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RESEARCH ARTICLE

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A variational approach to a quasilinear multiparameter elliptic system involving the p-Laplacian and nonlinear boundary condition

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Abstract We present a note on the paper by Brown and Wu (J Math Anal Appl 337:1326–1336, 2008). Indeed, we extend the multiplicity results for a class of semilinear elliptic system to the quasilinear elliptic system of the form:

$$\begin{cases} -\Delta_p u + m(x)\,|u|^{p-2}u = \frac{\alpha}{\alpha+\beta}\,|u|^{\alpha-2}\,u\,|v|^{\beta}, & x \in \Omega, \\ -\Delta_p v + m(x)\,|v|^{p-2}v = \frac{\beta}{\alpha+\beta}\,|u|^{\alpha}\,|v|^{\beta-2}\,v, & x \in \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial n} = \lambda\, a(x)|u|^{\gamma-2}u, \quad |\nabla v|^{p-2}\frac{\partial v}{\partial n} = \mu\, b(x)|v|^{\gamma-2}v, & x \in \partial\Omega. \end{cases}$$

Here Δ_p denotes the p-Laplacian operator defined by $\Delta_p z = div\,(|\nabla z|^{p-2}\nabla z)$, $p > 2$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $\alpha > 1, \beta > 1, 2 < \alpha+\beta < p < \gamma < p*$ $(p* = \frac{pN}{N-p}$ if $N > p$, $p* = \infty$ if $N \leq p)$, $\frac{\partial}{\partial n}$ is the outer normal derivative, $(\lambda, \mu) \in \mathbb{R}^2\backslash\{(0,0)\}$, the weight $m(x)$ is a positive bounded function, and $a(x), b(x) \in C(\partial\Omega)$ are functions which change sign in $\overline{\Omega}$.

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الملخص

نقدم ملاحظة على الورقة [13]. بالتأكيد، نمدد نتائج التعدد لصف النظام الإهليليجي نصف الخطي إلى النظام الإهليليجي شبه الخطي على الصيغة:

$$\begin{cases} -\Delta_p u + m(x)\,|u|^{p-2}u = \frac{\alpha}{\alpha+\beta}\,|u|^{\alpha-2}\,u\,|v|^{\beta}, & x \in \Omega, \\ -\Delta_p v + m(x)\,|v|^{p-2}v = \frac{\beta}{\alpha+\beta}\,|u|^{\alpha}\,|v|^{\beta-2}\,v, & x \in \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial n} = \lambda\,a(x)|u|^{\gamma-2}u, \quad |\nabla v|^{p-2}\frac{\partial v}{\partial n} = \mu\,b(x)|v|^{\gamma-2}v, & x \in \partial\Omega. \end{cases}$$

ترمز Δ_p هنا لمؤثر $\boldsymbol{p}$-لابلاس المعرف بِـ $\Delta_p z = div(|\nabla z|^{p-2}\nabla z)$ حيث $p > 2$ وَ $\Omega \subset \mathbb{R}^2$ مجال محدود مع حدود ملساء، وَ $\alpha > 1$، وَ $\beta > 1$، وَ $2 < \alpha + \beta < p < \gamma < p*$ ($p* = \frac{pN}{N-p}$ إذا كانت $N > p$، وَ $p* = \infty$ إذا كانت $p \geq N$)، وَ $\frac{\partial}{\partial n}$ المشتقة الخارجية الناظمية، وَ $(\lambda, \mu) \in \mathbb{R}^2\backslash\{(0,0)\}$، والوزن $m(x)$ دالة محدودة موجبة، وَ $a(x), b(x) \in C(\partial\Omega)$ دالتان تغيران الإشارة في $\overline{\Omega}$.

1 Introduction

We are concerned with the existence and multiplicity of nontrivial nonnegative solutions to the quasilinear elliptic system:

$$\begin{cases} -\Delta_p u + m(x)\,|u|^{p-2}u = \frac{\alpha}{\alpha+\beta}\,|u|^{\alpha-2}\,u\,|v|^{\beta}, & x \in \Omega, \\ -\Delta_p v + m(x)\,|v|^{p-2}v = \frac{\beta}{\alpha+\beta}\,|u|^{\alpha}\,|v|^{\beta-2}\,v, & x \in \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial n} = \lambda\,a(x)|u|^{\gamma-2}u, \quad |\nabla v|^{p-2}\frac{\partial v}{\partial n} = \mu\,b(x)|u|^{\gamma-2}v, & x \in \partial\Omega. \end{cases} \tag{1}$$

Here Δ_p denotes the p-Laplacian operator defined by $\Delta_p z = \operatorname{div}(|\nabla z|^{p-2}\nabla z)$, $p > 2, \Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $\alpha > 1, \beta > 1, 2 < \alpha + \beta < p < \gamma < p*$ ($p* = \frac{pN}{N-p}$ if $N > p$, $p* = \infty$ if $N \leq p$), $\frac{\partial}{\partial n}$ is the outer normal derivative, $(\lambda, \mu) \in \mathbb{R}^2\backslash\{(0, 0)\}$, the weight $m(x)$ is a positive bounded function and $a(x), b(x) \in C(\partial\Omega)$, with $a^{\pm} = \max\{\pm a, 0\} \not\equiv 0$, and $b^{\pm} = \max\{\pm b, 0\} \not\equiv 0$.

Problems involving the s-Laplace operator arise in some physical models like the flow of non-Newtonian fluids: pseudo-plastic fluids correspond to $s \in (1, 2)$ while dilatant fluids correspond to $s > 2$. The case $s = 2$ expresses Newtonian fluids [6]. On the other hand, quasilinear elliptic systems like (1) has an extensive practical background. It can be used to describe the multiplicate chemical reaction catalyzed by the catalyst grains under constant or variant temperature, it can be used in the theory of quasiregular and quasiconformal mappings in Riemannian manifolds with boundary (see [18,22]) and can be a simple model of tubular chemical reaction, more naturally, it can be a correspondence of the stable station of dynamical system determined by the reaction-diffusion system, see Ladde and Lakshmikantham et al. [19]. More naturally, it can be the populations of two competing species [16]. So, the study of positive solutions of elliptic systems has more practical meanings. We refer to [8,15] for additional results on elliptic systems.

We are motivated by the paper of Brown and Wu [13], in which Problem (1) was discussed when $m \equiv 1, p = 2$, and $1 < \gamma < 2 < \alpha + \beta < 2^*$. They have altogether proved that, there exists $C_0 > 0$ such that if the parameter λ, μ satisfy $0 < |\lambda|^{\frac{2}{2-q}} + |\mu|^{\frac{2}{2-q}} < C_0$, then Problem (1) for $m \equiv 1, p = 2$, and $1 < \gamma < 2 < \alpha + \beta < 2^*$, has at least two solutions (u_0^+, v_0^+) and (u_0^-, v_0^-) such that $u_0^{\pm} \geq 0, v_0^{\pm} \geq 0$ in Ω and $u_0^{\pm} \neq 0, v_0^{\pm} \neq 0$. In this paper, the method of [13] is extended for the system (1) but with $m \not\equiv 1, p > 2$, and $2 < \alpha + \beta < p < \gamma < p^*$. The change in γ completely changes the nature of the solution set of (1). When $p = 2$, for a single equation, similar problems (for Dirichlet or Neuman boundary condition) have been studied by Drabek et al. [7], Ambrosetti et al. [4], Brezis and Nirenberg [10], and Tehrani [20,21] using variational methods and by Amman and Lopez-Gomez [5] by using global bifurcation theory.

In recent years, several authors have used the Nehari manifold and fibering maps (i.e., maps of the form $t \longmapsto J_\lambda(tu)$ where J_λ is the Euler function associated with the equation) to solve semilinear and quasilinear problems (see [1–3,9,11–14,17,23–26]). By the fibering method, Drabek and Pohozaev [17], Bozhkov and Mitidieri [9] studied, respectively, the existence of multiple solution to a p-Laplacian single equation and (p, q)-Laplacian system. Brown and Zhang [14] have studied the following subcritical semilinear elliptic equation with a sign-changing weight function

$$\begin{cases} -\Delta u(x) = \lambda a(x)u + b(x)|u(x)|^{\gamma-2}u(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

where $\gamma > 2$. Exploiting the relationship between the Nehari manifold and fibering maps, they gave an interesting explanation of the well-known bifurcation result. In fact, the nature of the Nehari manifold changes as the parameter λ crosses the bifurcation value. Recently, in [11], the author considered the above problem with $1 < \gamma < 2$. In this work, we give a variational method which is similar to the fibering method (see [17,14] or [14]) to prove the existence of at least two nontrivial nonnegative solutions of Problem (1). In particular, by using the method of [13], we do this without the extraction of the Palais–Smale sequences in the Nehari manifold as in [1,3].

