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RESEARCH ARTICLE

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Nil-Armendariz rings relative to a monoid

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Abstract For a monoid M, we introduce M-*nil*-Armendariz rings, which are generalizations of *nil*-Armendariz rings and M-Armendariz rings, and we investigate their properties. We show that every NI ring is M-*nil*-Armendariz for any unique product monoid M, and if R is a 2-*primal* and M-Armendariz ring, then R is $M \times N$-*nil*-Armendariz, where N is a unique product monoid. Moreover, we study the relationship between the weak annihilator ideals of a ring R and those of the monoid ring $R[M]$ in case R is M-*nil*-Armendariz.

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الملخص

نقدم للمونوئيد M حلقات أرمانديز متلاشية-M تعتبر تعميمات لحلقات أرمانديز متلاشية وحلقات أرمانديز-M ونبحث خصائصها. نثبت أن كل حلقة-NI هي حلقة أرمانديز متلاشية-M لكل مونوئيد ضربي وحيد M، وأنه إذا كانت R أولية-2 وحلقة أرمانديز-M، فإن R حلقة أرمانديز متلاشية-$M \times N$ حيث N مونوئيد ضربي وحيد. بالإضافة إلى ذلك، ندرس العلاقة بين المثاليات المُعْدِمَة الضعيفة للحلقة R وتلك الخاصة بحلقة المونوئيد $R[M]$ في حال كانت R حلقة أرمانديز متلاشية-M.

1 Introduction

All rings considered here are associative with identity. Let R be a ring. The prime radical (i.e., the intersection of all prime ideals) of R and the set of all nilpotent elements in R are denoted by $P(R)$ and $nil(R)$, respectively. Due to Birkenmeier et al. [4], a ring R is called 2-*primal* if $P(R) = nil(R)$. A ring R is called an NI ring if $nil(R)$ forms an ideal, and a ring R is said to be semicommutative if for all $a, b \in R$, $ab = 0$ implies $aRb = 0$. Let I be an ideal of a ring R, I is called semicommutative if I is considered as a semicommutative ring without identity.

Rege and Chhawchharia [17] introduced the notion of an Armendariz ring. A ring R is called Armendariz if whenever polynomials $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_i b_j = 0$ for each i, j. The name Armendariz ring was chosen because Armendariz [3, Lemma 1] had noted that a reduced ring (i.e., a ring without nonzero nilpotent elements) satisfies this condition. Properties of Armendariz rings have been studied in Anderson and Camillo [1], Armendariz [3], Hong et al. [6], Huh et al. [7,8]; Kim and Lee [9], Lee and Wong [10], Matczuk [12], Moussavi and Hashemi [13] and Rege and Chhawchharia [17]. In [2], Antoine has studied a generalization of Armendariz rings, which he called *nil*-Armendariz rings. A ring R is called *nil*-Armendariz provided that whenever $f(x)g(x) \in nil(R)[x]$ for $f(x) = \sum_{i=0}^{m} a_i x^i$,

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$g(x) = \sum_{j=0}^{n} b_j x^j$ in $R[x]$, then $a_i b_j \in nil(R)$ for each i, j. Some properties of *nil*-Armendariz rings have been studied in [2].

Let M be a monoid. In the following, e will always stand for the identity of M. Following Liu [11], a ring R is called an M-Armendariz ring (an Armendariz ring relative to M), if whenever elements $\alpha = a_1g_1 + a_2g_2 + \cdots + a_ng_n$, $\beta = b_1h_1 + b_2h_2 + \cdots + b_mh_m \in R[M]$ satisfy $\alpha\beta = 0$, then $a_ib_j = 0$ for each i, j. If $M = \{e\}$, then every ring is M-Armendariz. If S is a semigroup with multiplication $st = 0$ for all $s, t \in S$, and $M = S^1$, then any ring is not M-Armendariz (see [11]). Let $M = (\mathbb{N} \cup \{0\}, +)$. Thus a ring R is M-Armendariz if and only if R is Armendariz. For more examples and details of M-Armendariz rings, see [11].

Let U be a subset of R. $U[M]$ means the set $\{a_1g_1 + a_2g_2 + \cdots + a_ng_n \in R[M] \mid a_i \in U, 1 \le i \le n\}$, that is, for any element $\alpha = a_1g_1 + a_2g_2 + \cdots + a_ng_n \in R[M]$, $\alpha \in U[M]$ if and only if $a_i \in U$ for each $1 \le i \le n$. In particular, $nil(R)[M]$ stands for the set $\{a_1g_1 + a_2g_2 + \cdots + a_ng_n \in R[M] \mid a_i \in nil(R), 1 \le i \le n\}$. For an element $\alpha = a_1g_1 + a_2g_2 + \cdots + a_ng_n \in R[M]$, we denote by C_α the set comprised of coefficients of α, that is, $C_\alpha = \{a_1, a_2, \ldots, a_n\}$, and for a subset $V \subseteq R[M]$, we define $C_V = \bigcup_{\alpha \in V} C_\alpha$.

Recall that a monoid M is called a $u.p.$-monoid (unique product monoid) if for any two nonempty finite subsets $A, B \subseteq M$, there exists an element $g \in M$ uniquely presented in the form ab where $a \in A$ and $b \in B$. The class of $u.p.$-monoids is quite large and important (see [5,14,16]). For example, this class includes the right or left ordered monoids, submonoids of a free group, and torsion-free nilpotent groups. Every $u.p.$-monoid M has no nonunity element of finite order.

