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RESEARCH ARTICLE

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Bayesian prediction under a class of multivariate distributions

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Abstract In this paper the prediction problem is studied under members of a class $\Im^*$ of multivariate distributions, constructed by AL-Hussaini and Ateya (Stat Pap 46:321–338, 2005; J Egypt Math Soc 14(1):45–54, 2006). More attention is given to bivariate compound Rayleigh distribution, which is a member of this class, as illustrative example.

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الملخص

تتم في هذه الورقة العلمية دراسة مسألة التنبؤ تحت عناصر العائلة $\Im^*$ المكونة من التوزيعات متعددة المتغيرات، والتي تم تكوينها من قبل الحسيني وعطية [7 – 8]. كمثال توضيحي، يُعطى اهتمام أكبر لتوزيع رايلي المركب ذي المتغيرين $(BVCR)$، والذي ينتمي لهذا الصف.

1 Introduction

This section deals with a class of continuous distributions $\Im$ and its multivariate version $\Im^*$, and the generation of a multivariate sample from $\Im^*$, and one and two-sample predictions.

Suppose that a class $\Im$ of distribution functions is of the form

$$\Im = \Big\{ F : F \equiv F_{X|\Theta}(x|\theta) = 1 - exp[-\theta\delta\lambda_\eta(x)],$$
$$0 \le a < x < b \le \infty, (\theta, \delta > 0, (\theta, \delta, \eta) \in \Omega) \Big\}, \quad (1.1)$$

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where a and b are non-negative real numbers such that a may assume the value zero and b the value infinity, $\lambda_\eta(x)$ is a continuous, monotone increasing, and differentiable function of x such that $\lambda_\eta(x) \to 0$ as $x \to a^+$, $\lambda_\eta(x) \to \infty$ as $x \to b^-$ and η is a parameter (could be a vector), (θ, δ, η) belongs to a parameter space Ω. This class covers some important distributions such as the Weibull, exponential, Rayleigh, compound Weibull, compound exponential (Lomax), compound Rayleigh, Pareto, power function, beta, Gompertz and compound Gompertz distributions, among others. The failure rate and survival functions corresponding to $F \in \Im$ are, respectively, $\delta\theta\lambda'_\eta(x)$ and $e^{-\theta\delta\lambda_\eta(x)}$, so that the probability density function (pdf) is given, for $0 \le a < x < b \le \infty$, by

$$f_{X|\Theta}(x|\theta) = \delta\theta\lambda'_\eta(x)exp[-\theta\delta\lambda_\eta(x)], \text{ where } \lambda'_\eta(x) = \frac{d(\lambda_\eta)}{dx}. \tag{1.2}$$

The class $\Im$ was used by AL-Hussaini and Osman [10], AL-Hussaini [4], Ahmad [1,2], Ahmad and Fawzy [3], AL-Hussaini and Ahmad [5,6], and Jafar et al. [13].

1.1 A class of multivariate distributions

AL-Hussaini and Ateya [7,8] constructed a class of multivariate distributions by compounding members of the class $\Im$ with the gamma distribution. The resulting multivariate distributions form a class $\Im^*$, given by

$$\Im^* = \left\{F^* : F^* \equiv F_{\boldsymbol{X}}(\boldsymbol{x}) = \int f_{\boldsymbol{X}}(\boldsymbol{u})d\boldsymbol{u}\right\},$$

where $\int \equiv \int_0^{x_1} \dots \int_0^{x_k}$, $\boldsymbol{u} = (u_1, \dots, u_k)$, $d\boldsymbol{u} = du_k \dots du_1$ and $f_{\boldsymbol{X}}(\boldsymbol{x})$ is the pdf of the random vector $\boldsymbol{X} = (X_1, \dots, X_k)$, given by

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}\left[\prod_{i=1}^{k} c_i\lambda'_{\eta_i}(x_i)\right]\left[1+\sum_{i=1}^{k} c_i\lambda_{\eta_i}(x_i)\right]^{-(\alpha+k)}, \tag{1.3}$$
$$c_i = \delta_i/\beta, \quad 0 \le a < x_i < b \le \infty, \quad i = 1, 2, \dots, k.$$

It was assumed that Θ is a positive random variable following the Gamma (α, β) distribution with pdf $g_\Theta(\theta)$ given by

$$g_\Theta(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}, \quad \theta > 0, \quad (\alpha > 0, \beta > 0). \tag{1.4}$$

The pdf $f_{\boldsymbol{X}}(\boldsymbol{x})$ in (1.3) may be obtained by writing

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \int_0^\infty \left[\prod_{i=1}^{k} f_{X_i|\Theta}(x_i|\theta)\right] g_\Theta(\theta)d\theta.$$

Maximum likelihood and Bayes estimation of the parameters of members of the class $\Im^*$ were obtained by AL-Hussaini and Ateya [7,8] and particularly when the underlying population distribution is bivariate compound Weibull or bivariate compound Gompertz.

