Arab J Math (2012) 1:341–346
DOI 10.1007/s40065-012-0039-9

RESEARCH ARTICLE

L. Oukhtite · A. Mamouni

Derivations satisfying certain algebraic identities on Jordan ideals

Received: 29 November 2010 / Accepted: 8 June 2011 / Published online: 26 July 2012

Abstract In this paper, we investigate commutativity of rings with involution in which derivations satisfy certain algebraic identities on Jordan ideals. Moreover, we extend some results for derivations of prime rings to Jordan ideals. Furthermore, an example is given to prove that the $*$-primeness hypothesis is not superfluous.

Mathematics Subject Classification 16W10 · 16W25 · 16U80

الملخص

نبحث في إبدالية الحلقات ذات الارتداد التي تحقق المشتقات فيها بعض المتطابقات الجبرية على مثاليات جوردان. بالإضافة إلى ذلك، نمدد بعض النتائج حول مشتقات الحلقات الأولية إلى مثاليات جوردان. نعطي كذلك مثالاً لإثبات أن الفرض الخاص بأولية-* ليس شرطاً زائداً عن اللزوم.

1 Introduction

Throughout this paper, R will denote an associative ring with center $Z(R)$. We will write for all $x, y \in R$, $[x, y] = xy - yx$ and $x \circ y = xy + yx$ for the Lie product and Jordan product, respectively. R is 2-torsion free if whenever $2x = 0$, with $x \in R$, then $x = 0$. R is prime if $aRb = 0$ implies $a = 0$ or $b = 0$. If R admits an involution $*$, then R is $*$-prime and if $aRb = aRb^* = 0$ yields $a = 0$ or $b = 0$. Note that every prime ring having an involution $*$ is $*$-prime but the converse is in general not true. Indeed, if R^o denotes the opposite ring of a prime ring R, then $R \times R^o$ equipped with the exchange involution $*_{ex}$, defined by $*_{ex}(x, y) = (y, x)$, is $*_{ex}$-prime but not prime. This example shows that every prime ring can be injected in a $*$-prime ring and from this point of view $*$-prime rings constitute a more general class of prime rings.

An additive subgroup J of R is said to be a Jordan ideal of R if $u \circ r \in J$, for all $u \in J$ and $r \in R$. A Jordan ideal J which satisfies $J^* = J$ is called a $*$-Jordan ideal. An additive mapping $d : R \to R$ is said to be a derivation if $d(xy) = d(x)y + xd(y)$ for all x, y in R.

Recently, many authors have studied commutativity of prime and semi-prime rings admitting suitably constrained derivations. Moreover, many of the obtained results extend other ones proven previously just for the action of derivations on the whole ring to actions of derivations on ideals. So, it is natural to ask what can we say about the commutativity of rings in which derivations satisfy certain identities on Jordan ideals?

L. Oukhtite (✉) · A. Mamouni
Département de Mathématiques, Groupe d'Algèbre et Applications, Faculté des Sciences et Techniques,
Université Moulay Ismaïl, B. P. 509 Boutalamine, Errachidia, Maroc
E-mail: oukhtitel@hotmail.com

A. Mamouni
E-mail: mamouni_1975@live.fr

2 Main results

Throughout, $(R, *)$ will be a 2-torsion-free ring with involution and $Sa_*(R) := \{r \in R | r^* = \pm r\}$ the set of symmetric and skew symmetric elements of R. We shall use without explicit mention the fact that if J is a nonzero Jordan ideal of a ring R, then $2[R, R]J \subseteq J$ and $2J[R, R] \subseteq J$ [9, Lemma 2.4]. Moreover, from the proof of Lemma 3 of [1], we have $4j^2R \subset J$ and $4Rj^2 \subset J$ for all $j \in J$. Since $4jrj = 2\{j(jr + rj) + (jr + rj)j\} - \{2j^2r + r2j^2\}$, it follows that $4jRj \subset J$ for all $j \in J$ (see the proof of Theorem 2.12 of [1]). We will make use of the following basic commutator identities: $[x, yz] = y[x, z] + [x, y]z$ and $[xy, z] = x[y, z] + [x, z]y$.

We begin with the following facts which will be used in the sequel.

Fact 2.1 Every $*$-prime ring is semiprime. Indeed, if $aRa = 0$ then $aRaRa^* = 0$ so that $a = 0$ or $aRa^* = 0$. But $aRa^* = 0$ together with $aRa = 0$ and force $a = 0$.

