Arab J Math (2012) 1:17–27
DOI 10.1007/s40065-012-0018-1

RESEARCH ARTICLE

David F. Anderson · Jodi Fasteen · John D. LaGrange

The subgroup graph of a group

Received: 1 December 2010 / Accepted: 6 March 2011 / Published online: 24 March 2012

Abstract Given any subgroup H of a group G, let $\Gamma_H(G)$ be the directed graph with vertex set G such that x is the initial vertex and y is the terminal vertex of an edge if and only if $x \neq y$ and $xy \in H$. Furthermore, if $xy \in H$ and $yx \in H$ for some $x, y \in G$ with $x \neq y$, then x and y will be regarded as being connected by a single undirected edge. In this paper, the structure of the connected components of $\Gamma_H(G)$ is investigated. All possible components are provided in the cases when $|H|$ is either two or three, and the graph $\Gamma_H(G)$ is completely classified in the case when H is a normal subgroup of G and G/H is a finite abelian group.

Mathematics Subject Classification (2010) 05C25 · 20B05

الملخص

لكل زمرة جزئية معطاة H من الزمرة G، لتكن $\Gamma_H(G)$ البيان المُوَجَّه الذي تحقق مجموعة رؤوسه G خاصية أن x رأس ابتدائي وَ y رأس طرفي لحرف إذا وفقط إذا كانت $y \neq x$ وَ $xy \in H$. علاوة على ذلك، إذا كانت $xy \in H$ وَ $yx \in H$ لبعض عناصر $x, y \in G$ بحيث يكون $x \neq y$، فإنه سوف يتم اعتبار أن x وَ y مترابطتان بوساطة حافة مفردة غير مُوَجَّهَة. يتم في هذه الورقة البحث في بناء مركبات $\Gamma_H(G)$ المترابطة. يتم تقديم جميع المركبات المحتملة في الحالات التي تكون فيها $|H|$ اثنين أو ثلاثة، ويتم تصنيف $\Gamma_H(G)$ بشكل كامل في حال كانت H زمرة جزئية ناظمية من G وَ $G/$ زمرة أبيلية منتهية.

D. F. Anderson (✉)
Mathematics Department, University of Tennessee, Knoxville,
TN 37996-1320, USA
E-mail: anderson@math.utk.edu

J. Fasteen
Fariborz Maseeh Department of Mathematics and Statistics, Portland State University,
Portland, OR 97207-0751, USA
E-mail: jfasteen@pdx.edu

J. D. LaGrange
Division of Natural and Behavioral Sciences, Lindsey Wilson College,
Columbia, KY 42728, USA
E-mail: lagrangej@lindsey.edu

1 Introduction

Given any subgroup H of a group G, let $\Gamma_H(G)$ be the directed graph with vertex set G such that x is the initial vertex and y is the terminal vertex of an edge if and only if $x \neq y$ and $xy \in H$. Furthermore, if $xy \in H$ and $yx \in H$ for some $x, y \in G$ with $x \neq y$, then x and y will be regarded as being connected by a single undirected edge. Thus, $\Gamma_H(G)$ has no loops or multiple edges. In fact, Theorem 2.9 shows that it is superfluous to allow loops in the definition of $\Gamma_H(G)$.

Recently, there has been considerable attention to associating graphs with algebraic structures. Much of this has been motivated by the *zero-divisor graph* $\Gamma(R)$ of a commutative ring R as introduced in [3]. Here, $\Gamma(R)$ is the (undirected) graph with vertices $Z(R)\backslash\{0\}$, the nonzero zero-divisors of R, and two distinct vertices x and y are adjacent if and only if $xy = 0$. For more about $\Gamma(R)$ and related graphs, see the recent survey article [1].

As motivation, let $G = R$ be a commutative ring with nonzero identity (viewed as a group under addition) and assume that $H = Z(R)$, its set of zero-divisors, is closed under addition. Recall that $Z(R)$ is always closed under multiplication, and thus $Z(R)$ is an ideal (in fact, a prime ideal) of R if and only if it is closed under addition. This would be the case, for example, when R is a finite local ring with maximal ideal M, and then $M = Z(R) = nil(R)$. This graph, called the *total graph of* R and denoted by $T(\Gamma(R))$, was introduced in [2]. Specifically, $T(\Gamma(R))$ has vertices of all elements of R, and two distinct vertices are adjacent if and only if $x + y \in Z(R)$. Thus, $\Gamma_{Z(R)}(R) = T(\Gamma(R))$ when $Z(R)$ is an ideal of R. Several results in [2] may be recovered from this paper; for example, [2, Theorem 2.2] follows from Theorem 6.2.

