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RESEARCH ARTICLE

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Finite groups with some $\mathscr{F}$-supplemented subgroups II

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Abstract In this paper, we obtain a new characterization of p-nilpotent groups under the assumption that some maximal subgroups of Sylow subgroup are $\mathscr{F}$-supplemented. As its applications, we generalize many known results.

Mathematics Subject Classification 20D10 · 20D20

الملخص

نحصل في هذه الورقة على تمييز جديد للزمر معدومة القوى-p في ظل افتراض أن بعض زمر سيلو الجزئية الأعظمية هي مُكَمَّلة-$\mathscr{F}$. كتطبيقات لهذا التمييز، نُعَمِّم الكثير من النتائج المعروفة.

1 Introduction

This article deals only with finite groups. The notion and terminologies used in this paper are standard. The reader is referred to the monograph of Guo [6].

A formation $\mathscr{F}$ is said to be saturated if it contains every group G with $G/\Phi(G) \in \mathscr{F}$. A formation $\mathscr{F}$ is said to be S-closed if every subgroup of G belongs to $\mathscr{F}$ whenever $G \in \mathscr{F}$. It is well known that the class of all p-nilpotent groups $\mathscr{N}_p$ and the class of all supersolvable groups $\mathscr{U}$ are S-closed saturated formation. A chief factor A/B of a group G is called $\mathscr{F}$-central if $[A/B](G/C_G(A/B)) \in \mathscr{F}$. The symbol $Z_\infty^{\mathscr{F}}(G)$ denotes the $\mathscr{F}$-hypercenter of a group G, that is, the product of all such normal subgroups of G whose G-chief factors are $\mathscr{F}$-central. We say a subgroup H of a group G has an $\mathscr{F}$-supplement T in G if G has a subgroup $T \in \mathscr{F}$ such that $G = HT$.

In the literature, groups with a system of special supplemented subgroups were studied by many authors; see, for example, [1,2,13,14]. In 2007, Guo [5] introduced the following concept of $\mathscr{F}$-supplemented subgroups again, which is also a generalization of c-normal, c-supplemented and $\mathscr{U}_c$-normal subgroups.

Definition 1.1 A subgroup H of a group G is said to be $\mathscr{F}$-supplemented in G if there exists a subgroup T of G such that $G = HT$ and $(H \cap T)H_G/H_G$ is contained in the $\mathscr{F}$-hypercenter $Z_\infty^{\mathscr{F}}(G/H_G)$ of G/H_G, where $\mathscr{F}$ is a formation of finite groups.

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In [18], by using some $\mathscr{F}$-supplemented subgroups, Yi et al. have given some conditions under which a finite group belongs to some saturated formations containing $\mathscr{U}$. The purpose of this paper is to go further into the influence of $\mathscr{F}$-supplemented subgroups on the structure of finite groups. Some new characterizations of p-nilpotency are obtained. We not only extend some results in [5] and [18], but also give more simple proofs.

2 Preliminaries

Lemma 2.1 [5, Lemma 2.2] *Let G be a group and $H \leq K \leq G$. Then*

(1) *If H is $\mathscr{F}$-supplemented in G and $\mathscr{F}$ is S-closed, then H is $\mathscr{F}$-supplemented in K.*
(2) *Suppose that H is normal in G. Then K/H is $\mathscr{F}$-supplemented in G/H if and only if K is $\mathscr{F}$-supplemented in G.*
(3) *Suppose that H is normal in G. Then, for every $\mathscr{F}$-supplemented subgroup E in G satisfying $(|H|, |E|) = 1$, HE/H is $\mathscr{F}$-supplemented in G/H.*

Lemma 2.2 [10, Lemma 2.3] *Let G be a group and p a prime such that $p^{n+1} \nmid |G|$ for some integer $n \geqslant 1$. If $(|G|, (p-1)(p^2-1)\cdots(p^n-1)) = 1$, then G is p-nilpotent.*

Lemma 2.3 [16, Lemma 2.8] *Let G be a group and p a prime dividing $|G|$ with $(|G|, p-1) = 1$. If N is normal in G of order p, then N lies in $Z(G)$.*

