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RESEARCH ARTICLE

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Warped product submanifolds of Lorentzian paracosymplectic manifolds

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Abstract In this paper we study the warped product submanifolds of a Lorentzian paracosymplectic manifold and obtain some nonexistence results. We show that a warped product semi-invariant submanifold in the form $M = M_\top \times_f M_\perp$ of a Lorentzian paracosymplectic manifold such that the characteristic vector field is normal to M is a usual Riemannian product manifold where totally geodesic and totally umbilical submanifolds of warped product are invariant and anti-invariant, respectively. We prove that the distributions involved in the definition of a warped product semi-invariant submanifold are always integrable. A necessary and sufficient condition for a semi-invariant submanifold of a Lorentzian paracosymplectic manifold to be warped product semi-invariant submanifold is obtained. We also investigate the existence and nonexistence of warped product semi-slant and warped product anti-slant submanifolds in a Lorentzian paracosymplectic manifold.

Mathematics Subject Classification 53B25 · 53C15 · 53C50

الملخص

ندرس في هذه الورقة مُتنوِّعات الضرب الملفوف الجزئية من مُتنوِّعة لورنتسية نظيرة بَيْنِيَّة التنامي مرافقة ونحصل على بعض نتائج عدم الوجود. نثبت أن متنوعة ضرب ملفوف نصف لا- متغايرة على الصيغة $M_\top \times_f M_\perp$ جزئية من متنوعة لورنتسية نظيرة بَيْنِيَّة التنامي مرافقة، بحيث يكون حقل الاتجاه المميز ناظمياً على M، هي متنوعة ضرب ريمان عادية حيث تكون المتنوعات المتقاصرة كلياً وتلك السُّرِّيَّة كلياً والجزئية من الضرب الملفوف لامتغايرة ومقابل-لامتغايرة، على التوالي. نثبت أن التوزيعات التي ينطوي عليها تعريف متنوعة الضرب الملفوف الجزئية نصف-المتغايرة هي دائماً قابلة للتكامل. يتم الحصول على شرط ضروري وكافٍ لتكون متنوعة نصف-متغايرة جزئية من متنوعة لورنتسية نظيرة بَيْنِيَّة التنامي مرافقة متنوعة ضرب ملفوف نصف-متغايرة جزئية. نبحث أيضاً وجود وعدم وجود متنوعات ضرب ملفوف نصف مائلة ومتنوعات ضرب ملفوف مقابل-مائلة جزئية من مُتنوِّعة لورنتسية نظيرة بَيْنِيَّة التنامي مرافقة.

1 Introduction

Warped product manifolds were introduced by Bishop and O'Neill [7] in 1969 as a generalization of Riemannian product manifolds. Warped products play some important roles in differential geometry as well as physics.

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The geometry of slant submanifolds has shown an increasing development in the last two decades. The theory of slant immersions in complex geometry was introduced by Chen [12,13] as a generalization of both holomorphic and totally real submanifolds. Later slant submanifolds have been studied by many geometers in various manifolds.

In 1996, Lotta [24] introduced the notion of slant submanifolds of an almost contact metric manifold. In [10,11] the authors studied and characterized slant submanifolds of K-contact and Sasakian manifolds.

On the other hand, in [5] Bejancu initiated the study of CR-submanifolds of an almost Hermitian manifold by generalizing invariant and anti-invariant submanifolds. Bejancu and Papaghiuc [6] extended this concept to submanifolds of almost contact metric manifolds and they called such submanifolds as semi-invariant submanifolds.

Recently, the study of semi-slant submanifolds was initiated by N. Papaghiuc as a generalization of CR-submanifolds [30]. In [9] Cabrerizo et al. defined and studied a contact version of semi-slant submanifolds (see also [1,20,23]).

Chen [14,15] initiated the study of CR-warped product in Kaehlerian manifolds and proved some nonexistence theorems for warped product CR-submanifolds of Kaehlerian manifolds. Hasegawa and Mihai [19] and Munteanu [26] studied the warped product contact CR-submanifolds in Sasakian manifolds.

As a generalization of warped product CR-submanifolds warped product semi-slant submanifolds are very important in differential geometry. Since every structure on a manifold may not allow defining warped product semi-slant submanifolds, the existence and nonexistence of these submanifolds are basic problems to study. In [33] Sahin proved the nonexistence of semi-slant warped product submanifolds of a Kaehler manifold. In [21] the authors studied the warped product submanifolds of a cosymplectic manifold which is locally product of a Kaehler manifold and a one dimensional manifold. Warped product semi-slant submanifolds in locally Riemannian product manifolds and Kenmotsu manifolds were studied by Atçeken [2,3], respectively.

An almost paracontact structure (φ, ξ, η) satisfying $\varphi^2 = I - \eta \otimes \xi$ and $\eta(\xi) = 1$ on a differentiable manifold, was introduced by Satō [35]. The structure is an analogue of the almost contact structure [8,34] and is closely related to almost product structure (in contrast to almost contact structure, which is related to almost complex structure). An almost contact manifold is always odd-dimensional but an almost paracontact manifold could be even-dimensional as well. In an almost paracontact manifold defined by Satō, the metric is always a Riemannian metric. In addition, in 1989, Matsumoto [27] replaced the structure vector field ξ by $-\xi$ in an almost paracontact manifold and associated a Lorentzian metric with the resulting structure and called it a Lorentzian almost paracontact manifold. Later on several authors studied Lorentzian almost paracontact manifolds, their different classes, such as Lorentzian paracosymplectic manifolds and Lorentzian para-Sasakian manifolds (see [25,31]) and their submanifolds (see [16,17,22,32,36,37]).

In [4], the author studied the warped product semi-invariant submanifolds in almost paracontact Riemannian manifolds.

In this paper we study warped product submanifolds of a Lorentzian paracosymplectic manifold and obtain to some nonexistence results. Section 2 is devoted to some basic definitions. In Sect. 3, we show that there does not exist a proper warped product submanifold in the form $M = N_1 \times_f N_2$ in a Lorentzian paracosymplectic manifold such that the characteristic vector field ξ is tangent to N_2. In Sect. 4, we study warped product semi-invariant submanifolds of a Lorentzian paracosymplectic manifold and give an example. We prove that the distributions involved in the definition of a warped product semi-invariant submanifold are always integrable. Also we obtain a necessary and sufficient condition for a semi-invariant submanifold of a Lorentzian paracosymplectic manifold to be warped product semi-invariant submanifold in terms of the shape operator. In Sect. 5, we show that there exist no proper warped product semi-slant submanifolds in the form $M = N_\top \times_f N_\theta$ (resp., $M = N_\theta \times_f N_\top$) with ξ belonging to $N_\top$ (resp., ξ belonging to N_θ) of a Lorentzian paracosymplectic manifold where $N_\top$ is an invariant submanifold and N_θ is a proper slant submanifold of the ambient manifold. The last section contains some nonexistence results for the proper warped product anti-slant submanifolds of a Lorentzian paracosymplectic manifold.