1.2 Generation of a multivariate random sample of size n from the class $\Im^*$

Assuming that $F_{X_i|\Theta}(x_i|\theta) = 1 - \exp[-\theta\delta_i\lambda_{\eta_i}(x_i)]$ and $g_\Theta(\theta) = \beta^\alpha\theta^{\alpha-1}e^{-\beta\theta}/\Gamma(\alpha)$, an observation x_{ij} is obtained by first generating θ_j from Gamma $(\alpha, \beta), u_i$ from U(0, 1) and then setting $x_{ij} = \lambda_{\eta_i}^{-1}(-(ln\ u_i)/\theta_j\delta_i)$, $j = 1, 2, \dots, n$, $i = 1, 2, \dots, k$. This is repeated until we obtain the required multivariate random sample.

1.3 One-sample prediction

Suppose that $x_1 < x_2 < \cdots < x_r$ be the informative type II censored sample, representing the first r ordered lifetimes of a random sample of size n drawn from a population with pdf $f_X(x)$, cumulative distribution function (cdf) $F_X(x)$ and reliability function (rf) $R(x)$. In one-sample scheme the Bayesian prediction intervals (BPI's) for the remaining unobserved future $(n-r)$ lifetimes are sought based on the first r observed ordered lifetimes.

For the remaining $(n-r)$ components, let $y_s = x_{r+s}$ denote the future lifetime of the sth component to fail, $1 \le s \le (n-r)$. The conditional density function of y_s given that the r components had already failed is

$$g_1(y_s|\boldsymbol{\theta}) \propto [R(x_r) - R(y_s)]^{(s-1)}[R(y_s)]^{n-r-s}[R(x_r)]^{-(n-r)} f_X(y_s|\boldsymbol{\theta}), \quad y_s > x_r, \tag{1.5}$$

$\boldsymbol{\theta}$ is the vector of parameters.

The predictive density function is given by

$$g_1^*(y_s|\boldsymbol{x}) = \int_{\boldsymbol{\Theta}} g_1(y_s|\boldsymbol{\theta})\pi^*(\boldsymbol{\theta}|\boldsymbol{x})d\boldsymbol{\theta}, \quad y_s > x_r, \tag{1.6}$$

where $\pi^*(\boldsymbol{\theta}|\boldsymbol{x})$ is the posterior density function of $\boldsymbol{\theta}$ given $\boldsymbol{x}$ and $\boldsymbol{x} = (x_1, \ldots, x_r)$.

A $(1-\tau)$ % BPI for y_s is an interval (L, U) such that

$$P(Y_s > L|\boldsymbol{x}) = \int_L^\infty g_1^*(y_s|\boldsymbol{x})dy_s = 1 - \frac{\tau}{2}, \quad L > x_r, \tag{1.7}$$

$$P(Y_s > U|\boldsymbol{x}) = \int_U^\infty g_1^*(y_s|\boldsymbol{x})dy_s = \frac{\tau}{2}, \quad U > x_r. \tag{1.8}$$

By solving Equations (1.7) and (1.8), we get the interval (L, U).

1.4 Two-sample prediction

Let $x_1 < x_2 < \cdots < x_r$ and $z_1 < z_2 < \cdots < z_m$ represent informative (type II censored) sample from a random sample of size n and a future ordered sample of size m, respectively. It is assumed that the two samples are independent and drawn from a population with pdf $f_X(x)$, cdf $F_X(x)$ and rf $R(x)$.

Our aim is to obtain the BPI's for z_s, $s = 1, 2, \ldots, m$. The conditional density function of z_s, given the vector of parameters $\boldsymbol{\theta}$, is

$$g_2(z_s|\boldsymbol{\theta}) \propto [1 - R(z_s)]^{(s-1)}[R(z_s)]^{m-s} f_X(z_s|\boldsymbol{\theta}), \quad z_s > 0, \tag{1.9}$$

$\boldsymbol{\theta}$ is the vector of parameters.
The predictive density function is given by

$$g_2^*(z_s|\boldsymbol{x}) = \int_{\boldsymbol{\Theta}} g_2(z_s|\boldsymbol{\theta})\pi^*(\boldsymbol{\theta}|\boldsymbol{x})d\boldsymbol{\theta}, \quad z_s > 0, \tag{1.10}$$

$\pi^*(\boldsymbol{\theta}|\boldsymbol{x})$ is the posterior density function of $\boldsymbol{\theta}$ given $\boldsymbol{x}$ and $\boldsymbol{x} = (x_1, \ldots, x_r)$.