Lemma 2.4 *Let p be a prime and G a group with $(|G|, p-1) = 1$. Suppose that P is a Sylow p-subgroup of G such that every maximal subgroup of P has a p-nilpotent supplement in G, then G is p-nilpotent.*

Proof If $p^2 \nmid |G|$, then G is p-nilpotent by Lemma 2.2. Now we assume that $p^2 || G|$. Let P_1 be a maximal subgroup of P. By the hypothesis, P_1 has a p-nilpotent supplement K_1 in G. Let $K_{1p'}$ be a normal Hall p'-subgroup of K_1. Then, obviously, $K_{1p'}$ is a Hall p'-subgroup of G. Hence $G = P_1K_1 = P_1N_G(K_{1p'})$. We claim that $K_{1p'}$ is normal in G. Indeed, if $K_{1p'}$ is not normal in G, then $P \cap N_G(K_{1p'}) < P$. It follows that P has a maximal subgroup P_2 such that $P \cap N_G(K_{1p'}) \leq P_2$. It is clear $P_1 \neq P_2$. By the hypothesis, P_2 has also a p-nilpotent supplement K_2 in G. By repeating the above argument, we can find a Hall p'-subgroup $K_{2p'}$ of G such that $G = P_2K_2 = P_2N_G(K_{2p'})$. If $p = 2$, then $K_{1p'}$ and $K_{2p'}$ are conjugate in G by applying a deep result of Gross (see [4, Main Theorem]). If $p > 2$, then G is a solvable group by Feit–Thompson Theorem and so $K_{1p'}$ and $K_{2p'}$ are conjugate in G. Since $K_{2p'}$ is normalized by K_2, there exists an element $g \in P_2$ such that $K_{2p'}^g = K_{1p'}$. Then $G = (P_2N_G(K_{2p'}))^g = P_2N_G(K_{1p'})$. This induces that $P = P \cap G = P \cap P_2N_G(K_{1p'}) = P_2(P \cap N_G(K_{1p'})) = P_2$. This contradiction completes the proof. □

Lemma 2.5 [11, Lemma 2.6] *Let H be a solvable normal subgroup of a group G $(H \neq 1)$. If every minimal normal subgroup of G which is contained in H is not contained in $\Phi(G)$, then the Fitting subgroup $F(H)$ of H is the direct product of minimal normal subgroups of G which are contained in H.*

Lemma 2.6 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. If there is a normal subgroup H of a group G such that $G/H \in \mathscr{F}$ and every cyclic subgroup of H with prime order or order 4 not having supersolvable supplement in G is $\mathscr{U}$-supplemented in G, then $G \in \mathscr{F}$.*

Proof Suppose that the assertion is false and let (G, H) be a counterexample for which $|G||H|$ is minimal. Let K be any proper subgroup of H. By Lemma 2.1(1), the hypothesis of the theorem still holds for (K, K). By the choice of G, K is supersolvable. By [6, Theorem 3.11.9], H is solvable. Since $G/H \in \mathscr{F}$, $G^{\mathscr{F}} \leq H$. Let M be a maximal subgroup of G such that $G^{\mathscr{F}} \nsubseteq M$ (that is, M is an $\mathscr{F}$-abnormal maximal subgroup of G). Then $G = MH$. We claim that the hypothesis holds for $(M, M \cap H)$. In fact, $M/M \cap H \cong MH/H = G/H \in \mathscr{F}$. Let $\langle x \rangle$ be any cyclic subgroup of $M \cap H$ with prime order or order 4. It is clear that $\langle x \rangle$ is also a cyclic subgroup of H with prime order or order 4. If $\langle x \rangle$ has a supersolvable supplement U in G, then $\langle x \rangle$ has a supersolvable supplement $U \cap M$ in M. If $\langle x \rangle$ is $\mathscr{U}$-supplemented in G, then $\langle x \rangle$ is also $\mathscr{U}$-supplemented in M by Lemma 2.1(1). Therefore, the hypothesis holds for $(M, M \cap H)$. By the choice of G, $M \in \mathscr{F}$. Then, by [6, Theorem 3.4.2], the following conditions hold: (1) $G^{\mathscr{F}}$ is a p-group, where $G^{\mathscr{F}}$ is the $\mathscr{F}$-residual of G; (2) $G^{\mathscr{F}}/\Phi(G^{\mathscr{F}})$ is a chief factor of G and $\exp(G^{\mathscr{F}}) = p$ or $\exp(G^{\mathscr{F}}) = 4$ (if $p = 2$ and $G^{\mathscr{F}}$ is non-abelian). Let L be an arbitrary cyclic subgroup of $G^{\mathscr{F}}$ with prime order or order 4. Suppose that L

