RESEARCH ARTICLE

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R-smash products of Hopf quasigroups

Abstract The theory of R-smash products for Hopf quasigroups is developed.

الملخص

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1 Introduction

Hopf quasigroups and Hopf coquasigroups were introduced in [4]. These are non-associative (or non-coassociative) generalisations of Hopf algebras, in which the antipode provides one with a certain level of control over the non-associativity. In particular, Hopf quasigroups are examples of unital coassociative H-bialgebras introduced in [5, Section 2]. They can be understood as linearisations of loops [1]. Also in [4], smash products of Hopf quasigroups were studied. It has been shown in [2] that a standard form of a smash product forces one to replace the conventional associativity of action (assumed in [4] from the onset) by a similar condition involving the antipode. In this note, which is a sequel to [2], we look at R-smash products [3] of Hopf quasigroups and, briefly, at W-smash coproducts of Hopf coquasigroups. This analysis reveals that some of the conventional requirements on the twisting map R need be replaced by similar conditions in which the antipode plays a prominent role (see Theorem 2.3 and Definition 2.1 for details).

All algebras and coalgebras are over a field $\Bbbk$ and they are assumed to be unital and counital, respectively, but are not assumed to be associative or coassociative unless stated otherwise. Unadorned tensor product symbol represents the tensor product of $\Bbbk$-vector spaces. We use the standard Sweedler notation for coproducts $\Delta(h) = h_{(1)} \otimes h_{(2)}$ (summation understood) even if the coproduct Δ is not assumed to be associative.

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2 *R*-Smash products of Hopf quasigroups

We begin by introducing the terminology used in this note.

Definition 2.1 Let H be an algebra with product μ_H and unit 1_H and a coalgebra with coproduct Δ_H and counit ε_H that are algebra morphisms. Similarly, let A be an algebra with product μ_A and unit 1_A and a coalgebra with coproduct Δ_A and counit ε_A that are algebra morphisms. Consider linear maps $S_H : H \to H$ and $R : H \otimes A \to A \otimes H$. The map R is said to be:

- *left normal* (resp. *right normal*) if

$$R \circ (\mathrm{id}_H \otimes 1_A) = 1_A \otimes \mathrm{id}_H, \quad (\text{resp. } R \circ (1_H \otimes \mathrm{id}_A) = \mathrm{id}_A \otimes 1_H),$$

  and it is said to be *normal* if it is both left and right normal;
- *left multiplicative* if

$$R \circ (\mathrm{id}_H \otimes \mu_A) = (\mu_A \otimes \mathrm{id}_H) \circ (\mathrm{id}_A \otimes R) \circ (R \otimes \mathrm{id}_A);$$

- *right S_H-multiplicative* if

$$R \circ (\mu_H \otimes \mathrm{id}_A) \circ (\mathrm{id}_H \otimes S_H \otimes \mathrm{id}_A) = (A \otimes \mu_H) \circ (R \otimes \mathrm{id}_H) \circ (\mathrm{id}_H \otimes R) \circ (\mathrm{id}_H \otimes S_H \otimes \mathrm{id}_A);$$

- *right S_H-normal* if

$$R \circ (S_H \otimes \mathrm{id}_A) \circ \mathsf{flip} \circ R \circ (1_H \otimes \mathrm{id}_A) = 1_H \otimes \mathrm{id}_A.$$

Dually, the map R is said to be:

- *left conormal* (resp. *right conormal*) if

$$(\varepsilon_A \otimes \mathrm{id}_H) \circ R = \mathrm{id}_H \otimes \varepsilon_A, \quad (\text{resp. } (\mathrm{id}_A \otimes \varepsilon_H) \circ R = \varepsilon_H \otimes \mathrm{id}_A),$$

  and it is said to be *conormal* if it is both left and right conormal;
- *left comultiplicative* if

$$(\Delta_A \otimes \mathrm{id}_H) \circ R = (\mathrm{id}_A \otimes R) \circ (R \otimes \mathrm{id}_A) \circ (\mathrm{id}_H \otimes \Delta_A);$$