2 Preliminaries

Let $\overline{M}$ be an m-dimensional differentiable manifold equipped with a triple (φ, ξ, η), where φ is a (1, 1) tensor field, ξ is a vector field, η is a 1-form on $\overline{M}$ such that [27]

$$\eta(\xi) = -1, \tag{2.1}$$
$$\varphi^2 = I + \eta \otimes \xi, \tag{2.2}$$

where I denotes the identity map of $T_p\overline{M}$ and $\otimes$ is the tensor product. Equations (2.1) and (2.2) imply that

$$\eta \circ \varphi = 0, \quad \varphi\xi = 0, \quad \mathrm{rank}(\varphi) = m - 1. \tag{2.3}$$

Then $\overline{M}$ admits a Lorentzian metric g, such that, for all $X, Y \in \chi(\overline{M})$,

$$g(\varphi X, \varphi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.4}$$

and $\overline{M}$ is said to admit a Lorentzian almost paracontact structure (φ, ξ, η, g). Then we get

$$g(X, \xi) = \eta(X), \tag{2.5}$$
$$\Phi(X, Y) = g(X, \varphi Y) = g(\varphi X, Y) = \Phi(Y, X), \tag{2.6}$$
$$(\overline{\nabla}_X \Phi)(Y, Z) = g(Y, (\overline{\nabla}_X \varphi)Z) = (\overline{\nabla}_X \Phi)(Z, Y), \tag{2.7}$$

where $\overline{\nabla}$ is the covariant differentiation with respect to g. It is clear that Lorentzian metric g makes ξ a timelike unit vector field, i.e, $g(\xi, \xi) = -1$. The manifold $\overline{M}$ equipped with a Lorentzian almost paracontact structure (φ, ξ, η, g) is called a Lorentzian almost paracontact manifold (for short, LAP-manifold) [27,28].

In Equations (2.1) and (2.2) if we replace ξ by $-\xi$, we obtain an almost paracontact structure on $\overline{M}$ defined by Satō [35].

A Lorentzian almost paracontact manifold endowed with the structure (φ, ξ, η, g) is called a Lorentzian paracontact manifold (for short LP-manifold) [27] if

$$\Phi(X, Y) = \frac{1}{2}((\overline{\nabla}_X \eta)Y + (\overline{\nabla}_Y \eta)X). \tag{2.8}$$

A Lorentzian almost paracontact manifold endowed with the structure (φ, ξ, η, g) is called a Lorentzian para-Sasakian manifold (for short, LP-Sasakian) [27] if

$$(\overline{\nabla}_X \varphi)Y = \eta(Y)X + g(X, Y)\xi + 2\eta(X)\eta(Y)\xi. \tag{2.9}$$

In a Lorentzian para-Sasakian manifold the 1-form η is closed and $\overline{\nabla}_X \xi = \varphi X$, for any $X \in \chi(\overline{M})$.

A Lorentzian almost paracontact manifold is called a Lorentzian paracosymplectic manifold [31] if

$$\overline{\nabla}\varphi = 0. \tag{2.10}$$

A Lorentzian paracosymplectic manifold is locally isometric to the Lorentzian manifold

$$(\overline{M} = R \times M^+ \times M^-, g = -dt^2 + g_+ + g_-), \tag{2.11}$$

which is a direct product of the real line and Riemannian manifolds M^+ and M^-. Moreover, if we put

$$\xi = \partial_t, \quad \eta = -dt, \quad \varphi = Id_{TM_+} - Id_{TM_-},$$

then it can be easily shown that the structure (φ, ξ, η) is a paracosymplectic structure on the Lorentzian manifold given by (2.11).

Let M be an isometrically immersed submanifold of a Lorentzian almost paracontact manifold $\overline{M}$. We denote the Levi–Civita connections on M and $\overline{M}$ by ∇ and $\overline{\nabla}$, respectively. Then the Gauss and Weingarten formulae are given by:

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.12}$$
$$\overline{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{2.13}$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$, where $\nabla^{\perp}$ is the connection in the normal bundle $TM^{\perp}$, h is the second fundamental form of M and A_N is the shape operator. The second fundamental form h and the shape operator A_N are related by:

$$g(A_N X, Y) = g(h(X, Y), N), \tag{2.14}$$

where the induced Riemannian metric on M is denoted by the same symbol g.

Consider that M is an isometrically immersed submanifold of a Lorentzian almost paracontact manifold $\overline{M}$. For any $X \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$, we put

$$\varphi X = tX + nX, \tag{2.15}$$
$$\varphi N = BN + CN, \tag{2.16}$$

where tX (resp., nX) is tangential (resp., normal) part of φX and BN (resp., CN) is tangential (resp., normal) part of φN. The submanifold M is called an invariant submanifold if n is identically zero, that is, $\varphi X = tX \in \Gamma(TM)$ for any $X \in \Gamma(TM)$. On the other hand, M is called an anti-invariant submanifold if t is identically zero, that is, $\varphi X = nX \in \Gamma(TM^{\perp})$ for any $X \in \Gamma(TM)$.

From (2.6) and (2.15), one can easily see that

$$g(X, tY) = g(tX, Y), \tag{2.17}$$

for any $X, Y \in \Gamma(TM)$.

Now, assume that M is an isometrically immersed submanifold of a Lorentzian almost paracontact manifold $\overline{M}$ such that the characteristic vector field ξ belongs to the tangent bundle of the submanifold. Then M is said to be a semi-invariant submanifold [6] if it is endowed with the pair of orthogonal distribution $(D, D^{\perp})$ satisfying the conditions

(i) $TM = D \oplus D^{\perp} \oplus \langle \xi \rangle$,
(ii) the distribution D is invariant under φ, i.e., $\varphi(D) = D$,
(iii) the distribution $D^{\perp}$ is anti-invariant under φ, i.e., $\varphi(D^{\perp}) \subset TM^{\perp}$.

Let M be an isometrically immersed submanifold of a Lorentzian almost paracontact manifold $(\overline{M}, \varphi, \xi, \eta, g)$ such that the characteristic vector field ξ is tangent to M. Hence, if we denote by D the orthogonal distribution to ξ in TM, we can consider the orthogonal direct decomposition $TM = D \oplus \langle \xi \rangle$. In this case, it is obvious that $g(X, X) > 0$ for any vector field $X \neq 0$ in D. For each nonzero vector X tangent to M at the point $p \in M$ such that X is not proportional to ξ_p, we denote by $\theta(X)$ the angle between φX and T_pM. Since $\varphi\xi = 0$, θ agrees with the angle between φX and D_p. Then M is called slant submanifold if the angle $\theta(X)$ is constant, which does not depend on the choice of $p \in M$ and $X \in T_pM - \langle \xi_p \rangle$. The constant angle θ is then called the slant angle of M in $\overline{M}$. The invariant and anti-invariant submanifolds of a Lorentzian almost paracontact manifold are slant submanifolds with $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. A slant submanifold, which is neither invariant nor anti-invariant is said to be a proper slant submanifold.

A useful characterization of slant submanifolds in a Lorentzian almost paracontact manifold is given in the following.

Theorem 2.1 *Let M be an immersed submanifold of a Lorentzian almost paracontact manifold $(\overline{M}, \varphi, \xi, \eta, g)$ such that $\xi \in \Gamma(TM)$. Then M is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$t^2 = \lambda(I + \eta \otimes \xi). \tag{2.18}$$

Furthermore, in such case, if θ is the slant angle of M, then $\lambda = \cos^2\theta$ [22].