A graph Γ is called *strongly connected* if, for any two distinct vertices v and w of Γ, there exists a set $\{u_1, \ldots, u_n\}$ of vertices of Γ with $u_1 = v$ and $u_n = w$ such that u_i is the initial vertex and u_{i+1} is the terminal vertex of an edge in Γ for each $i \in \{1, \ldots, n-1\}$. Note that any graph has an *undirected underlying graph* that is obtained by regarding every vertex as both an initial vertex and a terminal vertex of each of its incident edges. A graph is called *connected* if its undirected underlying graph is strongly connected. At the outset, a *component* of $\Gamma_H(G)$ is defined as any subgraph of $\Gamma_H(G)$ that is maximal with respect to being connected. However, Theorem 2.3 shows that every component in a subgroup graph is strongly connected. A component will be called *undirected* if every edge in the component is undirected.

Note that we may have $\Gamma_H(G) \cong \Gamma_{H'}(G')$, but $G \ncong G'$. As a specific example, let $G = \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$ and $G' = \mathbb{Z}_2 \oplus \mathbb{Z}_9$ with $H = H' = \{0\}$. Then $\Gamma_H(G)$ and $\Gamma_{H'}(G')$ each consists of two complete graphs K^1 and eight complete bipartite graphs $K^{1,1}$. A less trivial example is provided in Example 6.1. Also, it can happen that $\Gamma_H(G) \cong \Gamma_{H'}(G)$, but $H \ncong H'$. For example, let $G = \mathbb{Z}_4 \oplus \mathbb{Z}_2$ with $H = \{0, 2\} \oplus \mathbb{Z}_2$ and $H' = \mathbb{Z}_4 \oplus \{0\}$. Then $\Gamma_H(G)$ and $\Gamma_{H'}(G)$ each consists of two complete graphs K^4, but $H \ncong H'$.

In Sect. 2, the components of $\Gamma_H(G)$ are determined, and structural properties are revealed. These results are applied in Sect. 3 to provide sufficient conditions for H to be normal in G. In Sects. 4 and 5, we take a first step in completely classifying $\Gamma_H(G)$ by giving every possible component of $\Gamma_H(G)$ in the cases when $|H| = 2$ and $|H| = 3$. The results in Sect. 6 completely classify $\Gamma_H(G)$ in the case when H is a normal subgroup of G and G/H is a finite abelian group.

Throughout, the notation $x \to y$ will be used to indicate that there is an (possibly undirected) edge between x and y, and that $xy \in H$. If $S \subseteq G$, then $\Gamma_H(S)$ will denote the subgraph of $\Gamma_H(G)$ that is induced by S. As usual, $\mathbb{Z}^+$, $\mathbb{Z}_n$, S_n, A_n, K^n, and $K^{m,n}$ will denote the positive integers, the group of integers modulo n, the symmetric group of degree n, the alternating group of degree n, the complete graph on n vertices, and the complete bipartite graph with bipartitions of orders m and n, respectively. Moreover, $N(H) = \{g \in G \mid gHg^{-1} = H\}$ will denote the normalizer of H in G. For any undefined terminology or notation, see [5] or [6] for group theory and [4] for graph theory.

2 The components of $\Gamma_H(G)$

In this section, we determine the connected components of $\Gamma_H(G)$. Moreover, it is shown that the normalizer of H in G is revealed by $\Gamma_H(G)$, and so are the elements $g \in G$ such that $g^2 \in H$. Formulas for computing the degree of any vertex and the order of any component are also established.

Let K be a subgroup of a group G. The following theorem shows that, for a subgroup H of K, any component of $\Gamma_H(K)$ is also a component of $\Gamma_H(G)$.

