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RESEARCH ARTICLE

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Kaplansky classes of complexes

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Abstract In this paper, we introduce and study Kaplansky classes of complexes. We give some results by which one can construct many Kaplansky classes of complexes. We also give some relations between Kaplansky classes of complexes and cotorsion pairs.

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الملخص

نقدم وندرس في هذه الورقة صفوف كابلانسكي للعقد. نعطي بعض النتائج التي تمكننا من تكوين العديد من صفوف كابلانسكي للعقد. نعطي أيضاً بعض العلاقات بين صفوف كابلانسكي للعقد وأزواج الالتواء المرافق.

1 Introduction

The term "Kaplansky class" first appeared in [11]. There it is defined for modules over a ring R as a class $\mathcal{F}$ for which there is a cardinal number $\aleph$ with the following property: given $F \in \mathcal{F}$ and $x \in F$, there exists an $S \in \mathcal{F}$ such that $x \in S \subseteq F$, $|S| \leq \aleph$ and $F/S \in \mathcal{F}$. This definition is based on a result of Kaplansky [19] which states that if P is a projective R-module and $x \in P$ then there is a countably generated submodule S of P with $x \in S$ and with S and P/S projective (or equivalently, with S a summand of P). Recently, Bican, EI Bashir and Enochs [4] proved the so called "Flat Covers Conjecture" by showing that the class of flat R-modules is a Kaplansky class. In [16], Gillespie introduced the notion of Kaplansky classes in a Grothendieck category, and studied in detail the connection between Kaplansky classes and model categories.

In [20], the authors studied covers and envelopes by #-$\mathcal{F}$ complexes using Kaplansky classes in the category of complexes: a class $\mathcal{F}$ of complexes is called a Kaplansky class if there exists a cardinal number $\aleph$ such that for every $C \in \mathcal{F}$ and every $x \in C^k (k \in \mathbb{Z}$ arbitrary), there exists a subcomplex T of C such that $x \in T^k, T, C/T \in \mathcal{F}$ and $|T| \leq \aleph$. In this paper, we further study Kaplansky classes in the category of complexes. We first give some results by which one can construct many Kaplansky classes of complexes (see Theorems 3.3 and 3.5).

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Theorem 1.1 *Let $\aleph \geq |R|$ be a cardinal number. If $\mathcal{F}$ is a strongly $\aleph$-Kaplansky class of R-modules, then the following statements hold.*

(1) *$\widetilde{\mathcal{F}}$ is an $\aleph$-Kaplansky class of complexes.*
(2) *If $\mathcal{F}$ is closed under direct limits, then $\mathcal{E} \cap \widetilde{\#\mathcal{F}}$ is an $\aleph$-Kaplansky class of complexes.*

We also study some relations between Kaplansky classes of complexes and cotrosion pairs by showing the following result (see Theorem 3.10).

Theorem 1.2 *Let $\mathcal{F}$ be a Kaplansky class of complexes closed under extensions and well ordered direct limits and all projective complexes are in $\mathcal{F}$. Then the pair $(\mathcal{F}, \mathcal{F}^{\perp})$ is a perfect cotorsion pair.*

In particular, as an application of the above results, we prove the following result (see Example 4.9).

Theorem 1.3 *Let R be a commutative Noetherian ring and C a semidualizing R-module. Then $(\widetilde{\mathcal{A}_C}, (\widetilde{\mathcal{A}_C})^{\perp})$ and $(\mathcal{E} \cap \widetilde{\#\mathcal{A}_C}, (\mathcal{E} \cap \widetilde{\#\mathcal{A}_C})^{\perp})$ are perfect cotorsion pairs, where $\mathcal{A}_C$ is the Auslander class with respect to C.*

This paper consists of four sections. In Sect. 2, we will recall some notions which are necessary for our proofs of the main results of this paper. In Sect. 3, we will study Kaplansky classes of complexes. In particular, we will prove Theorem 1.1 and Theorem 1.2 as above. In Sect. 4, we will consider some consequences of the results in Sect. 3. In particular, Theorem 1.3 will be proved in this section.