has a supersolvable supplement T in G. Clearly, $G^{\mathscr{F}} = G^{\mathscr{F}} \cap LT = L(G^{\mathscr{F}} \cap T)$. Since $G^{\mathscr{F}}/\Phi(G^{\mathscr{F}})$ is abelian, $(G^{\mathscr{F}} \cap T)\Phi(G^{\mathscr{F}})/\Phi(G^{\mathscr{F}}) \trianglelefteq G/\Phi(G^{\mathscr{F}})$. Notice that $G^{\mathscr{F}}/\Phi(G^{\mathscr{F}})$ is a chief factor of G, we have $G^{\mathscr{F}} \cap T \leq \Phi(G^{\mathscr{F}})$ or $G^{\mathscr{F}} = (G^{\mathscr{F}} \cap T)\Phi(G^{\mathscr{F}}) = G^{\mathscr{F}} \cap T$. If the former holds, then $L = G^{\mathscr{F}} \trianglelefteq G$. Since $G/G^{\mathscr{F}} \in \mathscr{F}$, $G \in \mathscr{F}$ by [5, Lemma 2.3], a contradiction. Therefore $G^{\mathscr{F}} = G^{\mathscr{F}} \cap T$, and so $T = G$ is supersolvable, a contradiction. Hence every cyclic subgroup of H with prime order or order 4 is $\mathscr{U}$-supplemented in G. Now we can get the final contradiction with the same argument in the proof of [5, Theorem 3.2]. □

Lemma 2.7 [12, Theorem B] *Let $\mathscr{F}$ be any formation, and G a group. If $H \trianglelefteq G$ and $F^*(H) \leq Z_\infty^{\mathscr{F}}(G)$, then $H \leq Z_\infty^{\mathscr{F}}(G)$.*

Lemma 2.8 [3, IV, 3.11] *If $\mathscr{F}_1$ and $\mathscr{F}_2$ are two saturated formations such that $\mathscr{F}_1 \subseteq \mathscr{F}_2$, then $Z_\infty^{\mathscr{F}_1}(G) \leq Z_\infty^{\mathscr{F}_2}(G)$.*

3 Main results

Theorem 3.1 *Let G be a group and p a prime such that $(|G|, (p-1)(p^2-1)\cdots(p^n-1)) = 1$ for some integer $n \geqslant 1$. If there exists a Sylow p-subgroup P of G such that every n-maximal subgroup (if it exists) of P not having a p-nilpotent supplement in G is $\mathscr{N}_p$-supplemented in G, then G is p-nilpotent.*

Proof Suppose that the theorem is false and let G be a counterexample of minimal order.

(1) $O_{p'}(G) = 1$.
Assume that $O_{p'}(G) \neq 1$. Since P is a Sylow p-subgroup of G, $PO_{p'}(G)/O_{p'}(G)$ is a Sylow p-subgroup of $G/O_{p'}(G)$. Let $M/O_{p'}(G)$ be an n-maximal subgroup of $PO_{p'}(G)/O_{p'}(G)$. Then $M = P_nO_{p'}(G)$, where P_n is an n-maximal subgroup of P. If P_n has a p-nilpotent supplement K in G, then $M/O_{p'}(G)$ has a p-nilpotent supplement $KO_{p'}(G)/O_{p'}(G)$ in $G/O_{p'}(G)$. If P_n is $\mathscr{N}_p$-supplemented in G, then $M/O_{p'}(G)$ is $\mathscr{N}_p$-supplemented in $G/O_{p'}(G)$ by Lemma 2.1(3). Therefore $G/O_{p'}(G)$ satisfies the hypothesis of the theorem. The minimal choice of G yields that $G/O_{p'}(G)$ is p-nilpotent, and so G is p-nilpotent, a contradiction.