- *right S_H-comultiplicative* if

$$(\mathrm{id}_A \otimes S_H \otimes \mathrm{id}_H) \circ (\mathrm{id}_A \otimes \Delta_H) \circ R = (\mathrm{id}_A \otimes S_H \otimes \mathrm{id}_H) \circ (R \otimes \mathrm{id}_H) \circ (\mathrm{id}_H \otimes R) \circ (\Delta_H \otimes A);$$

- *right S_H-conormal* if

$$(\mathrm{id}_A \otimes \varepsilon_H) \circ R \circ \mathsf{flip} \circ (\mathrm{id}_A \otimes S_H) \circ R = \varepsilon_H \otimes \mathrm{id}_A.$$

The action of R on elements is denoted by

$$R(h \otimes a) = \sum_R a_R \otimes h^R = \sum_r a_r \otimes h^r, \text{ etc.},$$

for all $h \in H$ and $a \in A$. The reader is encouraged to write down all the above requirements on R in terms of this notation. For example, R is left multiplicative if
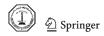
$$\sum_R (ab)_R \otimes h^R = \sum_{R,r} a_R b_r \otimes h^{Rr}, \tag{2.1}$$

and is right S_H-multiplicative if

$$\sum_R a_R \otimes (g S_H(h))^R = \sum_{R,r} a_{Rr} \otimes g^r S_H(h)^R, \tag{2.2}$$

for all $a, b \in A$ and $g, h \in H$, etc.

We are particularly interested in the case in which H and A are Hopf quasigroups or Hopf coquasigroups, and S_H is the antipode of H. We will concentrate on the former case, as the latter can be treated dually. Recall

from [4] that $(H, \mu_H, 1_H, \Delta_H, \varepsilon_H, S_H)$ as in Definition 2.1 is called a *Hopf quasigroup* provided Δ_H is coassociative and the following *Hopf quasigroup identities* are fulfilled

$$S_H(h_{(1)})(h_{(2)}g) = g\varepsilon(h) = h_{(1)}(S_H(h_{(2)})g), \qquad (2.3)$$

$$(gh_{(1)})S_H(h_{(2)}) = g\varepsilon(h) = (gS_H(h_{(1)}))h_{(2)}, \qquad (2.4)$$

for all $g, h \in H$. The identities (2.3)–(2.4) ensure that a Hopf quasigroup is an H-*bialgebra* with left division $h \backslash g = S_H(h)g$ and right division $g/h = gS_H(h)$ (see [5, Definition 2]). It is proven in [4] that the antipode S_H is antimultiplicative and anticomultiplicative and it immediately follows from the Hopf quasigroup identities that S_H enjoys the standard antipode property.

Definition 2.2 Let H and A be Hopf quasigroups, $R : H \otimes A \to A \otimes H$ a $\Bbbk$-linear map. An *R-smash product* of H and A is a Hopf quasigroup $A \rtimes_R H$ equal to $A \otimes H$ as a vector space, with tensor product coproduct, unit and counit, and the multiplication

$$\mu = (\mu_A \otimes \mu_H) \circ (\mathrm{id}_A \otimes R \otimes \mathrm{id}_H) \qquad (2.5)$$

and antipode

$$S = R \circ (S_H \otimes S_A) \circ \mathsf{flip}. \qquad (2.6)$$

The aim of this note is to determine necessary and sufficient conditions for R to produce an R-smash product of Hopf quasigroups. These are listed in the following theorem, which is a Hopf quasigroup version of [3, Corollary 4.6]

Theorem 2.3 *Let H, A be Hopf quasigroups, $R : H \otimes A \to A \otimes H$ a $\Bbbk$-linear map. If R is left multiplicative and left conormal, then the following statements are equivalent:*

(1) *$A \rtimes_R H$ is an R-smash product Hopf quasigroup for H and A;*
(2) *The map R is a coalgebra map that is normal, right S_H-multiplicative and right S_H-conormal.*

Before the proof of Theorem 2.3 is given, we state and prove three lemmata.

