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RESEARCH ARTICLE

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A family of symmetric second degree semiclassical forms of class $s = 2$

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Abstract A regular form (linear functional) u is called semiclassical, if there exist two nonzero polynomials Φ and Ψ such that $(\Phi u)' + \Psi u = 0$ with Φ monic and $\deg \Psi > 0$. Such a form is said to be of second degree if there are polynomials B, C and D such that its Stieltjes function $S(u)$ satisfies $BS^2(u) + CS(u) + D = 0$. Recently, all the symmetric second degree semiclassical forms of class $s \leq 1$ were determined. In this paper, by means of the quadratic decomposition, we determine all the symmetric semiclassical forms of class $s = 2$, which are also of second degree when Φ vanishes at zero. These forms generalize those of class $s = 1$.

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الملخص

تسمى الصيغة (الدالي الخطي) المنتظم u نصفَ كلاسيكية، إذا وجدت كثيرتا حدود غير صفريتين ϕ وَ ψ تحققان $(\phi u)' + \psi u = 0$، حيث ϕ واحدي و درجة $\psi > 0$. يقال أن مثل هذه الصيغة من الدرجة الثانية إذا وجدت كثيرات حدود B وَ C وَ D، بحيث تحقق دالة ستيلتجس $S(u)$ الخاصة بها $BS^2(u) + CS(u) + D = 0$. تم مؤخراً تحديد جميع الصيغ نصف الكلاسيكية المتناظرة من الدرجة الثانية للصف $s \leq 1$. في هذه الورقة، وباستخدام التحليل التربيعي، نحدد جميع الصيغ نصف الكلاسيكية المتناظرة من الدرجة الثانية للصف $s = 2$، عندما تتلاشى ϕ عند الصفر. تعمم هذه الصيغ تلك للصف $s = 1$.

1 Introduction and basic background

Second degree forms have been introduced since 1995 [13]. These forms are characterized by the fact that their formal Stieltjes function $S(u)$ satisfies a quadratic equation $BS^2(u) + CS(u) + D = 0$ where $B \neq 0$ and C are polynomials and D is a polynomial defined in terms of the previous ones. They have been studied in [7,16] and [17] in the framework of the orthogonality on several intervals. Later on, in [12] and [13] an algebraic approach to such second degree forms as an extension of the Tchebychev forms is given. Notice that every second degree form is semiclassical, i.e., there exist two polynomials $\Phi(x)$ and $\Psi(x)$, where $\Phi(x)$ is monic and $deg \Psi > 0$, such that $(\Phi(x)u)' + \Psi(x)u = 0$ [11,13]. In [3], the authors determine all the classical forms (i.e., semiclassical of class $s = 0$) which are of second degree. Hermite, Laguerre and Bessel are not of second degree. Only Jacobi forms which satisfy a certain condition possess this property. Later on, in [2], Beghdadi determines all the symmetric second degree semiclassical forms of class $s = 1$.

The aim of this work is to approach the problem of determining all the symmetric semiclassical forms of class $s = 2$ which are of second degree when $\Phi(0) = 0$. The first section is devoted to the preliminary

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results and notations used in the sequel. In the second section, we prove that a symmetric semiclassical form u is a second degree if and only if its odd part $x\sigma u$ is also second degree form. Using this result, we give all the forms which we look for. Three canonical cases for the polynomial Φ arise: $\Phi(x) = x^2$, $\Phi(x) = x^4$ and $\Phi(x) = x^2(x^2 - 1)$. As it turned out, we obtained explicitly a family of nonsymmetric second degree semiclassical forms of class $s = 1$ which generalize the classical ones.

In the sequel, we will recall some basic definitions and results. The field of complex numbers is denoted by $\mathbb{C}$. The vector space of polynomials with coefficients in $\mathbb{C}$ is represented as $\mathcal{P}$ and its dual space is represented as $\mathcal{P}'$. We denote by $\langle u, f\rangle$ the effect of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $(u)_n := \langle u, x^n\rangle$, $n \geq 0$, the moments of u. For any linear form u, any polynomial h, let $Du = u'$ and hu be the forms defined by duality:

$$\langle u', f\rangle := -\langle u, f'\rangle, \quad \langle hu, f\rangle := \langle u, hf\rangle, \quad f \in \mathcal{P}. \tag{1}$$

We recall the definition of right-multiplication of a form by a polynomial:

$$(uh)(x) := \left\langle u, \frac{xh(x) - \xi h(\xi)}{x - \xi} \right\rangle, \quad u \in \mathcal{P}', \quad h \in \mathcal{P}. \tag{2}$$

By duality, we obtain the Cauchy's product of two forms:

$$\langle uv, f\rangle := \langle u, vf\rangle, \quad u, v \in \mathcal{P}', f \in \mathcal{P}. \tag{3}$$

Consequently,

$$(uv)_n = \sum_{i+j=n} (u)_i (v)_j, \quad n \geq 0. \tag{4}$$

We define [14] the form $(x - c)^{-1}u$, $c \in \mathbb{C}$, through

$$\langle (x - c)^{-1}u, f\rangle := \langle u, \theta_c f\rangle \tag{5}$$

with

$$\big(\theta_c f\big)(x) = \frac{f(x) - f(c)}{x - c}, \quad u \in \mathcal{P}', f \in \mathcal{P}. \tag{6}$$

From the definitions, it results $(u\theta_0 f)(x) = \langle u, \frac{f(x)-f(\xi)}{x-\xi}\rangle$, $u \in \mathcal{P}'$, $f \in \mathcal{P}$.

We introduce the operator $\sigma : \mathcal{P} \longrightarrow \mathcal{P}$ defined by $(\sigma f)(x) = f(x^2)$ for all $f \in \mathcal{P}$. By transposition, we define σu:

$$\langle \sigma u, f\rangle = \langle u, \sigma f\rangle \quad u \in \mathcal{P}', f \in \mathcal{P}. \tag{7}$$

Consequently, $(\sigma u)_n = (u)_{2n}$.

We will also use the so-called formal Stieltjes function associated with $u \in \mathcal{P}'$ that is defined by

$$S(u)(z) = -\sum_{n\geq 0} \frac{(u)_n}{z^{n+1}}. \tag{8}$$

The following auxiliary results will be used in the sequel [14,15].