A $(1-\tau)$ % BPI for z_s is an interval (L, U) such that

$$P(Z_s > L|\boldsymbol{x}) = \int_L^\infty g_2^*(z_s|\boldsymbol{x})dz_s = 1 - \frac{\tau}{2}, \tag{1.11}$$

$$P(Z_s > U|\boldsymbol{x}) = \int_U^\infty g_2^*(z_s|\boldsymbol{x})dz_s = \frac{\tau}{2}. \tag{1.12}$$

By solving Equations (1.11) and (1.12), we get the interval (L, U).

2 Bayesian prediction intervals for future bivariate observations

The main goal in this section is to study the one-sample and two-sample prediction problems in case of bivariate informative observations.

While ordering a set of univariate random variables is a clear and straight-forward matter as it can be done by simply ordering the set of random variables, such ordering is not as clear if we are dealing with a set of random vectors.

Barnett [11] classified the principles used for ordering multivariate data into four principles: marginal, reduced (aggregate), partial and conditional (sequential) ordering. An interesting detailed discussion of such principles with illustrative examples are given in Barnett's paper.

In our paper, we wish to predict bivariate random vectors. The first components of the predicted random vectors are based on the ordered first components of the informative sample, as it is done in the univariate case. To predict the second components, we compute the norms of each vector of the informative sample, order the norms and then predict the future norms as it is done in the univariate case. The relation between the components of vectors and norms enables us to obtain the second components of the predicted vectors. In other words, we obtain the second component of a predicted vector from the knowledge of the values of the first component and the norm of the vector. Ateya [9] used this point of view to obtain the BPI's of future observations from bivariate truncated generalized Cauchy distribution.

2.1 One-sample prediction

Let $(x_1, y_1), \dots, (x_r, y_r)$ be the first r bivariate informative observations from a random sample of size n of bivariate observations. Suppose that the first components of such informative vectors are ordered, that is $x_1 < x_2 < \cdots < x_r$ and that their norms are given by $z_1, z_2, \dots, z_r$.

To obtain $BPI's$ for the remaining future vectors, denoted by $(x_1^*, y_1^*), \dots, (x_{n-r}^*, y_{n-r}^*)$, where $x_1^* < x_2^* < \cdots < x_{n-r}^*$ and norms $z_1^* < z_2^* < \cdots < z_{n-r}^*$ we apply the following steps:

1. based on ordered $z_1, z_2, \dots, z_r$, denoted by $z_{1:r}, z_{2:r}, \dots, z_{r:r}$ compute the BPI's for z_s^*, $s = 1, 2, \dots, (n-r)$, say (L_{1s}, U_{1s}),
2. based on $x_1 < x_2 < \cdots < x_r$ compute the BPI's for x_s^*, $s = 1, 2, \dots, (n-r)$, say (L_{2s}, U_{2s}),
3. from (1) and (2), compute the BPI's for y_s^*, $s = 1, 2, \dots, (n-r)$ which are $([L_{1s}^2 - L_{2s}^2]^{1/2}, [U_{1s}^2 - U_{2s}^2]^{1/2})$. This is true, since $z_s^* = (x_s^{*\,2} + y_s^{*\,2})^{1/2}$,
4. from (2) and (3), the $BPI's$ for (x_s^*, y_s^*), $s = 1, 2, \dots, (n-r)$ is $(L_{2s}, [L_{1s}^2 - L_{2s}^2]^{1/2}), (U_{2s}, [U_{1s}^2 - U_{2s}^2]^{1/2})$.

2.2 Two-sample prediction

In this case the first r bivariate informative observations $(x_1, y_1), \dots, (x_r, y_r)$ from a random sample of size n is such that $x_1 < x_2 < \cdots < x_r$ with norms $z_1, z_2, \dots, z_r$. An independent future sample of size m is $(x_1^*, x_1^*), \dots, (x_m^*, x_m^*)$, where $x_1^* < x_2^* < \cdots < x_m^*$ and norms $z_1^* < z_2^* < \cdots < z_m^*$. To obtain the BPI's of the future sample, we apply the following steps:

1. based on ordered $z_1, z_2, \dots, z_r$, denoted by $z_{1:r}, z_{2:r}, \dots, z_{r:r}$ compute the $BPI's$ for z_s^*, $s = 1, 2, \dots, m$, say (L_{1s}, U_{1s}),
2. based on $x_1 < x_2 < \cdots < x_r$ compute the BPI's for x_s^*, $s = 1, 2, \dots, m$, say (L_{2s}, U_{2s}),
3. from (1) and (2), compute the BPI's for y_s^*, $s = 1, 2, \dots, m$ which are $([L_{1s}^2 - L_{2s}^2]^{1/2}, [U_{1s}^2 - U_{2s}^2]^{1/2})$.
4. from (2) and (3), the BPI's for (x_s^*, y_s^*), $s = 1, 2, \dots, m$ is $(L_{2s}, [L_{1s}^2 - L_{2s}^2]^{1/2}), (U_{2s}, [U_{1s}^2 - U_{2s}^2]^{1/2})$.