Lemma 2.4 *Let H, A be Hopf quasigroups, $R : H \otimes A \to A \otimes H$ a $\Bbbk$-linear map. If R is a left conormal coalgebra map, then, for all $h \in H, a \in A$,*

$$\sum_R a_R \otimes h^R_{(1)} \otimes h^R_{(2)} = \sum_R a_R \otimes h_{(1)}{}^R \otimes h_{(2)} = \sum_R a_R \otimes h_{(1)} \otimes h_{(2)}{}^R, \qquad (2.7)$$

hence

$$R(h \otimes a) = \sum_R a_R \varepsilon_H(h_{(1)}{}^R) \otimes h_{(2)} = \sum_R a_R \otimes h_{(1)} \varepsilon_H(h_{(2)}{}^R). \qquad (2.8)$$

Furthermore, R is left comultiplicative.

Proof Equations (2.7) follow by applying $\mathrm{id}_A \otimes \mathrm{id}_H \otimes \varepsilon_A \otimes \mathrm{id}_H$ or $\varepsilon_A \otimes \mathrm{id}_H \otimes \mathrm{id}_A \otimes \mathrm{id}_H$ to the formula expressing the comultiplicativity of R, i.e. to

$$\sum_R a_{R(1)} \otimes h^R_{(1)} \otimes a_{R(2)} \otimes h^R_{(2)} = \sum_{R,r} a_{(1)R} \otimes h_{(1)}{}^R \otimes a_{(2)r} \otimes h_{(2)}{}^r, \qquad (2.9)$$

and by using the left conormality of R. Equations (2.8) then follow from (2.7) by applying $\mathrm{id}_A \otimes \varepsilon_H \otimes \mathrm{id}_H$ and $\mathrm{id}_A \otimes \mathrm{id}_H \otimes \varepsilon_H$.

Finally, apply $\mathrm{id}_A \otimes \varepsilon_H \otimes \mathrm{id}_A \otimes \mathrm{id}_H$ to (2.9) and use (2.8) to compute

$$\sum_R a_{R(1)} \otimes a_{R(2)} \otimes h^R = \sum_{R,r} a_{(1)R} \varepsilon_H(h_{(1)}{}^R) \otimes a_{(2)r} \otimes h_{(2)}{}^r = \sum_{R,r} = a_{(1)R} \otimes a_{(2)r} \otimes h^{Rr}.$$

Thus, R is left comultiplicative as required. □

Lemma 2.5 *Let H, A be Hopf quasigroups, $R : H \otimes A \to A \otimes H$ a $\Bbbk$-linear map. If R is a left conormal coalgebra map, then:*

(1) *R is right S_H-multiplicative if and only if, for all $a \in A$, $g, h \in H$,*

$$\sum_{R,r} a_{Rr} \varepsilon_H(g^r S_H(h)^R) = \sum_R a_R \varepsilon_H((g S_H(h))^R). \tag{2.10}$$

(2) *For all $a \in A$, $h \in H$, the conditions*

$$\sum_{R,r} a_{Rr} \varepsilon_H(S_H(h_{(1)})^r h_{(2)}{}^R) = \varepsilon_H(h)a = \sum_{R,r} a_{Rr} \varepsilon_H(h_{(1)}{}^r S_H(h_{(2)})^R) \tag{2.11}$$

are equivalent to

$$\sum_{R,r} a_{Rr} \otimes S_H(h_{(1)})^r h_{(2)}{}^R = a \otimes h = \sum_{R,r} a_{Rr} \otimes h_{(1)}{}^r S_H(h_{(2)})^R.$$

(3) *If R is right normal, then R is right S_H-multiplicative and right S_H-conormal if and only if it satisfies* (2.10) *and* (2.11).