Lemma 1.1 *For any $f \in \mathcal{P}$ and $u, v \in \mathcal{P}'$*

$$(fu)' = fu' + f'u, \tag{9}$$

$$(u\theta_0 f)(x) = (\theta_0(uf))(x), \tag{10}$$

$$f(x)(\sigma u) = \sigma(f(x^2)u), \tag{11}$$

$$\sigma u' = 2(\sigma(xu))', \tag{12}$$

$$\sigma(uv) = (\sigma u)(\sigma v) + x^{-1}(\sigma(xu)\sigma(xv)), \tag{13}$$

$$S(uv)(z) = -zS(u)(z)S(v)(z). \tag{14}$$

The form u is called regular if there exists a polynomial sequence $\{B_n\}_{n\geq 0}$, $\deg B_n = n$, such that $\langle u, B_n B_m\rangle = r_n\delta_{nm}$, $r_n \neq 0$, $n \geq 0$.

In this case $\{B_n\}_{n\geq 0}$ is said to be orthogonal with respect to u. It satisfies the recurrence relation (see, for instance, the monograph by Chihara [4])

$$\begin{aligned} B_0(x) = 1, \quad B_1(x) = x - \beta_0, \\ B_{n+2}(x) = (x - \beta_{n+1})B_{n+1}(x) - \gamma_{n+1}B_n(x), \quad n \geq 0. \end{aligned} \tag{15}$$

The regularity of u means that we must have $\gamma_{n+1} \neq 0$, $n \geq 0$.

In this paper, we suppose that the forms are normalized (i.e., $(u)_0 = 1$).

Definition 1.2 [13] The form u is called a second degree form if it is regular and if there exist two polynomials B and C such that

$$B(z)S^2(u)(z) + C(z)S(u)(z) + D(z) = 0, \tag{16}$$

where D is a polynomial depending on B, C, and u given by

$$D(z) = (u\theta_0 C)(z) - (u^2\theta_0^2 B)(z). \tag{17}$$

The regularity of u means that we must have $B \neq 0$; $C^2 - 4BD \neq 0$ and $D \neq 0$.
The following expressions are equivalent to (16), [13]:

$$B(x)u^2 = xC(x)u, \quad \langle u^2, \theta_0 B\rangle = \langle u, C\rangle. \tag{18}$$

In the sequel, we shall suppose B to be monic.
The polynomials B and C, given in (16) or by (18), are not unique, because B and C can be multiplied by an arbitrary polynomial. If in (16) the polynomials B , C and D are coprime, then the pair (B, C) is called a primitive pair. The primitive pair is unique.
Let us recall that a form u is called semiclassical when it is regular and there exist two polynomials Φ and Ψ, where $\Phi(x)$ is monic and $\deg(\Psi) \geq 1$, such that

$$(\Phi u)' + \Psi u = 0. \tag{19}$$

The class of the semiclassical form v is $s = \max(\deg\Psi - 1, \deg\Phi - 2)$ if and only if the following condition is satisfied

$$\prod_c \left(|\Phi'(c) + \Psi(c)| + \left|\left\langle u, \theta_c\Psi + \theta_c^2\Phi\right\rangle\right|\right) > 0, \tag{20}$$

where c goes over the zeros set of Φ [14].

When $s = 0$, u is called a classical form.

As a result, if u is a semiclassical form of class s satisfying (19), then the shifted form $\hat{u} = (h_{a^{-1}} \circ \tau_{-b})u$, $a \in \mathbb{C}^*$, $b \in \mathbb{C}$ is of class s satisfying the equation

$$(\hat{\Phi}\hat{u})' + \hat{\Psi}\hat{u} = 0 \tag{21}$$

with

$$\hat{\Phi}(x) = a^{-t}\Phi(ax + b), \quad \hat{\Psi}(x) = a^{1-t}\Psi(ax + b), \quad t = deg(\Phi) \tag{22}$$

where, for each polynomial f

$$\langle \tau_b u, f\rangle := \langle u, \tau_{-b} f\rangle := \langle u, f(x + b)\rangle, \langle h_a u, f\rangle := \langle u, h_a f\rangle := \langle u, f(ax)\rangle.$$

A second degree form u is a semiclassical form and satisfies (19), with [13]

$$\begin{aligned} k\Phi(x) &= B(x)(C^2(x) - 4B(x)D(x)) \\ k\Psi(x) &= -\frac{3}{2}B(x)(C^2(x) - 4B(x)D(x))', k \neq 0, \end{aligned} \tag{23}$$

where k is a normalization factor.

The second degree character is kept by shifting. Indeed, if u is a second degree form satisfying (18), then $\hat{u}$ is also second degree form [13]. It satisfies

$$\hat{B}(x)\hat{u}^2 = x\hat{C}(x)\hat{u}, \quad \langle \hat{u}^2, \theta_0 \hat{B}\rangle = \langle \hat{u}, \hat{C}\rangle. \tag{24}$$

with

$$\hat{B}(x) = a^{-r}B(ax+b), \quad \hat{C}(x) = a^{1-r}C(ax+b), \quad r = deg(B). \tag{25}$$

Lemma 1.3 [2] *Let u be a second degree semiclassical form satisfying (19)–(20). The class of u is* $s = \deg \Phi - 2 = \deg \Psi - 1$.

We finish this section by recalling this important result.

Theorem 1.4 [3] *Among the classical forms, only the Jacobi forms $\mathcal{J}(k-\frac{1}{2}, l-\frac{1}{2})$ are second degree forms, provided $k+l \geq 0$, $k, l \in \mathbb{Z}$ which satisfy*

$$\left((x^2-1)\mathcal{J}\left(k-\frac{1}{2}, l-\frac{1}{2}\right)\right)' + \left(-(k+l+1)x + k - l\right)\mathcal{J}\left(k-\frac{1}{2}, l-\frac{1}{2}\right) = 0.$$

2 Symmetric second degree semiclassical forms

2.1 Algebraic properties

We recall that a form u is called symmetric if $(u)_{2n+1} = 0, n \geq 0$. The conditions $(u)_{2n+1} = 0, n \geq 0$, are equivalent to the fact that the corresponding sequence of monic orthogonal polynomials (MOPS) $\{B_n\}_{n\geq 0}$ satisfies the recurrence relation (15) with $\beta_n = 0, n \geq 0$ [4].