This paper is divided into three sections, organized as follows. In Sect. 2, we give some notation, preliminaries, properties of the Nehari manifold and set up the variational framework of the problem. In Sect. 3, we give our main results.

2 Variational setting

Let $W_0^{1,p} = W_0^{1,p}(\Omega)$ denote the usual Sobolev space. In the Banach space $W = W_0^{1,p} \times W_0^{1,p}$ we introduce the norm

$$\|(u, v)\|_W = \left(\int_\Omega (|\nabla u|^p + m(x)|u|^p)\mathrm{d}x + \int_\Omega (|\nabla v|^p + m(x)|v|^p)\mathrm{d}x \right)^{\frac{1}{p}}$$

which is equivalent to be the standard one. Throughout this paper, we set C_1 and C_2 be the best Sobolev and the best Sobolev trace constants for the embedding of $W_0^{1,p}(\Omega)$ in $L^\gamma(\partial\Omega)$ and $W_0^{1,p}(\Omega)$ in $L^{\alpha+\beta}(\Omega)$, respectively. First, we give the definition of the weak solution of (1).

Definition 2.1 We say that $(u, v) \in W$ is a weak solution to (1) if for all $(w_1, w_2) \in W$ we have

$$\begin{aligned} &\int_\Omega |\nabla u|^{p-2}\, \nabla u\,.\nabla w_1\, \mathrm{d}x + \int_\Omega m(x)\, |u|^{p-2}\, u\, w_1\, \mathrm{d}x \\ &\quad = \lambda \int_{\partial\Omega} a(x)\, |u|^{\gamma-2} u\, w_1\, \mathrm{d}x + \frac{\alpha}{\alpha+\beta} \int_\Omega |u|^{\alpha-2}\, u\, |v|^\beta\, w_1\, \mathrm{d}x, \\ &\int_\Omega |\nabla v|^{p-2}\, \nabla v\,.\nabla w_2\, \mathrm{d}x + \int_\Omega m(x)\, |v|^{p-2}\, v\, w_2\, \mathrm{d}x \\ &\quad = \mu \int_{\partial\Omega} b(x)\, |v|^{\gamma-2} v\, w_2\, \mathrm{d}x + \frac{\beta}{\alpha+\beta} \int_\Omega |u|^\alpha\, |v|^{\beta-2}\, v\, w_2\, \mathrm{d}x. \end{aligned}$$

It is clear that Problem (1) has a variational structure. Let $\mathcal{I}_{\lambda,\mu} : W \to \mathbb{R}$ be the corresponding energy functional of Problem (1) is defined by

$$\mathcal{I}_{\lambda,\mu} = \frac{1}{p} M(u, v) - \frac{1}{\gamma} N(u, v) - \frac{1}{\alpha+\beta} R(u, v),$$

where

$$M(u, v) = \int_\Omega (|\nabla u|^p + m(x)|u|^p)\mathrm{d}x + \int_\Omega (|\nabla v|^p + m(x)|v|^p)\mathrm{d}x,$$

$$N(u, v) = \lambda \int_{\partial\Omega} a(x)\,|u|^\gamma\,\mathrm{d}x + \mu \int_{\partial\Omega} b(x)\,|v|^\gamma\,\mathrm{d}x, \quad \text{and} \quad R(u, v) = \int_\Omega |u|^\alpha\,|v|^\beta\,\mathrm{d}x.$$

It is well known that the weak solutions of Eq. (1) are the critical points of the energy functional $\mathcal{I}_{\lambda,\mu}$. Let J be the energy functional associated with an elliptic problem on a Banach space X. If J is bounded below and J has a minimizer on X, then this minimizer is a critical point of J. So, it is a solution of the corresponding elliptic problem. However, the energy functional $\mathcal{I}_{\lambda,\mu}$, is not bounded below on the whole space W, but is bounded on an appropriate subset, and a minimizer on this set (if it exists) gives rise to solution to (1).

Then we introduce the following notation: for any functional $f : W \longrightarrow \mathbb{R}$ we denote by $f'(u, v)\,(h_1, h_2)$ the Gateaux derivative of f at $(u, v) \in W$ in the direction of $(h_1, h_2) \in W$, and

$$f^{(1)}(u, v)h_1 = f'(u + \epsilon h_1, v)\mid_{\epsilon=0}, \quad f^{(2)}(u, v)h_2 = f'(u, v + \delta h_2)\mid_{\delta=0}.$$

Consider the Nehari minimization problem for $(\lambda, \mu) \in \mathbb{R}^2\backslash\{(0, 0)\}$,

$$\alpha_0(\lambda, \mu) = \inf\{\mathcal{I}_{\lambda,\mu}(u, v) \;:\; (u, v) \in \mathcal{S}_{\lambda,\mu}\},$$

where $\mathcal{S}_{\lambda,\mu} = \{(u, v) \in W\backslash\{(0, 0)\} : \langle \mathcal{I}'_{\lambda,\mu}(u, v), (u, v)\rangle = \langle \mathcal{I}^{(1)}_{\lambda,\mu}(u, v)u, \mathcal{I}^{(2)}_{\lambda,\mu}(u, v), v\rangle = 0\}$. It is clear that all critical points of $\mathcal{I}_{\lambda,\mu}$ must lie on $\mathcal{S}_{\lambda,\mu}$ which is known as the Nehari manifold (see [16]). We will see below that local minimizers of $\mathcal{I}_{\lambda,\mu}$ on $\mathcal{S}_{\lambda,\mu}$ are usually critical points of $\mathcal{I}_{\lambda,\mu}$. It is easy to see that $(u, v) \in \mathcal{S}_{\lambda,\mu}$ if and only if

$$M(u, v) - N(u, v) = R(u, v). \tag{2}$$

Note that $\mathcal{S}_{\lambda,\mu}$ contains every nonzero solution of Problem (1).

Define

$$\mathcal{G}_{\lambda,\mu}(u, v) = \langle \mathcal{I}'_{\lambda,\mu}(u, v), (u, v)\rangle.$$

Then for $(u, v) \in \mathcal{S}_{\lambda,\mu}$,

$$\begin{aligned}
\langle \mathcal{G}'_{\lambda,\mu}(u, v), (u, v)\rangle &= p\,M(u, v) - \gamma\,N(u, v) - (\alpha+\beta)\,R(u, v) && (3)\\
&= (p-\alpha-\beta)\,M(u, v) + (\alpha+\beta-\gamma)\,N(u, v) && (4)\\
&= (p-\gamma)\,M(u, v) + (\gamma-\alpha-\beta)\,R(u, v) && (5)\\
&= (p-\gamma)\,N(u, v) + (p-\alpha-\beta)\,R(u, v). && (6)
\end{aligned}$$

Now, we split $\mathcal{S}_{\lambda,\mu}$ into three parts:

$$\mathcal{S}^+_{\lambda,\mu} = \{u \in \mathcal{S}_{\lambda,\mu} : \langle \mathcal{G}'_{\lambda,\mu}(u, v), (u, v)\rangle > 0\},$$

$$\mathcal{S}^0_{\lambda,\mu} = \{u \in \mathcal{S}_{\lambda,\mu} : \langle \mathcal{G}'_{\lambda,\mu}(u, v), (u, v)\rangle = 0\},$$

$$\mathcal{S}^-_{\lambda,\mu} = \{u \in \mathcal{S}_{\lambda,\mu} : \langle \mathcal{G}'_{\lambda,\mu}(u, v), (u, v)\rangle < 0\}.$$

To state our main result, we now present some important properties of $\mathcal{S}^+_{\lambda,\mu}$, $\mathcal{S}^0_{\lambda,\mu}$, and $\mathcal{S}^-_{\lambda,\mu}$.

Lemma 2.2 *There exists $\zeta_0 > 0$ such that for*

$$0 < (|\lambda|\|a\|_\infty)^{\frac{p}{\gamma-p}} + (|\mu|\|b\|_\infty)^{\frac{p}{\gamma-p}} < \zeta_0,$$

we have $\mathcal{S}^0_{\lambda,\mu} = \emptyset$.