2 *Nil*-Armendariz rings relative to a monoid

Definition 2.1 Let M be a monoid and R a ring. We say that R is M-*nil*-Armendariz (a *nil*-Armendariz ring relative to M), if whenever $\alpha = a_1g_1 + a_2g_2 + \cdots + a_ng_n$, $\beta = b_1h_1 + b_2h_2 + \cdots + b_mh_m \in R[M]$ satisfy $\alpha\beta \in nil(R)[M]$, then $a_ib_j \in nil(R)$ for each i, j.

If $M = \{e\}$, then every ring is M-*nil*-Armendariz. A subring of an M-*nil*-Armendariz ring is M-*nil*-Armendariz. Let $M = (\mathbb{N} \cup \{0\}, +)$. Then a ring R is M-*nil*-Armendariz if and only if R is *nil*-Armendariz. If $R[M]$ is reduced, then R is M-Armendariz if and only if R is M-*nil*-Armendariz.

Lemma 2.2 ([11, Proposition 1.6]) *Suppose that R is M-Armendariz. If* $\alpha_1, \alpha_2, \ldots, \alpha_n \in R[M]$ *are such that* $\alpha_1\alpha_2\cdots\alpha_n = 0$, *then* $a_1a_2\cdots a_n = 0$, *where* a_i *is a coefficient of* α_i.

The following results shows that our definition of M-*nil*-Armendariz ring extends Liu's definition of M-Armendariz ring [11].

Theorem 2.3 *All M-Armendariz rings are M-nil-Armendariz.*

Proof We first show that $nil(R)[M] \subseteq nil(R[M])$ when R is M-Armendariz. Let $\alpha = a_1g_1 + a_2g_2 + \cdots + a_ng_n \in nil(R)[M]$ and $k \ge 1$ be such that $a_i^k = 0$ for all $1 \le i \le n$. In the following, we use essentially the same method in the proof of [2, Lemma 2.6] to claim that $\alpha^{nk} = 0$.

The coefficients of α^{nk} can be written as sums of monomials of length nk in the a_i's. Consider one of such monomials, $a_{i_1}a_{i_2}\cdots a_{i_{nk}}$ where $1 \le i_j \le n$. It would contain at least k, a_{j_0} for some $1 \le j_0 \le n$. Suppose $a_{i_{r_1}} = \cdots = a_{i_{r_k}} = a_{j_0}$ for some $1 \le r_1 \le \cdots \le r_k \le nk$. For all $i_s \ne i_{r_t}$, $1 \le t \le k$, let

$$\alpha'_{i_s} = 1e - a_{i_s}g \quad \text{and} \quad \alpha''_{i_s} = 1e + a_{i_s}g + a_{i_s}^2g^2 + \cdots + a_{i_s}^{k-1}g^{k-1},$$

where $g \in M$ and e is the identity element of M. Observe that $\alpha'_{i_s}\alpha''_{i_s} = 1e = e$ and that a_{i_s} is a product of coefficients of α'_{i_s} and α''_{i_s}. Now, we can write the monomial as

$$a_{i_1}a_{i_2}\cdots a_{i_{r_1-1}}a_{j_0}a_{i_{r_1+1}}\cdots a_{i_{r_2-1}}a_{j_0}\cdots a_{i_{r_k-1}}a_{j_0}a_{i_{r_k+1}}\cdots a_{i_{nk}}.$$

Since $a_{j_0}^k = 0$, we have

$$e_{i_1}\cdot e_{i_2}\cdots e_{i_{r_1-1}}\cdot(a_{j_0}e)\cdot e_{i_{r_1+1}}\cdots e_{i_{r_2-1}}(a_{j_0}e)\cdot e_{i_{r_2+1}}\cdots e_{i_{r_k-1}}(a_{j_0}e)\cdot e_{i_{r_k+1}}\cdots e_{i_{nk}} = 0, \tag{1}$$

where

$$e_{i_1} = e_{i_2} = \cdots = e_{i_{r_1-1}} = e_{i_{r_1+1}} = \cdots = e_{i_{r_k-1}} = e_{i_{r_k+1}} = \cdots = e_{i_{nk}} = e.$$

By replacing each e_{i_s} by the product $\alpha'_{i_s}\alpha''_{i_s}$, we have that

$$\alpha'_{i_1}\alpha''_{i_1}\cdots\alpha''_{i_{r_1}-1}(a_{j_0}e)\alpha'_{i_{r_1}+1}\cdots\alpha''_{i_{r_k}-1}(a_{j_0}e)\alpha'_{i_{r_k}+1}\cdots\alpha''_{i_{r_{nk}}}=0. \quad (2)$$

Now since R is M-Armendariz, by Lemma 2.2, we can choose a coefficient from each of the elements in Equation (2) and the product will be 0. Hence $a_{i_1}a_{i_2}\cdots a_{i_{nk}}=0$. Therefore, we have proved that all monomials appearing in the coefficients of α^{nk} are 0. Hence $\alpha\in nil(R[M])$. Therefore $nil(R)[M]\subseteq nil(R[M])$ is proved.