As an immediate consequence of Theorem 2.1 and (2.17) we have:

Corollary 2.2 *Let M be a slant submanifold of a Lorentzian almost paracontact manifold $(\overline{M}, \varphi, \xi, \eta, g)$ with $\xi \in \Gamma(TM)$. Then*

$$g(tX, tY) = \cos^2\theta\{g(X, Y) + \eta(X)\eta(Y)\}, \tag{2.19}$$
$$g(nX, nY) = \sin^2\theta\{g(X, Y) + \eta(X)\eta(Y)\}, \tag{2.20}$$

for all $X, Y \in \Gamma(TM)$, where θ is the slant angle [22].

Furthermore, let M be a submanifold of a Lorentzian almost paracontact manifold $\overline{M}$ such that $\xi \in \Gamma(TM)$. If there exist two differentiable distributions D_1 and D_2 on M such that $TM = D_1 \oplus D_2 \oplus \langle \xi \rangle$, D_1 is an invariant (resp., anti-invariant) distribution and D_2 is a slant distribution with the slant angle $\theta \neq 0$, then M is called a semi-slant (resp., anti-slant) submanifold of $\overline{M}$ [30]. Particularly, if dim $D_1 = 0$ and $\theta \neq \frac{\pi}{2}$ then a semi-slant submanifold reduces to a proper slant submanifold. Thus, semi-slant submanifolds can be considered as a generalization of slant submanifolds.

Let M be a semi-slant submanifold of a Lorentzian paracosymplectic manifold. By using Gauss–Weingarten formulae, (2.14) and (2.16) in (2.10) we have:

$$(\nabla_X t)Y = A_{nY}X + Bh(X, Y) \tag{2.21}$$

and

$$(\nabla_X n)Y = Ch(X, Y) - h(X, tY), \tag{2.22}$$

for all $X, Y \in \Gamma(TM)$. Here, the covariant derivatives of t and n are defined by:

$$(\nabla_X t)Y = \nabla_X tY - t\nabla_X Y,$$
$$(\nabla_X n)Y = \nabla_X^{\perp} nY - n\nabla_X Y.$$

3 Warped and doubly warped submanifolds

The notion of warped product manifolds was introduced by Bishop and O'Neill [7]. Let (B, g_B) and (F, g_F) be two semi-Riemannian manifolds and $b : B \to (0, \infty)$ be a smooth function. The warped product $M = B \times_b F$ of B and F is the product manifold $B \times F$ with the metric tensor

$$g = g_B \oplus b^2 g_F,$$

given by

$$g(X, Y) = g_B(d\pi(X), d\pi(Y)) + (b \circ \pi)^2 g_F(d\sigma(X), d\sigma(Y)),$$

where $X, Y \in \Gamma(T(B \times F))$ and $\pi : B \times F \to B$ and $\sigma : B \times F \to F$ are the canonical projections.

For warped product manifolds we have the following proposition [29].

Proposition 3.1 *Let $M = B \times_b F$ be a warped product manifold. If $X, Y \in \Gamma(TB)$ and $U, V \in \Gamma(TF)$ then*

(i) $\nabla_X Y \in \Gamma(TB)$,
(ii) $\nabla_X U = \nabla_U X = X(\ln b)U$,
(iii) $\nabla_U V = \nabla'_U V - g(U, V)\mathrm{grad}(\ln b)$,

where ∇ and ∇' denote the Levi–Civita connections on M and F, respectively.

In this case B is totally geodesic in M and F is totally umbilical in M [29].

As a generalization of the warped product of two semi-Riemannian manifolds, doubly warped product manifolds were introduced by Ehrlich [18]. A doubly warped product of semi-Riemannian manifolds (B, g_B) and (F, g_F) with warping functions $b : B \to (0, \infty)$ and $f : F \to (0, \infty)$ is a product manifold $B \times F$ endowed with a metric tensor

$$g = f^2 g_B \oplus b^2 g_F.$$

More explicitly, if $X, Y \in \Gamma(T(B \times F))$ then

$$g(X, Y) = (f \circ \sigma)^2 g_B(d\pi(X), d\pi(Y)) + (b \circ \pi)^2 g_F(d\sigma(X), d\sigma(Y)),$$

where $\pi : B \times F \to B$ and $\sigma : B \times F \to F$ are the canonical projections. We denote the doubly warped product of semi-Riemannian manifolds (B, g_B) and (F, g_F) by ${}_fB \times_b F$. If either $b = 1$ or $f = 1$, but not both, then ${}_fB \times_b F$ becomes a warped product of semi-Riemannian manifolds B and F. If both $b = 1$ and $f = 1$, then we have a product manifold. If neither b nor f is constant, then we have a proper (nontrivial) doubly warped product manifold (see also [38]).

In this case we have

$$\nabla_X U = X(\ln b)U + U(\ln f)X, \tag{3.1}$$

for any $X \in \Gamma(TB)$ and $U \in \Gamma(TF)$ [38].

Now, we first give a useful lemma for later use.

Lemma 3.2 *Let M be an immersed submanifold of a Lorentzian paracosymplectic manifold $(\overline{M}, \varphi, \xi, \eta, g)$ such that $\xi \in \Gamma(TM)$. Then we have*

$$\nabla_X \xi = 0, \tag{3.2}$$
$$h(X, \xi) = 0, \tag{3.3}$$

for all $X \in \Gamma(TM)$.

Proof Since $\overline{M}$ is a Lorentzian paracontact manifold, by using (2.10) we get

$$\overline{\nabla}_X \xi = 0, \tag{3.4}$$

for any $X \in \Gamma(TM)$. From Gauss formula in the last equation we complete the proof. □

Let us consider a doubly warped product of two semi-Riemannian manifolds N_1 and N_2 embedded into a Lorentzian paracosymplectic manifold $(\overline{M}, \varphi, \xi, g)$ with the characteristic vector field ξ belonging to the submanifold $M = {}_{f_2}N_1 \times_{f_1} N_2$.

Theorem 3.3 *Let $M = {}_{f_2}N_1 \times_{f_1} N_2$ be a doubly warped product submanifold of a Lorentzian paracosymplectic manifold $(\overline{M}, \varphi, \xi, g)$. Then*

(i) *f_1 is constant if $\xi \in \Gamma(TN_2)$,*
(ii) *f_2 is constant if $\xi \in \Gamma(TN_1)$.*

Proof (i) Assume that $\xi \in \Gamma(TN_2)$. Then for any $X \in \Gamma(TN_1)$ from (3.1) and (3.2) we get

$$X(\ln f_1)\xi + \xi(\ln f_2)X = 0.$$

This implies that $X(\ln f_1) = 0, \forall X \in \Gamma(TN_1)$. Hence f_1 is constant.