Proof Suppose otherwise, thus for $\zeta_0 = \left[\frac{(p-\alpha-\beta)}{(\gamma-\alpha-\beta)\, C_1^{\gamma}}\right]^{\frac{p}{\gamma-p}} \left[\frac{(\gamma-p)}{(\gamma-\alpha-\beta)\, C_2^{\alpha+\beta}}\right]^{\frac{p}{p-\alpha-\beta}}$, there exists (λ, μ) with $0 < (|\lambda|\|a\|_\infty)^{\frac{p}{\gamma-p}} + (|\mu|\|b\|_\infty)^{\frac{p}{\gamma-p}} < \zeta_0$ such that $\mathcal{S}^0_{\lambda,\mu} \neq \emptyset$. Then for $(u, v) \in \mathcal{S}^0_{\lambda,\mu}$ we have

$$0 = \langle \mathcal{G}'_{\lambda,\mu}(u, v), (u, v)\rangle = (p-\alpha-\beta)\, M(u, v) + (\alpha+\beta-\gamma)\, N(u, v) \tag{7}$$

$$= (p-\gamma)\, M(u, v) + (\gamma-\alpha-\beta)\, R(u, v). \tag{8}$$

By the Sobolev trace imbedding theorem,

$$\begin{aligned} N(u, v) &= \lambda \int_{\partial\Omega} a(x)\, |u|^{\gamma}\, \mathrm{d}x + \mu \int_{\partial\Omega} b(x)\, |v|^{\gamma}\, \mathrm{d}x \\ &\le |\lambda|\, \|a\|_\infty\, \|u\|_\gamma^\gamma + |\mu|\, \|b\|_\infty\, \|v\|_\gamma^\gamma \\ &\le C_1^\gamma\, |\lambda|\, \|a\|_\infty\, \|u\|_{1,p}^\gamma + C_1^\gamma\, |\mu|\, \|b\|_\infty\, \|v\|_{1,p}^\gamma \\ &\le C_1^\gamma \left[(|\lambda|\|a\|_\infty)^{\frac{p}{\gamma-p}} + (|\mu|\, \|b\|_\infty)^{\frac{p}{\gamma-p}}\right]^{\frac{\gamma-p}{p}} \|(u, v)\|_W^\gamma, \end{aligned} \tag{9}$$

and

$$R(u, v) \le C_2^{\alpha+\beta}\, ||(u, v)||_W^{\alpha+\beta}. \tag{10}$$

Indeed, by condition $\alpha + \beta < \frac{pN}{N-p}$, we have

$$\frac{pN}{\alpha\,(N-p)} - \frac{pN}{pN - \beta(N-p)} > 0.$$

So, there exists ϵ_0 such that

$$0 < \epsilon_0 < \frac{pN}{\alpha\,(N-p)} - \frac{pN}{pN - \beta(N-p)},$$

which implies

$$\frac{\beta\,(p^* - \alpha\,\epsilon_0)}{p^* - \alpha\,(\epsilon_0 + 1)} < 2p^* = \frac{pN}{N-p}.$$

Then using the Hölder inequality and the Sobolev inequality, we get

$$\begin{aligned} R(u, v) &= \int_\Omega |u|^\alpha\, |v|^\beta\, \mathrm{d}x \\ &\le \left(\int_\Omega [(|u|)^\alpha]^{\frac{p^*}{\alpha} - \epsilon_0}\, \mathrm{d}x\right)^{\frac{\alpha}{p^* - \epsilon_0\,\alpha}} \left(\int_\Omega [(|v|)^\beta]^{\frac{p^* - \epsilon_0\,\alpha}{p^* - (\epsilon_0+1)\,\alpha}}\, \mathrm{d}x\right)^{\frac{p^* - (\epsilon_0+1)\,\alpha}{2^* - \epsilon_0\,\alpha}} \\ &\le C_2^{\alpha+\beta}\, ||(u, v)||_W^{\alpha+\beta}. \end{aligned}$$

By using (9)–(10) in (7)–(8) we get

$$||(u, v)||_W \ge \left[\frac{(p-\alpha-\beta)}{(\gamma-\alpha-\beta)\, C_1^\gamma}\right]^{\frac{1}{\gamma-p}} \frac{1}{\left((|\lambda|\|a\|_\infty)^{\frac{p}{\gamma-p}} + (|\mu|\, \|b\|_\infty)^{\frac{p}{\gamma-p}}\right)^{1/p}},$$

and

$$||(u, v)||_W \le \left[\frac{(\gamma-\alpha-\beta)\, C_2^{\alpha+\beta}}{\gamma-p}\right]^{\frac{1}{p-\alpha-\beta}}.$$

This implies $(|\lambda|\|a\|_\infty)^{\frac{p}{\gamma-p}} + (|\mu|\|b\|_\infty)^{\frac{p}{\gamma-p}} \geq \zeta_0$, which is a contradiction. Thus, we can conclude that there exists $\zeta_0 > 0$ such that for $0 < (|\lambda|\|a\|_\infty)^{\frac{p}{\gamma-p}} + (|\mu|\|b\|_\infty)^{\frac{p}{\gamma-p}} < \zeta_0$, we have $\mathcal{S}^0_{\lambda,\mu} = \emptyset$. □

By Lemma (2.2), for each $(\lambda, \mu) \in \mathbb{R}^2\backslash\{(0, 0)\}$ with $0 < (|\lambda|\|a\|_\infty)^{\frac{p}{\gamma-p}} + (|\mu|\|b\|_\infty)^{\frac{p}{\gamma-p}} < \zeta_0$, we write $\mathcal{S}_{\lambda,\mu} = \mathcal{S}^+_{\lambda,\mu} \cup \mathcal{S}^-_{\lambda,\mu}$ and define

$$\alpha_0^+(\lambda, \mu) = \inf_{(u,v)\in\mathcal{S}^+_{\lambda,\mu}} \mathcal{I}_{\lambda,\mu}(u, v);\ \ \alpha_0^-(\lambda, \mu) = \inf_{(u,v)\in\mathcal{S}^-_{\lambda,\mu}} \mathcal{I}_{\lambda,\mu}(u, v).$$

Lemma 2.3 *We have*

(i) *If* $(u, v) \in \mathcal{S}^+_{\lambda,\mu}$, *then* $R(u, v) > 0$;
(ii) *If* $(u, v) \in \mathcal{S}^-_{\lambda,\mu}$, *then* $N(u, v) > 0$.

Proof (i) We consider the following two cases:
Case (i-a): $N(u, v) \leq 0$. We have

$$R(u, v) = M(u, v) - N(u, v) > 0.$$

Case (i-b): $N(u, v) > 0$. Since $(u, v) \in \mathcal{S}^+_{\lambda,\mu}$, by (6), we have

$$(p - \gamma)\, N(u, v) + (p - \alpha - \beta)\, R(u, v) > 0,$$

which implies

$$R(u, v) > \frac{\gamma - p}{p - \alpha - \beta} N(u, v) > 0.$$

(ii) We consider the following two cases:
Case (ii-a): $R(u, v) = 0$. Since $(u, v) \in \mathcal{S}_{\lambda,\mu}$ we have

$$N(u, v) = M(u, v) > 0.$$

Case (ii-b): $R(u, v) \neq 0$. Since $(u, v) \in \mathcal{S}^-_{\lambda,\mu}$, by (1), we have

$$(p - \alpha - \beta)\, M(u, v) + (\alpha + \beta - \gamma)\, N(u, v) < 0,$$

which implies

$$N(u, v) > \frac{p - \alpha - \beta}{\gamma - \alpha - \beta} M(u, v) > 0.$$

It follows that the conclusion is true. □

As proved in Binding et al. [7] or in Brown and Zhang [14], we have the following lemma.

Lemma 2.4 *Suppose that* (u_0, v_0) *is a local minimizer for* $\mathcal{I}_{\lambda,\mu}$ *on* $\mathcal{S}_{\lambda,\mu}$. *If* $(u_0, v_0) \notin \mathcal{S}^0_{\lambda,\mu}$, *then* (u_0, v_0) *is a critical point of* $\mathcal{I}_{\lambda,\mu}$.

Then we have the following result.

Lemma 2.5 $\mathcal{I}_{\lambda,\mu}$ *is coercive and bounded below on* $\mathcal{S}_{\lambda,\mu}$.