We next show that R is M-*nil*-Armendariz when R is M-Armendariz. Suppose that $\alpha=a_1g_1+a_2g_2+\cdots+a_ng_n$ and $\beta=b_1h_1+b_2h_2+\cdots+b_mh_m\in R[M]$ are such that $\alpha\beta\in nil(R)[M]$. Then $\alpha\beta\in nil(R[M])$. Hence there exists some positive integer p such that $(\alpha\beta)^p=0$. Then by Lemma 2.2, we obtain $a_ib_j\in nil(R)$ for each i,j. Therefore R is M-*nil*-Armendariz. □

The converse of Theorem 2.3 need not hold by the following example:

Example 2.4 Let R be an M-Armendariz reduced ring, and M a monoid with $|M|\geq 2$ and let

$$T_n(R)=\left\{\begin{pmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\0&a_{22}&\cdots&a_{2n}\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&a_{nn}\end{pmatrix}\mid a_{ij}\in R\right\}.$$

Then $T_n(R)$ is not M-Armendariz for $n\geq 4$ by [11, Remark 1.8]. But $T_n(R)$ is M-*nil*-Armendariz by Theorem 2.5 below. Hence an M-*nil*-Armendariz ring is not a trivial extension of an M-Armendariz ring.

The following results will give more examples of M-*nil*-Armendariz rings:

Theorem 2.5 *Let M be a monoid with $|M|\geq 2$. Then the following conditions are equivalent:*

(1) *R is M-nil-Armendariz.*
(2) *$T_n(R)$ is M-nil-Armendariz.*

Proof (1) ⇒ (2) It is easy to see that there exists an isomorphism of rings $T_n(R)[M]\longrightarrow T_n(R[M])$ defined by

$$\sum_{i=1}^{n}\begin{pmatrix}a^i_{11}&a^i_{12}&\cdots&a^i_{1n}\\0&a^i_{22}&\cdots&a^i_{2n}\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&a^i_{nn}\end{pmatrix}g_i\longrightarrow\begin{pmatrix}\sum_{i=1}^{n}a^i_{11}g_i&\sum_{i=1}^{n}a^i_{12}g_i&\cdots&\sum_{i=1}^{n}a^i_{1n}g_i\\0&\sum_{i=1}^{n}a^i_{22}g_i&\cdots&\sum_{i=1}^{n}a^i_{2n}g_i\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&\sum_{i=1}^{n}a^i_{nn}g_i\end{pmatrix}.$$

Suppose that $\alpha=A_1g_1+A_2g_2+\cdots+A_ng_n$ and $\beta=B_1h_1+B_2h_2+\cdots+B_mh_m\in T_n(R)[M]$ are such that $\alpha\beta\in nil(T_n(R))[M]$, where $A_i,B_j\in T_n(R)$. We claim $A_iB_j\in nil(T_n(R))$ for each i,j. Assume that

$$A_i=\begin{pmatrix}a^i_{11}&a^i_{12}&\cdots&a^i_{1n}\\0&a^i_{22}&\cdots&a^i_{2n}\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&a^i_{nn}\end{pmatrix},\quad B_j=\begin{pmatrix}b^j_{11}&b^j_{12}&\cdots&b^j_{1n}\\0&b^j_{22}&\cdots&b^j_{2n}\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&b^j_{nn}\end{pmatrix},$$

and let $\alpha_s=\sum_{i=1}^{n}a^i_{ss}g_i$ and $\beta_s=\sum_{j=1}^{m}b^j_{ss}h_j\in R[M]$. By observing that

$$nil(T_n(R))=\begin{pmatrix}nil(R)&R&\cdots&R\\0&nil(R)&\cdots&R\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&nil(R)\end{pmatrix},$$

we have $\alpha_s\beta_s \in nil(R)[M]$ for each $1 \leq s \leq n$. Since R is an M-*nil*-Armendariz ring, there exists some positive integer m_{ijs} such that $(a^i_{ss}b^j_{ss})^{m_{ijs}} = 0$ for any s and any i, j. Let $m_{ij} = max\{m_{ijs} \mid 1 \leq s \leq n\}$. Then $((A_iB_j)^{m_{ij}})^n = 0$, and so $A_iB_j \in nil(T_n(R))$. Therefore, $T_n(R)$ is M-*nil*-Armendariz.

(2) ⇒ (1) Suppose that $T_n(R)$ is M-*nil*-Armendariz. Note that R is isomorphic to the subring

$$\left\{ \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \mid a \in R \right\}$$

of $T_n(R)$. Thus R is M-*nil*-Armendariz since each subring of an M-*nil*-Armendariz ring is also M-*nil*-Armendariz. □

Let R be a ring and let

$$S_n(R) = \left\{ \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R \right\},$$

and

$$T(R, n) = \left\{ \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ 0 & a_1 & \cdots & a_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \mid a_i \in R \right\},$$

and $T(R, R)$ be the trivial extension of R by R. Using the same method in the proof of Theorem 2.5, we have the following results:

Corollary 2.6 *Let M be a monoid with $|M| \geq 2$. Then the following conditions are equivalent:*

(1) *R is M-nil-Armendariz.*
(2) *$S_n(R)$ is M-nil-Armendariz.*
(3) *$T(R, n)$ is M-nil-Armendariz.*
(4) *$T(R, R)$ is M-nil-Armendariz.*
(5) *$R[x]/(x^n)$ is M-nil-Armendariz for each $n \geq 2$.*