Proof (1) Obviously, the right S_H-multiplicativity of R implies (2.10). Conversely, the right S_H-multiplicativity of R can be inferred from (2.10) by repetitive use of Equations (2.8) in Lemma 2.4:

$$\sum_R a_R \otimes (g S_H(h))^R \overset{(2.8)}{=} \sum_R a_R \varepsilon_H((g S_H(h))_{(1)}{}^R) \otimes (g S_H(h))_{(2)}$$

$$= \sum_R a_R \varepsilon_H((g_{(1)} S_H(h_{(2)}))^R) \otimes g_{(2)} S_H(h_{(1)})$$

$$\overset{(2.10)}{=} \sum_{R,r} a_{Rr} \varepsilon_H(g_{(1)}{}^r) \varepsilon_H(S_H(h_{(2)})^R) \otimes g_{(2)} S_H(h_{(1)})$$

$$\overset{(2.8)}{=} \sum_{R,r} a_{Rr} \varepsilon_H(g_{(1)}{}^r) \otimes g_{(2)} S_H(h)^R \overset{(2.8)}{=} \sum_{R,r} a_{Rr} \otimes g^r S_H(h)^R,$$

as required.

The second statement is proven by a similar repetitive use of Equations (2.8) in Lemma 2.4, and the proof is left to the reader. To prove (3), if R is right normal and right S_H-multiplicative, then

$$\sum_{R,r} a_{Rr} \varepsilon_H(h_{(1)}{}^r S_H(h_{(2)})^R) \overset{(2.10)}{=} \sum_R a_R \varepsilon_H((h_{(1)} S_H(h_{(2)}))^R)$$

$$= \sum_R a_R \varepsilon_H(h) \varepsilon_H(1_H{}^R) = a \varepsilon_H(h),$$

so the second of Equations (2.11) is automatically satisfied. Now, we need to use the multiplicativity of the counit and (2.8) in Lemma 2.4 to compute

$$\sum_{R,r} a_{Rr} \varepsilon_H(S_H(h_{(1)})^r h_{(2)}{}^R) = \sum_{R,r} a_{Rr} \varepsilon_H(h_{(2)}{}^R) \varepsilon_H(S_H(h_{(1)})^r) = \sum_{R,r} a_{Rr} \varepsilon_H(S_H(h^R)^r).$$
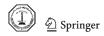
Hence, the first of Equations (2.11) is equivalent to right S_H-conormality of R. $\qquad\square$

Lemma 2.6 *Let H and A be Hopf quasigroups, $R : H \otimes A \to A \otimes H$ a left normal and left multiplicative map. If R is also a coalgebra map and is left conormal, then*

$$R \circ (\mathrm{id}_H \otimes S_A) = (S_A \otimes \mathrm{id}_H) \circ R. \tag{2.12}$$

Furthermore, the first of equalities (2.11) *implies that*

$$R \circ \mathsf{flip} \circ (S_A \otimes S_H) \circ R \circ \mathsf{flip} = S_A \otimes S_H. \tag{2.13}$$

Proof Take any $a \in A$ and $h \in H$. Then, using the left multiplicativity and left conormality of R to make a start and to finish, we can compute

$$\sum_R S_A(a_R) \otimes h^R = \sum_{R,\bar{R},r} S_A(a_{(1)\,R})(a_{(2)\,\bar{R}} S_A(a_{(3)})_r) \otimes h^{R\bar{R}r}$$

$$\overset{(2.9)}{=} \sum_{R,r} S_A(a_{(1)\,R(1)})(a_{(1)\,R(2)} S_A(a_{(2)})_r) \otimes h^{Rr} \overset{(2.3)}{=} \sum_r S_A(a)_r \otimes h^r.$$

This proves equality (2.12).