Theorem 2.1 *Let H and K be subgroups of a group G with $H \subseteq K$. Then $\Gamma_H(G)$ is the disjoint union of the induced subgraphs $\Gamma_H(K)$ and $\Gamma_H(G\backslash K)$ of $\Gamma_H(G)$.*

Proof Let $k \in K$ and $g \in G$. If $g \to k$, then $g \in Hk^{-1} \subseteq K$. Similarly, if $k \to g$, then $g \in k^{-1}H \subseteq K$. The theorem now follows since $G = K \cup (G \backslash K)$ and $K \cap (G \backslash K) = \emptyset$, and the above argument implies that there are no edges in $\Gamma_H(G)$ between any elements of K and $G \backslash K$. □

By Theorem 2.1 together with Cayley's theorem, any subgroup graph is isomorphic to a union of components of a subgroup graph of a symmetric group. In particular, components of subgroup graphs of symmetric groups can serve as models for components of all subgroup graphs. This is stated more formally in the following corollary.

Corollary 2.2 *Let H be a subgroup of a group G. Then there exists a subgroup H' of $S_{|G|}$ such that $\Gamma_H(G)$ is isomorphic to a union of components of $\Gamma_{H'}(S_{|G|})$.*

The next theorem determines the components of subgroup graphs and verifies that all components of subgroup graphs are strongly connected.

Theorem 2.3 *Let H be a subgroup of a group G and $g \in G$. Then the connected component of $\Gamma_H(G)$ containing g is $\Gamma_H(HgH \cup Hg^{-1}H)$. Furthermore, the (directed) distance from any element of $HgH \cup Hg^{-1}H$ to g is at most three, and the (directed) distance from g to any element of $HgH \cup Hg^{-1}H$ is also at most three.*

Proof Let $a \in HgH \cup Hg^{-1}H$. If $a = h_1gh_2$ for some $h_1, h_2 \in H$, then $\{a(h_2^{-1}g^{-1}), (h_2^{-1}g^{-1})g\} \subseteq H$. If $a = h_1g^{-1}h_2$ for some $h_1, h_2 \in H$, then $\{a(h_2^{-1}g), (h_2^{-1}g)g^{-1}, g^{-1}g\} \subseteq H$. Hence, the directed distance from any element of $HgH \cup Hg^{-1}H$ to g is at most three. Similarly, the directed distance from g to any element of $HgH \cup Hg^{-1}H$ is at most three.

To show that $\Gamma_H(HgH \cup Hg^{-1}H)$ is a component of $\Gamma_H(G)$, it remains to verify that if an edge connects a vertex a with any element of $HgH \cup Hg^{-1}H$, then $a \in HgH \cup Hg^{-1}H$. Let $b \in HgH \cup Hg^{-1}H$. Suppose that $b = hgh'$ for some $h, h' \in H$. If $a \in G$ with $ab = h'' \in H$, then $a = h''h'^{-1}g^{-1}h^{-1} \in Hg^{-1}H$. The case when $ba \in H$ is similar, and so are the cases when $b \in Hg^{-1}H$. □

Remark 2.4 Theorem 2.3 implies that any two distinct vertices belonging to the same component of $\Gamma_H(G)$ can be connected by a *directed* path having either one, two, or three edges. The vertices (1, 2, 3) and (1, 4, 3) of $\Gamma_{\{\iota,(1,3)(2,4)\}}(S_4)$ show that, in general, the minimum distance of three does not decrease when the "directed" hypothesis is dropped (see Fig. 1 in Example 2.16) and that all three cases of either one, two, or three edges are possible.

The next result shows that the normalizer of H in G is revealed by $\Gamma_H(G)$. Specifically, $N(H)$ is the union of all the undirected components of $\Gamma_H(G)$. Note that $gH = g^{-1}H$ if and only if $g^2 \in H$. Moreover, if $g \in N(H)$, then $HgH = gH = Hg$ and $Hg^{-1}H = g^{-1}H = Hg^{-1}$.

Theorem 2.5 *Let H be a subgroup of a group G and $g \in G$. Then $g \in N(H)$ if and only if the component of $\Gamma_H(G)$ containing g is undirected. In particular, H is normal in G if and only if $\Gamma_H(G)$ is undirected.*

Proof Suppose that $g \in N(H)$ and $a \to b$ for some vertices a and b contained in the same component as g. Note that $a \in HgH \cup Hg^{-1}H$ by Theorem 2.3. In particular, $a \in N(H)$ since $g \in N(H)$ and $H \subseteq N(H)$. Then $ba = a^{-1}(ab)a \in H$ since $ab \in H$ and H is a normal subgroup of $N(H)$; thus $b \to a$. Therefore, the component of $\Gamma_H(G)$ containing g is undirected.