Let M be a monoid with $|M| \geq 2$. Then by Theorem 2.5, we deduce that both the 2×2 upper triangular matrix ring $T_2(R) = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \mid a_{ij} \in R \right\}$, and the 2×2 lower triangular matrix ring $L_2(R) = \left\{ \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix} \mid a_{ij} \in R \right\}$ are M-*nil*-Armendariz if R is M-*nil*-Armendariz. Let R be a ring and M a monoid. Let $G_3(R) = \left\{ \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \mid a_{ij} \in R \right\}$. Then $G_3(R)$ is a subring of 3×3 full matrix ring $M_3(R)$ under usual addition and multiplication. In fact, $G_3(R)$ possesses the similar form of both the ring of all lower triangular matrices and the ring of all upper triangular matrices. A natural problem asks if the M-*nil*-Armendariz property of such subrings of $M_n(R)$ coincides with that of R. This inspires us to consider the M-*nil*-Armendariz property of $G_3(R)$.

Theorem 2.7 *Let M be a monoid with $|M| \geq 2$. Then the following conditions are equivalent:*

(1) *R is M-nil-Armendariz.*
(2) *$G_3(R)$ is M-nil-Armendariz.*

Proof (1) $\Rightarrow$ (2) We first show that $nil(G_3(R)) = \begin{pmatrix} nil(R) & 0 & 0 \\ R & nil(R) & R \\ 0 & 0 & nil(R) \end{pmatrix}$. Suppose that $\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \in \begin{pmatrix} nil(R) & 0 & 0 \\ R & nil(R) & R \\ 0 & 0 & nil(R) \end{pmatrix}$, and k is a positive integer such that $a_{11}^k = a_{22}^k = a_{33}^k = 0$. Then $\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}^{2k} = 0$. Hence $\begin{pmatrix} nil(R) & 0 & 0 \\ R & nil(R) & R \\ 0 & 0 & nil(R) \end{pmatrix} \subseteq nil(G_3(R))$. Now assume that $\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \in nil(G_3(R))$. Then there exists some positive integer k such that $\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}^{k} = 0$. Hence $a_{11}^k = a_{22}^k = a_{33}^k = 0$, and so $\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \in \begin{pmatrix} nil(R) & 0 & 0 \\ R & nil(R) & R \\ 0 & 0 & nil(R) \end{pmatrix}$. Therefore, $nil(G_3(R)) = \begin{pmatrix} nil(R) & 0 & 0 \\ R & nil(R) & R \\ 0 & 0 & nil(R) \end{pmatrix}$. Then by analogy with the proof of Theorem 2.5, we can show that $G_3(R)$ is M-*nil*-Armendariz.

(2) $\Rightarrow$ (1) It is trivial. □

Let M be a monoid with $|M| \geq 2$. From Theorem 2.5 and Theorem 2.7, one may suspect that if R is M-*nil*-Armendariz, then the $n \times n$ full matrix ring $M_n(R)$ is M-*nil*-Armendariz for $n \geq 2$. But the following example erases the possibility:

Example 2.8 Let M be a monoid with $|M| \geq 2$ and R a ring. Let $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} e + \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} g$ and $\beta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} e + \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} g$ be two elements in $M_2(R)[M]$. Then $\alpha\beta = 0$. But $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$ is not nilpotent. Thus $M_2(R)$ is not M-*nil*-Armendariz.

Theorem 2.9 *Let M be a monoid with $|M| \geq 2$. Then the finite direct sum of M-nil-Armendariz rings is M-nil-Armendariz.*

Proof It suffices to show that if R_1, R_2 are M-*nil*-Armendariz rings, then so is $R_1 \oplus R_2$. Let $\alpha = (a_1^1, b_1^1)g_1 + (a_2^1, b_2^1)g_2 + \cdots + (a_m^1, b_m^1)g_m$, and $\beta = (a_1^2, b_1^2)h_1 + (a_2^2, b_2^2)h_2 + \cdots + (a_n^2, b_n^2)h_n \in (R_1 \oplus R_2)[M]$ be such that $\alpha\beta \in nil(R_1 \oplus R_2)[M]$. Write $f_1 = a_1^1 g_1 + a_2^1 g_2 + \cdots a_m^1 g_m$, $g_1 = b_1^1 g_1 + b_2^1 g_2 + \cdots + b_m^1 g_m$, $f_2 = a_1^2 h_1 + a_2^2 h_2 + \cdots a_n^2 h_n$, $g_2 = b_1^2 h_1 + b_2^2 h_2 + \cdots + b_n^2 h_n$. Then $f_1 f_2 \in nil(R_1)[M]$ and $g_1 g_2 \in nil(R_2)[M]$. So by M-*nil*-Armendarizness of R_1 and R_2, $a_i^1 a_j^2 \in nil(R_1)$, $b_i^1 b_j^2 \in nil(R_2)$ for all i, j. Thus for each i, j, $(a_i^1, b_i^1)(a_j^2, b_j^2) \in nil(R_1 \oplus R_2)$. Therefore, $R_1 \oplus R_2$ is M-*nil*-Armendariz. □

Theorem 2.10 *Let M be a u.p.-monoid and R an NI ring. Then R is M-nil-Armendariz.*

Proof Let $\alpha = a_1 g_1 + a_2 g_2 + \cdots + a_n g_n$ and $\beta = b_1 h_1 + b_2 h_2 + \cdots + b_m h_m \in R[M]$ be such that $\alpha\beta \in nil(R)[M]$. Then $\overline{\alpha}\overline{\beta} = \overline{0}$, where $\overline{\alpha}, \overline{\beta}$ are the corresponding elements of α, β in $(R/nil(R))[M]$. Observe that $R/nil(R)$ is reduced and hence M-Armendariz by [11, Proposition 1.1]. Then we obtain $a_i b_j \in nil(R)$ for each i, j. Therefore R is M-$nil(R)$-Armendariz. □

Let $(M, \leq)$ be an ordered monoid. If for any $g, g', h \in M$, $g < g'$ implies that $gh < g'h$ and $hg < hg'$, then $(M, \leq)$ is called a strictly ordered monoid.