Proof If $(u, v) \in \mathcal{S}_{\lambda,\mu}$, it follows from (2) and the Sobolev embedding theorem

$$\begin{aligned}
\mathcal{I}_{\lambda,\mu}(u, v) &= \left(\frac{1}{p} - \frac{1}{\gamma}\right) M(u, v) - \left(\frac{1}{\alpha+\beta} - \frac{1}{\gamma}\right) R(u, v) \\
&\geq \left(\frac{1}{p} - \frac{1}{\gamma}\right) M(u, v) - \left(\frac{1}{\alpha+\beta} - \frac{1}{\gamma}\right) C_2^{\alpha+\beta} \|(u, v)\|_W^{\alpha+\beta} \\
&= \left(\frac{1}{p} - \frac{1}{\gamma}\right) M(u, v) - \left(\frac{1}{\alpha+\beta} - \frac{1}{\gamma}\right) C_2^{\alpha+\beta} (M(u, v))^{(\alpha+\beta)/p}.
\end{aligned}$$

Thus $\mathcal{I}_{\lambda,\mu}$ is coercive and bounded below on $\mathcal{S}_{\lambda,\mu}$. □

Lemma 2.6 *Let* $\zeta^* = (\frac{\alpha+\beta}{p})^{\frac{p}{p-\alpha-\beta}}\,\zeta_0$. *Then if* $0 < (|\lambda|\|a\|_\infty)^{\frac{p}{\gamma-p}} + (|\mu|\|b\|_\infty)^{\frac{p}{\gamma-p}} < \zeta^*$, *we have*

(i) $\alpha_0^+(\lambda,\mu) < 0$;
(ii) $\alpha_0^-(\lambda,\mu) \geq k_0$, *for some* $k_0 = k_0(\alpha,\beta,\gamma,C_1,C_2,a,b,\lambda,\mu) > 0$.

Proof (i) Let $(u,v) \in \mathcal{S}^+_{\lambda,\mu}$. By (4)

$$\frac{p-\alpha-\beta}{\gamma-\alpha-\beta}\,M(u,v) > N(u,v),$$

and so

$$\begin{aligned}
\mathcal{I}_{\lambda,\mu}(u,v) &= \left(\frac{1}{p}-\frac{1}{\alpha+\beta}\right)M(u,v)+\left(\frac{1}{\alpha+\beta}-\frac{1}{\gamma}\right)QN(u,v)\\
&\leq \left(\frac{\alpha+\beta-p}{p(\alpha+\beta)}\right)M(u,v)+\left(\frac{\gamma-\alpha-\beta}{\gamma(\alpha+\beta)}\right)\left[\frac{(p-\alpha-\beta)}{(\gamma-\alpha-\beta)}M(u,v)\right]\\
&= \left[\frac{\alpha+\beta-p}{p(\alpha+\beta)}+\frac{p-\alpha-\beta}{\gamma(\alpha+\beta)}\right]M(u,v)\\
&= \frac{(\gamma-p)(\alpha+\beta-p)}{p\gamma(\alpha+\beta)}\,M(u,v) < 0.
\end{aligned}$$

Thus $\alpha_0^+(\lambda,\mu) < 0$.
(ii) Let $(u,v) \in \mathcal{S}^-_{\lambda,\mu}$, by (4) and (9) we have

$$\begin{aligned}
M(u,v) &< \frac{\gamma-\alpha-\beta}{p-\alpha-\beta}\,N(u,v)\\
&\leq \frac{\gamma-\alpha-\beta}{p-\alpha-\beta}\,C_1^\gamma\left[(|\lambda|\|a\|_\infty)^{\frac{p}{\gamma-p}}+(|\mu|\,\|b\|_\infty)^{\frac{p}{\gamma-p}}\right]^{\frac{\gamma-p}{p}}\;\|(u,v)\|_W^\gamma.
\end{aligned}$$

This implies

$$||(u,v)||_W > \left(\frac{p-\alpha-\beta}{(\gamma-\alpha-\beta)\,C^\gamma}\right)^{\frac{1}{\gamma-p}}\frac{1}{\left((|\lambda|\|a\|_\infty)^{\frac{p}{\gamma-p}}+(|\mu|\,\|b\|_\infty)^{\frac{p}{\gamma-p}}\right)^{1/p}},\quad \text{for all } (u,v)\in\mathcal{S}^-_{\lambda,\mu}. \tag{11}$$

By the proof of Lemma (2.5) we have

$$\begin{aligned}
\mathcal{I}_{\lambda,\mu}(u,v) &\geq ||(u,v)||_W^{\alpha+\beta}\left[\left(\frac{1}{p}-\frac{1}{\gamma}\right)||(u,v)||_W^{p-\alpha-\beta}-C_2^{\alpha+\beta}\left(\frac{1}{\alpha+\beta}-\frac{1}{\gamma}\right)\right]\\
&> \left(\frac{p-\alpha-\beta}{(\gamma-\alpha-\beta)\,C_1^\gamma}\right)^{\frac{\alpha+\beta}{\gamma-p}}\frac{1}{\left((|\lambda|\|a\|_\infty)^{\frac{p}{\gamma-p}}+(|\mu|\,\|b\|_\infty)^{\frac{p}{\gamma-p}}\right)^{\frac{\alpha+\beta}{p}}}\\
&\quad\times\left[\left(\frac{\gamma-p}{p\gamma}\right)\left(\frac{p-\alpha-\beta}{(\gamma-\alpha-\beta)\,C_1^\gamma}\right)^{\frac{p-\alpha-\beta}{\gamma-p}}\frac{1}{\left((|\lambda|\|a\|_\infty)^{\frac{p}{\gamma-p}}+(|\mu|\,\|b\|_\infty)^{\frac{p}{\gamma-p}}\right)^{\frac{p-\alpha-\beta}{p}}}\right.\\
&\quad\left.-\,C_2^{\alpha+\beta}\left(\frac{1}{\alpha+\beta}-\frac{1}{\gamma}\right)\right].
\end{aligned}$$

Thus, if

$$0 < (|\lambda| \|a\|_\infty)^{\frac{p}{\gamma-p}} + (|\mu| \|b\|_\infty)^{\frac{p}{\gamma-p}} < \zeta^*,$$

then

$$\mathcal{I}_{\lambda,\mu} > k_0, \quad \text{for all } (u,v) \in \mathcal{S}^-_{\lambda,\mu},$$

for some $k_0 = k_0(\alpha, \beta, \gamma, C_1, C_2, a, b, \lambda, \mu) > 0$. This completes the proof. □

For each $(u, v) \in W$ with $N(u, v) > 0$, we write

$$t_{\max} = \left(\frac{(p-\alpha-\beta)\, M(u,v)}{(\gamma-\alpha-\beta)\, N(u,v)} \right)^{1/(\gamma-p)} > 0.$$

Then we have the following lemma.

Lemma 2.7 *For each $(u, v) \in W$ with $N(u, v)) > 0$ and*

$$0 < (|\lambda| \|a\|_\infty)^{\frac{p}{\gamma-p}} + (|\mu| \|b\|_\infty)^{\frac{p}{\gamma-p}} < \zeta_0,$$

we have

(i) *if $R(u, v) \le 0$, then there is a unique $t^- > t_{\max}$ such that $(t^- u, t^- v) \in \mathcal{S}^-_{\lambda,\mu}$ and*

$$\mathcal{I}_{\lambda,\mu}(t^- u, t^- v) = \sup_{t \ge 0} \mathcal{I}_{\lambda,\mu}(tu, tv);$$

(ii) *if $R(u, v) > 0$, then there are unique $0 < t^+ = t^+(u, v) < t_{\max} < t^-$ such that $(t^+ u, t^+ v) \in \mathcal{S}^+_{\lambda,\mu}$, $(t^- u, t^- v) \in \mathcal{S}^-_{\lambda,\mu}$ and*

$$\mathcal{I}_{\lambda,\mu}(t^+ u, t^+ v) = \inf_{0 \le t \le t_{\max}} \mathcal{I}_{\lambda,\mu}(tu, tv), \quad \mathcal{I}_{\lambda,\mu}(t^- u, t^{+-} v) = \sup_{t \ge 0} \mathcal{I}_{\lambda,\mu}(tu, tv).$$