2 Preliminaries

Throughout this paper, R denotes an associative ring with identity and all modules are unitary left R-modules unless otherwise stated. The letter $\aleph$ always denotes an infinite cardinal number.

2.1 Let $\mathcal{A}$ be an abelian category with enough projectives and injectives. Given a class $\mathcal{F}$ of objects of $\mathcal{A}$. Following [8], a morphism $\phi : X \longrightarrow F$ of $\mathcal{A}$ is called an $\mathcal{F}$-preenvelope of X if $F \in \mathcal{F}$ and $\mathrm{Hom}(F, F') \longrightarrow \mathrm{Hom}(X, F') \longrightarrow 0$ is exact for all $F' \in \mathcal{F}$. If, moreover, any $f : F \longrightarrow F$ such that $f\phi = \phi$ is an automorphism of F then $\phi : X \longrightarrow F$ is called an $\mathcal{F}$-envelope of X. An $\mathcal{F}$-precover and an $\mathcal{F}$-cover of X are defined dually. It is immediate that envelopes and covers, if they exist, are unique up to isomorphism.

2.2 Given a class $\mathcal{F}$ of objects of $\mathcal{A}$, write $\mathcal{F}^{\perp} = \{C \in \mathrm{Ob}(\mathcal{A}) |\ \mathrm{Ext}^1(F, C) = 0 \text{ for all } F \in \mathcal{F}\}$ and ${}^{\perp}\mathcal{F} = \{C \in \mathrm{Ob}(\mathcal{A}) |\ \mathrm{Ext}^1(C, F) = 0 \text{ for all } F \in \mathcal{F}\}$. A pair $(\mathcal{F}, \mathcal{G})$ of classes of objects of $\mathcal{A}$ is called a cotorsion pair (or cotorsion theory) [12] if $\mathcal{F}^{\perp} = \mathcal{G}$ and ${}^{\perp}\mathcal{G} = \mathcal{F}$. Two simple examples of cotorsion pairs in the category of R-modules are $(\mathcal{P}roj, R\text{-Mod})$ and $(R\text{-Mod}, \mathcal{I}nj)$, where $\mathcal{P}roj$ (resp., $\mathcal{I}nj$) is the class of projective (resp., injective) R-modules. A cotorsion pair $(\mathcal{F}, \mathcal{G})$ is said to be cogenerated by a set $\mathcal{X}$ of objects of $\mathcal{A}$ if $\mathcal{X}^{\perp} = \mathcal{G}$. A cotorsion pair $(\mathcal{F}, \mathcal{G})$ is said to be perfect if every object of $\mathcal{A}$ has an $\mathcal{F}$-cover and a $\mathcal{G}$-envelope.

2.3 A (cochain) complex $\cdots \longrightarrow X^{-1} \xrightarrow{\delta_X^{-1}} X^0 \xrightarrow{\delta_X^0} X^1 \longrightarrow \cdots$ of R-modules will be denoted by (X, δ_X) or simply X. The nth boundary (resp., cycle, homology) of X is defined as $\mathrm{Im}\delta^{n-1}$ (resp., $\mathrm{Ker}\delta^n$, $\mathrm{Ker}\delta^n/\mathrm{Im}\delta^{n-1}$) and it is denoted by $\mathrm{B}^n(X)$ (resp., $\mathrm{Z}^n(X)$, $\mathrm{H}^n(X)$). A complex X is called exact if $\mathrm{Z}^n(X) = \mathrm{B}^n(X)$ (or equivalently, $\mathrm{H}^n(X) = 0$) for any $n \in \mathbb{Z}$. We let $\mathcal{E}$ denote the class of exact complexes.

$\mathcal{C}$ will denote the abelian category of complexes of R-modules. This category has enough projectives and injectives (see, e.g., [15, Proposition 3.2]). If X and Y are both complexes of R-modules, then by a map (or morphism) $f : X \longrightarrow Y$ of complexes we mean a sequence of R-homomorphisms $f^n : X^n \longrightarrow Y^n$ such that $f^{n+1}\delta_X^n = \delta_Y^n f^n$ for each $n \in \mathbb{Z}$, and f is denoted by $\{f^n\}_{n\in\mathbb{Z}}$. Following [10], $\mathrm{Hom}_R(X, Y)$ denotes the set of maps of complexes from X to Y and $\mathrm{Ext}^i_R(X, Y)$ $(i \geq 1)$ are right derived functors of Hom (these extension functors are not the same as those defined in [3]).