3 One-sample prediction in case of (BVCR) distribution

If, in (1.3), $k = 2$, $\lambda_\eta(x) = x^2$, $\lambda_\eta(y) = y^2$, $\delta_1 = \delta_2 = 1$ so that $c_1 = c_2 = 1/\beta = c$, then (X, Y) has a bivariate compound Rayleigh (BVCR) pdf, given by

$$f_{X,Y}(x, y) = 4\alpha(\alpha+1)c^2xy[1 + c(x^2 + y^2)]^{-(\alpha+2)}, \quad x > 0, \quad y > 0. \tag{3.1}$$

The marginal pdf's of the random variables X and Y are given, respectively, by

$$f_X(x) = 2\alpha c x[1 + cx^2]^{-(\alpha+1)}, \quad x > 0, \tag{3.2}$$

$$f_Y(y) = 2\alpha c y[1 + cy^2]^{-(\alpha+1)}, \quad y > 0. \tag{3.3}$$

In this section we apply the steps given in Sect. 2.1.

Step 1

The norm Z of the vector (X, Y) is given by $Z = (X^2 + Y^2)^{1/2}$. In Appendix A the pdf and hence cdf and rf are derived. Such functions are given by

$$f_Z(z) = 2\alpha(\alpha + 1)c^2 z^3[1 + cz^2]^{-(\alpha+2)}, \quad z > 0, \tag{3.4}$$

$$F_Z(z) = 1 - \alpha c z^2[1 + cz^2]^{-(\alpha+1)} - [1 + cz^2]^{-\alpha}, \quad z > 0, \tag{3.5}$$

$$R(z) = \alpha c z^2[1 + cz^2]^{-(\alpha+1)} + [1 + cz^2]^{-\alpha}, \quad z > 0. \tag{3.6}$$

From (3.4) and (3.6), the conditional density of Z_s^* given (c, α) is obtained (see Appendix B), as

$$\begin{aligned} g_1(z_s^*|c,\alpha) \propto \sum^{*} B_{i,j,l,s} c^{k_3}\alpha^{k_4}(\alpha+1) z_s^{*(2(k_1-j)+3)}(1+cz_s^{*2})^{-\alpha k_1 - k_1 + j - \alpha - 2} \\ \times z_{r:r}^{2(k_2-l)}(1+cz_{r:r}^2)^{-\alpha k_2 - k_2 + l}, \end{aligned} \tag{3.7}$$

where

$$\sum^{*} = \sum_{i=0}^{s-1}\sum_{j=0}^{k_1}\sum_{l=0}^{k_2}, \; B_{i,j,l,s} = (-1)^i \binom{s-1}{i}\binom{k_1}{j}\binom{k_2}{l},$$

with $k_1 = n - r + i - s$, $k_2 = s - i - (n - r) - 1$, $k_3 = 1 - j - l$ and $k_4 = -j - l$.

Suppose that the prior belief of the experimenter is given by the pdf
$\pi(c, \alpha) = \pi_1(c|\alpha)\pi_2(\alpha)$, $c|\alpha \sim \text{Gamma}(c_1, \alpha)$ and $\alpha \sim \text{Gamma}(c_2, c_3)$.
So that

$$\pi(c, \alpha) \propto \alpha^{c_1+c_2-1} c^{c_1-1} e^{-\alpha(c+c_3)}. \tag{3.8}$$

The likelihood function of (c, α) given $Z_{1:r}, \ldots, Z_{r:r}$ is given by

$$\begin{aligned} L(c,\alpha|z_{1:r},\ldots,z_{r:r}) &\propto [R(z_{r:r})]^{n-r}\prod_{i_1=1}^{r} f(z_{i_1}) \\ &= 2^r\alpha^r c^{2r}(\alpha+1)^r\left(\prod_{i_1}^{r} z_{i_1}\right)^3\left(\prod_{i_1}^{r}(1+cz_{i_1}^2)\right)^{-(\alpha+2)}\sum_{l_1}^{n-r}\binom{n-r}{l_1}\alpha^{n-r-l_1}c^{n-r-l_1} \\ &\quad\times z_{r:r}^{2(n-r-l_1)}(1+cz_{r:r}^2)^{-\alpha(n-r)-(n-r)+l_1}. \end{aligned} \tag{3.9}$$