(ii) Similarly, for $\xi \in \Gamma(TN_1)$ and $Z \in \Gamma(TN_2)$ we have

$$\xi(\ln f_1)Z + Z(\ln f_2)\xi = 0,$$

which implies that f_2 is constant. This completes the proof. □

As an immediate consequence of the above theorem we have the following:

Corollary 3.4 *There does not exist a proper warped product submanifold $M = N_1 \times_f N_2$ in a Lorentzian paracosymplectic manifold $(\overline{M}, \varphi, \xi, \eta, g)$ such that ξ is tangent to N_2.*

Let $M = N_1 \times_f N_2$ be a proper warped product submanifold of a Lorentzian paracosymplectic manifold $(\overline{M}, \varphi, \xi, g)$ such that $\xi \in \Gamma(TM)$. Then we can write

$$\xi = \xi_1 + \xi_2, \quad \xi_1 \in \Gamma(TN_1), \quad \xi_1 \in \Gamma(TN_2). \tag{3.5}$$

From (3.2) we have

$$\nabla_X \xi = 0, \quad \forall X \in \Gamma(TN_1), \tag{3.6}$$
$$\nabla_Z \xi = 0, \quad \forall Z \in \Gamma(TN_2). \tag{3.7}$$

By using (3.5) and Proposition 3.1 in (3.6) we get

$$X(\ln f)\xi_2 = 0. \tag{3.8}$$

Since M is a proper warped product submanifold, (3.8) implies that $\xi_2 = 0$. Similarly, from (3.5), (3.6) and Proposition 3.1 we get

$$g(Z, \xi_2)\text{grad}(\ln f) = 0, \quad \forall Z \in \Gamma(TN_2). \tag{3.9}$$

Since grad$(\ln f)$ cannot be zero, then $g(Z, \xi_2) = 0, \forall Z \in \Gamma(TN_2)$, which implies that $\xi_2 = 0$.

Thus we have proved the following:

Corollary 3.5 *There does not exist a proper warped product submanifold $M = N_1 \times_f N_2$ in a Lorentzian paracosymplectic manifold $(\overline{M}, \varphi, \xi, \eta, g)$ such that ξ have both TN_1 and TN_2 components.*

Now, to study the warped product submanifolds $M = N_1 \times_f N_2$ with the structure vector field $\xi \in \Gamma(TN_1)$, we shall give some useful formulae.

Lemma 3.6 *Let $M = N_1 \times_f N_2$ be a proper warped product submanifold of a Lorentzian paracosymplectic manifold $(\overline{M}, \varphi, \xi, \eta, g)$ such that $\xi \in \Gamma(TN_1)$. Then we have*

$$\xi(\ln f) = 0, \tag{3.10}$$

$$A_{nZ}X = -Bh(X, Z), \tag{3.11}$$

$$g(h(X, Y), nZ) = -g(h(X, Z), nY), \tag{3.12}$$

$$g(h(X, W), nZ) = -g(h(X, Z), nW), \tag{3.13}$$

for any $X, Y \in \Gamma(TN_1)$ and $Z, W \in \Gamma(TN_2)$.

Proof From (3.2) and Proposition 3.1, Equation (3.10) is obvious. By using (2.10) and Proposition 3.1, we have

$$\begin{aligned} X(\ln f)tZ + h(X, tZ) - A_{nZ}X + \nabla^{\perp}_X nZ = X(\ln f)tZ + n\nabla_X Z \\ +Bh(X, Z) + Ch(X, Z) \end{aligned}$$

and therefore

$$h(X, tZ) - A_{nZ}X + \nabla^{\perp}_X nZ = Bh(X, Z) + Ch(X, Z), \tag{3.14}$$

for any $X \in \Gamma(TN_1)$ and $Z \in \Gamma(TN_2)$. From the tangential parts of (3.14) we get (3.11). By taking the product in (3.11) by $Y \in \Gamma(TN_1)$ and $W \in \Gamma(TN_2)$, we obtain (3.12) and (3.13), respectively. This completes the proof. □

4 Warped product semi-invariant submanifolds

Now, we shall investigate the warped product semi-invariant submanifolds of Lorentzian paracosymplectic manifolds.

Theorem 4.1 *Let $M = M_{\top} \times_f M_{\perp}$ be a warped product semi-invariant submanifold of a Lorentzian paracosymplectic manifold $(\overline{M}, \varphi, \xi, g)$ such that M_T is an invariant submanifold, $M_{\perp}$ is an anti-invariant submanifold of $\overline{M}$ and $\xi \in \Gamma(TM^{\perp})$. Then M is an usual Riemannian product manifold.*

Proof From Proposition 3.1, Gauss formula and (2.4) we have

$$g(\nabla_X Z, W) = g(\nabla_Z X, W) = g(\overline{\nabla}_Z X, W) = g(\varphi\overline{\nabla}_Z X, \varphi W),$$

for any $X \in \Gamma(TM_{\top})$ and $Z, W \in \Gamma(TM_{\perp})$. Since $\overline{M}$ is a Lorentzian paracosymplectic manifold, by using Gauss–Weingarten formulae, (2.14) and Proposition 3.1 in the last equation we get

$$\begin{aligned} X(\ln f)g(Z, W) &= g(\overline{\nabla}_Z\varphi X, \varphi W) = g(h(Z, \varphi X), \varphi W) \\ &= g(\overline{\nabla}_{\varphi X} Z, \varphi W) = g(\overline{\nabla}_{\varphi X}\varphi Z, W) \\ &= -g(A_{\varphi Z}\varphi X, W) = -g(h(\varphi X, W), \varphi Z) \\ &= -g(\overline{\nabla}_W\varphi X, \varphi Z) = -g(\varphi\overline{\nabla}_W X, \varphi Z) \\ &= -g(\overline{\nabla}_W X, Z) = -g(\overline{\nabla}_X W, Z) \\ &= -X(\ln f)g(W, Z), \end{aligned}$$

which implies that

$$X(\ln f)g(W, Z) = 0.$$

This completes the proof. □

Theorem 4.2 *Let $M = M_\top \times_f M_\perp$ be a warped product semi-invariant submanifold of a Lorentzian paracosymplectic manifold $(\overline{M}, \varphi, \xi, g)$ such that $M_\top$ is an invariant submanifold, $M_\perp$ is an anti-invariant submanifold of $\overline{M}$ and $\xi \in \Gamma(TM_\perp)$. Then M is a Lorentzian product manifold.*

Proof Choose $X \in \Gamma(TM_\top)$ and note that $\xi \in \Gamma(TM_\perp)$. From Proposition 3.1 and (3.2) we have

$$\nabla_X \xi = \nabla_\xi X = X(\ln f)\xi = 0,$$

which implies that f is constant. This completes the proof. □

Now we give an example for a submanifold of a Lorentzian paracontact manifold in the form $M = M_\perp \times_f M_\top$.

Example 4.3 Let $\overline{M}$ be the 5-dimensional real number space with a coordinate system (x_1, x_2, y_1, y_2, z). If we define

$$\eta = dz, \quad \xi = -\frac{\partial}{\partial z},$$

$$\varphi\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_i}, \quad (1 \le i \le 2)$$
$$\varphi\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial y_j}, \quad (1 \le j \le 2)$$

$$\varphi\left(\frac{\partial}{\partial z}\right) = 0,$$
$$g = (dx_i)^2 + (dy_j)^2 - \eta \otimes \eta,$$

on $\overline{M}$, then (φ, ξ, η, g) becomes a Lorentzian almost paracontact structure in $\overline{M}$.