Let $\mathcal{F}$ be a class of R-modules. A complex X is called a #-$\mathcal{F}$ complex [20] if all terms X^i are in $\mathcal{F}$ for $i \in \mathbb{Z}$, and a complex X is called an $\mathcal{F}$-complex [15] if X is exact and all cycle modules $\mathrm{Z}^i(X)$ are in $\mathcal{F}$ for $i \in \mathbb{Z}$. Then $\mathcal{F}$-complexes are #-$\mathcal{F}$ complexes whenever $\mathcal{F}$ is closed under extensions. The classes of #-$\mathcal{F}$ complexes and $\mathcal{F}$-complexes will be denoted by $\widetilde{\#\mathcal{F}}$ and $\widetilde{\mathcal{F}}$, respectively.

2.4 Given a class $\mathcal{F}$ of R-modules, we say that $\mathcal{F}$ is an $\aleph$-Kaplansky class if, for every $F \in \mathcal{F}$ and every $x \in F$, there exists an $S \in \mathcal{F}$ with $x \in S \subseteq F$ and $|S| \leq \aleph$ and $F/S \in \mathcal{F}$. Also $\mathcal{F}$ is called a Kaplansky class if it is an $\aleph$-Kaplansky class for some cardinal number $\aleph$.

Similarly, we give the following definition.

2.5 Given a class $\mathcal{F}$ of complexes, we say that $\mathcal{F}$ is an $\aleph$-Kaplansky class if, for every $F \in \mathcal{F}$ and every $x \in F^k (k \in \mathbb{Z}$ arbitrary), there exists a subcomplex T of F such that $x \in T^k$, T, $F/T \in \mathcal{F}$ and $|T| \leq \aleph$. Also $\mathcal{F}$ is called a Kaplansky class if it is an $\aleph$-Kaplansky class for some cardinal number $\aleph$.

3 Kaplansky classes of complexes

In this section, we study Kaplansky classes of complexes defined in (2.5). Doing so will lead us to the notion of strongly Kaplansky classes of modules, which we will need in order to prove Theorems 3.3 and 3.5.

Definition 3.1 Let $\mathcal{F}$ be a class of R-modules. We say that $\mathcal{F}$ is a strongly $\aleph$-Kaplansky class if, for every $M \in \mathcal{F}$ and every $K \leq M$ with $|K| \leq \aleph$, there exists an R-module S such that $K \leq S \leq M$ and $|S| \leq \aleph$, and S, $M/S \in \mathcal{F}$. Also $\mathcal{F}$ is called a strongly Kaplansky class if $\mathcal{F}$ is a strongly $\aleph$-Kaplansky class for some cardinal $\aleph$.

Remark 3.2 (1) Strongly $\aleph$-Kaplansky classes here are actually the $\aleph$-Kaplansky classes defined by Gillespie ([16]).

(2) If $\mathcal{F}$ is closed under pure submodules and pure quotients, then $\mathcal{F}$ is a strongly $\aleph$-Kaplansky class for any $\aleph \geq |R|$ by [12, Proposition 3.2.2].

Let $\aleph \geq |R|$ be a cardinal number. If $\mathcal{F}$ is an strongly $\aleph$-Kaplansky class of R-modules, then $\mathcal{F}$ is an $\aleph$-Kaplansky class. In the following, we show that if $\mathcal{F}$ is a strongly $\aleph$-Kaplansky class of R-modules then $\widetilde{\mathcal{F}}$ is an $\aleph$-Kaplansky class of complexes. However, we do not know whether $\mathcal{F}$ (strongly) $\aleph$-Kaplansky class implies that $\widetilde{\mathcal{F}}$ is (strongly) $\aleph$-Kaplansky.