In addition, the sequence $\{B_n\}_{n\geq 0}$ has the following quadratic decomposition

$$B_{2n}(x) = P_n(x^2), \quad B_{2n+1}(x) = xR_n(x^2), \quad n \geq 0. \tag{26}$$

The sequences $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ are respectively orthogonal with respect to σu and $x\sigma u$. We have for instance:

$$\begin{aligned} &P_{n+2}(x) = (x - \beta^P_{n+1})P_{n+1}(x) - \gamma^P_{n+1}P_n(x), \quad n \geq 0, \\ &P_1(x) = x - \beta^P_0, \quad P_0(x) = 1, \end{aligned} \tag{27}$$

with

$$\beta^P_0 = \gamma_1, \quad \beta^P_{n+1} = \gamma_{2n+2} + \gamma_{2n+3}, \quad \gamma^P_{n+1} = \gamma_{2n+1}\gamma_{2n+2}, \quad n \geq 0. \tag{28}$$

We have the following characterisations.

Proposition 2.1 [2] *The even part σu of a symmetric second degree form u is also second degree form.*

Proposition 2.2 *Let u be a regular and symmetric form. The following statements are equivalent:*

(a) *u is a second degree form*
(b) *The odd part $x\sigma u$ of u is a second degree form.*

Proof "$(a) \Longrightarrow (b)$" According to Proposition 2.1 and the fact that the multiplication by a polynomial preserves the quadratic property.

"$(b) \Longrightarrow (a)$" We denote by v the normalized form defined by $\gamma_1 v = x\sigma u$. We suppose that $x\sigma u$ is a second degree form. Then there exist two polynomials B_1 and C_1 such that

$$B_1(z)S^2(v)(z) + C_1(z)S(v)(z) + D_1(z) = 0, \tag{29}$$

where

$$D_1(z) = (v\theta_0 C_1)(z) - (v^2\theta_0^2 B_1)(z). \tag{30}$$

From (8) and the fact that u is a symmetric form, we have

$$S\big(v\big)(z^2) = \gamma_1^{-1} z S\big(u\big)(z) + \gamma_1^{-1}. \tag{31}$$

Make a change of variable $z \longrightarrow z^2$ in (29), multiply by γ_1^2 and substitute (31) in the resulting equation, we get (16) with

$$\begin{cases} B(z) = z^2 B_1(z^2), \\ C(z) = 2zB_1(z^2) + \gamma_1 z C_1(z^2), \\ D(z) = B_1(z^2) + \gamma_1 C_1(z^2) + \gamma_1^2 D_1(z^2). \end{cases} \tag{32}$$

From (6), we have $\big(u\theta_0(\xi C_1(\xi^2))\big)(z) = \big(uC_1(\xi^2)\big)(z)$. Using (2), we obtain

$$\begin{aligned}\big(u\theta_0(\xi C_1(\xi^2))\big)(z) &= \left\langle u, \frac{zC_1(z^2) - \xi C_1(\xi^2)}{z-\xi}\right\rangle \\ &= \left\langle u, z\xi(\theta_{z^2}C_1)(\xi^2) + \frac{z^2C_1(z^2) - \xi^2 C_1(\xi^2)}{z^2-\xi^2}\right\rangle.\end{aligned}$$

But $\langle u, z\xi(\theta_{z^2}C_1)(\xi^2)\rangle = 0$ since u is a symmetric form, then

$$\begin{aligned}\big(u\theta_0(\xi C_1(\xi^2))\big)(z) &= \left\langle u, \frac{z^2C_1(z^2) - \xi^2 C_1(\xi^2)}{z^2-\xi^2}\right\rangle \\ &= \left\langle \sigma u, \xi\frac{C_1(z^2) - C_1(\xi)}{z^2-\xi} + C_1(z^2)\right\rangle,\end{aligned}$$

by virtue of (7). Therefore,

$$\big(u\theta_0(\xi C_1(\xi^2))\big)(z) = \gamma_1(v\theta_0 C_1)(z^2) + C_1(z^2). \tag{33}$$

Replacing B_1 by C_1 in (33), we get

$$\big(u\theta_0(\xi B_1(\xi^2))\big)(z) = \gamma_1(v\theta_0 B_1)(z^2) + B_1(z^2). \tag{34}$$

From (6), we have $(u^2\theta_0^2(\xi^2 B(\xi^2)))(z) = (u^2 B(\xi^2))(z)$ and by (13), we have $\sigma u^2 = (\sigma u)^2$ because u is a symmetric form. Then, using the same process described above with (u^2, B_1) instead of (u, C_1), we get

$$\big(u^2\theta_0^2\left(\xi^2 B(\xi^2)\right)\big)(z) = B_1(z^2) + \left\langle \xi(\sigma u)^2, \frac{B_1(z^2) - B_1(\xi)}{z^2-\xi}\right\rangle.$$

But, from (6), we have

$$\frac{B_1(z^2) - B_1(\xi)}{z^2-\xi} = \frac{z^2(\theta_0 B_1)(z^2) - \xi(\theta_0 B_1)(\xi)}{z^2-\xi} = (\theta_0 B_1)(z^2) + \xi\frac{(\theta_0 B_1)(z^2) - (\theta_0 B_1)(\xi)}{z^2-\xi}.$$

Then, we get

$$\big(u^2\theta_0^2\left(\xi^2 B_1(\xi^2)\right)\big)(z) = B_1(z^2) + 2\gamma_1(\theta_0 B_1)(z^2) + \left\langle \xi^2(\sigma u)^2, \frac{(\theta_0 B_1)(z^2) - (\theta_0 B_1)(\xi)}{z^2-\xi}\right\rangle, \tag{35}$$

since $\langle(\sigma u)^2, \xi\rangle = \langle u^2, \xi^2\rangle = 2\gamma_1$, by (4) and (15).