Corollary 2.11 *Let M be a strictly totally ordered monoid and R an NI ring. Then R is M-nil-Armendariz.*

Corollary 2.12 *Let R be an NI ring. Then R is $\mathbb{Z}$-nil-Armendariz, that is, for any $\alpha = a_{-m}x^{-m} + a_{-(m-1)}x^{-(m-1)} + \cdots + a_p x^p$, $\beta = b_{-n}x^{-n} + b_{-(n-1)}x^{-(n-1)} + \cdots + b_q x^q \in R[x, x^{-1}]$, if $\alpha\beta \in nil(R)[x, x^{-1}]$, then $a_i b_j \in nil(R)$ for $-m \leq i \leq p$ and $-n \leq j \leq q$.*

Taking $M = \{\mathbb{N} \cup \{0\}, +\}$ in Corollary 2.11, it follows that every NI ring is *nil*-Armendariz. Thus Corollary 2.11 is a generalization of Antoine's [2, Proposition 2.1].

It was shown in Liu [11, Proposition 1.4], that if M is a strictly totally ordered monoid and I a reduced ideal of R such that R/I is an M-Armendariz ring, then R is M-Armendariz. The following result is a generalization of [11, Proposition 1.4].

Theorem 2.13 *Let M be a strictly totally ordered monoid and I an ideal of a ring R. If I is semicommutative and R/I is M-nil-Armendariz, then R is M-nil-Armendariz.*

Proof The proof is a simple mutatis mutandis argument using the proof of [11, Proposition 1.4]. □

Recall that a monoid M is called torsion-free if the following property holds: if $g, h \in M$ and $k \geq 1$ are such that $g^k = h^k$, then $g = h$.

Corollary 2.14 *Let M be a commutative cancellative and torsion-free monoid. If one of the following conditions holds, then R is M-nil-Armendariz.*

(1) *R is an NI ring.*
(2) *R/I is M-nil-Armendariz for some semicommutative ideal I of R.*

Proof If M is commutative cancellative and torsion-free, then by Ribenbiom [18], there exists a compatible strict total order $\leq$ on M. Now the results follow from Corollary 2.11 and Theorem 2.13. □

Anderson and Camillo [1, Theorem 2] have shown that a ring is Armendariz if and only if $R[x]$ is Armendariz. For Armendariz rings relative to monoids, Liu [11, Proposition 2.1], have shown that if M is a monoid and N a u.p. monoid, and R a reduced M-Armendariz ring, then $R[M]$ is N-Armendariz. As to a *nil*-Armendariz ring relative to a monoid, we have the following result:

Theorem 2.15 *Let M be a monoid and N a $u.p.$-monoid. If R is an M-Armendariz NI ring, then $R[M]$ is N-nil-Armendariz.*

Proof By Theorem 2.3, we obtain $nil(R)[M] \subseteq nil(R[M])$ when R is M-Armendariz. Following Lemma 2.2, we get $nil(R[M]) \subseteq nil(R)[M]$. Hence $nil(R)[M] = nil(R[M])$. Thus it is easy to see that $R[M]$ is an NI ring because R is an NI ring. Now the result follows from Theorem 2.10. □

Corollary 2.16 *Let M be a monoid and R an M-Armendariz NI ring. Then $R[M]$ is an NI ring and $nil(R)[M] = nil(R[M])$.*

Theorem 2.17 *Let M be a monoid and N a $u.p.$-monoid. If R is an M-Armendariz 2-primal ring, then $R[N]$ is M-nil-Armendariz.*

Proof Since R is an M-Armendariz 2-*primal* ring, by Corollary 2.16, we obtain $nil(R)[M] = nil(R[M])$.

Now we show that $nil(R)[N] = nil(R[N])$ when N is a $u.p.$-monoid and R is a 2-*primal* ring. Let $\alpha = a_1g_1 + a_2g_2 + \cdots + a_ng_n \in nil(R[N])$. There exists some positive integer k such that $\alpha^k = 0$. Consider $\beta = \alpha^{k-1}$. Then $\alpha\beta = \alpha^k = 0 \in nil(R)[N]$, and hence, since R is N-*nil*-Armendariz by Theorem 2.10, $a^{(1)}b \in nil(R)$ where $a^{(1)} \in C_\alpha$ and $b \in C_\beta$. Therefore for all $a^{(1)} \in C_\alpha$,

$$a^{(1)}\beta = a^{(1)}\alpha^{k-1} = a^{(1)}\alpha\alpha^{k-2} \in nil(R)[N].$$