Fact 2.2 If R is a semiprime ring and $x \in Z(R)$ is such that $x^2 = 0$, then $x = 0$.

Fact 2.3 If R is a semiprime ring and $x \in R$ is such that $[[R, R], x] = 0$, then $x \in Z(R)$. Indeed, from $[[rx, s], x] = 0$ it follows that $[r, x][x, s] = 0$. Replacing s by tr in the last equality, we find that $[r, x]t[x, r] = 0$ and thus $[x, r]R[x, r] = 0$ for all $r \in R$. Using semiprimeness of R we conclude that $x \in Z(R)$.

In order to prove our main theorem, we shall need the following lemmas.

Lemma 2.4 ([7, Theorem 3.3]) *Let R be a 2-torsion free $*$-prime ring and let d be a nonzero derivation of R. If $a \in Sa_*(R)$ is such that $d([R, a]) = 0$, then $a \in Z(R)$. In particular, if $d([x, y]) = 0$ for all $x, y \in R$, then R is commutative.*

Lemma 2.5 *Let R be a 2-torsion-free $*$-prime ring and J a nonzero $*$-Jordan ideal of R.*

(a) ([5, Lemma 2]) *If $aJb = a^*Jb = 0$, then $a = 0$ or $b = 0$.*

(b) ([5, Lemma 3]) *If $[J, J] = 0$, then $J \subseteq Z(R)$.*

(c) ([5, Lemma 4]) *If d is a derivation of R such that $d(J) = 0$, then $d = 0$ or $J \subseteq Z(R)$.*

(d) ([4, Lemma 3]) *If $J \subseteq Z(R)$, then R is commutative.*

(e) ([8, Lemma 8]) *If d is a nonzero derivation of R such that $d(x^2) = 0$ for all $x \in J$, then R is commutative.*

Theorem 2.6 *Let R be a 2-torsion-free $*$-prime ring and J a nonzero $*$-Jordan ideal of R. If R admits a nonzero derivation d such that $d([x, y]) = 0$ for all $x, y \in J$, then R is commutative.*

Proof Assume that $d([x, y]) = 0$ for all $x, y \in J$. Let us consider $\delta : R \longrightarrow R$ defined by $\delta(x) = d(x) + (* \circ d \circ *)(x)$. Obviously, δ is a derivation which commutes with $*$ and $\delta([x, y]) = 0$ for all $x, y \in J$. Hence

$$[\delta(x), y] = [\delta(y), x] \quad \text{for all } x, y \in J. \tag{1}$$

Replacing y by $2[y, u]v$ in Equation (1), because of $\delta([y, u]) = 0$, we find that

$$[\delta(x), [y, u]]v = [[y, u], x]\delta(v) \quad \text{for all } u, v, x, y \in J. \tag{2}$$

Substituting $2v[r, s]$ for v in Equation (2), where $r, s \in R$, and using Equation (2) we obtain $[[y, u], x]v\delta([r, s]) = 0$ for all $u, v, x, y \in J$ and $r, s \in R$ in such a way that

$$[[y, u], x]J\delta([r, s]) = 0 \quad \text{for all } u, x, y \in J, r, s \in R. \tag{3}$$

Since δ commutes with $*$, in view of Equation (3), it then follows that

$$[[y, u], x]J(\delta([r, s]))^* = 0 \quad \text{for all } u, x, y \in J, r, s \in R. \tag{4}$$

Using Lemma 2.4, Equations (3) and (4) assure that $\delta([r, s]) = 0$ for all $r, s \in R$ or $[[y, u], x] = 0$ for all $u, x, y \in J$.

If $\delta([r, s]) = 0$ for all $r, s \in R$, then [7, Theorem 3.3] assures the commutativity of R.
Now assume that

$$[[y, u], x] = 0 \quad \text{for all } u, x, y \in J. \tag{5}$$

Substituting $yu + uy$ for y in Equation (5) we find that

$$[y, u][u, x] + [u, x][y, u] = 0 \quad \text{for all } u, x, y \in J. \tag{6}$$

Since $[[y, u], [u, x]] = 0$ by Equation (5), then Equation (6) together with 2-torsion freeness yield

$$[y, u][u, x] = 0 \quad \text{for all } u, x, y \in J. \tag{7}$$

Replacing y by $2[r, s]y$ in Equation (7), where $r, s \in R$, we obtain $[[r, s], u]y[u, x] = 0$ for all $u, x, y \in J$ and $r, s \in R$ so that