The second assertion is proven by the following calculation, for all $a \in A, h \in H$,

$$\sum_{R,r} S_A(a_R)_r \otimes S_H(h^R)^r \overset{(2.12)}{=} \sum_{R,r} S_A(a_{Rr}) \otimes S_H(h^R)^r$$

$$\overset{(2.8)}{=} \sum_{R,r} S_A(a_{Rr}) \varepsilon_H(S_H(h_{(1)})_{(1)}{}^r h_{(2)}{}^R) \otimes S_H(h_{(1)})_{(2)}$$

$$= \sum_{R,r} S_A(a_{Rr}) \varepsilon_H(S_H(h_{(2)})^r h_{(3)}{}^R) \otimes S_H(h_{(1)}) \overset{(2.11)}{=} S_A(a) \otimes S_H(h).$$

Thus, the equality (2.13) holds as required.   □

Proof of Theorem 2.3 $(2) \Rightarrow (1)$ The normality of R immediately implies that $1_A \otimes 1_H$ is the unit of $A \rtimes_R H$. The left counitality of R together with the fact that a counit of a Hopf quasigroup is an algebra map imply that also the counit $\varepsilon_A \otimes \varepsilon_H$ of $A \rtimes_R H$ is an algebra homomorphism. The coproduct Δ of $A \rtimes_R H$ is obviously unital, and it is multiplicative since R is a coalgebra morphism. This part of the proof is not different from the standard Hopf algebra case (see [3]). It remains to check the Hopf quasigroup identities (2.3) and (2.4), which is done by explicit calculations.

For all $a, b \in A$ and $g, h \in H$,

$$S((a \otimes h)_{(1)})((a \otimes h)_{(2)}(b \otimes g)) \overset{(2.6),(2.5)}{=} \sum_{\bar{R},R,r} S_A(a_{(1)})_R (a_{(2)} b_r)_{\bar{R}} \otimes S_H(h_{(1)})^{R\bar{R}}(h_{(2)}{}^r g)$$

$$\overset{(2.1)}{=} \sum_{R,r} (S_A(a_{(1)})(a_{(2)} b_r))_R \otimes S_H(h_{(1)})^R (h_{(2)}{}^r g)$$

$$\overset{(2.3)}{=} \sum_{R,r} \varepsilon_A(a) b_{r\,R} \otimes S_H(h_{(1)})^R (h_{(2)}{}^r g)$$

$$\overset{(2.8)}{=} \sum_{R,r} \varepsilon_A(a) b_{r\,R} \varepsilon_H(S_H(h_{(2)})^R h_{(3)}{}^r) \otimes S_H(h_{(1)})(h_{(4)} g)$$

$$\overset{(2.11)}{=} \varepsilon_A(a) b \otimes S_H(h_{(1)})(h_{(2)} g) \overset{(2.3)}{=} \varepsilon_A(a) \varepsilon_H(h) b \otimes g.$$

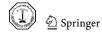This proves the first of Equations (2.3). Next

$$(a \otimes h)_{(1)}(S((a \otimes h)_{(2)})(b \otimes g)) \overset{(2.6),(2.1)}{=} \sum_{R,r,\bar{R}} a_{(1)} (S_A(a_{(2)})_R b_r)_{\bar{R}} \otimes h_{(1)}{}^{\bar{R}}(S_H(h_{(2)})^{Rr} g)$$

$$\overset{(2.1)}{=} \sum_{R,\bar{R}} a_{(1)} (S_A(a_{(2)}) b)_{R\bar{R}} \otimes h_{(1)}{}^{\bar{R}}(S_H(h_{(2)})^R g)$$

$$\overset{(2.8)}{=} \sum_{R,\bar{R}} a_{(1)} (S_A(a_{(2)}) b)_{R\bar{R}} \varepsilon_H(h_{(1)}{}^{\bar{R}}(S_H(h_{(4)})^R) \otimes h_{(2)}(S_H(h_{(3)}) g)$$

$$\overset{(2.3)}{=} \sum_{R,\bar{R}} a_{(1)} (S_A(a_{(2)}) b)_{R\bar{R}} \varepsilon_H(h_{(1)}{}^{\bar{R}}(S_H(h_{(2)})^R) \otimes g$$