Consider $\gamma = \alpha^{k-2}$. Then

$$a^{(1)}\alpha\gamma = a^{(1)}\beta \in nil(R)[N].$$

Since the coefficients of $a^{(1)}\alpha$ are $a^{(1)}a^{(2)}$ where $a^{(2)}$ is a coefficient of α, and because R is N-*nil*-Armendariz by Theorem 2.10, we obtain $a^{(1)}a^{(2)}c \in nil(R)$ where $a^{(1)}, a^{(2)} \in C_\alpha$ and $c \in C_\gamma$. Therefore, for all $a^{(1)} \in C_\alpha$, $a^{(2)} \in C_\alpha$, we obtain

$$a^{(1)}a^{(2)}\gamma = a^{(1)}a^{(2)}\alpha \cdot \alpha^{k-3} \in nil(R)[N].$$

Repeating the same way as above, we obtain $a^{(1)}a^{(2)} \cdots a^{(k)} \in nil(R)$ where $a^{(i)} \in C_\alpha$ for each $1 \leq i \leq k$, and so $a_i \in nil(R)$ for each $1 \leq i \leq n$. Thus $\alpha \in nil(R)[N]$, and so $nil(R[N]) \subseteq nil(R)[N]$.

Now we show that $nil(R[N]) \supseteq nil(R)[N]$. Assume that $\alpha = a_1g_1 + a_2g_2 + \cdots + a_ng_n \in nil(R)[N]$. Consider the finite subset $\{a_1, a_2, \ldots, a_n\} \subseteq nil(R)$. Since R is a 2-*primal* ring, there exists a positive integer p such that any product of p elements $a_{i_1}a_{i_2}\cdots a_{i_p}$ from $\{a_1, a_2, \ldots, a_n\}$ is zero. Hence we obtain $\alpha^p = 0$, and so $nil(R)[N] \subseteq nil(R[N])$. Therefore, $nil(R)[N] = nil(R[N])$ is proved.

Next we show that $R[N]$ is M-*nil*-Armendariz. Note that there exists an isomorphism of rings $R[N][M] \cong R[M][N]$ defined by

$$\sum_p\left(\sum_i a_{ip}n_i\right)m_p \longrightarrow \sum_i\left(\sum_p a_{ip}m_p\right)n_i.$$

Now suppose that $\alpha_i, \beta_j \in R[N]$ are such that

$$\left(\sum_i \alpha_i g_i\right)\left(\sum_j \beta_j h_j\right) \in nil(R[N])[M],$$

where $g_i, h_j \in M$. We show that $\alpha_i\beta_j \in nil(R[N])$ for all i, j. Assume that $\alpha_i = \sum_p a_{ip}n_p$ and $\beta_j = \sum_q b_{jq}n'_q$ where $n_p, n'_q \in N$ for all p and q. Then

$$\left(\sum_i\left(\sum_p a_{ip}n_p\right)g_i\right)\left(\sum_j\left(\sum_q b_{jq}n'_q\right)h_j\right) \in nil(R[N])[M] = nil(R)[N][M].$$

Thus in $R[M][N]$ we have

$$\left(\sum_p\left(\sum_i a_{ip}g_i\right)n_p\right)\left(\sum_q\left(\sum_j b_{jq}h_j\right)n'_q\right) \in nil(R)[M][N] = nil(R[M])[N].$$

By Theorem 2.15, $R[M]$ is N-*nil*-Armendariz. Thus

$$\left(\sum_i a_{ip}g_i\right)\left(\sum_j b_{jq}h_j\right) \in nil(R[M]) = nil(R)[M]$$

for all p, q. So $a_{ip}b_{jq} \in nil(R)$ for all i, j, p, q, since by Theorem 2.3, M-Armendariz rings are M-*nil*-Armendariz. Hence

$$\alpha_i\beta_j = \left(\sum_p a_{ip}n_p\right)\left(\sum_q b_{jq}n'_q\right) \in nil(R)[N] = nil(R[N])$$

for all i, j. Therefore, $R[N]$ is M-*nil*-Armendariz. □

Corollary 2.18 *Let M be a u.p.-monoid and R a 2-primal ring. Then $nil(R)[M] = nil(R[M])$.*

Corollary 2.19 *Let M be a monoid and R a 2-primal ring. If R is M-Armendariz, then $R[x]$ and $R[x, x^{-1}]$ are M-nil-Armendariz.*

Proof Note that $R[x] \cong R[\mathbb{N} \cup \{0\}]$ and $R[x, x^{-1}] \cong R[\mathbb{Z}]$. □

Theorem 2.20 *Let M be a monoid and N a u.p.-monoid. If R is an M-Armendariz NI ring, then R is $(M \times N)$-nil-Armendariz.*

Proof By analogy with the proof of [11, Theorem 2.3]. □

3 Weak annihilator ideals of M-*nil*-Armendariz rings

Let R be a ring. For a subset X of a ring R, we define $N_R(X) = \{a \in R \mid xa \in nil(R) \text{ for all } x \in X\}$, which is called the weak annihilator of X in R. If X is singleton, say $X = \{r\}$, we use $N_R(r)$ in place of $N_R(\{r\})$.