Conversely, suppose that the component of $\Gamma_H(G)$ containing g is undirected. Let $h \in H$. Then $(hg^{-1})g = h \in H$. Clearly $ghg^{-1} \in H$ if $g = hg^{-1}$. If $g \neq hg^{-1}$, then $hg^{-1} \to g$. Thus, $g \to hg^{-1}$ since the component of $\Gamma_H(G)$ containing g is undirected. Hence $ghg^{-1} \in H$, and thus $g \in N(H)$.

The "in particular" statement is clear. □

Remark 2.6 The structure of $\Gamma_H(N(H))$ is easy to describe. In particular, the structure of $\Gamma_H(G)$ is easy to describe when H is a normal subgroup of G. First, observe that for any $a, b, x, y \in N(H)$ with $a \in xH$ and $b \in yH$, the three relations $a \to b$, $x \to y$, and $yH = x^{-1}H$ are equivalent. Therefore, if $x^2 \in H$, then $\Gamma_H(xH) = \Gamma_H(x^{-1}H)$ is a component of $\Gamma_H(N(H))$ (and it is a component of $\Gamma_H(G)$ by Theorem 2.1) that is isomorphic to the complete graph $K^{|H|}$. If $x^2 \notin H$, then xH and $x^{-1}H$ are disjoint, and hence $\Gamma_H(xH \cup x^{-1}H)$ is a component of $\Gamma_H(N(H))$ which is isomorphic to the complete bipartite graph $K^{|H|,|H|}$.

Let $K = \{x \in N(H) | x^2 \in H\}$; note that K need not be a subgroup of G (e.g., $G = S_3$ and $H = \{e\}$). Then $\Gamma_H(N(H))$ is completely determined by $|K|$ and $|H|$. If $N(H)/H$ is finite, then the number of complete graph components is $\alpha(N(H), H) = |\{xH | x \in K\}|$ and the number of complete bipartite components is $\beta(N(H), H) = |\{xH | x \in N(H)\backslash K\}|/2$. Note that, by Cauchy's theorem, $\alpha(N(H), H) = 1$ if and only if $|N(H)/H|$ is odd; otherwise, $\alpha(N(H), H)$ is even (cf. Theorem 6.2).

The two extremes for $\Gamma_H(G)$ are when $H = \{e\}$ or $H = G$. By the above comments, $\Gamma_G(G)$ is the complete graph on $|G|$ vertices and $\Gamma_{\{e\}}(G)$ consists of $|K|$ complete graphs K^1 and $|G\backslash K|/2$ complete bipartite graphs $K^{1,1}$. Note that for any normal subgroup H of G, the graphs $\Gamma_H(G)$ and $\Gamma_{\{\overline{e}\}}(G/H)$ are easily obtained from one another. In fact, each component consisting of a single vertex xH in $\Gamma_{\{\overline{e}\}}(G/H)$ corresponds to a complete graph component in $\Gamma_H(G)$ on the elements of xH, and each component consisting of two vertices xH and yH in $\Gamma_{\{\overline{e}\}}(G/H)$ corresponds to a complete bipartite graph component in $\Gamma_H(G)$ with bipartitions given by xH and yH.

For the convenience of the reader, the results of Remark 2.6 are summarized in the following theorem.

Theorem 2.7 *Let H be subgroup of a group G, and set $K = \{x \in N(H) | x^2 \in H\}$. If $x \in N(H)$, then the component of $\Gamma_H(N(H))$ containing x is either a complete graph with vertex-set xH or is a complete bipartite graph with vertex-set $xH \cup x^{-1}H$. In particular, every component of $\Gamma_H(N(H))$ is isomorphic to either $K^{|H|}$ or $K^{|H|,|H|}$. Furthermore, if $N(H)/H$ is finite, then the number of complete graph components of $\Gamma_H(N(H))$ is given by $\alpha(N(H), H) = |\{xH | x \in K\}|$ and the number of complete bipartite components of $\Gamma_H(N(H))$ is $\beta(N(H), H) = |\{xH | x \in N(H)\backslash K\}|/2$.*

The *girth* of a graph Γ is the maximum integer n such that Γ contains no directed cycle having less than n edges (if Γ has no directed cycles, then its girth is defined as ∞). While Theorem 2.3 provides information regarding distance in subgroup graphs, the next result shows that the girth is small in components of subgroup graphs that contain cycles. In particular, the proof implies that every vertex of a component that contains a cycle is a vertex of some cycle that has either three or four edges.