$$[[r, s], u]J[u, x] = 0 \quad \text{for all } u, x \in J, r, s \in R. \tag{8}$$

Let $u \in J \cap Sa_*(R)$, from (8) it follows that

$$[[r, s], u]J[u, x]^* = 0 \quad \text{for all } x \in J, r, s \in R. \tag{9}$$

Using Equations (8) and (9), Lemma 2.4 forces $[u, J] = 0$ or $[[r, s], u] = 0$ for all $r, s \in R$.
Assume that $[[R, R], u] = 0$; since a $*$-prime ring is semiprime then Fact 2.3 forces $u \in Z(R)$ and, therefore, $[u, J] = 0$. Therefore, in both cases we find that

$$[u, J] = 0 \quad \text{for all } u \in J \cap Sa_*(R). \tag{10}$$

Let $x \in J$; as $x - x^* \in J \cap Sa_*(R)$ and $x + x^* \in J \cap Sa_*(R)$, then Equation (10) leads to

$$[x - x^*, j] = 0 \quad \text{and} \quad [x + x^*, j] = 0 \quad \text{for all } j \in J. \tag{11}$$

In the light of 2-torsion freeness, Equation (11) yields $[x, j] = 0$. Therefore, we conclude that

$$[x, j] = 0 \quad \text{for all } x, j \in J. \tag{12}$$

In view of Lemma 2.5, Equation (12) forces $J \subseteq Z(R)$. Once again using Lemma 2.5, we conclude that R is commutative. □

The following example proves that the $*$-primeness hypothesis in Theorem 2.6 is not superfluous.

Example 2.7 Let $R = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} | x, y, z \in \mathbb{Z} \right\}$ where $\mathbb{Z}$ is the ring of integers. Let us consider $d\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}^* = \begin{pmatrix} z & -y \\ 0 & x \end{pmatrix}$. Note that R is not $*$-prime for $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^* = 0$. Moreover, if we set $J = \left\{ \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} | y \in \mathbb{Z} \right\}$, then J is a $*$-Jordan ideal and $d([x, y]) = 0$ for all $x, y \in J$, but $d \neq 0$ and R is noncommutative. Hence in Theorem 2.6 the hypothesis of $*$-primeness is crucial.

Corollary 2.8 ([6, Theorem 1.3]) *Let R be a 2-torsion free $*$-prime ring and I a nonzero $*$-ideal of R. If R admits a nonzero derivation d such that $d([x, y]) = 0$ for all $x, y \in I$, then R is commutative.*

Corollary 2.9 ([7, Theorem 3.3]) *Let R be a 2-torsion free $*$-prime ring. If R admits a nonzero derivation d such that $d([x, y]) = 0$ for all $x, y \in R$, then R is commutative.*

Bell and Daif [2, Theorem 3] showed that, if a prime ring R admits a nonzero derivation d satisfying $d([x, y]) = 0$ for all x, y in a nonzero ideal I of R, then R is commutative. As an application of Theorem 2.6, the following theorem extends the result of [2] to Jordan ideals, but only with further assumption that the ring R be 2-torsion free.

Theorem 2.10 *Let R be a 2-torsion-free prime ring and J a nonzero Jordan ideal of R. If R admits a nonzero derivation d satisfying $d([x, y]) = 0$ for all $x, y \in J$, then R is commutative.*

Proof Assume that d is a nonzero derivation of R such that $d([x, y]) = 0$ for all $x, y \in J$. Let $\mathcal{D}$ be the additive mapping defined on $\mathcal{R} = R \times R^0$ by $\mathcal{D}(x, y) = (d(x), 0)$. Clearly, $\mathcal{D}$ is a nonzero derivation of $\mathcal{R}$. Moreover, if we set $\mathcal{J} = J \times J$, then $\mathcal{J}$ is a $*$ex-Jordan ideal of $\mathcal{R}$ and $\mathcal{D}([x, y]) = 0$ for all $x, y \in \mathcal{J}$. Since $\mathcal{R}$ is a $*$ex-prime ring, in view of Theorem 2.6 we deduce that $\mathcal{R}$ is commutative and a fortiori R is commutative. □

Corollary 2.11 *Let R be a 2-torsion free prime ring and I a nonzero ideal of R. If R admits a nonzero derivation d such that $d([x, y]) = 0$ for all $x, y \in I$, then R is commutative.*

Daif and Bell [3, Theorem 2.10 and Theorem 2.6] showed that, if a semiprime (resp., prime) ring R admits a derivation d satisfying $d[x, y] = \pm[x, y]$ for all x, y in R (resp., some nonzero ideal K of R), then R is commutative. The following theorem generalizes Theorems 2.6 and 2.10 of [3] for 2-torsion-free rings.