Obviously, for any subset X of a ring R, $N_R(X) = \{a \in R \mid xa \in nil(R) \text{ for all } x \in X\} = \{b \in R \mid bx \in nil(R) \text{ for all } x \in X\}$, and $r_R(X) \subseteq N_R(X)$ and $l_R(X) \subseteq N_R(X)$. If R is reduced, then $r_R(X) = N_R(X) = l_R(X)$ for any subset X of R. It is easy to see that for any subset $X \subseteq R$, $N_R(X)$ is an ideal of R in case $nil(R)$ is an ideal.

Given a ring R, we define

$$NAnn_R(2^R) = \{N_R(U) \mid U \subseteq R\}$$

and

$$NAnn_{R[M]}(2^{R[M]}) = \{N_{R[M]}(V) \mid V \subseteq R[M]\}.$$

For an element $\alpha \in R[M]$, C_α denotes the set consisting of coefficients of α and for a subset V of $R[M]$, C_V denotes the set $\cup_{\alpha \in V} C_\alpha$.

Theorem 3.1 *Let M be a monoid and R an M-Armendariz NI ring. Then*

$$\psi : NAnn_R(2^R) \longrightarrow NAnn_{R[M]}(2^{R[M]})$$

defined by $\psi(I) = I[M]$ for every $I \in NAnn_R(2^R)$ is bijective.

Proof As usual we shall identity R with the subring $R \cdot e \subseteq R[M]$ and identity M with $1 \cdot M \subseteq R[M]$.

Let $I = N_R(U) \in NAnn_R(2^R)$ where $U \subseteq R$. We show that $I[M] = N_R(U)[M] = N_{R[M]}(U)$. For any $\alpha = a_1g_1 + a_2g_2 + \cdots + a_ng_n \in N_R(U)[M]$ and any $u \in U$, we have $ua_i \in nil(R)$ for all $1 \le i \le n$. Then $u\alpha \in nil(R)[M])$ and so $u\alpha \in nil(R[M])$ by Corollary 2.16. Hence $N_R(U)[M] \subseteq N_{R[M]}(U)$. Conversely, let $\alpha = a_1g_1 + a_2g_2 + \cdots + a_ng_n \in N_{R[M]}(U)$. Then $u\alpha = ua_1g_1 + ua_2g_2 + \cdots + ua_ng_n \in nil(R[M])$ for all $u \in U$. Then by Corollary 2.16, we obtain $ua_i \in nil(R)$ for all $1 \le i \le n$. Thus $a_i \in N_R(U)$ for all $1 \le i \le n$ and so $\alpha \in N_R(U)[M]$. Hence $N_{R[M]}(U) \subseteq N_R(U)[M]$, and so $N_{R[M]}(U) = N_R(U)[M] = I[M]$. Therefore, ψ is well defined.

Suppose there exist $U \subseteq R$ and $U' \subseteq R$ such that $I = N_R(U) \in NAnn_R(2^R)$, and $I' = N_R(U') \in NAnn_R(2^R)$ and $I \neq I'$. Then it is easy to check that $I[M] \neq I'[M]$. Hence ψ is injective.

Now it is only necessary to show that ψ is surjective. Let $\alpha = a_1g_1 + a_2g_2 + \cdots + a_ng_n \in N_{R[M]}(V)$ with $N_{R[M]}(V) \in NAnn_{R[M]}(2^{R[M]})$. Then we have $\beta\alpha \in nil(R[M]) = nil(R)[M]$ for any $\beta = b_1h_1 + b_2h_2 + \cdots b_mh_m \in V$. Thus $b_ia_j \in nil(R)$ for all $1 \le i \le m$ and $1 \le j \le n$ since M-Armendariz rings are M-*nil*-Armendariz. Hence $a_j \in N_R(C_V)$ for all $1 \le j \le n$, and so $\alpha \in N_R(C_V)[M]$. Now it is easy to see that $N_{R[M]}(V) = N_R(C_V)[M] = \psi(N_R(C_V))$. Therefore, ψ is surjective. □

By [15], a ring R is said to be a nilpotent $p.p.$-ring if for any element $p \notin nil(R)$, we have $N_R(p)$ is generated as a right ideal by a nilpotent element.

Theorem 3.2 *Let M be a monoid and R an M-Armendariz NI ring. If R is a nilpotent $p.p.$-ring, then so is $R[M]$.*

Proof By Corollary 2.16, we have $nil(R[M]) = nil(R)[M]$. Let $\alpha = a_1g_1 + a_2g_2 + \cdots + a_ng_n \notin nil(R[M])$, and $\beta = b_1h_1 + b_2h_2 + \cdots + b_mh_m \in N_{R[M]}(\alpha)$. Then $\alpha\beta \in nil(R[M]) = nil(R)[M]$. Thus we have $a_ib_j \in nil(R)$ for each i, j since R is M-*nil*-Armendariz by Theorem 2.3. Since $\alpha \notin nil(R[M]) = nil(R)[M]$, there exists some $1 \le i \le n$ such that $a_i \notin nil(R)$. So there exists some $c \in nil(R)$ such that $N_R(a_i) = c \cdot R$ because R is a nilpotent $p.p.$-ring. Now we show that $N_{R[M]}(\alpha) = ce \cdot R[M]$. Since $b_j \in N_R(a_i)$ for all $1 \le j \le m$, $b_j = cr_j$ with $r_j \in R$. Hence $\beta = ce(r_1h_1 + r_2h_2 + \cdots + r_mh_m) \in ce \cdot R[M]$, and so $N_{R[M]}(\alpha) \subseteq ce \cdot R[M]$. Conversely, for any $\gamma = v_1e_1 + v_2e_2 + \cdots + v_pe_p \in R[M]$, since $c \in nil(R)$ and $nil(R)$ of an NI ring is an ideal, we obtain $a_icv_j \in nil(R)$ for each i, j, and so $\alpha \cdot ce \cdot \gamma \in nil(R)[M] = nil(R[M])$. Hence we obtain $ce \cdot R[M] \subseteq N_{R[M]}(\alpha)$. Therefore, $N_{R[M]}(\alpha) = ce \cdot R[M]$ where $ce \in nil(R[M])$. □