$$\overset{(2.11)}{=} \varepsilon_H(h) a_{(1)} (S_A(a_{(2)}) b) \otimes g \overset{(2.3)}{=} \varepsilon_A(a) \varepsilon_H(h) b \otimes g,$$

where also the normality was used to derive the penultimate equality. This proves the second of relations (2.3). It is the proof of (2.4) where the right S_H-multiplicativity of R, (2.2), is used. The first of identities (2.4) is proven by the following calculation

$$
\begin{aligned}
((b \otimes g)(a \otimes h)_{(1)}) S((a \otimes h)_{(2)}) &\overset{(2.5),(2.6)}{=} \sum_{R,r,\bar{R}} (ba_{(1)\,R}) S_A(a_{(2)})_{r\,\bar{R}} \otimes (g^R h_{(1)})^{\bar{R}} S_H(h_{(2)})^r \\
&\overset{(2.2)}{=} \sum_{R,r} (ba_{(1)\,R}) S_A(a_{(2)})_r \otimes ((g^R h_{(1)}) S_H(h_{(2)}))^r \\
&\overset{(2.4)}{=} \sum_{R,r} \varepsilon_H(h)(ba_{(1)\,R}) S_A(a_{(2)})_r \otimes g^{Rr} \\
&\overset{(2.12)}{=} \sum_{R,r} \varepsilon_H(h)(ba_{(1)\,R}) S_A(a_{(2)_r}) \otimes g^{Rr} \\
&= \sum_{R} \varepsilon_H(h)(ba_{R(1)}) S_A(a_{R(2)}) \otimes g^R \overset{(2.4)}{=} \varepsilon_A(a)\varepsilon_H(h) b \otimes g.
\end{aligned}
$$

The penultimate equality is a consequence of left comultiplicativity of R, which is asserted by Lemma 2.4. In derivation of the last equality, the left conormality of R was also used. Finally, and again using Lemma 2.4 in the penultimate equality and left conormality of R in the last one, we compute

$$
\begin{aligned}
((b \otimes g) S((a \otimes h)_{(1)}))(a \otimes h)_{(2)} &\overset{(2.6),(2.5)}{=} \sum_{R,r,\bar{R}} (b S_A(a_{(1)})_{Rr}) a_{(2)\,\bar{R}} \otimes (g^r S_H(h_{(1)})^R)^{\bar{R}} h_{(2)} \\
&\overset{(2.2)}{=} \sum_{R,\bar{R}} (b S_A(a_{(1)})_R) a_{(2)\,\bar{R}} \otimes (g S_H(h_{(1)}))^{R\bar{R}} h_{(2)} \\
&\overset{(2.12)}{=} \sum_{R,\bar{R}} (b S_A(a_{(1)\,R})) a_{(2)\,\bar{R}} \otimes (g S_H(h_{(1)}))^{R\bar{R}} h_{(2)} \\
&= \sum_{R} (b S_A(a_{R(1)})) a_{R(2)} \otimes (g S_H(h_{(1)}))^R h_{(2)} \overset{(2.4)}{=} \varepsilon_A(a)\varepsilon_H(h) b \otimes g.
\end{aligned}
$$

This completes the proof that $A \rtimes_R H$ is a Hopf quasigroup as required.

(1) $\Rightarrow$ (2) The fact that $1_A \otimes 1_H$ is the unit of the R-smash product Hopf quasigroup $A \rtimes_R H$ immediately implies the normality of R. The equalities, for all $a \in A$, $h \in H$,

$$
\Delta((1_A \otimes h)(a \otimes 1_H)) = \Delta(1_A \otimes h)\Delta(a \otimes 1_H), \quad \varepsilon((1_A \otimes h)(a \otimes 1_H)) = \varepsilon(1_A \otimes h)\varepsilon(a \otimes 1_H),
$$

resulting from the multiplicativity of coproduct and counit imply that R is a coalgebra map. This is no different from the standard Hopf algebra case.