Proof Fix $(u, v) \in W$ with $N(u, v) > 0$. Let

$$E(t) = t^{p-\alpha-\beta} M(u,v) - t^{\gamma-\alpha-\beta} N(u,v) \quad \text{for } t \ge 0.$$

Clearly, $E(0) = 0$, $E(t) \to -\infty$ as $t \to \infty$. Since

$$E'(t) = (p-\alpha-\beta) t^{p-1-\alpha-\beta} M(u,v) - (\gamma-\alpha-\beta) t^{\gamma-\alpha-\beta-1} N u, v),$$

we have $E'(t) = 0$ at $t = t_{\max}$, $E'(t) > 0$ for $t \in [0, t_{\max})$ and $E'(t) < 0$ for $t \in (t_{\max}, \infty)$. Then $E(t)$ achieves its maximum at $t_{\max}$, increasing for $t \in [0, t_{\max})$ and decreasing for $t \in (t_{\max}, \infty)$. Moreover,

$$\begin{aligned} E(t_{\max}) &= \left(\frac{(p-\alpha-\beta)\, M(u,v)}{(\gamma-\alpha-\beta)\, N(u,v)} \right)^{\frac{p-\alpha-\beta}{\gamma-p}} M(u,v) - \left(\frac{(p-\alpha-\beta)\, M(u,v)}{(\gamma-\alpha-\beta)\, N(u,v)} \right)^{\frac{\gamma-\alpha-\beta}{\gamma-p}} N(u,v) \\ &= \|(u,v)\|_W^{\alpha+\beta} \left[\left(\frac{p-\alpha-\beta}{\gamma-\alpha-\beta} \right)^{\frac{p-\alpha-\beta}{\gamma-p}} - \left(\frac{p-\alpha-\beta}{\gamma-\alpha-\beta} \right)^{\frac{\gamma-\alpha-\beta}{\gamma-p}} \right] \left(\frac{\|(u,v)\|_W^\gamma}{N(u,v))} \right)^{\frac{p-\alpha-\beta}{\gamma-p}} \\ &\ge \|(u,v)\|_W^{\alpha+\beta} \left(\frac{\gamma-p}{\gamma-\alpha-\beta} \right) \left(\frac{p-\alpha-\beta}{\gamma-\alpha-\beta} (C_1)^{-\gamma} \right)^{\frac{p-\alpha-\beta}{\gamma-p}} \\ &\quad \times \frac{1}{\left((|\lambda| \|a\|_\infty)^{\frac{p}{\gamma-p}} + (|\mu|\, \|b\|_\infty)^{\frac{p}{\gamma-p}} \right)^{\frac{p-\alpha-\beta}{p}}}. \end{aligned} \tag{12}$$

(i) $R(u,v)\ \le 0$: There is a unique $t^- > t_{\max}$ such that $E(t^-)=R(u,v)$ and $E'(t^-)<0$. Now,

$$\begin{aligned}&(p-\alpha-\beta)\,M(t^-u,t^-v)-(\gamma-\alpha-\beta)\,N(t^-u,t^-v)\\&\quad=(t^-)^{1+\alpha+\beta}\left[(p-\alpha-\beta)(t^-)^{p-1-\alpha-\beta}\,M(u,v)-(\gamma-\alpha-\beta)(t^-)^{\gamma-\alpha-\beta-1}N(u,v)\right]\\&\quad=(t^-)^{1+\alpha+\beta}\,E'(t^-)<0,\end{aligned}$$

and

$$\begin{aligned}\mathcal{G}_{\lambda,\mu}(t^-u,t^-v)&=(t^-)^p\,M(u,v)-(t^-)^\gamma\,N(u,v)-(t^-)^{\alpha+\beta}\,R(u,v)\\&=(t^-)^{\alpha+\beta}\left[(t^-)^{p-\alpha-\beta}M(u,v)-(t^-)^{\gamma-\alpha-\beta}N(u,v)-R(u,v)\right]\\&=(t^-)^{\alpha+\beta}\left[E(t^-)-R(u,v)\right]=0.\end{aligned}$$

Thus, $(t^-u,t^-v)\in\mathcal{S}^-_{\lambda,\mu}$. Since for $t>t_{\max}$, we have

$$\begin{aligned}(p-\alpha-\beta)\,M(tu,tv)-(\gamma-\alpha-\beta)\,N(tu,tv)&<0,\\ \frac{d^2}{dt^2}\mathcal{J}_{\lambda,\mu}(tu,tv)&<0,\end{aligned}$$

and

$$\frac{d}{dt}\mathcal{J}_{\lambda,\mu}(tu,tv)=t^{p-1}\,M(u,v)-t^{\gamma-1}\,N(u,v)-t^{\alpha+\beta-1}\,R(u,v)=0\quad\text{for } t=t^-.$$

Thus, $\mathcal{I}_{\lambda,\mu}(t^-u,t^-v)=\sup_{t\ge 0}\mathcal{I}_{\lambda,\mu}(tu,tv)$.

(ii) $R(u,v)>0$. By (12) and

$$\begin{aligned}E(0)=0<\ &R(u,v)\\ \le\ &C_2^{\alpha+\beta}\,\|(u,v)\|_W^{\alpha+\beta}\\ <\ &\|(u,v)\|_W^{\alpha+\beta}\left(\frac{\gamma-p}{\gamma-\alpha-\beta}\right)\left(\frac{p-\alpha-\beta}{\gamma-\alpha-\beta}\,(C_1)^{-\gamma}\right)^{\frac{p-\alpha-\beta}{\gamma-p}}\\ &\times\frac{1}{\left((|\lambda|\|a\|_\infty)^{\frac{p}{\gamma-p}}+(|\mu|\,\|b\|_\infty)^{\frac{p}{\gamma-p}}\right)^{\frac{p-\alpha-\beta}{p}}}\\ \le\ &E(t_{\max})\end{aligned}$$

for $0<(|\lambda|\|a\|_\infty)^{\frac{p}{\gamma-p}}+(|\mu|\,\|b\|_\infty)^{\frac{p}{\gamma-p}}<\zeta_0$, there are unique t^+ and t^- such that $0<t^+<t_{\max}<t^-$,

$$\begin{aligned}E(t^+)=R(u,v)&=E(t^-),\\ E'(t^+)>0&>E'(t^-).\end{aligned}$$

We have $(t^+u,t^+v)\in\mathcal{S}^+_{\lambda,\mu}$, $(t^-u,t^-v)\in\mathcal{S}^-_{\lambda,\mu}$, and $\mathcal{I}_{\lambda,\mu}(t^-u,t^-v)\ge\mathcal{I}_\lambda(tu,tv)\ge\mathcal{I}_\lambda(t^+u,t^+v)$ for each $t\in[t^+,t^-]$ and $\mathcal{I}_\lambda(t^+u,t^+v)\le\mathcal{I}_\lambda(tu,tv)$ for each $t\in[0,t^+]$. Thus,

$$\mathcal{I}_{\lambda,\mu}(t^+u,t^+v)=\inf_{0\le t\le t_{\max}}\mathcal{I}_{\lambda,\mu}(tu,tv),\quad \mathcal{I}_{\lambda,\mu}(t^-u,t^{+-}v)=\sup_{t\ge 0}\mathcal{I}_{\lambda,\mu}(tu,tv).$$

This completes the proof. □

For each $u\in W$ with $R(u,v)>0$, we write

$$\bar{t}_{\max}=\left(\frac{(\gamma-\alpha-\beta)\,R(u,v)}{(\gamma-p)\,M(u,v)}\right)^{1/(p-\alpha-\beta)}>0. \tag{13}$$

Then we have the following lemma.