Now, assume that M is an immersed submanifold of $\overline{M}$ given by

$$\Omega(v, \theta, \beta, u) = (v\cos\theta, v\sin\theta, v\cos\beta, v\sin\beta, \sqrt{2}u).$$

Then one can easily see that the tangent bundle of M is spanned by the vectors

$$W_1 = (\cos\theta, \sin\theta, \cos\beta, \sin\beta, 0),$$
$$W_2 = (-v\sin\theta, v\cos\theta, 0, 0, 0),$$
$$W_3 = (0, 0, -v\sin\beta, v\cos\beta, 0),$$
$$W_4 = (0, 0, 0, 0, \sqrt{2}).$$

On the other hand, since

$$\varphi W_1 = (\cos\theta, \sin\theta, -\cos\beta, -\sin\beta, 0),$$
$$\varphi W_2 = (-v\sin\theta, v\cos\theta, 0, 0, 0),$$
$$\varphi W_3 = (0, 0, v\sin\beta, -v\cos\beta, 0),$$
$$\varphi W_4 = (0, 0, 0, 0, 0),$$

then φW_1 and φW_4 are orthogonal to M, φW_2 and φW_3 are tangent to M and we can take

$$D_1 = \mathrm{Span}\{W_2, W_3\} \quad \text{and} \quad D_2 = \mathrm{Span}\{W_1, W_4\}.$$

In this case, D_1 is an invariant distribution and D_2 is an anti-invariant distribution in M. Thus M becomes a semi-invariant submanifold. Moreover, the induced metric tensor of M is given by:

$$g = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & v^2 & 0 & 0 \\ 0 & 0 & v^2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix},$$

that is,

$$g = 2(dv^2 - du^2) + v^2(d\theta^2 + d\beta^2) = 2g_{M_\perp} + v^2 g_{M_\top}.$$

Thus M is a warped product semi-invariant submanifold of $\overline{M}$ with warping function $f = v^2$.

Let $M = M_\perp \times_f M_\top$ be a warped product semi-invariant submanifold of a Lorentzian almost paracosymplectic manifold, where $M_\perp$ is an anti-invariant submanifold and $M_\top$ is an invariant submanifold of $\overline{M}$.

Now, we investigate the geometric properties of the leaves of the warped product semi-invariant submanifolds of a Lorentzian paracosymplectic manifold.

Theorem 4.4 *Let $M = M_\perp \times_f M_T$ be a warped product semi-invariant submanifold of a Lorentzian paracosymplectic manifold $(\overline{M}, \varphi, \xi, g)$. Then the invariant distribution D_1 and the anti-invariant distribution D_2 are always integrable.*

Proof From (2.10), Gauss formula, (2.15), (2.16) and Proposition 3.1 we have

$$\begin{aligned} \overline{\nabla}_X \varphi U &= \varphi \overline{\nabla}_X U \\ \nabla_X tU + h(X, tU) &= t\nabla_X U + n\nabla_X U + Bh(U, X) + Ch(U, X) \\ X(\ln f)tU + h(X, tU) &= X(\ln f)tU + Bh(U, X) + Ch(U, X), \end{aligned} \tag{4.1}$$

for any $X \in \Gamma(D_2)$, $U \in \Gamma(D_1)$. By equating the tangential and the normal components of (4.1) we obtain

$$Bh(X, U) = 0 \tag{4.2}$$

and

$$h(X, tU) = Ch(X, U). \tag{4.3}$$

From (2.21) and (4.2) we have

$$A_{nX} U = -X(\ln f)tU. \tag{4.4}$$

Since the distribution D_2 is totally geodesic in M and it is anti-invariant in $\overline{M}$, then from Gauss–Weingarten formulae we have:

$$\begin{aligned} \overline{\nabla}_X \varphi Y &= \varphi \overline{\nabla}_X Y \\ \overline{\nabla}_X nY &= \varphi \nabla_X Y + \varphi h(X, Y) \\ -A_{nY} X + \nabla^\perp_X nY &= t\nabla_X Y + n\nabla_X Y + Bh(X, Y) + Ch(X, Y), \end{aligned} \tag{4.5}$$

for any $X, Y \in \Gamma(D_2)$. By equating the tangential parts of the last equation we get

$$A_{nY} X = -Bh(X, Y). \tag{4.6}$$

By changing the role of X and Y in (4.6) we obtain

$$A_{nY} X = A_{nX} Y. \tag{4.7}$$

Furthermore, since A is self-adjoint from Gauss formula and (2.6) we have

$$\begin{aligned} g(A_{nX} Y, Z) &= g(h(Y, Z), nX) \\ &= g(\overline{\nabla}_Z Y, \varphi X) \\ &= g(\overline{\nabla}_Z \varphi Y, X) \\ &= -g(A_{nY} Z, X) \\ &= -g(A_{nY} X, Z), \end{aligned} \tag{4.8}$$

for any $X, Y \in \Gamma(D_2)$, $Z \in \Gamma(D_1)$. From (4.6) to (4.8) we obtain

$$A_{nX}Y = 0 \quad \text{and} \quad Bh(X, Y) = 0. \tag{4.9}$$

On the other hand for any $U, V \in \Gamma(D_1)$ we have

$$\begin{aligned}
\overline{\nabla}_U \varphi V &= \varphi \overline{\nabla}_U V \\
\overline{\nabla}_U tV &= \varphi \nabla_U V + \varphi h(U, V) \\
h(U, tV) + \nabla_U tV &= \varphi(\nabla'_U V - g(U, V)\text{grad}(\ln f)) + Bh(U, V) + Ch(U, V) \\
h(U, tV) + \nabla_U tV &= t(\nabla'_U V) - g(U, V)n(\text{grad}(\ln f)) + Bh(U, V) + Ch(U, V).
\end{aligned}$$

By equating the tangential and normal parts of the last equation we get

$$\nabla'_U tV - g(tV, U)\text{grad}(\ln f) = t(\nabla'_U V) + Bh(U, V) \tag{4.10}$$

and

$$h(U, tV) = -g(U, V)n(\text{grad}(\ln f)) + Ch(U, V). \tag{4.11}$$

(4.11) implies that

$$h(U, tV) = h(V, tU). \tag{4.12}$$

Finally, from (2.22), (4.12) and the symmetry of h we have

$$\begin{aligned}
n([V, U]) &= n(\nabla_V U - \nabla_U V) \\
&= \nabla^\perp_V nU - (\nabla_V n)U - \nabla^\perp_U nV + (\nabla_U n)V \\
&= (\nabla_U n)V - (\nabla_V n)U \\
&= Ch(U, V) - h(U, tV) - Ch(V, U) + h(V, tU) \\
&= 0,
\end{aligned}$$

which implies that $[V, U] \in \Gamma(D_1)$.

By a similar way, from (2.21) and (4.7) we get

$$\begin{aligned}
t([X, Y]) &= t(\nabla_X Y - \nabla_X Y) \\
&= \nabla_X tY - (\nabla_X t)Y - \nabla_Y tX + (\nabla_Y t)X \\
&= (\nabla_Y t)X - (\nabla_X t)Y \\
&= A_{nX}Y + Bh(Y, X) - A_{nY}X - Bh(X, Y) \\
&= 0.
\end{aligned}$$

Thus $[X, Y] \in \Gamma(D_2)$ for any $X, Y \in \Gamma(D_2)$. This completes the proof. □

Since the distributions D_1 and D_2 are always integrable, we denote by $M_\top$ and $M_\perp$ the integral submanifolds of D_1 and D_2, respectively.