Since the posterior density $\pi^*(c, \alpha|z_{1:r}, \ldots, z_{r:r}) \propto \pi(c, \alpha)L(c, \alpha|z_{1:r}, \ldots, z_{r:r})$, it follows, from (3.7) to (3.9) that

$$\begin{aligned} g_1(z_s^*|c,\alpha)\pi^*(c,\alpha|z_{1:r},\ldots,z_{r:r}) &= A\sum^{**} B^*_{i,j,l,s,l_1} c^{n+r+c_1-j-l-l_1} \\ &\times\alpha^{n+c_1+c_2-j-l-l_1-1}(\alpha+1)^{r+1}\left(\prod_{i_1}^{r} z_{i_1}\right)^3\left(\prod_{i_1}^{r}(1+cz_{i_1}^2)\right)^{-(\alpha+2)} z_s^{*(2(k_1-j)+3)} \\ &\times(1+cz_s^{*2})^{-\alpha k_1-k_1+j-\alpha-2} z_{r:r}^{2(s-i-l_1-l-1)}(1+cz_{r:r}^2)^{-\alpha(s-i-1)-s+i+l_1+l+1} \\ &\times\exp[-\alpha c-\alpha c_3], \end{aligned} \tag{3.10}$$

where A is a normalizing constant and

$$\sum^{**} = \sum^{*} \sum_{l_1=0}^{n-r}, \quad B^{*}_{i,j,l,s,l_1} = B_{i,j,l,s} \binom{n-r}{l_1}.$$

It then follows, from (1.6) and (3.10) that the predictive density function of Z_s^* is given by

$$g_1^*(z_s^*|z_{1:r},\ldots,z_{r:r}) = \int_0^\infty \int_0^\infty g_1(z_s^*|c,\alpha)\pi^*(c,\alpha|z_{1:r},\ldots,z_{r:r})dc\,d\alpha. \tag{3.11}$$

To obtain $(1-\tau)$ % BPI for Z_s^*, say (L_{1s}, U_{1s}), we solve the following two nonlinear equations, numerically,

$$P(Z_s^* > L_{1s}|z_{1:r},\ldots,z_{r:r}) = \int_{L_{1s}}^\infty g_1^*(z_s^*|z_{1:r},\ldots,z_{r:r})dz_s^* = 1-\frac{\tau}{2}, L_{1s} > z_{r:r}, \tag{3.12}$$

$$P(Z_s^* > U_{1s}|z_{1:r},\ldots,z_{r:r}) = \int_{U_{1s}}^\infty g_1^*(z_s^*|z_{1:r},\ldots,z_{r:r})dz_s^* = \frac{\tau}{2}, U_{1s} > z_{r:r}. \tag{3.13}$$

Step 2
By using the pdf (3.2) and its cdf, the predictive density function of X_s^* can be written as follows

$$g_1^*(x_s^*|x_1,\ldots,x_r) = \int_0^\infty \int_0^\infty g_1(x_s^*|c,\alpha)\pi^*(c,\alpha|x_1,\ldots,x_r)dc\,d\alpha, \tag{3.14}$$

where

$$\begin{aligned} g_1(x_s^*|c,\alpha)\pi^*(c,\alpha|x_1,\ldots,x_r) &= A_1 \sum_{i=0}^{s-1} B_{i,s} c^{c_1+r} \alpha^{c_1+c_2+r} \left(\prod_{i_1}^{r} x_{i_1}\right) \\ &\times \left(\prod_{i_1}^{r}(1+cx_{i_1}^2)\right)^{-(\alpha+1)} x_s^*(1+cx_s^{*2})^{(-\alpha(n-r+i-s+1)-1)}(1+cx_r^2)^{-\alpha(s-i-1)} \\ &\times \exp[-\alpha c - \alpha c_3], \end{aligned} \tag{3.15}$$

where A_1 is a normalizing constant and $B_{i,s} = (-1)^i \binom{s-1}{i}$.

To obtain $(1-\tau)$ % BPI for X_s^*, say (L_{2s}, U_{2s}), we solve the following two nonlinear equations, numerically,

$$P(X_s^* > L_{2s}|x_1,\ldots,x_r) = \int_{L_{2s}}^\infty g_1^*(x_s^*|x_1,\ldots,x_r)dx_s^* = 1-\frac{\tau}{2}, \quad L_{2s} > x_r, \tag{3.16}$$

$$P(X_s^* > U_{2s}|x_1,\ldots,x_r) = \int_{U_{2s}}^\infty g_1^*(x_s^*|x_1,\ldots,x_r)dx_s^* = \frac{\tau}{2}, \quad U_{2s} > x_r. \tag{3.17}$$

Step 3
From Steps 2 and 3, a $(1-\tau)$ % BPI for Y_s^* is $([L_{1s}^2 - L_{2s}^2]^{1/2}, [U_{1s}^2 - U_{2s}^2]^{1/2})$.