Theorem 3.3 *Let $\aleph \geq |R|$ be a cardinal number. If $\mathcal{F}$ is a strongly $\aleph$-Kaplansky class of R-modules, then $\widetilde{\mathcal{F}}$ is an $\aleph$-Kaplansky class of complexes.*

Proof Let $\mathcal{F}$ be a strongly $\aleph$-Kaplansky class of R-modules. Then, for any $M \in \mathcal{F}$ and any $K \leq M$ with $|K| \leq \aleph$, there exists an R-module S such that $K \leq S \leq M$ and $|S| \leq \aleph$, and S, $M/S \in \mathcal{F}$. Now let

$$F \equiv \cdots \longrightarrow F^{-2} \xrightarrow{\delta^{-2}} F^{-1} \xrightarrow{\delta^{-1}} F^{0} \xrightarrow{\delta^{0}} F^{1} \xrightarrow{\delta^{1}} F^{1} \xrightarrow{\delta^{1}} F^{2} \longrightarrow \cdots$$

be any complex in $\widetilde{\mathcal{F}}$. Without loss of generality, we may assume that $x \in F^0$. Then $\delta^0(Rx) \leq \mathrm{Ker}\delta^1 \in \mathcal{F}$ and $|\delta^0(Rx)| \leq \aleph$, and so there exists an R-module S^1 such that $\delta^0(Rx) \leq S^1 \leq \mathrm{Ker}\delta^1$ and $|S^1| \leq \aleph$, and S^1, $(\mathrm{Ker}\delta^1)/S^1 \in \mathcal{F}$.

Consider the R-module $\mathrm{Ker}(\delta^1|_{S^1}) = S^1$. Since $\mathcal{F}$ is exact, there exists an R-module $A^0 \leq F^0$ such that $|A^0| \leq \aleph$, $\delta^0(A^0) = S^1$ and $Rx \leq A^0$. That is, we get an exact sequence $A^0 \xrightarrow{\delta^0} S^1 \xrightarrow{\delta^1} 0$. Note that $\mathrm{Ker}(\delta^0|_{A^0}) \leq \mathrm{Ker}\delta^0 \in \mathcal{F}$ and $|\mathrm{Ker}(\delta^0|_{A^0})| \leq |A^0| \leq \aleph$, then there exists an R-module S^0 such that $\mathrm{Ker}(\delta^0|_{A^0}) \leq S^0 \leq \mathrm{Ker}\delta^0$ and $|S^0| \leq \aleph$, and S^0, $(\mathrm{Ker}\delta^0)/S^0 \in \mathcal{F}$. Then we get an exact sequence $A^0 + S^0 \xrightarrow{\delta^0} S^1 \xrightarrow{\delta^1} 0$, where $\mathrm{Ker}(\delta^0|_{A^0+S^0}) = \mathrm{Ker}\delta^0 \cap (A^0 + S^0) = (\mathrm{Ker}\delta^0 \cap A^0) + S^0 = S^0$ since $\mathrm{Ker}(\delta^0|_{A^0}) \leq S^0 \leq \mathrm{Ker}\delta^0$.

By continuing in this way, this time for $\mathrm{Ker}(\delta^0|_{A^0+S^0})$, we can get an exact sequence

$$T \equiv \cdots \longrightarrow A^{-1} + S^{-1} \xrightarrow{\delta^{-1}} A^0 + S^0 \xrightarrow{\delta^0} S^1 \xrightarrow{\delta^1} 0 \longrightarrow \cdots .$$

Now, by the construction above, one can check easily that T is a subcomplex of F such that $x \in T_0$, T, $F/T \in \widetilde{F}$ and $|T| \leq \aleph$. This implies that $\widetilde{\mathcal{F}}$ is an $\aleph$-Kaplansky class of complexes. □

With Remark 3.2(2), we get the following corollary.