Theorem 2.8 *Let H be a subgroup of a group G. Then any component of $\Gamma_H(G)$ is either isomorphic to K^1 or K^2, or it contains a cycle of length three or four. In particular, if $g \notin N(H)$, then the component of $\Gamma_H(G)$ containing g has a cycle.*

Proof Let $g \in G$ be a vertex of a component of $\Gamma_H(G)$. If $g \in N(H)$, then Theorem 2.7 shows that the component containing g is either a complete graph or a complete bipartite graph. Hence, the component containing g is either isomorphic to K^1 or K^2, or it has a cycle of length three or four.

Suppose that $g \notin N(H)$. Then there exists an $h \in H$ such that $gh \neq hg$. In particular, $g \neq h^{-1}gh$.

Case 1 Suppose that $g^2 \in H$. If $gh \neq h^{-1}g$, then the elements g, gh, $h^{-1}gh$, and $h^{-1}g$ are distinct. Thus $g \to gh \to h^{-1}gh \to h^{-1}g \to g$, showing that the component containing g has a cycle of length four. If $gh = h^{-1}g$, then $g \to gh \to h^{-1}gh \to g$ shows that the component containing g has a cycle of length three.

Case 2 Suppose that $g^2 \notin H$. If $gh^{-1} \neq hg^{-1}$, then the elements g, g^{-1}, gh^{-1}, and hg^{-1} are distinct. Thus $g \to g^{-1} \to gh^{-1} \to hg^{-1} \to g$, showing that the component containing g has a cycle of length four. If $gh^{-1} = hg^{-1}$, then $g \to g^{-1} \to gh^{-1} \to g$ shows that the component containing g has a cycle of length three.

The "in particular" statement follows immediately. □

Let H be a subgroup of a finite group G and $g \in G$. Define $N_g = \{v \in G | v \to g \text{ and } g \to v\}$. That is, N_g is the set of vertices of $\Gamma_H(G)$ that are adjacent to g via an undirected edge. Also, recall that the *degree* of a vertex g, denoted by $\deg(g)$, is the cardinality of the set $\{v \in G | \text{ either } v \to g \text{ or } g \to v\}$.

The following result shows that the set $K = \{g \in G | g^2 \in H\}$ is revealed by $\Gamma_H(G)$. In particular, it is redundant to allow loops in the definition of $\Gamma_H(G)$.

Theorem 2.9 *Let G be a finite group and $g \in G$. Then*

$$deg(g) = \begin{cases} 2|H| - |N_g|, & \text{if } g^2 \notin H \\ 2|H| - |N_g| - 2, & \text{if } g^2 \in H \end{cases}.$$

Proof Let $A = \{a \in G | ga \in H \text{ or } ag \in H\}$. Then $|A| = |g^{-1}H \cup Hg^{-1}| = |g^{-1}H| + |Hg^{-1}| - |g^{-1}H \cap Hg^{-1}| = 2|H| - |g^{-1}H \cap Hg^{-1}|$. Also,

$$|N_g| = \begin{cases} |g^{-1}H \cap Hg^{-1}|, & \text{if } g^2 \notin H \\ |g^{-1}H \cap Hg^{-1}| - 1, & \text{if } g^2 \in H \end{cases}.$$

Hence

$$\deg(g) = \begin{cases} |A|, & \text{if } g^2 \notin H \\ |A| - 1, & \text{if } g^2 \in H \end{cases} = \begin{cases} 2|H| - |N_g|, & \text{if } g^2 \notin H \\ 2|H| - (|N_g| + 1) - 1, & \text{if } g^2 \in H \end{cases}.$$

□

Remark 2.10 Given any vertex g of $\Gamma_H(G)$, the *out-degree* of g, denoted by $\deg^+(g)$, is the cardinality of the set $\{v \in G | g \to v\}$. Notice that $\deg^+(g) = \begin{cases} |g^{-1}H|, & \text{if } g^2 \notin H \\ |g^{-1}H| - 1, & \text{if } g^2 \in H \end{cases}$. One defines the *in-degree* of g analogously, and it is clear that $\deg^-(g) = \begin{cases} |Hg^{-1}|, & \text{if } g^2 \notin H \\ |Hg^{-1}| - 1, & \text{if } g^2 \in H \end{cases}$. In particular, it follows that

$$\deg^-(g) = \deg^+(g) = \begin{cases} |H|, & \text{if } g^2 \notin H \\ |H| - 1, & \text{if } g^2 \in H \end{cases}$$

for any $g \in G$.