(2) $Z_\infty^{\mathscr{N}_p}(G) = 1$.
Suppose that $Z_\infty^{\mathscr{N}_p}(G) \neq 1$. Then we may take a minimal normal N of G which is contained in $Z_\infty^{\mathscr{N}_p}(G)$. By Step (1), N is not a p'-group. Consequently, the order of N is p. By Lemma 2.1(2), G/N satisfies the hypothesis of the theorem. Thus the minimal choice of G yields that G/N is p-nilpotent. By Lemma 2.3, $G/Z(G)$ is p-nilpotent, which implies that G is p-nilpotent, a contradiction.

(3) $O_p(G) \neq 1$.
By Lemma 2.2, $p^{n+1}||P|$ and so there exists a non-identity n-maximal subgroup of P. By Lemma 2.4, P has an n-maximal subgroup H which has no p-nilpotent supplement in G. Thus by the hypothesis, G has a non-p-nilpotent subgroup T of G such that $G = HT$ and $(H \cap T)H_G/H_G$ is contained in the $\mathscr{N}_p$-hypercenter $Z_\infty^{\mathscr{N}_p}(G/H_G)$ of G/H_G. If $O_p(G) = 1$, then $H_G = 1$. It follows that $H \cap T \leq Z_\infty^{\mathscr{N}_p}(G) = 1$. Then $|T|_p = p^n$ and so T is p-nilpotent by Lemma 2.2, a contradiction. Therefore $O_p(G) \neq 1$.

(4) Every n-maximal subgroup of P has a p-nilpotent supplement in G.
Let N be a minimal normal subgroup of G contained in $O_p(G)$. Clearly, N is an elementary abelian p-subgroup. Invoking Lemma 2.1(2) and the minimal choice of G, G/N is p-nilpotent. Since $\mathscr{N}_p$ is a saturated formation, we have that N is a unique minimal normal subgroup of G contained in $O_p(G)$ and $N \nleq \Phi(G)$. Hence there exists a maximal subgroup M of G such that $G = N \rtimes M$. It follows that $M \cong G/N$ is p-nilpotent. It is easy to see that $O_p(G) \cap M$ is normal in G. Then the uniqueness of N yields that $N = O_p(G)$. Let P_n be an arbitrary n-maximal subgroup of P. We will show P_n has a p-nilpotent supplement in G. If not, then by the hypothesis, G has a non-p-nilpotent subgroup T of G such that $G = P_nT$ and $(P_n \cap T)(P_n)_G/(P_n)_G$ is contained in the $\mathscr{N}_p$-hypercenter $Z_\infty^{\mathscr{N}_p}(G/(P_n)_G)$ of $G/(P_n)_G$. If $(P_n)_G \neq 1$, then $N \leq (P_n)_G \leq P_n$, and so P_n has a p-nilpotent supplement M in G, a contradiction. Thus we may assume $(P_n)_G = 1$. Consequently, we have $P_n \cap T \leq Z_\infty^{\mathscr{N}_p}(G) = 1$. In view of Lemma 2.2, T is p-nilpotent, a contradiction.

(5) Final contradiction.
By Step (4), every maximal subgroup of P has a p-nilpotent supplement in G, and so G is p-nilpotent by Lemma 2.4, a contradiction. □

Corollary 3.2 *Let p be the smallest prime dividing the order of G and P a Sylow p-subgroup of G. If every maximal subgroup of P not having a p-nilpotent supplement in G is $\mathscr{N}_p$-supplemented in G, then G is p-nilpotent.*

Corollary 3.3 [7, Theorem 3.4] *Let G be a group and P a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If all maximal subgroups of P are c-normal in G, then G is p-nilpotent.*

Corollary 3.4 [8, Theorem 3.4] *Let G be a group and P a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If all maximal subgroups of P are c-supplemented in G, then G is p-nilpotent.*