Theorem 4.5 *Let M be a submanifold of a Lorentzian paracosymplectic manifold $(\overline{M}, \varphi, \xi, g)$. Then M is a semi-invariant submanifold if and only if $nt = 0$.*

Proof Let M be a semi-invariant submanifold of a Lorentzian paracosymplectic manifold $(\overline{M}, \varphi, \xi, \eta, g)$. We denote the orthogonal projections on the invariant distribution D_1 and the anti-invariant distribution D_2 by P_1 and P_2, respectively. Then we have:

$$P_1 + P_2 = I, \quad (P_1)^2 = P_1, \quad (P_2)^2 = P_2, \quad P_1 P_2 = P_2 P_1 = 0. \tag{4.13}$$

If the characteristic vector field ξ is tangent to M, then from

$$\begin{aligned}
X + \eta(X)\xi &= t^2 X + BnX \\
0 &= ntX + CnX \\
0 &= tBZ + BCZ \\
Z &= nBZ + C^2 Z,
\end{aligned} \tag{4.14}$$

for any $X \in \Gamma(TM)$, $Z \in \Gamma(TM^{\perp})$. By using (2.15) we can write:

$$tX + nX = tP_1X + tP_2X + nP_1X + nP_2X,$$

for any $X \in \Gamma(TM)$. By equating the tangential and normal parts of the last equation we get:

$$\begin{aligned} tX &= tP_1X + tP_2X, \\ nX &= nP_1X + nP_2X. \end{aligned} \tag{4.15}$$

Since D_1 is invariant and D_2 is anti-invariant, we get

$$nP_1 = 0 \quad \text{and} \quad tP_2 = 0.$$

Thus from (4.15) we have:

$$tP_1 = t \quad \text{and} \quad nP_2 = n,$$

which implies that

$$ntX = nP_2tX = nP_2tP_1X = 0,$$

for all $X \in \Gamma(TM)$. From the last equation and the second equation of (4.14) we also get

$$Cn = 0. \tag{4.16}$$

Conversely, assume that M be a submanifold of a Lorentzian paracosymplectic manifold $(\overline{M}, \varphi, \xi, \eta, g)$ satisfying $nt = 0$. From (2.4), (2.6) and the second equation in (4.14) we have:

$$\begin{aligned} g(X, \varphi Z) &= g(\varphi X, Z) \\ g(X, BZ) &= g(nX, Z) \\ g(X, \varphi BZ) &= g(\varphi nX, Z) \\ g(X, tBZ) &= g(CnX, Z) = 0, \end{aligned}$$

for all $X \in \Gamma(TM)$, $Z \in \Gamma(TM^{\perp})$. It is obvious from the last equation that $tB = 0$ and so by using (4.14) we get $BC = 0$. Moreover, from (4.14) we also have:

$$t^3 = t \quad \text{and} \quad C^3 = C. \tag{4.17}$$

By putting

$$P_1 = t^2 \quad \text{and} \quad P_2 = I - t^2, \tag{4.18}$$

we obtain

$$P_1 + P_2 = I, \quad (P_1)^2 = P_1, \quad (P_2)^2 = P_2, \quad P_1P_2 = P_2P_1 = 0,$$

which implies that P_1 and P_2 are orthogonal complementary projections defining complementary distributions D_1 and D_2. Since it is assumed that $nt = 0$ then from (4.17) and (4.18) we conclude

$$\begin{aligned} tP_1 = t, \quad & tP_2 = 0, \\ P_2tP_1 = 0, \quad & nP_1 = 0, \end{aligned}$$

which implies that D_1 is an invariant distribution and D_2 is an anti-invariant distribution. This completes the proof. □

Theorem 4.6 *Let M be a semi-invariant submanifold of a Lorentzian paracosymplectic manifold $(\overline{M}, \varphi, \xi, g)$. Then M is a warped product semi-invariant submanifold if and only if the shape operator of M satisfies*

$$A_{\varphi X}U = -X(\mu)\varphi U, \quad X \in \Gamma(D_2), U \in \Gamma(D_1), \tag{4.19}$$

for some function μ on M such that $W(\mu) = 0$, $W \in \Gamma(D_1)$.

Proof Let $M = M_\perp \times_f M_\top$ be a warped product semi-invariant submanifold of a Lorentzian paracosymplectic manifold. From (4.4) we have:

$$A_{\varphi X} U = -X(\ln f)\varphi U,$$

for any $X \in \Gamma(D_2)$ and $U \in \Gamma(D_1)$. Since f is a function on $M_\perp$, putting $\mu = \ln f$ implies that $W(\mu) = 0$, for all $W \in \Gamma(D_1)$.

Conversely, let M be a semi-invariant submanifold of $\overline{M}$ and μ be a function on M satisfying (4.19) such that $W(\mu) = 0$, for all $W \in \Gamma(D_1)$. Since $\overline{M}$ is a Lorentzian paracosymplectic manifold, from (4.9) we have:

$$g(\nabla_X Y, \varphi V) = g(\overline{\nabla}_X Y, \varphi V) = g(\overline{\nabla}_X \varphi Y, V) = -g(A_{\varphi Y} X, V) = 0,$$

for any $X, Y \in \Gamma(D_2)$ and $V \in \Gamma(D_1)$. So, the anti-invariant distribution D_2 is totally geodesic in M. On the other hand from (4.4) we have:

$$\begin{aligned} g(\nabla_U V, X) &= g(\overline{\nabla}_U V, X) = -g(V, \overline{\nabla}_U X) \\ &= -g(\varphi V, \overline{\nabla}_U \varphi X) = -g(\varphi V, \overline{\nabla}_U wX) \\ &= g(A_{wX} U, \varphi V) \\ &= -g(X(\mu)\varphi U, \varphi V) \\ &= -X(\mu) g(U, V), \end{aligned}$$

for any $U, V \in \Gamma(D_1)$ where $\mu = \ln f$. Since the distribution D_1 of M is always integrable and $W(\mu) = 0$ for all $W \in \Gamma(TM_\top)$ then the integral submanifold of D_1 is a totally umbilical submanifold in M and its mean curvature vector field is nonzero and parallel. Since a warped product manifold $M = M_\perp \times_f M_\top$ is characterized by the fact that $M_\perp$ and $M_\top$ are totally geodesic and totally umbilical submanifolds of M, respectively, we complete the proof. □

5 Warped product semi-slant submanifolds

Let M be a warped product semi-slant submanifold of a Lorentzian paracosymplectic manifold $(\overline{M}, \varphi, \xi, g)$. From Corollary 3.4, there do not exist warped product semi-slant submanifolds $N_\top \times_f N_\theta$ with $\xi \in \Gamma(TN_\theta)$ and $N_\theta \times_f N_\top$ with $\xi \in \Gamma(TN_\top)$ of $\overline{M}$ where $N_\top$ is an invariant submanifold and N_θ is a proper slant submanifolds of $\overline{M}$. Thus we have the following two cases:

(i) $M = N_\top \times_f N_\theta$ with $\xi \in \Gamma(TN_T)$,
(ii) $M = N_\theta \times_f N_\top$ with $\xi \in \Gamma(TN_\theta)$.

