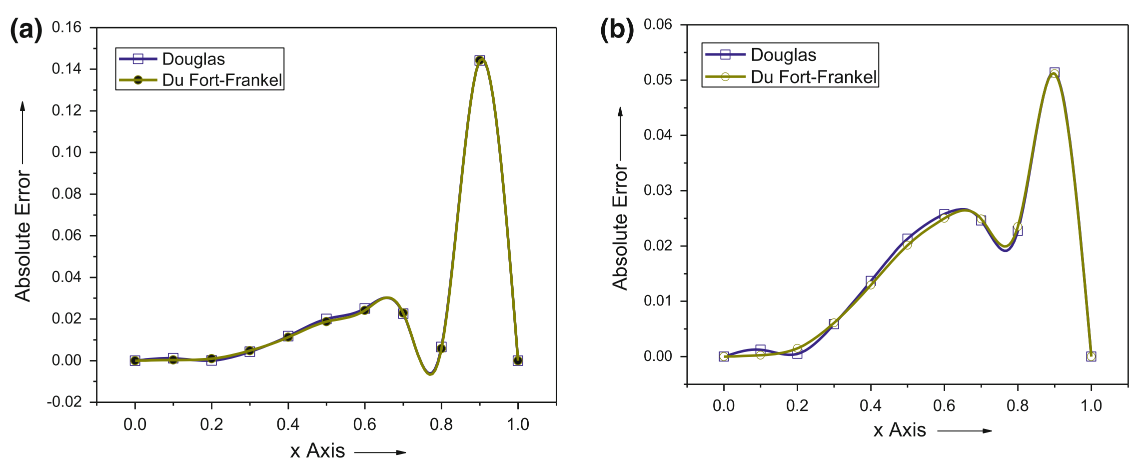

Fig. 5 Comparison of the absolute error for $N = 40$, $T = 0.1$, $\nu_d = 0.125$ and $k = 0.001$ for **a** Problem 4.1 and **b** Problem 4.2

In Table 2 we have compared our computed numerical solutions to the exact solutions for $N = 10$, $k = 0.001$ and $\nu_d = 2$. In Table 4 we compare the numerical solutions with the exact solutions for different values of ν_d $(0.5, 0.125, 0.03125)$ and N $(20, 40, 80)$. In Tables 6, 9 and 12 we have displayed the numerical

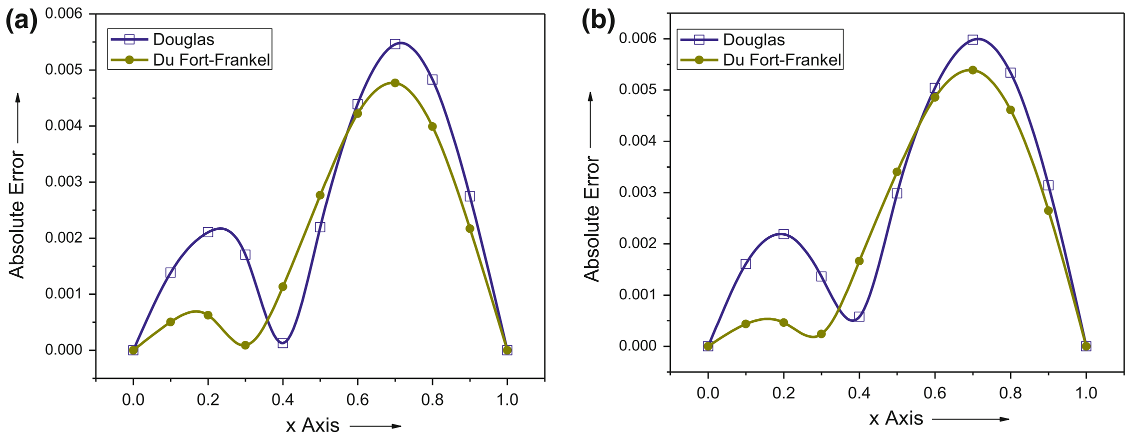

Fig. 6 Comparison of the absolute error for $N = 80$, $T = 0.1$, $\nu_d = 0.03125$ and $k = 0.001$ for **a** Problem 4.1 and **b** Problem 4.2

and analytical solutions for very small ν_d values 10^{-2}, 10^{-3} and 10^{-4}. In Tables 15 and 16 we show that our results are as good as the results of Kutluay et al. [5] for $\nu_d = 2$, $h = 0.025$, $k = 0.0001$ and $\nu_d = 0.02$, $h = 0.0125$, $k = 0.001$.

4.3 Problem 3

Consider Equation (1.1) with boundary condition (4.1) and initial condition as $w(x, 0) = \frac{2\pi \sin \pi x}{2+\cos \pi x}$. The exact solution (2.3) of the equation (1.1) can be obtained in the similar fashion with the Fourier coefficients as follows:

$$A_0 = \int_0^1 \left(\frac{2 + \cos \pi x}{3}\right)^{\frac{2}{\nu_d}} dx, \quad A_n = 2 \int_0^1 \left(\frac{2 + \cos \pi x}{3}\right)^{\frac{2}{\nu_d}} \cos n\pi x \, dx.$$

In Table 1 we have compared our computed numerical solutions to the exact solutions for $N = 10$, $k = 0.001$ and $\nu_d = 2$. In Table 4 we compare the numerical solutions with the exact solutions for different values of ν_d $(0.5, 0.125, 0.03125)$ and N $(20, 40, 80)$. In Tables 7, 10 and 13 we have displayed the numerical and analytical solutions for very small values of ν_d, e.g., 10^{-2}, 10^{-3} and 10^{-4}.

5 Figures

In this section we describe the physical properties of the solutions using $2D$ and $3D$ plots (Figs. 1, 2, 3). In Figs. 4, 5 and 6 we verify the accuracy of the present method by comparing the absolute error of our result to the absolute error of Douglas finite difference scheme [7] for different N $(10, 40, 80)$, ν_d $(2, 0.125, 0.03125)$ and $T = 0.1$ for Problems 4.1 and 4.2. The truncation errors of Du Fort–Frankel method is $O(h^2 + k^2 + (k/h)^2)$ and that of Douglas scheme is $O(h^4 + k^2)$. Since we have kept values of time step k smaller than h, i.e., $k < h$ so truncation error is more for Douglas method. This can easily be observed in Fig. 4a, b when $h = 0.1$ and $k = 0.001$. Similar trends can be observed in Figs. 5 and 6. In these two figures for chosen values of h and k we obtain same accuracy (order of truncation error is $O(10^{-6})$ for both) so we observe that Du Fort–Frankel method gives values as accurate as that of Douglas.

6 Conclusions

Since exact solutions fail to converge if ν_d or T is small. Therefore, while computing numerical solutions for small values of ν_d we have kept value of T high so that we can compute exact solutions and thus numerical solutions are verified. Computed results show that if we can keep the ratio k/h sufficiently small, good results can be obtained. Since Du Fort–Frankel method does not require matrix inversion, it is easy to program and takes less time to compute.

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