Lemma 2.8 *For each $u \in W$ with $R(u, v) > 0$, we have*

(i) *if $N(u, v) \leq 0$, then there is a unique $t^+ < \bar{t}_{\max}$ such that $(t^+u, t^+v) \in \mathcal{S}^+_{\lambda,\mu}$ and*

$$\mathcal{I}_{\lambda,\mu}(t^+u, t^+v) = \inf_{t\geq 0} \mathcal{I}_{\lambda,\mu}(tu, tv);$$

(ii) *if $N(u, v) > 0$, then there are unique $0 < t^+ = t^+(u) < \bar{t}_{\max} < t^-$ such that $(t^+u, t^+v) \in \mathcal{S}^+_{\lambda,\mu}$, $(t^-u, t^-v) \in \mathcal{S}^-_{\lambda,\mu}$ and*

$$\mathcal{I}_{\lambda,\mu}(t^+u, t^+v) = \inf_{0\leq t\leq \bar{t}_{\max}} \mathcal{I}_{\lambda,\mu}(tu, tv), \quad \mathcal{I}_{\lambda,\mu}(t^-u, t^-v) = \sup_{t\geq 0} \mathcal{I}_{\lambda,\mu}(tu, tv).$$

Proof Fix $(u, v) \in W$ with $R(u, v) > 0$. Let

$$\overline{E}(t) = t^{p-\gamma}\, M(u, v) - t^{\alpha+\beta-\gamma}\, R(u, v) \quad \text{for } t > 0. \tag{14}$$

Clearly, $\overline{E}(t) \to -\infty$ as $t \to 0^+$. Since

$$\overline{E}'(t) = (p-\gamma)\, t^{p-\gamma-1}\, P(u, v) - (\alpha+\beta-\gamma)\, t^{\alpha+\beta-\gamma-1}\, R(u, v),$$

we have $\overline{E}'(t) = 0$ at $t = \bar{t}_{\max}$, $\overline{E}'(t) > 0$ for $t \in [0, \bar{t}_{\max})$ and $\overline{E}'(t) < 0$ for $t \in (\bar{t}_{\max}, \infty)$. Then $\overline{E}(t)$ achieves its maximum at $\bar{t}_{\max}$, increasing for $t \in [0, \bar{t}_{\max})$ and decreasing for $t \in (\bar{t}_{\max}, \infty)$. Using the argument in Lemma (2.7) we can obtain the result of Lemma 2.8 □

3 Existence of solutions

Now we can state our main results.

Theorem 3.1 *If the parameters λ, μ satisfy $0 < (|\lambda|\|a\|_\infty)^{\frac{p}{\gamma-p}} + (|\mu|\|b\|_\infty)^{\frac{p}{\gamma-p}} < \zeta^*$, then Problem (1) has at least two solutions (u_0^+, v_0^+) and (u_0^-, v_0^-) such that $u_0^\pm \geq 0$, $v_0^\pm \geq 0$ in Ω and $u_0^\pm \neq 0$, $v_0^\pm \neq 0$.*

Theorem 3.2 *Suppose that $a(x) \geq 0\, (\leq 0)$, then there exists a positive constant ζ_1 such that if $\lambda \leq 0\, (\geq 0)$ and μ satisfies $0 < |\mu| < \zeta_1$, then Problem (1) has at least two solutions (u_0^+, v_0^+) and (u_0^-, v_0^-) such that $u_0^\pm \geq 0$, $v_0^\pm \geq 0$ in Ω and $u_0^\pm \neq 0$, $v_0^\pm \neq 0$.*

The proof of Theorem (3.2) is similar to that of Theorem (3.1) and for this reason, will be omitted here.

Remark 3.3 Our ideas can also be applied to the following elliptic system:

$$\begin{cases} -\Delta_p u + m(x)\,|u|^{p-2}u = \lambda\, a(x)|u|^{\gamma-2}u, & x \in \Omega, \\ -\Delta_p v + m(x)\,|v|^{p-2}v = \mu\, b(x)|v|^{\gamma-2}v, & x \in \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial n} = \frac{\alpha}{\alpha+\beta}\,|u|^{\alpha-2}\,u\,|v|^{\beta}, \quad |\nabla v|^{p-2}\frac{\partial v}{\partial n} = \frac{\beta}{\alpha+\beta}\,|u|^{\alpha}\,|v|^{\beta-2}\,v, & x \in \partial\Omega, \end{cases}$$

where $p, \alpha, \beta, \gamma, m(x), a(x)$ and $b(x)$ are as before. The results presented here have analogous statements for the latter problem. The proofs of the multiplicity results are similar to the ones performed for Problem (1) so we leave the details to the reader.

The proof of the Theorem (3.1) will be a consequence of the next two propositions.

Proposition 3.4 *If $0 < (|\lambda|\|a\|_\infty)^{\frac{p}{\gamma-p}} + (|\mu|\|b\|_\infty)^{\frac{p}{\gamma-p}} < \zeta^*$, then the functional $\mathcal{I}_{\lambda,\mu}$ has a minimizer (u_0^+, v_0^+) in $\mathcal{S}^+_{\lambda,\mu}$ and it satisfies*

(i) $\mathcal{I}_{\lambda,\mu}(u_0^+, v_0^+) = \alpha_0^+(\lambda, \mu)$;

(ii) *(u_0^+, v_0^+) is a nontrivial nonnegative solution of Problem (1) such that $u_0^+ \geq 0$, $v_0^+ \geq 0$ in Ω and $u_0^+ \neq 0$, $v_0^+ \neq 0$.*

Proof By Lemma (2.5), $\mathcal{I}_{\lambda,\mu}$ is coercive and bounded below on $\mathcal{S}_{\lambda,\mu}$. Let $\{(u_n, v_n\} \subseteq \mathcal{S}^+_{\lambda,\mu}$ be a minimizing sequence for $\mathcal{I}_{\lambda,\mu}$, i.e.,

$$\lim_{n\to\infty} \mathcal{I}_{\lambda,\mu}(u_n, v_n) = \inf_{u\in\mathcal{S}^+_{\lambda,\mu}} \mathcal{I}_{\lambda,\mu}(u, v).$$

Then by Lemma (2.5) and the Rellich theorem, there exist a subsequence $\{(u_n, v_n)\}$ and $(u_0^+, v_0^+) \in W$ such that (u_0^+, v_0^+) is a solution of Problem (1) and

$$\begin{aligned} u_n &\rightharpoonup u_0^+ \quad \text{weakly in } W_0^{1,p}, \\ u_n &\to u_0^+ \quad \text{strongly in } L^\gamma(\Omega) \text{ and in } L^{\alpha+\beta}(\Omega), \\ v_n &\rightharpoonup v_0^+ \quad \text{weakly in } W_0^{1,p}, \\ v_n &\to v_0^+ \quad \text{strongly in } L^\gamma(\Omega) \text{ and in } L^{\alpha+\beta}(\Omega). \end{aligned}$$

This implies

$$R(u_n, v_n) \to R(u_0^+, v_0^+) \quad \text{as } n \to \infty,$$

$$N(u_n, v_n) \to N(u_0^+, v_0^+) \quad \text{as } n \to \infty.$$

Since

$$\mathcal{I}_{\lambda,\mu}(u_n, v_n) = \left(\frac{1}{p} - \frac{1}{\gamma}\right) M(u_n, v_n) - \left(\frac{1}{\alpha+\beta} - \frac{1}{\gamma}\right) R(u_n, v_n),$$

and by Theorem 2.6 (i)

$$\mathcal{I}_{\lambda,\mu}(u_n, v_n) \to \alpha_0^+(\lambda, \mu) < 0 \quad \text{as} \quad n \to \infty.$$

Letting $n \to \infty$, we see that $R(u_0, v_0) > 0$. In particular $u_0^+ \neq 0$, $v_0^+ \neq 0$. Now we prove that $u_n \to u_0^+$ strongly in $W_0^{1,p}$, $v_n \to v_0^+$ strongly in $W_0^{1,p}$. Suppose otherwise, then either

$$\|u_0^+\|_{1,p} < \liminf_{n\to\infty} \|u_n\|_{1,p} \quad \text{or} \quad \|v_0^+\|_{1,p} < \liminf_{n\to\infty} \|v_n\|_{1,p}. \tag{15}$$

Fix $(u, v) \in W$ with $R(u, v) > 0$. Let

$$K_{(u,v)}(t) = \overline{E}(t) - N(u, v),$$

where $\overline{E}(t)$ is as in (14). Clearly, $K_{(u,v)}(t) \to -\infty$ as $t \to 0^+$, and

$$K_{(u,v)}(t) \to -N(u, v) \quad \text{as } t \to \infty.$$

Since $K'_{(u,v)}(t) = \overline{E}'(t)$, by an argument similar to the one in the proof of Lemma (2.8) we have that the function $K_{(u,v)}(t)$ achieves its maximum at $\bar{t}_{\max}$, is increasing for $t \in (0, \bar{t}_{\max})$ and decreasing for $t \in (\bar{t}_{\max}, \infty)$, where

$$\bar{t}_{\max} = \left(\frac{(\gamma - \alpha - \beta)\, R(u, v)}{(\gamma - p)\, M(u, v)}\right)^{1/(p-\alpha-\beta)} > 0,$$

is as in (13). Since $R(u_0^+, v_0^+) > 0$, by Lemma (2.8) there is unique $t_0^+ < \bar{t}_{\max}$ such that $(t_0^+ u_0^+, t_0^+ v_0^+) \in \mathcal{S}^+_{\lambda,\mu}$ and

$$\mathcal{I}_{\lambda,\mu}(t_0^+ u_0^+, t_0^+ v_0^+) = \inf_{0\le t\le \bar{t}_{\max}} \mathcal{I}_{\lambda,\mu}(t u_0^+, t v_0^+).$$