Theorem 5.1 *There do not exist proper warped product semi-slant submanifolds $M = N_\top \times_f N_\theta$ of a Lorentzian paracosymplectic manifold $(\overline{M}, \varphi, \xi, \eta, g)$ such that $N_\top$ is an invariant submanifold, N_θ is a proper slant submanifold of $\overline{M}$ and $\xi \in \Gamma(TN_\top)$.*

Proof Let $M = N_\top \times_f N_\theta$ be a proper warped product semi-slant submanifold of a Lorentzian paracosymplectic manifold $(\overline{M}, \varphi, \xi, \eta, g)$ such that $\xi \in \Gamma(TN_\top)$. From (2.10), Gauss formula, (2.15), (2.16) and Proposition 3.1 we have

$$\begin{aligned} tX(\ln f)Z + h(Z, tX) = {} & X(\ln f)tZ + X(\ln f)nZ \\ & + Bh(Z, X) + Ch(Z, X), \end{aligned} \tag{5.1}$$

for any $X \in \Gamma(TN_\top)$ and $Z \in \Gamma(TN_\theta)$. Equating the tangential and normal parts of (5.1) we get

$$tX(\ln f)Z = X(\ln f)tZ + Bh(Z, X) \tag{5.2}$$

and

$$h(Z, tX) = X(\ln f)nZ + Ch(Z, X). \tag{5.3}$$

On the other hand by virtue of (3.13) we have

$$g(h(X, Z), nZ) = 0, \tag{5.4}$$

which implies that

$$0 = g(h(X, Z), \varphi Z) = g(\varphi h(X, Z), Z) = g(Bh(X, Z), Z),$$

for all $Z \in \Gamma(TN_\theta)$. Thus we get

$$Bh(X, Z) \in \Gamma(TN_\top). \tag{5.5}$$

From (5.2) we can write:

$$tX(\ln f)g(Z, tZ) = X(\ln f)g(tZ, tZ) + g(Bh(Z, X), tZ).$$

By using (2.19) and (5.5) in the last equation above we get

$$tX(\ln f)g(Z, tZ) = (\cos^2\theta)X(\ln f)g(Z, Z). \tag{5.6}$$

Moreover, from Proposition 3.1 we have

$$\begin{aligned} X(\ln f)g(Z, Z) &= g(\nabla_Z X, Z) = g(\overline{\nabla}_Z X, Z) = -g(X, \overline{\nabla}_Z Z) \\ &= -g(\varphi X, \varphi\overline{\nabla}_Z Z) + \eta(X)\eta(\overline{\nabla}_Z Z) \\ &= -g(\varphi X, \overline{\nabla}_Z \varphi Z) \\ &= -g(\varphi X, \nabla_Z tZ - A_{nZ} Z) \\ &= -g(tX, \nabla_Z tZ) + g(tX, A_{nZ} Z) \\ &= g(tX, g(Z, tZ)\text{grad}(\ln f)) + g(h(tX, Z), nZ). \end{aligned} \tag{5.7}$$

Using (5.4) in (5.7) we get

$$X(\ln f)g(Z, Z) = tX(\ln f)g(Z, tZ). \tag{5.8}$$

Thus, making use of (5.6) and (5.8), we get

$$(\sin^2\theta)X(\ln f)\|Z\|^2 = 0,$$

for all $X \in \Gamma(TN_\top)$ and $Z \in \Gamma(TN_\theta)$. The last equation implies that either $\theta = 0$ or f is a constant function on $N_\top$. Since $M = N_\top \times_f N_\theta$ is assumed to be a proper warped product semi-slant submanifold, f must be constant on $N_\top$. This completes the proof. □

Theorem 5.2 *There do not exist proper warped product semi-slant submanifolds $M = N_\theta \times_f N_\top$ of a Lorentzian paracosymplectic manifold $(\overline{M}, \varphi, \xi, \eta, g)$ such that N_θ is a proper slant submanifold, $N_\top$ is an invariant submanifold of $\overline{M}$ and $\xi \in \Gamma(TN_\theta)$.*

Proof Let $M = N_\theta \times_f N_\top$ be a proper warped product semi-slant submanifold of a Lorentzian paracosymplectic manifold $(\overline{M}, \varphi, \xi, \eta, g)$ such that $\xi \in \Gamma(TN_\theta)$. From Proposition 3.1 we can write

$$\nabla_Z X = \nabla_X Z = Z(\ln f)X, \tag{5.9}$$

for any $Z \in \Gamma(TN_\theta)$ and $X \in \Gamma(TN_\top)$. Since $\overline{M}$ is a Lorentzian paracosymplectic manifold from (3.2) we have

$$\xi(\ln f) = 0. \tag{5.10}$$

Making use of $g(X, Z) = 0$, the Gauss formula and (5.9) it is easy to see that

$$g(\overline{\nabla}_X X, Z) = g(\nabla_X X, Z) = -g(X, \nabla_X Z) = -Z(\ln f)g(X, X). \tag{5.11}$$

From (2.4), Gauss formula and (5.10) we get

$$\begin{aligned} g(\overline{\nabla}_X X, Z) &= g(\nabla_X X, Z) = g(\varphi\overline{\nabla}_X X, \varphi Z) - \eta(\overline{\nabla}_X X)\eta(Z) \\ &= g(\nabla_X \varphi X, tZ) + g(h(X, \varphi X), nZ). \end{aligned} \tag{5.12}$$

Then we have:

$$g(h(X, fX), nZ) - (fZ)(\ln f)g(X, tX) = -Z(\ln f)g(X, X), \tag{5.13}$$

by virtue of (5.11) and (5.12).

On the other hand, since $N_\top$ is an invariant submanifold by using (2.10), (2.15), (2.16) and Lemma 3.1 we obtain

$$\begin{aligned} \overline{\nabla}_Z \varphi X &= \varphi \overline{\nabla}_Z X \\ h(Z, tX) &= Bh(Z, X) + Ch(Z, X), \end{aligned} \tag{5.14}$$

which implies that

$$Bh(Z, X) = 0 \tag{5.15}$$

and

$$h(Z, tX) = Ch(Z, X). \tag{5.16}$$

Similarly we have

$$\begin{aligned} \overline{\nabla}_X \varphi Z &= \varphi \overline{\nabla}_X Z \\ tZ(\ln f)X + h(X, tZ) - A_{nZ}X + \nabla^{\perp}_X nZ &= Z(\ln f)tX \\ &\quad + Bh(Z, X) + Ch(Z, X). \end{aligned}$$

By equating the tangential parts of the last equation and using (5.15) we obtain:

$$tZ(\ln f)X - A_{nZ}X = Z(\ln f)tX. \tag{5.17}$$

Thus from (2.19) and (5.17) we have:

$$g(h(X, tX), nZ) = tZ(\ln f)g(X, tX) - (\cos^2\theta)Z(\ln f)g(X, X). \tag{5.18}$$

By writing (5.18) in (5.13) we conclude

$$(\sin^2\theta)Z(\ln f)\|X\|^2 = 0,$$

which implies that either $\theta = 0$ or $Z(\ln f) = 0$, for all $Z \in \Gamma(TN_\theta)$. Since N_θ is a proper semi-slant submanifold then $\theta = 0$ is impossible. Hence, f must be constant on N_θ. This completes the proof. □

6 Warped product anti-slant submanifolds

Let M be a warped product anti-slant submanifold of a Lorentzian paracosymplectic manifold $(\overline{M}, \varphi, \xi, g)$. From Corollary 3.4, there do not exist warped product anti-slant submanifolds of type $N_\perp \times_f N_\theta$ with $\xi \in \Gamma(TN_\theta)$ and $N_\theta \times_f N_\perp$ with $\xi \in \Gamma(TN_\perp)$ of $\overline{M}$ where $N_\perp$ is an anti-invariant submanifold and N_θ is a proper slant submanifold of $\overline{M}$. Thus we have the following two cases:

(i) $M = N_\perp \times_f N_\theta$ with $\xi \in \Gamma(TN_\perp)$,
(ii) $M = N_\theta \times_f N_\perp$ with $\xi \in \Gamma(TN_\theta)$.