4 Two-sample prediction in case of (BVCR) distribution

In this case we apply the steps in Sect. 2.2 as follows

Step 1
Substituting from (3.4) and (3.6) in (1.9) and then using (3.8) and (3.9) we can write

$$\begin{aligned} g_2(z_s^*|c,\alpha)\pi^*(c,\alpha|z_{1:r},\ldots,z_{r:r}) &= A\sum^{**} B^*_{i,j,s,m} c^{n+r+c_1-l_1+k-j+1} \\ &\times \alpha^{n+c_1+c_2+k-j-l_1}(\alpha+1)^{r+1}\left(\prod_{i_1}^{r} z_{i_1}\right)^3 \left(\prod_{i_1}^{r}(1+cz_{i_1}^2)\right)^{-(\alpha+2)} z_s^{*(2(k-j)+3)} \\ &\times (1+cz_s^{*2})^{-\alpha k-k+j-\alpha-2} z_{r:r}^{2(n-r-l_1)}(1+cz_{r:r}^2)^{-\alpha(n-r)-(n-r)+l_1} \\ &\times \exp[-\alpha c-\alpha c_3], \end{aligned} \tag{4.1}$$

where

$$\sum^{**} = \sum_{i=0}^{s-1}\sum_{j=0}^{k}\sum_{l_1=0}^{n-r},\ B^*_{i,j,s,m} = (-1)^i\binom{s-1}{i}\binom{k}{j}\binom{n-r}{l_1},\quad k=m-s+i,$$

and A is a normalizing constant.

It then follows that the predictive density function of Z_s^* is given by

$$g_2^*(z_s^*|z_{1:r},\ldots,z_{r:r}) = \int_0^\infty\int_0^\infty g_1(z_s^*|c,\alpha)\pi^*(c,\alpha|z_{1:r},\ldots,z_{r:r})dc\,d\alpha. \tag{4.2}$$

To obtain $(1-\tau)$ % BPI for Z_s^*, say (L_{1s}, U_{1s}), we solve the following two nonlinear equations, numerically,

$$P(Z_s^* > L_{1s}|z_{1:r},\ldots,z_{r:r}) = \int_{L_{1s}}^\infty g_2^*(z_s^*|z_{1:r},\ldots,z_{r:r})dz_s^* = 1-\frac{\tau}{2},\quad L_{1s}>0, \tag{4.3}$$

$$P(Z_s^* > U_{1s}|z_{1:r},\ldots,z_{r:r}) = \int_{U_{1s}}^\infty g_2^*(z_s^*|z_{1:r},\ldots,z_{r:r})dz_s^* = \frac{\tau}{2},\quad U_{1s}>0. \tag{4.4}$$

Step 2
Using the pdf (3.2), its cdf and the same prior as in (3.8) the predictive density function of X_s^* is given by

$$g_2^*(x_s^*|x_1,\ldots,x_r) = \int_0^\infty\int_0^\infty g_2(x_s^*|c,\alpha)\pi^*(c,\alpha|x_1,\ldots,x_r)dc\,d\alpha, \tag{4.5}$$

where

$$\begin{aligned} g_2(x_s^*|c,\alpha)\pi^*(c,\alpha|x_1,\ldots,x_r) &= A_1\sum_{i=0}^{s-1} B_{i,s}c^{r+c_1}\alpha^{c_1+c_2+r}\left(\prod_{i_1}^{r} x_{i_1}\right) \\ &\times \left(\prod_{i_1}^{r}(1+cx_{i_1}^2)\right)^{-(\alpha+1)} x_s^*(1+cx_s^{*2})^{(-\alpha(m+i-s+1)-1)}(1+cx_r^2)^{-\alpha(n-r)} \\ &\times \exp[-\alpha c-\alpha c_3], \end{aligned} \tag{4.6}$$

where A_1 is a normalizing constant and

$$B_{i,s} = (-1)^i \binom{s-1}{i}.$$

To obtain $(1-\tau)$ % BPI for X_s^*, say (L_{2s}, U_{2s}), we solve the following two nonlinear equations, numerically,

$$P(X_s^* > L_{2s}|x_1,\ldots,x_r) = \int_{L_{2s}}^{\infty} g_2^*(x_s^*|x_1,\ldots,x_r)dx_s^* = 1-\frac{\tau}{2}, \quad L_{2s} > 0, \tag{4.7}$$

$$P(X_s^* > U_{2s}|x_1,\ldots,x_r) = \int_{U2s}^{\infty} g_2^*(x_s^*|x_1,\ldots,x_r)dx_s^* = \frac{\tau}{2}, \quad U_{2s} > 0. \tag{4.8}$$

Step 3
From Steps 2 and 3, a $(1-\tau)$ % BPI for Y_s^* is $([L_{1s}^2 - L_{2s}^2]^{1/2}, [U_{1s}^2 - U_{2s}^2]^{1/2})$.