Now, using (4) and taking into account $x\sigma u = \gamma_1 v$, we prove that

$$\xi^2(\sigma u)^2 = (\xi\sigma u)^2 + 2\xi^2\sigma u = \gamma_1^2 v^2 + 2\gamma_1\xi v.$$

Then, (35) becomes

$$(u^2\theta_0^2(\xi^2 B(\xi^2)))(z) = B_1(z^2) + 2\gamma_1(\theta_0 B_1)(z^2) + \gamma_1^2(v^2\theta_0^2 B_1)(z^2) + 2\gamma_1\left\langle v, \frac{\xi(\theta_0 B_1)(z^2) - \xi(\theta_0 B_1)(\xi)}{z^2-\xi}\right\rangle.$$

But,

$$\left\langle v, \frac{\xi(\theta_0 B_1)(z^2) - \xi(\theta_0 B_1)(\xi)}{z^2 - \xi} \right\rangle = -(\theta_0 B_1)(z^2) + (v\theta_0 B_1)(z^2).$$

Therefore, we deduce

$$\left(u^2\theta_0^2(\xi^2 B_1(\xi^2))\right)(z) = B_1(z^2) + \gamma_1^2 \left(v^2\theta_0^2 B_1\right)(z^2) + 2\gamma_1 \left(v\theta_0 B_1\right)(z^2). \tag{36}$$

Thus, on account of (30), (32)–(34) and (36), we conclude that the polynomials B, C and D given by (32) verify the relation (17).
Hence u is also a second degree form. □

Using Proposition 2.1, Beghdadi gives all the symmetric second degree semiclassical forms of class $s = 1$:

Theorem 2.3 [2] *Among the symmetric semiclassical forms of class $s = 1$, only the forms denoted by $\mathcal{I}(k - \frac{1}{2}, l - \frac{1}{2})$ are second degree forms, provided $k + l \geq 0, l \neq 0, k, l \in \mathbb{Z}$ which satisfy*

$$\left(x(x^2-1)\mathcal{I}\left(k - \frac{1}{2}, l - \frac{1}{2}\right)\right)' + \left(-2(k+l+1)x^2 + 2l + 1\right)\mathcal{I}\left(k - \frac{1}{2}, l - \frac{1}{2}\right) = 0.$$

The form $\mathcal{I} = \mathcal{I}(k - \frac{1}{2}, l - \frac{1}{2})$ possesses the following representation [2]:

$$\langle \mathcal{I}, f \rangle = \frac{\Gamma(k+l+1)}{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(l+\frac{1}{2}\right)} \int_{-1}^{1} \frac{x^{2l}(1-x^2)^k}{\sqrt{1-x^2}} f(x)dx, \quad k \geq 0, l > 0.$$

Remark Unfortunately, we are not able to determine all the symmetric second degree semiclassical forms of class $s = 2$ by Proposition 2.1, especially because σu is among the second degree semiclassical forms of class $s = 1$ which are unknown.

2.2 Symmetric second degree semiclassical forms of class $s = 2$: case $\Phi(0) = 0$

Let us begin with an example $\mathcal{V}$ among the symmetric forms which is a second degree semiclassical form of class $s = 2$ satisfying (19) with $\Phi(0) = 0$. This example is given in [1]. The form $\mathcal{V}$ satisfies (16) with

$$B(z) = z^4(z^2 - 1), \quad C(z) = 2z^3(z^2 - 1), \quad D(z) = z^2(z^2 - 1) - \lambda^2, \tag{37}$$

and (19) with

$$\Phi(x) = x^2(x^2 - 1), \quad \Psi(x) = -x^3. \tag{38}$$

The corresponding MOPS of $\mathcal{V}$ satisfies (15) with

$$\gamma_1 = \lambda, \quad \gamma_{2n+2} = \frac{1}{4^{1-\frac{\delta_{n,0}}{2}}} \frac{1 - 2(n+1)\lambda}{1 - 2n\lambda}, \quad \gamma_{2n+3} = \frac{1}{4} \frac{1 - 2n\lambda}{1 - 2(n+1)\lambda}, \quad n \geq 0. \tag{39}$$

Now, we state the following result which is essential for this work.

Proposition 2.4 [2] *Let u be a symmetric semiclassical form of class s, satisfying (19). If s is even then Φ is even and Ψ is odd. If s is odd then Φ is odd and Ψ is even.*

In the sequel, we suppose $s = 2$, u is symmetric, and $\Phi(0) = 0$. Then, according to the above proposition, u satisfies (19) with

$$\Phi(x) = c_4x^4 + c_2x^2, \quad \Psi(x) = a_3x^3 + a_1x, \quad |c_4| + |a_3| \neq 0.$$

Then, using the fact that Φ is monic and the semiclassical character is kept by shifting, we distinguish three canonical cases for Φ: $\Phi(x) = x^2$, $\Phi(x) = x^4$, $\Phi(x) = x^2(x^2 - 1)$.

First case: $\Phi(x) = x^2$

According to Lemma 1.3, this case is excluded because $s = 2 \neq \deg \Phi - 2$.

Second case: $\Phi(x) = x^4$

Let $\Psi(x) = a_3x^3 + a_1x$. After multiplying (19) by x, applying the operator σ and using (11)–(12), we obtain

$$(x^2(x\sigma u))' + \frac{1}{2}\big((a_3 - 1)x + a_1\big)(x\sigma u) = 0.$$

Then $x\sigma u = \gamma_1\mathcal{B}(\alpha)$ where $\mathcal{B}(\alpha)$ is the classical Bessel form with $a_3 = -4\alpha + 1$ and $a_1 = -4$. Recall that the form $\mathcal{B}(\alpha)$ satisfies (19) with

$$\Phi(x) = x^2, \quad \Psi(x) = -2(\alpha x + 1), \quad \alpha \neq -\frac{n}{2}, \quad n \in \mathbb{N}.$$

Since $\mathcal{B}(\alpha)$ is not a second degree form [3], according to Proposition 2.2, we conclude that u is not a second degree form.