Theorem 6.1 *There do not exist proper warped product anti-slant submanifolds $M = N_\perp \times_f N_\theta$ of a Lorentzian paracosymplectic manifold $(\overline{M}, \varphi, \xi, \eta, g)$ such that $N_\perp$ is an anti-invariant submanifold, N_θ is a proper slant submanifold of $\overline{M}$ and $\xi \in \Gamma(TN_\perp)$.*

Proof Let $M = N_\perp \times_f N_\theta$ be a proper warped product anti-slant submanifold of a Lorentzian paracosymplectic manifold $(\overline{M}, \varphi, \xi, \eta, g)$ such that ξ is tangent to $N_\perp$ and $X \in \Gamma(TN_\perp)$, $Z \in \Gamma(TN_\theta)$. From Proposition 3.1, Gauss–Weingarten formulae, (2.4) and (2.10) we get

$$\begin{aligned} X(\ln f)g(Z,Z) &= g(\nabla_Z X, Z) = g(\overline{\nabla}_Z X, Z) \\ &= -g(X, \overline{\nabla}_Z Z) \\ &= -g(\varphi X, \varphi\overline{\nabla}_Z Z) + \eta(X)\eta(\overline{\nabla}_Z Z) \\ &= -g(\varphi X, \overline{\nabla}_Z \varphi Z) \\ &= -g(\varphi X, h(Z, tZ) + \nabla^\perp_Z nZ), \end{aligned}$$

which implies that

$$X(\ln f)g(Z,Z) = -g(nX, h(Z,tZ)) - g(nX, \nabla^\perp_Z nZ). \tag{6.1}$$

On the other hand, by using (2.15) and (2.16) we have:

$$\begin{aligned} \overline{\nabla}_Z \varphi Z &= \varphi\overline{\nabla}_Z Z \\ \nabla_Z tZ + h(Z,tZ) - A_{nZ}Z + \nabla^\perp_Z nZ &= t\nabla^{N_\theta}_Z Z + n\nabla^{N_\theta}_Z Z \\ &\quad -g(Z,Z)n(grad(\ln f)) \\ &\quad +Bh(Z,Z) + Ch(Z,Z), \end{aligned} \tag{6.2}$$

where ∇^{N_θ} denotes the Levi–Civita connection on N_θ. From the normal parts of (6.2) we obtain:

$$\begin{aligned} g(\nabla^\perp_Z nZ, nX) &= -g(h(Z,tZ), nX) + g(n\nabla^{N_\theta}_Z Z, nX) \\ &\quad -g(Z,Z)g(n(\text{grad}(\ln f)), nX) + g(Ch(Z,Z), nX), \end{aligned}$$

which implies that

$$\begin{aligned} g(\nabla^\perp_Z nZ, nX) &= -g(h(Z,tZ), nX) - (\sin^2\theta)X(\ln f)g(Z,Z) \\ &\quad +g(h(Z,Z), X) + \eta(h(Z,Z))\eta(X) \\ &= -g(h(Z,tZ), nX) - (\sin^2\theta)X(\ln f)g(Z,Z), \end{aligned} \tag{6.3}$$

by virtue of (2.4) and (2.20). From (6.1) and (6.3) we conclude

$$(\cos^2\theta)X(\ln f)\|Z\|^2 = 0.$$

Hence, either $\theta = \frac{\pi}{2}$ or $X(\ln f) = 0$. Since N_θ is a proper slant submanifold then $\theta \neq \frac{\pi}{2}$. So $X(\ln f) = 0$, for all $X \in \Gamma(TN_\perp)$, which implies that f is constant on $N_\perp$. The proof is complete. □

Theorem 6.2 *There do not exist proper warped product anti-slant submanifolds $M = N_\theta \times_f N_\perp$ of a Lorentzian paracosymplectic manifold $(\overline{M}, \varphi, \xi, \eta, g)$ such that N_θ is a proper slant submanifold, $N_\perp$ is an anti-invariant submanifold of $\overline{M}$ and $\xi \in \Gamma(TN_\theta)$.*

Proof Assume that $M = N_\theta \times_f N_\perp$ is a proper warped product anti-slant submanifold of a Lorentzian paracosymplectic manifold $(\overline{M}, \varphi, \xi, \eta, g)$ with $\xi \in \Gamma(TN_\theta)$. Let $X \in \Gamma(TN_\perp)$, $Z \in \Gamma(TN_\theta)$. Then we have:

$$\begin{aligned} \overline{\nabla}_X \varphi Z &= \varphi\overline{\nabla}_X Z \\ tZ(\ln f)X + h(X,tZ) - A_{nZ}X + \nabla^\perp_X nZ &= Z(\ln f)nX + Bh(Z,X) \\ &\quad +Ch(Z,X). \end{aligned} \tag{6.4}$$

From the normal components of (6.4) we get

$$h(X,tZ) + \nabla^\perp_X nZ = Z(\ln f)nX + Ch(Z,X), \tag{6.5}$$

which, by using (3.13), implies that

$$g(\nabla^\perp_X nZ, nX) = Z(\ln f)g(nX,nX) + g(Ch(Z,X), nX). \tag{6.6}$$

By a similar way we have:

$$\overline{\nabla}_Z \varphi X = \varphi \overline{\nabla}_Z X$$
$$-A_{nX} Z + \nabla^{\perp}_Z nX = Z(\ln f) nX + Bh(Z, X) + Ch(Z, X), \tag{6.7}$$

which gives

$$\nabla^{\perp}_Z nX = Z(\ln f) nX + Ch(Z, X). \tag{6.8}$$

Thus, from (3.13), (6.5) and (6.8) we obtain

$$g(\nabla^{\perp}_X nZ, nX) = g(\nabla^{\perp}_Z nX, nX).$$

Since $N_{\perp}$ is anti-invariant then by taking into account the Proposition 3.1 we reach

$$\begin{aligned} g(\nabla^{\perp}_X nZ, nX) &= g(\nabla^{\perp}_Z nX, nX) = g(\overline{\nabla}_Z nX, nX) \\ &= g(\overline{\nabla}_Z \varphi X, \varphi X) \\ &= Z(\ln f) g(X, X). \end{aligned} \tag{6.9}$$

If we write (6.9) in (6.6) and use (2.20) we get

$$\begin{aligned} Z(\ln f) g(X, X) &= Z(\ln f) g(nX, nX) + g(Ch(Z, X), nX) \\ &= (\sin^2 \theta) Z(\ln f) g(X, X) + g(\varphi h(Z, X), \varphi X) \\ &= (\sin^2 \theta) Z(\ln f) g(X, X). \end{aligned}$$

Thus we conclude

$$(\cos^2 \theta) Z(\ln f) \|X\|^2 = 0,$$

which implies that either $\theta = \frac{\pi}{2}$ or $X(\ln f) = 0$. Since N_θ is a proper slant submanifold then $\theta \neq \frac{\pi}{2}$. So f must be constant on N_θ. The proof is complete. □

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