5 Numerical example

In this section we follow the steps

Table 1 One-sample prediction 95 % BPI's for z_s^*, y_s^* and x_s^*, $s = 1, 2, 3$

r	c_1, c_2, c_3	α, c		z_1^*	z_2^*	z_3^*
			1	9,743	9,865	9,897
10			2	(3.9064, 5.6565)	(4.4398, 6.6373)	(4.8985, 7.8809)
			3	1.7501	2.1975	2.9824
			1	9,633	9,742	9,799
20	1, 1.5, 2	0.76, 1.3	2	(3.8761, 5.4953)	(4.4523, 6.4451)	(4.8723, 7.1942)
			3	1.6192	1.9928	2.3219
			1	9,580	9,612	9,687
45			2	(3.7670, 4.8779)	(4.3687, 6.1819)	(4.7585, 6.8615)
			3	1.1109	1.8132	2.1030
r	c_1, c_2, c_3	α, c		x_1^*	x_2^*	x_3^*
			1	9,611	9,841	9,884
10			2	(2.4110, 3.0393)	(2.7269, 3.7051)	(3.1654, 4.4564)
			3	0.6283	0.9782	1.2910
			1	9,588	9,623	9,716
20	1, 1.5, 2	0.76, 1.3	2	(2.3720, 2.9688)	(2.5971, 3.4690)	(3.0912, 4.1933)
			3	0.5968	0.8719	1.1021
			1	9,541	9,592	9,610
45			2	(2.2891, 2.7694)	(2.4870, 3.2379)	(2.9714, 3.9531)
			3	0.4803	0.7509	0.9817
r	c_1, c_2, c_3	α, c		y_1^*	y_2^*	y_3^*
			1	9,740	9,804	9,867
10			2	(3.0736, 4.7706)	(3.5036, 5.5069)	(3.7389, 6.4999)
			3	1.6970	2.0033	2.7610
			1	9,689	9,708	9,768
20	1, 1.5, 2	0.76, 1.3	2	(3.0655, 4.6243)	(3.6164, 5.4319)	(3.7661, 5.8457)
			3	1.5588	1.8154	2.0796
			1	9,588	9,650	9,712
45			2	(2.9917, 4.0155)	(3.5917, 5.2661)	(3.7167, 5.6083)
			3	1.0238	1.6744	1.8916

1, Number of samples which cover the BPI's from 10,000 samples; 2, BPI's for z_s^*, x_s^*, y_s^*; 3, length of the BPI's

Table 2 Two-sample prediction 95 % BPI's for z_s^*, y_s^* and x_s^*, $s = 1, 2, 3$

r	c_1, c_2, c_3	α, c		z_1^*	z_2^*	z_3^*
			1	9,698	9,778	9,865
10			2	(1.4319, 2.1823)	(2.2627, 3.4651)	(3.3804, 5.2912)
			3	0.7501	1.2014	1.9108
			1	9,579	9,645	9,703
20	1, 1.5, 2	0.76, 1.3	2	(1.4053, 2.0401)	(2.2816, 3.1608)	(3.2239, 4.5159)
			3	0.6348	0.8792	1.2920
			1	9,498	9,514	9,639
45			2	(1.7919, 1.9721)	(2.2502, 3.0318)	(3.1705, 4.1634)
			3	0.1801	0.7816	0.9925
r	c_1, c_2, c_3	α, c		x_1^*	x_2^*	x_3^*
			1	9,753	9,799	9,836
10			2	(0.8941, 1.2541)	(1.3730, 1.9512)	(2.1106, 2.9016)
			3	0.3601	0.5782	0.7910
			1	9,655	9,698	9,713
20	1, 1.5, 2	0.76, 1.3	2	(0.8714, 1.2152)	(1.2537, 1.6696)	(2.0943, 2.7255)
			3	0.3438	0.4159	0.63111
			1	9,581	9,630	9,703
45			2	(0.8680, 0.6083)	(1.2301, 1.6013)	(2.0805, 2.5665)
			3	0.2403	0.3709	0.5861
r	c_1, c_2, c_3	α, c		y_1^*	y_2^*	y_3^*
			1	9,863	9,870	9,949
10			2	(1.1184, 1.7859)	(1.7985, 3.2524)	(2.6405, 4.4264)
			3	0.6676	1.4539	1.7840
			1	9,797	9,813	9,901
20	1, 1.5, 2	0.76, 1.3	2	(1.1025, 1.6387)	(1.9062, 2.6839)	(2.4510, 3.6011)
			3	0.5362	0.7776	1.1496
			1	9,678	9,690	9,762
45			2	(1.0681, 1.5677)	(1.8842, 2.5744)	(2.3924, 3.2783)
			3	0.4816	0.6902	0.8859

1, Number of samples which cover the BPI's from 10,000 samples; 2, BPI's for z_s^*, x_s^*, y_s^*; 3, length of the BPI's