Developing the second of the Hopf quasigroup conditions (2.4) for $A \rtimes_R H$ as in the first part of the proof of the theorem, and using Lemma 2.6 and (2.8) repeatedly, we arrive at the following equality:

$$
\sum_{R,r,\bar{R}} (b S_A(a_{(1)\,Rr})) a_{(2)\,\bar{R}} \varepsilon_H(g_{(1)}{}^r S_H(h_{(1)})_{(1)}{}^R) \otimes (g_{(2)} S_H(h_{(1)})_{(2)})^{\bar{R}} h_{(2)} = \varepsilon_A(a)\varepsilon_H(h) b \otimes g.
$$

Apply $\mathrm{id}_A \otimes \varepsilon_H$ to both sides of this equation, set $b = \sum_{\hat{R},\hat{r}} a_{(1)\,\hat{R}\hat{r}} \varepsilon_H(g_{(1)}{}^{\hat{r}} S_H(h)_{(1)}{}^{\hat{R}})$, and shift Sweedler's indices as required to obtain:

$$
\begin{aligned}
\sum_{R,r,\bar{R},\hat{R},\hat{r}} (a_{(1)\,\hat{R}\hat{r}} \varepsilon_H(g_{(1)}{}^{\hat{r}} S_H(h)_{(1)}{}^{\hat{R}}) S_A(a_{(2)\,Rr})) a_{(3)\,\bar{R}} \varepsilon_H(g_{(2)}{}^r S_H(h)_{(2)}{}^R) \varepsilon_H((g_{(3)} S_H(h)_{(3)})^{\bar{R}}) \\
= \sum_{\hat{R},\hat{r}} a_{\hat{R}\hat{r}} \varepsilon_H(g^{\hat{r}} S_H(h))^{\hat{R}}).
\end{aligned}
$$

The fact that R is a coalgebra map, Equatuion (2.9), implies

$$\sum_{R,r,\bar{R}} (a_{(1)Rr(1)} S_A(a_{(1)Rr(2)})) a_{(2)\bar{R}} \varepsilon_H(g_{(1)}{}^r S_H(h)_{(1)}{}^R) \varepsilon_H((g_{(2)} S_H(h)_{(2)})^{\bar{R}})$$

$$= \sum_{\hat{R},\hat{r}} a_{\hat{R}\hat{r}} \varepsilon_H(g^{\hat{r}} S_H(h))^{\hat{R}}).$$

Finally, the antipode property combined with the left conormality of R yield equation (2.10). Therefore, R is right S_H-multiplicative by Lemma 2.5.

Finally, making the same steps in the proof of the first Hopf quasigroup identity for $A \bowtie_R H$ as in the first part of the proof of the theorem, we conclude that the first of conditions (2.3) implies

$$\sum_{R,r} \varepsilon_A(a) b_{r\,R} \varepsilon_H(S_H(h_{(2)})^R h_{(3)}{}^r) \otimes S_H(h_{(1)})(h_{(4)}g) = \varepsilon_A(a) \varepsilon_H(h) b \otimes g.$$

Applying $\mathrm{id}_A \otimes \varepsilon_H$ and evaluating this identity at $a = 1_A$ and $g = 1_H$, one immediately obtains the first of Equations (2.11). In view of Lemma 2.5, this is tantamount to right S_H-conormality of R.   □

Remark 2.7 In [3, Corollary 4.6], which is a predecessor of Theorem 2.3, it is assumed that the R is compatible with the antipodes so that the equality (2.13) is satisfied. As explained in Lemma 2.6 this follows from other hypotheses made in Theorem 2.3, most notably from right S_H- and left conormality of R, which are not assumed in [3].