Third case: $\Phi(x) = x^2(x^2 - 1)$

This case is mentioned in [6] and [18], when the authors gave all the symmetric semiclassical forms of class $s = 2$ with $\Phi(x) = x^2(x^2 - 1)$. These forms satisfy

$$(x^2(x^2 - 1)u)' + \big((-2\alpha - 2\beta - 3)x^3 + (2\beta + 1)x\big)\, u = 0, \quad \gamma_1(\alpha + \beta + 1) \neq \beta. \tag{40}$$

Taking into account [18], we have

$$\begin{cases} \gamma_1 = \lambda, \quad \gamma_{2n+2} = \dfrac{(n + \beta + 1)(n + \alpha + \beta + 1)d_{n+1}(\lambda)}{(2n + \alpha + \beta + 1)(2n + \alpha + \beta + 2)d_n(\lambda)}, n \geq 0, \\ \gamma_{2n+3} = \dfrac{(n + 1)(n + \alpha + 1)d_n(\lambda)}{(2n + \alpha + \beta + 2)(2n + \alpha + \beta + 3)d_{n+1}(\lambda)}, n \geq 0. \end{cases} \tag{41}$$

with

$$d_n(\lambda) = \begin{cases} \lambda\dfrac{\Gamma(\beta + 1)\Gamma(\alpha + \beta + 1)\Gamma(n + 1)\Gamma(n + \alpha + 1)}{\Gamma(\alpha + 1)\Gamma(n + \beta + 1)\Gamma(n + \alpha + \beta + 1)} + \dfrac{\beta}{\alpha + \beta + 1} - \lambda, \beta(\alpha + \beta + 1) \neq 0, n \geq 0, \\ 1 - \lambda \sum_{k=0}^{n-1} \dfrac{(2k + 1)\Gamma(\alpha + k + 1)\Gamma(\beta + 1)}{\Gamma(\alpha + 1)\Gamma(\beta + k + 2)}, \alpha + \beta = -1, n \geq 0, \\ \dfrac{1}{\alpha + 1} - \lambda \displaystyle\sum_{k=0}^{n-1} \dfrac{2k + \alpha + 2}{(k + 1)(k + \alpha + 1)}, \beta = 0, n \geq 0, \left(\sum_{0}^{-1} = 0\right). \end{cases} \tag{42}$$

The regularity condition is

$$\alpha \neq -n - 1, \quad \beta \neq -n - 1, \quad \alpha + \beta \neq -n - 1, \lambda \neq 0, \quad d_n(\lambda) \neq 0 \quad n \in \mathbb{N}.$$

In the sequel, we denote by $\mathcal{L}(\alpha, \beta, \lambda)$ the form u which satisfies (40).

We have $\mathcal{V} = \mathcal{L}(-\frac{1}{2}, -\frac{1}{2}, \lambda)$.

Theorem 2.5 *Among the symmetric semiclassical forms of class $s = 2$ satisfying (19) with $\Phi(0) = 0$, only the forms $\mathcal{L}(p - \frac{1}{2}, q - \frac{1}{2}, \lambda)$ are second degree forms, provided $p + q \geq 0$, $\lambda^{-1} \neq \frac{2(p+q)}{2q-1}$, $p, q \in \mathbb{Z}$.*

Proof After multiplying (40) by x, applying the operator σ and using (11)–(12), we obtain

$$(x(x-1)(x\sigma u))' + (-(\alpha+\beta+2)x+\beta+1)(x\sigma u) = 0. \tag{43}$$

Let us make the suitable shift for $(x\sigma u)$

$$\widehat{(x\sigma u)} = \left(h_{(-\frac{1}{2})^{-1}} \circ \tau_{-\frac{1}{2}}\right)(x\sigma u).$$

Using (22), $\widehat{(x\sigma u)}$ satisfies (21) with

$$\hat{\Phi}(x) = x^2 - 1, \quad \hat{\Psi}(x) = -(\alpha+\beta+2)x + \alpha - \beta. \tag{44}$$

Therefore, we have $\left(h_{(-\frac{1}{2})^{-1}} \circ \tau_{-\frac{1}{2}}\right)(x\sigma\mathcal{L}(\alpha,\beta,\lambda)) = \lambda\mathcal{J}(\alpha,\beta)$
where $\mathcal{J}(a,b)$ is the classical Jacobi form with Pearson equation

$$\left((x^2-1)\mathcal{J}(a,b)\right)' + \left(-(a+b+2)x + a - b\right)\mathcal{J}(a,b) = 0.$$

According to Theorem 1.4, Proposition 2.2 and the fact that the shifted form of a second degree form is also second degree form, we obtain: $\mathcal{L}(\alpha,\beta,\lambda)$ is a second degree semiclassical form of class $s=2$ if and only if $\alpha = p - \dfrac{1}{2}, \beta = q - \dfrac{1}{2}, \lambda^{-1} \neq \dfrac{2(p+q)}{2q-1}, p+q \geq 0, p,q \in \mathbb{Z}$. □

Let us now give the polynomial coefficients B, C and D of (16) corresponding to these forms. For this, we need the following lemmas.

Lemma 2.6 [3] *Let u and v be two regular forms satisfying the following relation:*

$$M(x)u = N(x)v, \tag{45}$$

where $M(x)$ and $N(x)$ are two polynomials.
If u is a second degree form satisfying (16), then v is also a second degree form and satisfies

$$\tilde{B}(z)S^2(v)(z) + \tilde{C}(z)S(v)(z) + \tilde{D}(z) = 0, \tag{46}$$

with

$$\begin{cases} \tilde{B}(z) = B(z)N^2(z), \\ \tilde{C}(z) = N(z)\{2B(z)((v\theta_0 N)(z) - (u\theta_0 M)(z)) + M(z)C(z)\}, \\ \tilde{D}(z) = B(z)((v\theta_0 N)(z) - (u\theta_0 M)(z))^2 \\ \qquad\quad + M(z)C(z)((v\theta_0 N)(z) - (u\theta_0 M)(z)) + M^2(z)D(z). \end{cases} \tag{47}$$

Lemma 2.7 *We have*

$$x^2\mathcal{L}(\alpha,\beta,\lambda) = \mu\mathcal{L}(\alpha,\beta+1,\lambda), \tag{48}$$

$$(x^2-1)\mathcal{L}(\alpha,\beta,\lambda) = \mu\mathcal{L}(\alpha+1,\beta,\lambda), \tag{49}$$

where μ is the normalization factor.