Corollary 3.4 *If $\mathcal{F}$ is closed under pure submodules and pure quotients, then $\widetilde{\mathcal{F}}$ is an $\aleph$-Kaplansky class of complexes for any $\aleph \geq |R|$.*

Theorem 3.5 *Let $\aleph \geq |R|$ be a cardinal. If $\mathcal{F}$ is a strongly $\aleph$-Kaplansky class of R-modules closed under direct limits, then $\mathcal{E} \cap \widetilde{\#\mathcal{F}}$ is an $\aleph$-Kaplansky class of complexes.*

Proof Let $\mathcal{F}$ be a strongly $\aleph$-Kaplansky class of R-modules. Then, for every $M \in \mathcal{F}$ and every $K \leq M$ with $|K| \leq \aleph$, there exists an R-module S such that $K \leq S \leq M$, $|S| \leq \aleph$ and $S, M/S \in \mathcal{F}$. Now let

$$F \equiv \cdots \longrightarrow F^{-2} \xrightarrow{\delta^{-2}} F^{-1} \xrightarrow{\delta^{-1}} F^0 \xrightarrow{\delta^0} F^1 \xrightarrow{\delta^1} F^2 \longrightarrow \cdots$$

be any complex in $\mathcal{E} \cap \widetilde{\#\mathcal{F}}$. Without loss of generality, we may assume that $x \in F^0$. Note that $Rx \leq F^0 \in \mathcal{F}$ and $|Rx| \leq \aleph$, then there exists an R-module S_1^0 such that $|S_1^0| \leq \aleph$, $Rx \leq S_1^0 \leq F^0$ and $S_1^0, F^0/S_1^0 \in \mathcal{F}$. Since $|\mathrm{Ker}(\delta^0|_{S_1^0})| \leq |S_1^0| \leq \aleph$, there exists an R-module $A_1^{-1} \leq F^{-1}$ such that $|A_1^{-1}| \leq \aleph$ and $\delta^{-1}(A_1^{-1}) = \mathrm{Ker}(\delta^0|_{S_1^0})$. Similarly, consider $\mathrm{Ker}(\delta^{-1}|_{A_1^{-1}})$, then we get $A_1^{-2} \leq F^{-2}$ such that $|A_1^{-2}| \leq \aleph$ and $\delta^{-2}(A_1^{-2}) = \mathrm{Ker}(\delta^{-1}|_{A_1^{-1}})$. Continue this process, we get an exact sequence

$$F1 \equiv \cdots \longrightarrow A_1^{-2} \xrightarrow{\delta^{-2}} A_1^{-1} \xrightarrow{\delta^{-1}} S_1^0 \xrightarrow{\delta^0} \delta^0(S_1^0) \longrightarrow 0 \longrightarrow \cdots$$

of R-modules such that $|(F1)^i| \leq \aleph$ for any $i \in \mathbb{Z}$. Obviously, $F1 \leq F$.

Note that $\delta^0(S_1^0) \leq F^1 \in \mathcal{F}$ and $|\delta^0(S_1^0)| \leq \aleph$, then there is an R-module S_2^1 such that $|S_2^1| \leq \aleph$, $\delta^0(S_1^0) \leq S_2^1 \leq F^1$ and $S_2^1, F^1/S_2^1 \in \mathcal{F}$. Since $|\mathrm{Ker}(\delta^1|_{S_2^1})| \leq |S_2^1| \leq \aleph$, there exists an R-module A_2^0 such that $|A_2^0| \leq \aleph$, $\delta^0(A_2^0) = \mathrm{Ker}(\delta^1|_{S_2^1})$ and $S_1^0 \leq A_2^0$ (since $\delta^0(S_1^0) \leq \mathrm{Ker}(\delta^1|_{S_2^1})$). Continue this process, we get an exact sequence

$$F2 \equiv \cdots \longrightarrow A_2^{-2} \xrightarrow{\delta^{-2}} A_2^{-1} \xrightarrow{\delta^{-1}} A_2^0 \xrightarrow{\delta^0} S_2^1 \xrightarrow{\delta^1} \delta^1(S_2^1) \longrightarrow 0 \longrightarrow \cdots$$

of R-modules such that $|(F2)^i| \leq \aleph$ for any $i \in \mathbb{Z}$. Obviously, $F1 \leq F2 \leq F$.