Theorem 3.3 *Let M be a monoid and R an M-Armendariz NI ring. If for any nonempty subset $X \not\subseteq nil(R)$, $N_R(X)$ is generated as a right ideal by a nilpotent element, then for any nonempty subset $U \not\subseteq nil(R[M])$, $N_{R[M]}(U)$ is generated as a right ideal by a nilpotent element.*

Proof Let U be a nonempty subset of $R[M]$ with $U \nsubseteq nil(R[M])$. Suppose $\beta = b_1h_1 + b_2h_2 + \cdots + b_mh_m \in N_{R[M]}(U)$. Then $\alpha\beta \in nil(R[M]) = nil(R)[M]$ for each $\alpha = a_1g_1 + a_2g_2 + \cdots + a_ng_n \in U$. Thus $a_ib_j \in nil(R)$ for each i, j since R is M-*nil*-Armendariz by Theorem 2.3. Hence $b_j \in N_R(C_U)$ for each $1 \leq j \leq m$. If $C_U \subseteq nil(R)$, then $U \subseteq nil(R)[M] = nil(R[M])$, a contradiction. Thus there exists $c \in nil(R)$ such that $N_R(C_U) = c \cdot R$. Now we show that $N_{R[M]}(U) = ce \cdot R[M]$. Since $b_j \in N_R(C_U) = c \cdot R$ for all $1 \leq j \leq m$, there exists $r_j \in R$ such that $b_j = cr_j$ for all $1 \leq j \leq m$. Hence $\beta = ce(r_1h_1 + r_2h_2 + \cdots + r_mh_m) \in ce \cdot R[M]$, and so $N_{R[M]}(U) \subseteq ce \cdot R[M]$. Conversely, for any $\gamma = v_1e_1 + v_2e_2 + \cdots + v_pe_p \in R[M]$, and any $\alpha = a_1g_1 + a_2g_2 + \cdots + a_ng_n \in U$, since $c \in nil(R)$ and $nil(R)$ of an NI ring is an ideal, we obtain $a_icv_j \in nil(R)$ for each i, j, and so $\alpha \cdot ce \cdot \gamma \in nil(R)[M] = nil(R[M])$ by Corollary 2.16. Hence we obtain $ce \cdot R[M] \subseteq N_{R[M]}(U)$. Therefore, $N_{R[M]}(U) = ce \cdot R[M]$ where $ce \in nil(R[M])$. □

Using the same method in the proof of Theorem 3.2 or Theorem 3.3, we obtain the following result:

Theorem 3.4 *Let M be a monoid and R an M-Armendariz NI ring. If for any principally right ideal $p \cdot R \nsubseteq nil(R)$, $N_R(p \cdot R)$ is generated as a right ideal by a nilpotent element, then for any principally right ideal $\alpha \cdot R[M] \nsubseteq nil(R[M])$, $N_{R[M]}(\alpha \cdot R[M])$ is generated as a right ideal by a nilpotent element.*

Theorem 3.5 *Let M be a $u.p.$-monoid and R a 2-primal ring. Then we have the following:*

(1) *If R is a nilpotent $p.p.$-ring, then so is $R[M]$.*
(2) *If for any nonempty subset $X \nsubseteq nil(R)$, $N_R(X)$ is generated as a right ideal by a nilpotent element, then for any nonempty subset $U \nsubseteq nil(R[M])$, $N_{R[M]}(U)$ is generated as a right ideal by a nilpotent element.*
(3) *If for any principally right ideal $p \cdot R \nsubseteq nil(R)$, $N_R(p \cdot R)$ is generated as a right ideal by a nilpotent element, then for any principally right ideal $\alpha \cdot R[M] \nsubseteq nil(R[M])$, $N_{R[M]}(\alpha \cdot R[M])$ is generated as a right ideal by a nilpotent element.*

Proof It is trivial. □

Corollary 3.6 *Let M be a strictly totally ordered-monoid and R a 2-primal ring. Then we have the following:*

(1) *If R is a nilpotent $p.p.$-ring, then so is $R[M]$.*
(2) *If for any nonempty subset $X \nsubseteq nil(R)$, $N_R(X)$ is generated as a right ideal by a nilpotent element, then for any nonempty subset $U \nsubseteq nil(R[M])$, $N_{R[M]}(U)$ is generated as a right ideal by a nilpotent element.*
(3) *If for any principally right ideal $p \cdot R \nsubseteq nil(R)$, $N_R(p \cdot R)$ is generated as a right ideal by a nilpotent element, then for any principally right ideal $\alpha \cdot R[M] \nsubseteq nil(R[M])$, $N_{R[M]}(\alpha \cdot R[M])$ is generated as a right ideal by a nilpotent element.*

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