Theorem 2.12 *Let R be a 2-torsion-free $*$-prime ring and J a nonzero $*$-Jordan ideal of R. If R admits a nonzero derivation d such that $d([x, y]) = [x, y]$ for all $x, y \in J$; then R is commutative.*

Proof Assume that

$$d([x, y]) = [x, y] \quad \text{for all } x, y \in J. \tag{13}$$

Suppose that $Z(R) \cap J = 0$; replacing x by $4xy^2$ in Equation (13) we get

$$[x, y]d(y^2) = 0 \quad \text{for all } x, y \in J. \tag{14}$$

Writing $2[r, s]x$ instead of x in Equation (14) we find that $[[r, s], y]xd(y^2) = 0$ and thus

$$[[r, s], y]Jd(y^2) = 0 \quad \text{for all } y \in J \text{ and } r, s \in R. \tag{15}$$

For $y \in J \cap Sa_*(R)$, in view of (15), Lemma 2.4 yields $[[r, s], y] = 0$ or $d(y^2) = 0$.
Assume

$$[[r, s], y] = 0 \quad \text{for all } r, s \in R. \tag{16}$$

In the light of semiprimeness, Equation (16) together with Fact 3 force $y \in Z(R)$. Hence $y = 0$ and, therefore, $d(y^2) = 0$ for all $y \in J \cap Sa_*(R)$.

Let $y \in J$. Since $y + y^*, y - y^* \in J \cap Sa_*(R)$, then $d((y + y^*)^2) = 0$ and $d((y - y^*)^2) = 0$, which forces $d(y^2) = -d((y^*)^2)$.

Replacing y by y^* in Equation (21) we arrive at

$$([[r, s], y])^* Jd(y^2) = 0 \quad \text{for all } y \in J \text{ and } r, s \in R. \tag{17}$$

Using Equation (15) together with Equation (17), Lemma 2.4 forces

$$d(y^2) = 0 \quad \text{for all } y \in J.$$

According to [8, Lemma 3], the last equality implies that R is commutative. But $J \cap Z(R) = 0$ forces $J = 0$; which contradicts our hypothesis. Hence,

$$J \cap Z(R) \neq 0$$

Suppose that $d(u) = 0$ for all $u \in J \cap Z(R)$. Substituting $4u^2r$ for x in Equation (19) where $r \in R$ we get $u^2(d([r, y]) - [r, y]) = 0$ and, therefore,

$$u^2R(d([r, y]) - [r, y]) = 0 \quad \text{for all } y \in J \text{ and } r \in R. \tag{18}$$

Since $u^* \in J \cap Z(R)$, similarly we then obtain

$$(u^2)^*R(d([r, y]) - [r, y]) = 0 \quad \text{for all } y \in J \text{ and } r \in R. \tag{19}$$

Combining Equations (18) and (19), the $*$-primeness forces

$$d([r, y]) = [r, y] \quad \text{for all } y \in J \text{ and } r \in R. \tag{20}$$

Replacing r by ry in Equation (20) we obtain

$$[r, y]d(y) = 0 \quad \text{for all} \quad y \in J \quad \text{and} \quad r \in R. \tag{21}$$

Substituting rs for r in Equation (21), we find that $[r, y]sd(y) = 0$ so that

$$[r, y]Rd(y) = 0 \quad \text{for all} \quad y \in J \quad \text{and} \quad r \in R. \tag{22}$$

For $y \in J \cap Sa_*(R)$, Equation (22) yields $d(y) = 0$ or $y \in Z(R)$.

Let $y \in J$; since $y - y^* \in J \cap Sa_*(R)$, then $y - y^* \in Z(R)$ or $d(y - y^*) = 0$.

If $y - y^* \in Z(R)$, then $[r, y] = [r, y^*]$ for all $r \in R$. Hence, Equation (22) becomes $[r, y^*]Rd(y) = 0$ so that

$$[r, y]^* Rd(y) = 0 \quad \text{for all} \quad y \in J \quad \text{and} \quad r \in R. \tag{23}$$

Using Equation (22) together with Equation (23), we conclude that either $y \in Z(R)$ or $d(y) = 0$.