Note that $A_2^0 \leq F^0 \in \mathcal{F}$ and $|A_2^0| \leq \aleph$, then there is an R-module S_3^0 such that $|S_3^0| \leq \aleph$, $A_2^0 \leq S_3^0 \leq F^0$ and $S_3^0, F^0/S_3^0 \in \mathcal{F}$. Using the similar argument as above we get an exact sequence

$$F3 \equiv \cdots \longrightarrow A_3^{-2} \xrightarrow{\delta^{-2}} A_3^{-1} \xrightarrow{\delta^{-1}} S_3^0 \xrightarrow{\delta^0} S_2^1 + \delta^0(S_3^0) \xrightarrow{\delta^1} \delta^1(S_2^1) \longrightarrow 0 \longrightarrow \cdots$$

of R-modules such that $|(F3)^i| \leq \aleph$ for any $i \in \mathbb{Z}$. Obviously, $F1 \leq F2 \leq F3 \leq F$.

Note that $A_3^{-1} \leq F^{-1} \in \mathcal{F}$ and $|A_3^{-1}| \leq \aleph$, then there is an R-module S_4^{-1} such that $|S_4^{-1}| \leq \aleph$, $A_3^{-1} \leq S_4^{-1} \leq F^{-1}$ and $S_4^{-1}, F^{-1}/S_4^{-1} \in \mathcal{F}$. Using the similar argument as above we get an exact sequence

$$F4 \equiv \cdots \longrightarrow A_4^{-2} \xrightarrow{\delta^{-2}} S_4^{-1} \xrightarrow{\delta^{-1}} S_3^0 + \delta^{-1}(S_4^{-1}) \xrightarrow{\delta^0} S_2^1 + \delta^0(S_3^0) \xrightarrow{\delta^1} \delta^1(S_2^1) \longrightarrow 0 \longrightarrow \cdots .$$

of R-modules such that $|(F4)^i| \leq \aleph$ for any $i \in \mathbb{Z}$. Obviously, $F1 \leq F2 \leq F3 \leq F4 \leq F$.

Now we repeat the procedure and get exact sequences

$$F5 \equiv \cdots \longrightarrow A_5^{-2} \xrightarrow{\delta^{-2}} A_5^{-1} \xrightarrow{\delta^{-1}} S_5^0 \xrightarrow{\delta^0} S_2^1 + \delta^0(S_5^0) \xrightarrow{\delta^1} \delta^1(S_2^1) \longrightarrow 0 \longrightarrow \cdots ,$$

$$F6 \equiv \cdots \longrightarrow A_6^{-2} \xrightarrow{\delta^{-2}} A_6^{-1} \xrightarrow{\delta^{-1}} A_6^0 \xrightarrow{\delta^0} S_6^1 \xrightarrow{\delta^1} \delta^1(S_6^1) \longrightarrow 0 \longrightarrow \cdots , \text{ and}$$

$$F7 \equiv \cdots \longrightarrow A_7^{-2} \xrightarrow{\delta^{-2}} A_7^{-1} \xrightarrow{\delta^{-1}} A_7^0 \xrightarrow{\delta^0} A_7^1 \xrightarrow{\delta^1} S_7^2 \xrightarrow{\delta^2} \delta^2(S_7^2) \longrightarrow 0 \longrightarrow \cdots$$

of R-modules such that $|(F5)^i| \leq \aleph$, $|(F6)^i| \leq \aleph$ and $|(F7)^i| \leq \aleph$ for any $i \in \mathbb{Z}$. And $F1 \leq \cdots \leq F7 \leq F$.

If we continue this "Zig-Zag" procedure, we can get exact sequences Fm for all $m \in \mathbb{N}$ such that $Fi \leq Fj$ for $i \leq j$, $x \in (Fm)^0$ and $|Fm| \leq \aleph$ for all $m \in \mathbb{N}$ (since $|(Fm)^i| \leq \aleph$ for any $i \in \mathbb{Z}$). Furthermore, for each $i \in \mathbb{Z}$, there are infinitely many m with $(Fm)^i \in \mathcal{F}$.