Proof The form $u = \mathcal{L}(\alpha,\beta,\lambda)$ satisfies (40). Multiplying by x^2, we obtain

$$(x^2(x^2-1)(x^2u))' + (-(2\alpha+2\beta+5)x^3 + (2\beta+3)x)(x^2u) = 0. \tag{50}$$

Hence (48). Multiplying (40) by (x^2-1), we obtain

$$(x^2(x^2-1)((x^2-1)u))' + (-(2\alpha+2\beta+5)x^3 + (2\beta+1)x)((x^2-1)u) = 0. \tag{51}$$

Hence (49). □

Using Lemma 2.6, Lemma 2.7, and the fact that $\mathcal{V} = \mathcal{L}(-\frac{1}{2},-\frac{1}{2},\lambda)$ and satisfies (16) with (37), the elements B, C and D in (16) are given here in every case:

Proposition 2.8 *Let us consider* $u = \mathcal{L}(p - \frac{1}{2}, q - \frac{1}{2}, \lambda)$, *where* p *and* q *are integers provided* $p + q \geq 0$ *and* $\lambda^{-1} \neq \frac{2(p+q)}{2q-1}$. *Then, we have the following:*

(1) *For* $p \geq 0$,

(i) *if* $q \geq 0$,*then*

$$u = \mu(x^2 - 1)^p x^{2q} \mathcal{V}, \tag{52}$$

$$\begin{cases} B(z) = z^4(z^2 - 1), \\ C(z) = -2\mu z^4(z^2 - 1)\mathcal{X}(z) + 2\mu(z^2 - 1)^{p+1} z^{2q+3}, \\ D(z) = \mu^2 z^4(z^2 - 1)\mathcal{X}^2(z) - 2\mu^2(z^2 - 1)^{p+1} z^{2q+3}\mathcal{X}(z) \\ \qquad + \mu^2(z^2 - 1)^{2p} z^{4q}(z^2(z^2 - 1) - \lambda^2), \end{cases} \tag{53}$$

where

$$\mathcal{X}(z) = (\mathcal{V}\theta_0((\xi^2 - 1)^p \xi^{2q}))(z), \mu = (\langle \mathcal{V}, (x^2 - 1)^p x^{2q} \rangle)^{-1}.$$

(ii) *if* $q \leq -1$ *and* $p + q \geq 0$, *then*

$$x^{-2q} u = \mu(x^2 - 1)^p \mathcal{V}, \tag{54}$$

$$\begin{cases} B(z) = (z^2 - 1) z^{-4q+4}, \\ C(z) = z^{-2q}\{2z^4(z^2 - 1)\mathcal{Y}(z) + 2\mu z^3(z^2 - 1)^{p+1}\}, \\ D(z) = z^4(z^2 - 1)\mathcal{Y}^2(z) + 2\mu z^3(z^2 - 1)^{p+1}\mathcal{Y}(z) + \mu^2(z^2 - 1)^{2p}(z^2(z^2 - 1) - \lambda^2), \end{cases} \tag{55}$$

where

$$\mathcal{Y}(z) = (u\xi^{-2q-1})(z) - \mu(\mathcal{V}\theta_0((\xi^2 - 1)^p))(z) \ , \quad \mu = \frac{\langle u, x^{-2q} \rangle}{\langle \mathcal{V}, (x^2 - 1)^p \rangle} \ .$$

(2) *For* $p \leq -1$ *and* $q \geq 1$ *such that* $p + q \geq 0$, *we have*

$$(x^2 - 1)^{-p} u = \mu x^{2q} \mathcal{V} \ , \tag{56}$$

$$\begin{cases} B(z) = z^4(z^2 - 1)^{-2p+1}, \\ C(z) = (z^2 - 1)^{-p}\{2z^4(z^2 - 1)\mathcal{Z}(z) + 2\mu(z^2 - 1)z^{2q+3}\}, \\ D(z) = z^4(z^2 - 1)\mathcal{Z}^2(z) + 2\mu(z^2 - 1)z^{2q+3}\mathcal{Z}(z) + \mu^2 z^{4q}\left(z^2(z^2 - 1) - \lambda^2\right), \end{cases} \tag{57}$$

where

$$\mathcal{Z}(z) = \left(u\theta_0\left((\xi^2 - 1)^{-p}\right)\right)(z) - \mu\left(\mathcal{V}\xi^{2q-1}\right)(z) \ , \quad \mu = \frac{\left\langle u, (x^2 - 1)^{-p} \right\rangle}{\left\langle \mathcal{V}, x^{2q} \right\rangle}.$$

Integral representation

The form $u = \mathcal{L}(\alpha, \beta, \lambda)$ has the following representation [6,15] (for $\Re(\alpha) > -1$, $\Re(\beta) > 0$)

$$\langle u, f \rangle = \lambda \frac{\Gamma(\alpha + \beta + 2)}{\Gamma(\alpha + 1)\Gamma(\beta + 1)} \int_{-1}^{1} |x|^{2\beta - 1}(1 - x^2)^\alpha f(x) dx + \left(1 - \frac{\lambda(\alpha + \beta + 1)}{\beta}\right) f(0). \tag{58}$$

From Theorem 2.5, we deduce the following:

A symmetric semiclassical form of class $s = 2$ satisfying (19) with $\Phi(0) = 0$ is a second degree form and positive definite if the weight function has the following expression:

$$\begin{aligned} w(x) &= \lambda \frac{\Gamma(p + q + 1)}{\Gamma(p + \frac{1}{2})\Gamma(q + \frac{1}{2})} \frac{x^{2q-2}(1 - x^2)^p Y(1 - x^2)}{\sqrt{1 - x^2}} + \left(1 - \frac{2\lambda(p + q)}{2q - 1}\right)\delta_0, \\ & p \in \mathbb{N}, q \in \mathbb{N}^*, \lambda \in \left]0, \frac{2q - 1}{2(p + q)}\right] \end{aligned} \tag{59}$$

where Y is the characteristic function of $\mathbb{R}^+$.

The case $p = 0$ and $q = 0$ is $\mathcal{V}$. This form is not positive definite, and has the integral representation [1] $\mathcal{V} = \delta_0 + \lambda Pf \dfrac{1}{\pi} \dfrac{Y(1-x^2)}{x^2\sqrt{1-x^2}}$ (see [1]), with the definition

$$\left\langle Pf \frac{Y(1-x^2)}{x^2\sqrt{1-x^2}}, f \right\rangle = \lim_{\epsilon \to 0} \left(\int_{-1}^{-\epsilon} \frac{f(x)\sqrt{1-x^2}}{x^2} dx + \int_{\epsilon}^{1} \frac{f(x)\sqrt{1-x^2}}{x^2} dx \right).$$

Particular cases:

(1) If $p = q = 1$ and $\lambda = \dfrac{1}{8}$ then $u = \frac{1}{2}\delta_0 + \frac{1}{2}\mathcal{U}$ where $\mathcal{U}$ is a Tchebychev form of second kind. Let us recall that its sequence $\{B_n\}_{n\geq 0}$ satisfies (15) with

$$\beta_n = 0, \quad \gamma_{2n+1} = \frac{n+1}{4(n+2)}, \quad \gamma_{2n+2} = \frac{n+3}{4(n+2)}, \quad n \geq 0.$$

In a very interesting work [5], J. Charris, G. Salas and V. Silva studied this sequence of orthogonal polynomials. This sequence is a particular case of a more general sequence considered in Example 1 presented in [10]. According to Theorem 2.5 we deduce that it is a second degree form.