If $d(y - y^*) = 0$, then $d(y) = d(y^*)$ and replacing y by y^* in Equation (22) we obtain $[r, y]^* Rd(y) = 0$ which combined with Equation (22) leads to $d(y) = 0$ or $y \in Z(R)$. Accordingly, in all the cases we find that $d(y) = 0$ or $y \in Z(R)$ for all $y \in J$. Let $J_1 = \{x \in J | x \in Z(R)\}$ and $J_2 = \{x \in J | d(x) = 0\}$. Clearly each of J_1 and J_2 is additive subgroup of J such that $J = J_1 \cup J_2$. But, a group can not be the set-theoretic union of its two proper subgroups. Hence $J = J_1$ or $J = J_2$; that is, $J \subseteq Z(R)$ or $d(J) = 0$, and, in the second case, [5, Lemma 4] assures that $J \subseteq Z(R)$ as well. Thus R is commutative by [4, Lemma 3].

Now suppose there exists $u \in J \cap Z(R)$ such that $d(u) \neq 0$. Replacing x by $4x^2u$ in Equation (19) we obtain $[x^2, y]d(u) = 0$ which, in the light of $d(u) \in Z(R)$, forces

$$[x^2, y]Rd(u) = 0 \quad \text{for all} \quad x, y \in J. \tag{24}$$

Since J is a $*$-ideal, then Equation (24) leads to

$$[x^2, y]^* Rd(u) = 0 \quad \text{for all} \quad x, y \in J. \tag{25}$$

Combining Equation (24) and Equation (25) we conclude that

$$[x^2, y] = 0 \quad \text{for all} \quad x, y \in J. \tag{26}$$

Replacing y by $4ry^2$ in Equation (26) we obtain

$$[x^2, r]y^2 = 0 \quad \text{for all} \quad x, y \in J \quad \text{and} \quad r \in R. \tag{27}$$

Substituting rt for r in Equation (27) we find that $[x^2, r]ty^2 = 0$ and thus

$$[x^2, r]Ry^2 = 0 \quad \text{and} \quad [x^2, r]^* Ry^2 = 0 \quad \text{for all} \quad x, y \in J \quad \text{and} \quad r \in R. \tag{28}$$

Since R is $*$-prime, then Equation (28) yields $x^2 \in Z(R)$ for all $x \in J$ or $y^2 = 0$ for all $y \in J$. Hence, in both the cases we have $x^2 \in Z(R)$ for all $x \in J$. Therefore, $xy + yx \in Z(R)$ and replacing x by $2x^2$, 2-torsion freeness forces $x^2 y \in Z(R)$ for all $x, y \in J$. Accordingly,

$$x^2[y, r] = 0 \quad \text{for all} \quad x, y \in J \quad \text{and} \quad r \in R.$$

Since $x^2 \in Z(R)$, then we get

$$x^2 R[y, r] = 0 \quad \text{and} \quad x^2 R[y, r]^* = 0 \quad \text{for all} \quad x, y \in J \quad \text{and} \quad r \in R. \tag{29}$$

Once again using the $*$-primeness, from Equation (29) either $J \subseteq Z(R)$, in which case [4, Lemma 3] assures the commutativity of R, or $x^2 = 0$ for all $x \in J$. But the last case forces $d(x^2) = 0$ for all $x \in J$ and in view of [8, Lemma 3] this yields that R is commutative. □

References

1. Awtar, R.: Lie and Jordan structure in prime rings with derivations. Proc. Amer. Math. Soc **41**, 67–74 (1973)
2. Bell, H.E.; Daif, M.N.: On derivations and commutativity in prime rings. Acta Mathematica Hungarica **66**, 337–343 (1995)
3. Daif, M.N.; Bell, H.E.: Remarks on derivations on semiprime rings. Int. J. Math. Math. Sci **15**, 205–206 (1992)
4. Oukhtite, L.: Posner's second theorem for Jordan ideals in rings with involution. Expo. Math. **29**(4), 415–419 (2011)
5. Oukhtite, L.: On Jordan ideals and derivations in rings with involution. Comment. Math. Univ. Carol. **51**(3), 389–395 (2010)
6. Oukhtite, L.; Salhi, S.: On derivations in σ-prime rings. Int. J. Algebra **1**(5), 241–246 (2007)
7. Oukhtite, L.; Salhi, S.: On commutativity of σ-prime rings. Glas. Mat. Ser. III **41**(1), 57–64 (2006)
8. Oukhtite, L.; Mamouni, A.: Generalized derivations centralizing on Jordan ideals of rings with involution. submitted
9. Zaidi, S.M.A.; Ashraf, M.; Ali, S.: On Jordan ideals and left (θ, θ)-derivations in prime rings. Int. J. Math. Math. Sci. **2004**(37), 1957–1969 (2004)