Example 2.8 Let H and A be Hopf quasigroups. Recall from [2] that A is a *left H-quasimodule Hopf quasigroup*, if

(a) A is a left H-quasimodule, i.e. there exists a $\mathbb{k}$-linear map $H \otimes M \to M$, $h \otimes m \mapsto h \cdot m$, such that, for all $a \in M$ and $h \in H$,

$$1_H \cdot m = m, \quad h_{(1)} \cdot (S_H(h_{(2)}) \cdot m) = \varepsilon_H(h) m = S_H(h_{(1)}) \cdot (h_{(2)} \cdot m);$$

(b) the H-action satisfies the following compatibility conditions:

$$(h_{(1)} \cdot a)(h_{(2)} \cdot b) = h \cdot (ab), \quad h \cdot 1_A = \varepsilon_H(h) 1_A,$$

$$\Delta_A(h \cdot a) = h_{(1)} \cdot a_{(1)} \otimes h_{(2)} \cdot a_{(2)}, \quad \varepsilon_A(h \cdot a) = \varepsilon_H(h) \varepsilon_A(a),$$

for all $h \in H, a, b \in A$.

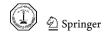Let A be a left H-quasimodule Hopf quasigroup such that, for all $g, h \in H$ and $a \in A$,

$$h_{(1)} \otimes h_{(2)} \cdot a = h_{(2)} \otimes h_{(1)} \cdot a, \quad g \cdot (S_H(h) \cdot a) = (g S_H(h)) \cdot a. \tag{2.14}$$

Then the map

$$R : H \otimes A \to H \otimes A, \quad h \otimes a \mapsto h_{(1)} \cdot a \otimes h_{(2)},$$

is a coalgebra map which is left multiplicative, normal, left conormal, right S_H-multiplicative and right S_H-conormal. Consequently, there is an R-smash product Hopf quasigroup for H and A, $A \bowtie_R H$. $A \bowtie_R H$ coincides with the smash product $A \bowtie H$ described in [2].

That R satisfies all assumptions of Theorem 2.3 can be checked by straightforward calculations which are left to the reader. We only mention in passing that for the right S_H-conormality of R both Equations (2.14) are needed.

Dually to a Hopf quasigroup, an algebra and coalgebra H with coproduct and counit that are algebra maps is called a *Hopf coquasigroup* provided the product is associative and there exists a linear map $S_H : H \to H$ such that, for all $h \in H$,

$$S_H(h_{(1)})h_{(2)(1)} \otimes h_{(2)(2)} = 1_H \otimes h = h_{(1)}S_H(h_{(2)(1)}) \otimes h_{(2)(2)}$$

and

$$h_{(1)(1)} \otimes h_{(1)(2)}S_H(h_{(2)}) = h \otimes 1_H = h_{(1)(1)} \otimes S_H(h_{(1)(2)})h_{(2)}.$$

When written in terms of commutative diagrams, the definitions of a Hopf quasigroup and Hopf coquasigroup are formally dual to each other in the sense that one is obtained by reversing all the arrows in the definition of the other. Thus the theory of R-smash coproducts for Hopf coquasigroups can be obtained by dualising the theory of R-smash products for Hopf quasigroups. By this means one can first state

Definition 2.9 Let H and A be Hopf coquasigroups and let $W : H \otimes A \to A \otimes H$ be a $\Bbbk$-linear map. By a W-*smash coproduct* of H and A we mean a Hopf coquasigroup $H_W \blacktriangleright\!\!\!\prec A$ that is equal to $H \otimes A$ as a vector space with tensor product unit, multiplication and counit, and with comultiplication and antipode

$$\Delta = (\mathrm{id}_H \otimes W \otimes \mathrm{id}_A) \circ (\Delta_H \otimes \Delta_A), \quad S = \mathsf{flip} \circ (S_A \otimes S_H) \circ W.$$

Then, dualising Theorem 2.3, we obtain the following Hopf coquasigroup version of [3, Corollary 4.8]

Theorem 2.10 *Let H, A be Hopf coquasigroups, $W : H \otimes A \to A \otimes H$ a $\Bbbk$-linear map. If W is left comultiplicative and left normal, then the following statements are equivalent:*

(1) $H_W \blacktriangleright\!\!\!\prec A$ *is a W-smash coproduct Hopf coquasigroup of H and A;*
(2) *The map W is an algebra map that is conormal, right S_H-comultiplicative and right S_H-normal.*

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