(2) If $\lambda^{-1} = \frac{2(p+q)}{2q-1}$ then $u = \mathcal{I}(p-\frac{1}{2}, q-\frac{3}{2})$. This means that the second degree forms $u = \mathcal{L}(p-\frac{1}{2}, q-\frac{1}{2}, \lambda)$ generalize the symmetric second degree forms of class $s = 1$.

In fact, from (40), $u = \mathcal{L}\left(p-\frac{1}{2}, q-\frac{1}{2}, \lambda\right)$ satisfies (19) with

$$\Phi(x) = x^2(x^2-1), \quad \Psi(x) = (-2p-2q-1)x^3 + 2qx. \tag{60}$$

We have $\Phi'(0) + \Psi(0) = 0$ and $\langle u, \theta_0\Psi + \theta_0^2\Phi \rangle = -2\lambda(p+q) + 2q - 1$.

Then, if $-2\lambda(p+q) + 2q - 1 = 0$ we can simplify (19)–(60) by x and we necessarily have $p + q \neq 0$ because $\lambda(2q-1) \neq 0$. Therefore, $\gamma_1 = \lambda = \frac{2q-1}{2(p+q)}$ and u verifies (19) with

$$\Phi(x) = x(x^2-1), \quad \Psi(x) = -2(p+q)x^2 + (2q-1).$$

Here, $\Phi'(0) + \Psi(0) + \langle u, \theta_0\Psi + \theta_0^2\Phi \rangle = 2(q-1)$.

Hence, for $(p, q) = (k, l+1)$, we get the statement of Theorem 2.3.

2.3 The study of $\sigma\mathcal{L}\left(p-\frac{1}{2}, q-\frac{1}{2}, \lambda\right)$

In this part, the focus will be put on σu: the even part of $u = \mathcal{L}(p-\frac{1}{2}, q-\frac{1}{2}, \lambda)$, provided $p + q \geq 0$, $p, q \in \mathbb{Z}$.

The linear form u verifies the functional equation

$$(x^2(x^2-1)u)' + ((-2p-2q-1)x^3 + 2qx)u = 0.$$

Multiplication by x gives

$$(x^3(x^2-1)u)' + ((-2p-2q-2)x^4 + (2q+1)x^2)u = 0.$$

Applying the operator σ in both hand sides of the above equation and using (11)–(12), we obtain

$$\left(\Phi^P(x)\sigma u\right)' + \Psi^P(x)\sigma u = 0 \tag{61}$$

where $\Phi^P(x) = x^2(x-1), \quad \Psi^P(x) = -(p+q+1)x^2 + (q+\frac{1}{2})x$.

We have $\Psi^P(0) + (\Phi^P)'(0) = 0$ and $\langle \sigma u, \theta_0\Psi^P + \theta_0^2\Phi^P \rangle = -(p+q)\lambda + q - \frac{1}{2}$.

Then, from Proposition 2.1 and the standard criterion (20), we obtain the following cases:

(i) If $2(p+q)\lambda \neq 2q-1$ then σu is a nonsymmetric second degree form of class $s = 1$.

(ii) If $\lambda^{-1} = \frac{2(p+q)}{2q-1}$ then $(h_{(-\frac{1}{2})^{-1}} \circ \tau_{-\frac{1}{2}})\sigma u = \mathcal{J}(p-\frac{1}{2}, q-\frac{3}{2})$: the classical second degree forms. Indeed, in this case, we necessarily have $p+q \neq 0$. Then, for $(p,q) = (k, l+1)$, we obtain the statement of Theorem 1.4.

From (27) and (28), the coefficients $\{\beta_n^P, \gamma_{n+1}^P\}_{n\geq 0}$ of $\{P_n\}_{n\geq 0}$ are

$$\beta_0^P = \gamma_1, \quad \beta_{n+1}^P = \gamma_{2n+2} + \gamma_{2n+3}, \quad \gamma_{n+1}^P = \gamma_{2n+1}\gamma_{2n+2},$$

where γ_n, $n \geq 1$ are given by (41) and $(\alpha, \beta) = (p - \frac{1}{2}, q - \frac{1}{2})$.

Proposition 2.9 *Let $u = \mathcal{L}(p - \frac{1}{2}, q - \frac{1}{2}, \lambda)$, where p and q are integer numbers with $p + q \geq 0$. Then, the second degree form σu satisfies*

$$\bar{B}(z)S^2\big(\sigma u\big)(z) + \bar{C}(z)S\big(\sigma u\big)(z) + \bar{D}(z) = 0 \tag{62}$$

with:
(1) *For $p \geq 0$,*

(i) *if $q \geq 0$, then*

$$\begin{cases} \bar{B}(z) = z^3(z-1), \\ \bar{C}(z) = -2\mu z^3(z-1)\bar{\mathcal{X}}(z) + 2\mu(z-1)^{p+1}z^{q+2}, \\ \bar{D}(z) = \mu^2 z^3(z-1)\bar{\mathcal{X}}^2(z) - 2\mu^2(z-1)^{p+1}z^{q+2}\bar{\mathcal{X}}(z) + \mu^2(z-1)^{2p}z^{2q}\left(z(z-1)-\lambda^2\right), \end{cases} \tag{63}$$

where

$$\bar{\mathcal{X}}(z) = \left((\sigma\mathcal{V})\theta_0\left((\xi-1)^p\xi^q\right)\right)(z), \mu = \left(\left\langle \mathcal{V}, (x^2-1)^p x^{2q}\right\rangle\right)^{-1}.$$