Corollary 3.5 [9, Theorem 3.1] *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If every maximal subgroup of P is c-supplemented in G, then G is p-nilpotent.*

Corollary 3.6 [14, Theorem 3.1] *Let p be a prime dividing the order of a group G with $(|G|, p-1) = 1$. Suppose that every maximal subgroup of P is c-supplemented in G and $G \in C_{p'}$, then $G/O_p(G)$ is p-nilpotent and $G \in D_{p'}$.*

Corollary 3.7 *If every maximal subgroup of any noncyclic Sylow subgroup of G not having a supersolvable supplement in G is $\mathscr{U}$-supplemented in G, then G is a Sylow tower group of supersolvable type.*

Proof Let p be the smallest prime dividing $|G|$ and P a Sylow p-subgroup of G. If P is cyclic, then G is p-nilpotent obviously. If P is not cyclic, then G is also p-nilpotent by Corollary 3.2 and Lemma 2.8. Let U be the normal p-complement of G. By Lemma 2.1(1), U satisfies the hypothesis of the corollary. Therefore, it follows by induction that U, and so G is a Sylow tower group of supersolvable type. □

Corollary 3.8 *Let p be a prime and G a group with $(|G|, (p-1)(p^2-1)\cdots(p^n-1)) = 1$ for some integer $n \geqslant 1$. Suppose that G has a normal subgroup H such that G/H is p-nilpotent and H has a Sylow p-subgroup P such that every n-maximal subgroup (if it exists) of P not having a p-nilpotent supplement in G is $\mathscr{N}_p$-supplemented in G. Then G is p-nilpotent.*

Proof We distinguish two cases:

Case I. $P = H$.

Since G/P is p-nilpotent, we can let K/P be the normal p-complement of G/P. By The Schur–Zassenhaus Theorem, there exists a Hall p'-subgroup $K_{p'}$ of K such that $K = PK_{p'}$. By Lemma 2.1(1), every n-maximal subgroup (if it exists) of P not having a p-nilpotent supplement in K is $\mathscr{N}_p$-supplemented in K. Applying Theorem 3.1, we have K is p-nilpotent and so $K = H \times K_{p'}$. Hence, $K_{p'}$ is a normal p-complement of G and G is p-nilpotent.

Case II. $P < H$.

By Lemma 2.1(1), every n-maximal subgroup (if it exists) of P not having a p-nilpotent supplement in H is $\mathscr{N}_p$-supplemented in H. A new application of Theorem 3.1 yields that H is p-nilpotent. Now, let $H_{p'}$ be the normal p-complement of H. Obviously, $H_{p'} \trianglelefteq G$. By Lemma 2.1(3), it is easy to see that every n-maximal subgroup (if it exists) of $PH_{p'}/H_{p'}$ not having a p-nilpotent supplement in $G/H_{p'}$ is $\mathscr{N}_p$-supplemented in $G/H_{p'}$. Thus, $G/H_{p'}$ satisfies the hypotheses for the normal subgroup $H/H_{p'}$. By induction $G/H_{p'}$ is p-nilpotent, and so G is p-nilpotent. □

Theorem 3.9 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. A group $G \in \mathscr{F}$ if and only if there is a normal subgroup H of G such that $G/H \in \mathscr{F}$ and every maximal subgroup of any noncyclic Sylow subgroup of H not having a supersolvable supplement in G is $\mathscr{U}$-supplemented in G.*

Proof The necessity is obvious. We only need to prove the sufficiency. Suppose that the assertion is false and let G be a counterexample of minimal order.