Let $T = \varinjlim Fm$. Then $T \leq F$ is exact such that $x \in T^0$, $T^i \in \mathcal{F}$ and $|T^i| \leq \aleph_0 \cdot \aleph = \aleph$ for any $i \in \mathbb{Z}$, and so $|T| \leq \aleph$. Finally, $F/T = F/\varinjlim Fm = \varinjlim F/Fm$. Then, for each $i \in \mathbb{Z}$, $(F/T)^i \in \mathcal{F}$ since there are infinitely many m such that $(F/Fm)^i \in \mathcal{F}$ by construction. Thus $F/T \in \mathcal{E} \cap \widetilde{\#\mathcal{F}}$. □

Corollary 3.6 *$\mathcal{E}$ is an $\aleph$-Kaplansky class of complexes for any $\aleph \geq |R|$.*

With Remark 3.2(3), we get the following result.

Corollary 3.7 *If $\mathcal{F}$ is closed under pure submodules, pure quotients and direct limits, then $\mathcal{E} \cap \widetilde{\#\mathcal{F}}$ is an $\aleph$-Kaplansky class of complexes for any $\aleph \geq |R|$.*

In the following, we consider some relations between Kaplansky classes and cotorsion pairs.

Recall that a continuous chain of subcomplexes of a given complex C is a set of subcomplexes of C, $\{C_\alpha |\ \alpha < \lambda\}$ (for some ordinal number λ), such that C_α is a subcomplex of C_β for all $\alpha \leq \beta < \lambda$, and that $C_\gamma = \sum_{\alpha<\gamma} C_\alpha$ whenever $\gamma < \lambda$ is a limit ordinal.

The next lemma was originally stated and proved for the category of modules (see [6] or [7]). In fact, it is also true in the category of complexes or more general categories (without modifying the proofs given in [6] or [7]) (see, for example, [12, Proposition 3.1.1]).

Lemma 3.8 *Let X and Y be complexes of R-modules. If X is the direct union of a continuous chain $\{X_\alpha | \alpha < \lambda\}$ of subcomplexes for an ordinal number λ such that $\mathrm{Ext}^1_R(X_0, Y) = 0$ and $\mathrm{Ext}^1_R(X_{\alpha+1}/X_\alpha, Y) = 0$ for all $\alpha < \lambda$, then $\mathrm{Ext}^1_R(X, Y) = 0$.*

The following lemma can be proved using a similar method as proved in [11, Theorem 2.8].

Lemma 3.9 *Let $\mathcal{F}$ be a Kaplansky class of complexes closed under extensions and well ordered direct limits. Then the pair $(\mathcal{F}, \mathcal{F}^\perp)$ is cogenerated by a set. Furthermore, every complex has an $\mathcal{F}^\perp$-envelope.*

The "module version" of the next result was given in [11, Theorem 2.9].

Theorem 3.10 *Let $\mathcal{F}$ be a Kaplansky class of complexes closed under extensions and well ordered direct limits and all projective complexes are in $\mathcal{F}$. Then the pair $(\mathcal{F}, \mathcal{F}^\perp)$ is a perfect cotorsion pair.*

Proof Immediately by Lemma 3.9 and [12, Corollaries 3.1.10, 3.1.11 and 3.1.12]. □

4 Applications

In this section, we consider some consequences of the results in Sect. 3.

Example 4.1 $(\mathcal{E}, \mathcal{E}^\perp)$ is a perfect cotorsion pair.

Proof By Corollary 3.6, $\mathcal{E}$ is a Kaplansky class of complexes, and so $(\mathcal{E}, \mathcal{E}^\perp)$ is a perfect cotorsion pair by Theorem 3.10. □

Let $\mathcal{F}$ be the class of flat R-modules. Then we have the next result.

Example 4.2 $(\widetilde{\mathcal{F}}, \widetilde{\mathcal{F}}^\perp)$ and $(\mathcal{E} \cap \widetilde{\#\mathcal{F}}, (\mathcal{E} \cap \widetilde{\#\mathcal{F}})^\perp)$ are perfect cotorsion pairs.