One-sample prediction

1. given the set of prior parameters, generate the parameters (c, α),
2. generate θ_j from Gamma $(\alpha, 1/c)$ and u_i from U(0, 1), $j = 1, 2, \ldots, n$ and $i = 1, 2$,
3. the bivariate sample will be in the form $(x_j, y_j) = (\sqrt{-\ln(u_1)/\theta_j}, \sqrt{-\ln(u_2)/\theta_j})$, $j = 1, 2, \ldots, n$,
4. for the first r informative observations from the previous sample, compute the norms, $z_1, z_2, \ldots, z_r$,
5. based on the first r ordered norms, $z_{1:r}, z_{2:r}, \ldots, z_{r:r}$, compute the BPI's for z_s^*, $s = 1, 2, \ldots, n - r$, say (L_{1s}^*, U_{1s}^*) as mentioned in step 1 of Sect. 3,
6. based on $x_1 < x_2 < \cdots < x_r$, compute the BPI's for x_s^*, $s = 1, 2, \ldots, n - r$, say (L_{2s}^*, U_{2s}^*) as mentioned in Step 2 in Sect. 3,
7. from Steps 5 and 6, the BPI for y_s^*, $s = 1, 2, \ldots, n - r$ is $(L_{2s}, [L_{1s}^2 - L_{2s}^2]^{1/2})$, $(U_{2s}, [U_{1s}^2 - U_{2s}^2]^{1/2})$.

Two-sample prediction

1. based on the first r ordered norms, $z_{1:r}, z_{2:r}, \ldots, z_{r:r}$, compute the BPI's for z_s^*, $s = 1, 2, \ldots, m$, say (L_{1s}^*, U_{1s}^*) as mentioned in step 1 in Sect. 4,
2. based on $x_1 < x_2 < \cdots < x_r$, compute the BPI's for x_s^*, $s = 1, 2, \ldots, m$, say (L_{2s}^*, U_{2s}^*) as mentioned in Step 2 in Sect. 4,
3. from Steps 1 and 2, the BPI for y_s^*, $s = 1, 2, \ldots, m$ is $(L_{2s}, [L_{1s}^2 - L_{2s}^2]^{1/2})$, $(U_{2s}, [U_{1s}^2 - U_{2s}^2]^{1/2})$.

In Tables 1 and 2, a 95 % BPI's are computed in case of the one- and two-sample predictions, respectively, with the same parameters c, α, hyperparameters c_1, c_2, c_3 and using informative samples of different sizes, r.

6 Results and discussion

In Tables 1 and 2 we take different sizes for the informative sample, 10, 20 and 45 and predict the first three future observations.

In these tables, we observe that

1. the length of the BPI's and the number of samples which cover these intervals increase by increasing s and decrease by increasing the informative sample size,
2. the results become better as the informative sample size r gets larger.
3. In all cases, the simulated percentage coverages are at least 95 %.
4. There is no particular reason for choosing the hyperparameters (c_1, c_2, c_3) as (1, 1.5, 2).
5. If the hyperparameters are unknown, they can be estimated by using the empirical Bayes method (see Maritz and Lwin [14]) or the hierarchical method (see Bernardo and Smith [12]).

Appendix A

Proof of Equations (3.4)–(3.6)

From the joint density function of the random variables X and Y which is given by (3.1) and using the transforms $X = Z\cos\Theta$ and $Y = Z\sin\Theta$ we get the joint density function of the random variables Z and Θ in the form

$$f_{Z,\Theta}(z,\theta) = 4\alpha(\alpha+1)c^2z^3\sin\theta\cos\theta[1+cz^2]^{-(\alpha+2)}, \quad z>0, \quad 0\le\theta\le\pi/2. \tag{A.1}$$

Integrating (A.1) with respect to θ, we get the density function of Z as in (3.4).
The (cdf) of the random variable Z is given by

$$F_Z(z) = 2\alpha(\alpha+1)c^2\int_0^z u^3[1+cu^2]^{-(\alpha+2)}du. \tag{A.2}$$

The cdf (3.5) is obtained by integrating by parts the integral in (A.2). The rf is then obtained as in (3.6), since $R(z) = 1 - F_Z(z)$.

Appendix B

Proof of Equation (3.7)

From (1.5), (3.4) and (3.6) we have

$$\begin{aligned} g_1(z_s^*|c,\alpha) &\propto [R(z_{r:r}) - R(z_s^*)]^{(s-1)}[R(z_s^*)]^{n-r-s}[R(z_{r:r}^*)]^{-(n-r)}f_Z(z_s^*) \\ &= \sum_{i=0}^{s-1}(-1)^i\binom{s-1}{i}[R(z_s^*)]^{n-r-s+i}[R(z_{r:r})]^{s-i-(n-r)-1}f_Z(z_s^*), \end{aligned} \tag{B.1}$$

where the reliability function $R(z)$, given by (3.6) yields

$$^k = \sum_{i=0}^{k}\binom{k}{i}c^{k-i}\alpha^{k-i}z^{2(k-i)}(1+cz^2)^{-\alpha k-k+i}. \tag{B.2}$$

Using (B.2) and (3.4) in (B.1) we get (3.7).

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