Then

$$\begin{aligned} K_{(u_0^+,v_0^+)}(t_0^+) &= (t_0^+)^{p-\gamma} M(u_0^+, v_0^+) - (t_0^+)^{\alpha+\beta-\gamma} R(u_0^+, v_0^+) - N(u_0^+, v_0^+) \\ &= (t_0^+)^{-\gamma} \left(M(t_0^+ u_0^+, t_0^+ v_0^+) - R(t_0^+ u_0^+, t_0^+ v_0^+) - N(t_0^+ u_0^+, t_0^+ v_0^+) \right) \\ &= 0. \end{aligned} \tag{16}$$

By (15) and (16) we obtain

$$K_{(u_n,v_n)}(t_0^+) > 0 \quad \text{for n sufficiently large.}$$

Since $(u_n, v_n) \in \mathcal{S}^+_{\lambda,\mu}$, we have $\bar{t}_{\max}(u_n) > 1$. Moreover,

$$K_{(u_n,v_n)}(1) = M(u_n, v_n) - R(u_n, v_n) - N(u_n, v_n) = 0,$$

and $K_{(u_n,v_n)}(t)$ is increasing for $t \in (0, \bar{t}_{\max}(u_n, v_n))$. This implies $K_{(u_n,v_n)}(t) < 0$ for all $t \in (0, 1]$ and n sufficiently large. We obtain $1 < t_0^+ \le \bar{t}_{\max}(u_0, v_0)$. But $(t_0^+ u_0^+, t_0^+ v_0^+) \in \mathcal{S}^+_{\lambda,\mu}$ and

$$\mathcal{I}_{\lambda,\mu}(t_0^+ u_0^+, t_0^+ v_0^+) = \inf_{0 \le t \le \bar{t}_{\max}(u_0^+, v_0^+)} \mathcal{I}_{\lambda,\mu}(t u_0^+, t v_0^+).$$

This implies

$$\mathcal{I}_{\lambda,\mu}(t_0^+ u_0^+, t_0^+ v_0^+) < \mathcal{I}_{\lambda,\mu}(u_0^+, v_0^+) < \lim_{n\to\infty} \mathcal{I}_{\lambda,\mu}(u_n, v_n) = \alpha_0^+(\lambda, \mu),$$

which is a contradiction. Hence

$$\begin{aligned} u_n &\to u_0^+ \quad \text{strongly in } W_0^{1,p}, \\ v_n &\to v_0^+ \quad \text{strongly in } W_0^{1,p}. \end{aligned}$$

This implies

$$\mathcal{I}_{\lambda,\mu}(u_n, v_n) \to \mathcal{I}_{\lambda,\mu}(u_0^+, v_0^+) = \alpha_0^+(\lambda, \mu) \quad \text{as } n \to \infty.$$

Thus (u_0^+, v_0^+) is a minimizer for $\mathcal{I}_{\lambda,\mu}$ on $\mathcal{S}^+_{\lambda,\mu}$. Since $\mathcal{I}_{\lambda,\mu}(u_0^+, v_0^+) = \mathcal{I}_{\lambda,\mu}(|u_0^+|, |v_0^+|)$ and $(|u_0^+|, |v_0^+|) \in \mathcal{S}^+_{\lambda,\mu}$, by Lemma (2.4) we may assume that (u_0^+, v_0^+) is a nontrivial nonnegative solution of Equation (1). □

Next, we establish the existence of a local minimum for $\mathcal{I}_{\lambda,\mu}$ on $\mathcal{S}^-_{\lambda,\mu}$.

Proposition 3.5 *If* $0 < (|\lambda| \|a\|_\infty)^{\frac{p}{\gamma-p}} + (|\mu| \|b\|_\infty)^{\frac{p}{\gamma-p}} < \zeta^*$, *then the functional* $\mathcal{I}_{\lambda,\mu}$ *has a minimizer* (u_0^-, v_0^-) *in* $\mathcal{S}^-_{\lambda,\mu}$ *and it satisfies*

(i) $\mathcal{I}_{\lambda,\mu}(u_0^-, v_0^-) = \alpha_0^-(\lambda, \mu)$;
(ii) (u_0^-, v_0^-) *is a nontrivial nonnegative solution of Problem* (1), *such that* $u_0^- \ge 0$, $v_0^- \ge 0$ *in* Ω *and* $u_0^- \ne 0$, $v_0^- \ne 0$.

Proof Let $\{(u_n, v_n)\}$ be a minimizing sequence for $\mathcal{I}_{\lambda,\mu}$ on $\mathcal{S}^-_{\lambda,\mu}$, i.e.,

$$\lim_{n\to\infty} \mathcal{I}_{\lambda,\mu}(u_n, v_n) = \inf_{u \in \mathcal{M}^-_{\lambda,\mu}} \mathcal{I}_{\lambda,\mu}(u, v).$$

Then by Lemma (2.5) and the Rellich theorem, there exist a subsequence $\{(u_n, v_n)\}$ and $(u_0^-, v_0^-) \in W$ such that (u_0^-, v_0^-) is a solution of Problem (1) and

$$\begin{aligned} u_n &\rightharpoonup u_0^- \quad \text{weakly in } W_0^{1,p}, \\ u_n &\to u_0^- \quad \text{strongly in } L^\gamma(\Omega) \text{ and in } L^{\alpha+\beta}(\Omega), \\ v_n &\rightharpoonup v_0^- \quad \text{weakly in } W_0^{1,p}, \\ v_n &\to v_0^- \quad \text{strongly in } L^\gamma(\Omega) \text{ and in } L^{\alpha+\beta}(\Omega). \end{aligned}$$

This implies

$$R(u_n, v_n) \to R(u_0^-, v_0^-) \quad \text{as } n \to \infty,$$

$$N(u_n, v_n) \to N(u_0^-, v_0^-) \quad \text{as } n \to \infty.$$

Moreover, by (4) we obtain

$$N(u_n, v_n) > \frac{(2-\alpha-\beta)}{(\gamma-\alpha-\beta)} M(u_n, v_n); \tag{17}$$

By (11) and (17) there exists a positive number η_0 such that

$$N(u_n, v_n) > \eta_0.$$

This implies

$$N(u_0^-, v_0^-) \geq \eta_0. \tag{18}$$

In particular $u_0^- \neq 0$, $v_0^- \neq 0$. Arguing by contradiction, we may assume that, $v_0^+ \equiv 0$. Then as u_0^+ is a nonzero solution of

$$\begin{cases} -\Delta u + m(x)\, u = 0, & x \in \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial n} = \lambda\, a(x)|u|^{\gamma-2} u, & x \in \partial\Omega, \end{cases}$$

we have

$$M(u_0^+, 0) = N(u_0^+, 0) > 0.$$

Since $c^{\pm} = \max\{\pm a, 0\} \not\equiv 0$, we may choose $z \in W^{1,2} \backslash \{0\}$ such that

$$M(z, 0) = N(0, z) > 0,$$

and

$$R(u_0^+, z) \geq 0.$$

Now

$$N(u_0^+, z) = N(u_0^+, 0) + N(0, z) > 0,$$

and so by Lemma (2.8) there is a unique $0 < t^+ < \bar{t}_{\max}$ such that $(t^+ u_0^+, t^+ z) \in \mathcal{S}^+_{\lambda,\mu}$. Moreover,

$$\bar{t}_{\max} = \left(\frac{(\gamma-\alpha-\beta)\, R(u_0^+, z)}{(\gamma - p)\, M(u_0^+, z)} \right)^{1/(p-\alpha-\beta)} = \left(\frac{\gamma-\alpha-\beta}{\gamma-p} \right)^{1/(p-\alpha-\beta)} > 1,$$

and

$$\mathcal{I}_{\lambda,\mu}(t^+ u_0^+, t^+ z) = \inf_{0 \leq t \leq \bar{t}_{\max}} \mathcal{I}_{\lambda,\mu}(t u_0^+, t z).$$