Proof By Corollaries 3.4 and 3.7, $\widetilde{\mathcal{F}}$ and $\mathcal{E} \cap \widetilde{\#\mathcal{F}}$ are Kaplansky classes of complexes, and so $(\widetilde{\mathcal{F}}, \widetilde{\mathcal{F}}^\perp)$ and $(\mathcal{E} \cap \widetilde{\#\mathcal{F}}, (\mathcal{E} \cap \widetilde{\#\mathcal{F}})^\perp)$ are perfect cotorsion pairs by Theorem 3.10. □

Let $\mathcal{L}$ be the class of R-modules of finite injective dimension n. An analysis of the proof of [2, Proposition 2.5] shows the next result.

Lemma 4.3 *For every $M \in \mathcal{L}$ and every $K \leq M$ with $|K| \leq \aleph$, there exists an R-module S such that $K \leq S \leq M$ and $S, M/S \in \mathcal{F}$, and $|S| \leq \aleph'$ for some cardinal number $\aleph'$ which is only dependent of the cardinal number $\aleph$.*

Now using a similar proof as proved in Theorem 3.3, we get the following result by Lemma 4.3.

Proposition 4.4 *$\widetilde{\mathcal{L}}$ is a Kaplansky class of complexes.*

Following from [14], a complex X is said to be Gorenstein injective if there is an exact sequence

$$\cdots \longrightarrow X^{-1} \longrightarrow X^{0} \longrightarrow X^{1} \longrightarrow \cdots$$

of complexes such that $\mathrm{Hom}(I, -)$ exacts the sequence for any injective complexes I and $X \cong \mathrm{Ker}(X^0 \longrightarrow X^1)$. By [14, Theorems 3.1.3 and 3.2.5], over an n-Gorenstein ring, $\widetilde{\mathcal{L}}^{\perp}$ is actually the class of Gorenstein injective complexes, and $\widetilde{\mathcal{L}}$ is closed under extensions and well ordered direct limits. Thus, by Lemma 3.9, we get the following result that can be found in [14, Theorem 3.2.9].

Example 4.5 Let R be a Gorenstein ring. Then every complex has a Gorenstein injective envelope.

Let R be a commutative Noetherian ring. A semidualizing module for R is a finitely generated R-module C such that $\mathrm{Ext}_R^{\geq 1}(C, C) = 0$ and the natural homothety map $\chi_C^R : R \longrightarrow \mathrm{Hom}_R(C, C)$ is an isomorphism. The definition of semidualizing modules can go back to [13] (where the more general PG-modules are studied) and [18], but a more recent reference is [5, Definition 2.1]. Examples of semidualizng modules include the rank 1 free module and a dualizing (canonical) module, when one exists.

Next we recall the definition of the "module version" of the Auslander class with respect to a semidualizing module C (see, for example, [9, Definition 1.5]).

Definition 4.6 The Auslander class $\mathcal{A}_C = \mathcal{A}_C(R)$ is the full subcategory of R-Mod whose objects are specified as follows: an R-module M belongs to $\mathcal{A}_C$ if

1. $\mathrm{Tor}_{\geq 1}^R(C, M) = 0$;
2. $\mathrm{Ext}_R^{\geq 1}(C, C \otimes_R M) = 0$;
3. The natural map $\gamma_M^C : M \longrightarrow \mathrm{Hom}_R(C, C \otimes_R M)$ is an isomorphism.

Remark 4.7 By [9, Proposition 3.6 and Theorem 3.11], $\mathcal{A}_C$ is closed under direct limits and extensions, and all projective R-modules belong to $\mathcal{A}_C$.

An analysis of the proof of [9, Proposition 3.10] shows the following result.

Proposition 4.8 *$\mathcal{A}_C$ is a strongly $\aleph$-Kaplansky class for any $\aleph \geq |R|$.*

Now we can give the next result by Remark 4.7, Proposition 4.8 and Theorems 3.3, 3.5 and 3.10.

Example 4.9 $(\widetilde{\mathcal{A}_C}, (\widetilde{\mathcal{A}_C})^{\perp})$ and $(\mathcal{E} \cap \widetilde{\#\mathcal{A}_C}, (\mathcal{E} \cap \widetilde{\#\mathcal{A}_C})^{\perp})$ are perfect cotorsion pairs.

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