By Lemma 2.1(1), every maximal subgroup of any noncyclic Sylow subgroup of H not having a supersolvable supplement in H is $\mathscr{U}$-supplemented in H. By Corollary 3.7, H is a Sylow tower group of supersolvable type. Let q be the largest prime divisor of $|H|$ and let Q be a Sylow q-subgroup of H. Then Q is normal in G. Obviously, $(G/Q)/(H/Q) \cong G/H \in \mathscr{F}$. It is easy to see that every maximal subgroup of any noncyclic Sylow subgroup of H/Q not having a supersolvable supplement in G/Q is $\mathscr{U}$-supplemented in G/Q by Lemma 2.1(3). By the minimality of G, we have $G/Q \in \mathscr{F}$. If Q is cyclic, then $G \in \mathscr{F}$ by [5, Lemma 2.3], a contradiction. Hence, we may assume that Q is noncyclic. Let N be a minimal normal subgroup of G contained in Q. We can also prove $G/N \in \mathscr{F}$ easily. Since $\mathscr{F}$ is a saturated formation, N is the unique minimal normal subgroup of G contained in Q and $N \nleq \Phi(G)$. So there exists a maximal subgroup M of G such that $G = NM$ and $N \cap M = 1$. By Lemma 2.5, we have $Q = F(Q) = N$. Let M_q be a Sylow q-subgroup of M. Then $G_q = NM_q$ is a Sylow q-subgroup of G. Let $N_1 = N \cap Q_1$, where Q_1 is a maximal subgroup of G_q containing M_q. Then $G_q = NQ_1$, N_1 is a maximal subgroup of N and $N_1 \trianglelefteq G_q$. Let T be any supplement of N_1 in G, then $N_1T = G$ and $N = N \cap N_1T = N_1(N \cap T)$. This implies that $N \cap T \neq 1$. But since $N \cap T$ is normal in G and N is a minimal normal subgroup of G, $N \cap T = N$. So $T = G$ is the unique supplement of N_1 in G. Since G is not supersolvable, N_1 is $\mathscr{U}$-supplemented in G. From $T = G$, we have $N_1 \leq N \cap Z_\infty^{\mathscr{U}}(G)$. By the minimality of N, $Z_\infty^{\mathscr{U}}(G) \cap N = 1$ or $N \leq Z_\infty^{\mathscr{U}}(G)$. If the latter holds, then $N_1 = 1$ and so $|N| = q$. By [5, Lemma 2.3], $G \in \mathscr{F}$, a contradiction. Therefore $Z_\infty^{\mathscr{U}}(G) \cap N = 1$. We have also $|N| = q$, the same contradiction as above. □

Theorem 3.10 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$ and let G be a group. Then $G \in \mathscr{F}$ if and only if there is a normal subgroup H such that $G/H \in \mathscr{F}$ and every cyclic subgroup of $F^*(H)$ with prime order or order 4 not having supersolvable supplement in G is $\mathscr{U}$-supplemented in G.*

Proof The necessity is obvious. We only need to prove the sufficiency. If some subgroup of $F^*(H)$ has a supersolvable supplement in G, then $G/F^*(H) \in \mathscr{F}$ and in this case $G \in \mathscr{F}$ by Lemma 2.6. Hence we may assume that every cyclic subgroup of $F^*(H)$ with prime order or order 4 is $\mathscr{U}$-supplemented in G. By Lemma 2.1(1), every cyclic subgroup of $F^*(H)$ with prime order or order 4 is $\mathscr{U}$-supplemented in $F^*(H)$. By [5, Theorem 3.2], $F^*(H)$ is supersolvable. In particular, $F^*(H)$ is solvable and so $F^*(H) = F(H)$. Now by [18, Lemma 3.3.3], $F^*(H) \leq Z_\infty^{\mathscr{U}}(G)$ and since $Z_\infty^{\mathscr{U}}(G) \leq Z_\infty^{\mathscr{F}}(G)$ by Lemma 2.8, we have $F^*(H) \leq Z_\infty^{\mathscr{F}}(G)$. By Lemma 2.7, $H \leq Z_\infty^{\mathscr{F}}(G)$. It follows from $G/H \in \mathscr{F}$ that $G \in \mathscr{F}$. □

Corollary 3.11 [17, Theorem 3.2] *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathscr{F}$. If all minimal subgroups and all cyclic subgroups of $F^*(H)$ are c-normal in G, then $G \in \mathscr{F}$.*

Corollary 3.12 [15, Theorem 1.2] *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathscr{F}$. If all minimal subgroups and all cyclic subgroups with order 4 of $F^*(H)$ are c-supplemented in G, then $G \in \mathscr{F}$.*

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