This implies

$$\mathcal{I}_{\lambda,\mu}(t^+ u_0^+, t^+ z) \leq \mathcal{I}_{\lambda,\mu}(u_0^+, z) < \mathcal{I}_{\lambda,\mu}(u_0^+, 0) = \alpha_0^+(\lambda, \mu),$$

which is a contradiction and hence $u_0^- \neq 0$, $v_0^- \neq 0$. Now we prove that $u_n \to u_0^-$ strongly in $W_0^{1,p}$, $v_n \to v_0^-$ strongly in $W_0^{1,p}$. Suppose otherwise, then either

$$\|u_0^-\|_{1,p} < \liminf_{n\to\infty} \|u_n\|_{1,p} \quad \text{or} \quad \|v_0^-\|_{1,p} < \liminf_{n\to\infty} \|v_n\|_{1,p}. \tag{19}$$

By Lemma (2.7) there is a unique t_0^- such that $(t_0^- u_0^-, t_0^- v_0^-) \in \mathcal{S}_{\lambda,\mu}^-$. Since $(u_n, v_n) \in \mathcal{S}_{\lambda,\mu}^-$, $\mathcal{I}_{\lambda,\mu}(u_n, v_n) \geq \mathcal{I}_{\lambda,\mu}(tu_n, tv_n)$ for all $t \geq 0$, we have

$$\mathcal{I}_{\lambda,\mu}(t_0^- u_0^-, t_0^- v_0^-) < \lim_{n\to\infty} \mathcal{I}_{\lambda,\mu}(t_0^- u_n, t_0^- v_n) \leq \lim_{n\to\infty} \mathcal{I}_{\lambda,\mu}(u_n, v_n) = \alpha_0^-(\lambda, \mu),$$

and this is a contradiction. Hence

$$u_n \to u_0^- \quad \text{strongly in } W_0^{1,p},$$
$$v_n \to v_0^- \quad \text{strongly in } W_0^{1,p}.$$

This implies

$$\mathcal{I}_{\lambda,\mu}(u_n, v_n) \to \mathcal{I}_{\lambda,\mu}(u_0^-, v_0^-) = \alpha_0^-(\lambda, \mu) \quad \text{as } n \to \infty.$$

Thus (u_0^-, v_0^{+-}) is a minimizer for $\mathcal{I}_{\lambda,\mu}$ on $\mathcal{S}_{\lambda,\mu}^-$. Since $\mathcal{I}_{\lambda,\mu}(u_0^-, v_0^-) = \mathcal{I}_{\lambda,\mu}(|u_0^-|, |v_0^-|)$ and $(|u_0^-|, |v_0^-|) \in \mathcal{S}_{\lambda,\mu}^-$, by Lemma (2.4) we may assume that (u_0^-, v_0^-) is a nontrivial nonnegative solution of Equation (1). □

Proof of Theorem 3.1 By Propositions (3.4), (3.5), we obtain that Equation (1) has two nontrivial nonnegative solutions (u_0^+, v_0^+) and (u_0^-, v_0^-) such that $(u_0^+, v_0^+) \in \mathcal{S}_{\lambda,\mu}^+$ and $(u_0^-, v_0^-) \in \mathcal{S}_{\lambda,\mu}^-$. It remains to show that the solutions found in Propositions (3.4) and (3.5) are distinct. Since $\mathcal{S}_{\lambda,\mu}^+ \cap \mathcal{S}_{\lambda,\mu}^- = \emptyset$, this implies that (u_0^+, v_0^+) and (u_0^-, v_0^-) are distinct. This concludes the proof. □

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References

1. Adriouch, K.; El Hamidi, A.: The Nehari manifold for systems of nonlinear elliptic equations. Nonlinear Anal. **64**(5), 2149–2167 (2006)
2. Afrouzi, G.A.; Rasouli, S.H.: A remark on the existence and multiplicity result for a nonlinear elliptic problem involving the p-Laplacian. Nonlinear Differ. Equ. Appl. **16**, 717–730 (2009)
3. Alves, C.O.; El Hamidi, A.: Nehari manifold and existence of positive solutions to a class of quasilinear problems. Nonlinear Anal. **60**, 611–624 (2005)
4. Ambrosetti, A.; Brezis, H.; Cerami, G.: Combined effects of concave and convex nonlinearities in some elliptic problems. J. Funct. Anal. **122**, 519–543 (1994)
5. Amman, H.; Lopez-Gomez, J.: A priori bounds and multiple solution for superlinear indefinite elliptic problems. J. Differ. Equ. **146**, 336–374 (1998)
6. Atkinson, C.; El Kalli, K.: Some boundary value problems for the Bingham model. J. Non-Newtonian Fluid Mech. **41**, 339–363 (1992)
7. Binding, P.A.; Drabek, P.; Huang, Y.X.: On Neuman boundary value problems for some quasilinear equations. Nonlinear Anal. **42**, 613–629 (2000)
8. Bouchekif, M.; Nasri, Y.: On a nonhomogeneous elliptic system with changing sign data. Nonlinear Anal. **65**, 1476–1487 (2006)
9. Bozhkov, Y.; Mitidieri, E.: Existence of multiple solutions for quasilinear systems via fibering method. J. Differ. Equ. **190**, 239–267 (2003)
10. Brezis, H.; Nirenberg, L.: Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. Commun. Pure. Appl. Math. **36**, 437–477 (1983)
11. Brown, K.J.: The Nehari manifold for a semilinear elliptic equation involving a sublinear term. Calc. Var. **22**, 483–494 (2005)
12. Brown, K.J.; Wu, T.F.: A fibering map approach to a semilinear elliptic boundary value problem. Electron. J. Differ. Equ. **69**, 1–9 (2007)
13. Brown, K.J.; Wu, T.F.: A semilinear elliptic system involving nonlinear boundary condition and sign changing weight function. J. Math. Anal. Appl. **337**, 1326–1336 (2008)
14. Brown, K.J.; Zhang, Y.: The Nehari manifold for a semilinear elliptic problem with a sign changing weight function. J. Differ. Equ. **193**, 481–499 (2003)
15. Clément, Ph.; Fleckinger, J.; Mitidieri, E.; de Thelin, F.: Existence of positive solutions for quasilinear elliptic systems. J. Differ. Equ. **166**(2), 455–477 (2000)
16. Dancer, E.N.: Competing species systems with diffusion and large interaction. Rend. Semin. Mat. Fis. Milano. **65**, 23–33 (1995)
17. Drabek, P.; Pohozaev, S.I.: Positive solutions for the p-Laplacian: application of the fibering method. Proc. R. Soc. Edinburgh Sect A. **127**, 721–747 (1997)

18. Escobar, J.F.: Uniqueness theorems on conformal deformations of metrics, Sobolev inequalities, and an eigenvalue estimate. Commun. Pure Appl. Math. **43**, 857–883 (1990)
19. Ladde, G.S.; Lakshmikantham, V.; Vatsale, A.S.: Existence of coupled quasisolutions of systems of nonlinear elliptic boundary value problems. Nonlinear Anal. **8**(5), 501–515 (1984)
20. Tehrani, H.T.: A multiplicity result for the jumping nonlinearity problem. J. Differ. Equ. **118**(1), 305–472 (2003)
21. Tehrani, H.T.: On indefinite superlinear elliptic equations. Calc. Var. **4**, 139–153 (1996)
22. Tolksdorf, P.: Regularity for a more general class of quasilinear elliptic equations. J. Differ. Equ. **51**, 126–150 (1984)
23. Wu, T.F.: The Nehari manifold for a semilinear elliptic system involving sign-changing weight function. Nonlinear Anal. (2012) (in press)
24. Wu, T.F.: On semilinear elliptic equations involving concave-convex nonlinearities and sign-changing weight function. J. Math. Anal. Appl. **318**, 253–270 (2006)
25. Wu, T.F.: A semilinear elliptic problem involving nonlinear boundary condition and sign-changing potential. Electron. J. Differ. Equ. **131**, 1–15 (2006)
26. Wu, T.F.: Multiplicity results for a semilinear elliptic equation involving sign-changing weight function. Rocky Mt. J. Math. (2012) (in press)