(ii) *if $q \leq -1$ and $p + q \geq 0$, then*

$$\begin{cases} \bar{B}(z) = (z-1)z^{-2q+3}, \\ \bar{C}(z) = z^{1-q}\left\{2z^2(z-1)\bar{\mathcal{Y}}(z) + 2\mu z(z-1)^{p+1}\right\}, \\ \bar{D}(z) = z^3(z-1)\bar{\mathcal{Y}}^2(z) + 2\mu z^2(z-1)^{p+1}\bar{\mathcal{Y}}(z) + \mu^2(z-1)^{2p}\left(z(z-1)-\lambda^2\right), \end{cases} \tag{64}$$

where

$$\bar{\mathcal{Y}}(z) = \left((\sigma u)\xi^{-q-1}\right)(z) - \mu\left((\sigma\mathcal{V})\theta_0\left((\xi-1)^p\right)\right)(z), \quad \mu = \frac{\left\langle u, x^{-2q}\right\rangle}{\left\langle \mathcal{V}, (x^2-1)^p\right\rangle}.$$

(2) *For $p \leq -1$ and $q \geq 1$ such that $p + q \geq 0$, we have*

$$\begin{cases} \bar{B}(z) = z^3(z-1)^{-2p+1}, \\ \bar{C}(z) = z(z-1)^{-p}\left\{2z^2(z-1)\bar{\mathcal{Z}}(z) + 2\mu(z-1)z^{q+1}\right\}, \\ \bar{D}(z) = z^3(z-1)\bar{\mathcal{Z}}^2(z) + 2\mu(z-1)z^{q+2}\bar{\mathcal{Z}}(z) + \mu^2 z^{2q}\left(z(z-1)-\lambda^2\right), \end{cases} \tag{65}$$

where

$$\bar{\mathcal{Z}}(z) = \left((\sigma u)\theta_0\left((\xi-1)^{-p}\right)\right)(z) - \mu\left((\sigma\mathcal{V})\xi^{q-1}\right)(z), \mu = \frac{\left\langle u, (x^2-1)^{-p}\right\rangle}{\left\langle \mathcal{V}, x^{2q}\right\rangle}.$$

Proof From Proposition 2.8, we notice that the polynomial coefficients of the second degree equation (16) satisfied by $u = \mathcal{L}(p - \frac{1}{2}, q - \frac{1}{2}, \lambda)$ are such that B and D are even and C is odd. Then, there exist B^e, C^o and D^e such that

$$B(z) = B^e(z^2), \quad C(z) = zC^o(z^2), \quad D(z) = D^e(z^2). \tag{66}$$

From (8) and the fact that u is a symmetric form, we have

$$S(u)(z) = zS(\sigma u)(z^2). \tag{67}$$

Substituting (66) and (67) in (16) and making a change of variable $z^2 \longrightarrow z$, we get (62) with,

$$\begin{cases} \bar{B}(z) = zB^e(z), \\ \bar{C}(z) = zC^o(z), \\ \bar{D}(z) = D^e(z). \end{cases} \tag{68}$$

From (2), (6) and (11), we easily prove that for a symmetric form w, we have

$$\left(w\theta_0 f(\xi^2)\right)(z) = z\left((\sigma w)\theta_0 f\right)(z^2), \quad f \in \mathcal{P}. \tag{69}$$

Hence, the desired result is obtained by using (69) and the expressions of B, C and D given in the three different cases of Proposition 2.8. □

Integral representation

From (58)–(59), we get

$$\begin{aligned}\langle \sigma u, f(x)\rangle &= \langle u, f(x^2)\rangle \\ &= \left(1 - \frac{2\lambda(p+q)}{2q-1}\right) f(0) + 2\lambda \frac{\Gamma(p+q+1)}{\Gamma(p+\frac{1}{2})\Gamma(q+\frac{1}{2})} \int_0^{+1} \frac{x^{2q-2}(1-x^2)^p}{\sqrt{1-x^2}} f(x^2)dx.\end{aligned}$$

Then, we obtain after a change of variables

$$\langle \sigma u, f\rangle = \left(1 - \frac{2\lambda(p+q)}{2q-1}\right) f(0) + \lambda \frac{\Gamma(p+q+1)}{\Gamma(p+\frac{1}{2})\Gamma(q+\frac{1}{2})} \int_0^{+1} \frac{x^{q-1}(1-x)^p}{\sqrt{x(1-x)}} f(x)dx,\ p, q \in \mathbb{N}, q \neq 0. \tag{70}$$

Notice that this form is a particular case of the so-called Koornwinder linear functionals (see [8]).

Remark Thanks to Proposition 2.2, we carry out the complete description of the symmetric second degree semiclassical forms of class $s = 2$ when $\Phi(0) = 0$. Unfortunately, the case when $\Phi(0) \neq 0$ is not covered by this Proposition and the description of these forms remains open.

Notice that this last set is not empty. Indeed, let us define the normalized form $\mathcal{W}$ by $\mathcal{W} = \mathcal{U} + \lambda\delta_1 + \lambda\delta_{-1}$, $\lambda \in \mathbb{C} - \{0\}$ where $\mathcal{U}$ is a Tchebychev form of second kind. This form is symmetric and semiclassical of class $s = 2$ satisfying (19) with $\Phi(x) = (x^2-1)^2$ and $\Psi(x) = -5x(x^2-1)$. It is a particular case of the so-called Koornwinder linear functionals (see [6,8] and [9] for more information).

Moreover, it is well known that $\mathcal{U}$ is a second degree form verifying the quadratic equation (see [11])

$$S^2\big(\mathcal{U}\big)(z) + 4zS\big(\mathcal{U}\big)(z) + 4 = 0. \tag{71}$$

From $(\mathcal{W})_{2n} = (\mathcal{U})_{2n} + 2\lambda$, $(\mathcal{W})_{2n+1} = 0$, $n \geq 0$, we get $S(\mathcal{U})(z) = S(\mathcal{W})(z) + \frac{2\lambda z}{z^2-1}$. Then, substituting in (71), we obtain after multiplying by $(z^2-1)^2$

$$(z^2-1)^2 S^2(\mathcal{W})(z) + 4z(z^2-1)(z^2+\lambda-1)S(\mathcal{W})(z) + 4(2\lambda+1)z^4 + 4(\lambda^2-2\lambda-2))z^2 + 4 = 0.$$

Hence, $\mathcal{W}$ is a symmetric second degree semiclassical form of class $s = 2$ satisfying (19) with $\Phi(0) \neq 0$.

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