Let $g \in G$. Any subgroup H of G acts on the set of cosets $X = \{(hg)H | h \in H\}$ by $h' * (hgH) = (h'hg)H$. Given any $x \in X$, let $H_x = \{h \in H | h * x = x\}$.

It is straightforward to check that $H_{gH} = H \cap gHg^{-1}$ for all $g \in G$. Moreover, it is well known that HgH and $Hg^{-1}H$ both have cardinalities equal to $|H|^2/|H \cap (gHg^{-1})|$ (this follows from a more general result for *double cosets* [6, Theorem 1.7.1]). In particular, $|HgH| = |Hg^{-1}H| = |H|^2/|H_{gH}|$. These observations together with Theorem 2.3 imply the validity of the next result. Note that the proof provided below is independent of the known facts about double cosets.

Theorem 2.11 *Let H be a subgroup of a finite group G and $g \in G$. If C is the vertex set of the component of $\Gamma_H(G)$ containing g, then*

$$|C| = \begin{cases} 2|H|^2/|H_{gH}|, & \text{if } HgH \cap Hg^{-1}H = \emptyset \\ |H|^2/|H_{gH}|, & \text{otherwise} \end{cases}.$$

Proof Given any $h \in H$, it is straightforward to check that the mapping $H_{gH} \to \{h' | (hg)H = (h'g)H\}$ defined by the rule $\alpha \mapsto h\alpha$ is a bijection. Thus,

$$|HgH| = |\cup_{h \in H} (hg)H| = |H|^2/|H_{gH}|.$$

Also, the mapping $H_{gH} \to H_{g^{-1}H}$ defined by the rule $\alpha \mapsto g^{-1}\alpha g$ is bijective, and it follows that $|Hg^{-1}H| = |H|^2/|H_{gH}|$. Therefore, if $HgH \cap Hg^{-1}H = \emptyset$, then

$$|HgH \cup Hg^{-1}H| = 2|H|^2/|H_{gH}| - |HgH \cap Hg^{-1}H| = 2|H|^2/|H_{gH}|.$$

But it is easy to check that $HgH \cap Hg^{-1}H \neq \emptyset$ if and only if $HgH = Hg^{-1}H$ (cf. [6, Lemma 1.7.1]). Therefore,

$$|HgH \cup Hg^{-1}H| = 2|H|^2/|H_{gH}| - |HgH| = |H|^2/|H_{gH}|$$

if $HgH \cap Hg^{-1}H \neq \emptyset$. The result now follows by Theorem 2.3. □

Since H_{gH} is a subgroup of H, the next corollary follows immediately from Theorem 2.11 and Lagrange's theorem.

Corollary 2.12 *Let H be a subgroup of a finite group G. Then $|H|$ divides the order of any component of $\Gamma_H(G)$.*

Corollary 2.13 *Let H be a subgroup of a finite group G, and let C be the set of vertices of a component of $\Gamma_H(G)$. If $|H| = |C|$, then $\Gamma_H(C) \cong K^{|H|}$. In particular, if $|C|$ is prime and $|H| > 1$, then $\Gamma_H(C) \cong K^{|H|}$. In this case, $C \subseteq N(H)$.*

Proof If $|H| = |C|$, then Theorem 2.11 implies that the equality $H_{gH} = H$ holds for each $g \in C$. Hence $g \in N(H)$ for all $g \in C$, and thus $\Gamma_H(C) \cong K^{|H|}$ by Theorem 2.7. The "in particular" statement then follows from Corollary 2.12. □

Remark 2.14 If $g \in N(H)$, then $HgH \cap Hg^{-1}H = gH \cap g^{-1}H$, and it follows that $g^2 \notin H$ if and only if $HgH \cap Hg^{-1}H = \emptyset$. Furthermore,

$$H_{gH} = \{h \in H \mid hgH = gH\} = \{h \in H \mid g^{-1}hg \in H\} = H.$$

In this case, the result in Theorem 2.11 reduces to

$$|C| = \begin{cases} 2|H|, & \text{if } g^2 \notin H \\ |H|, & \text{otherwise} \end{cases}.$$

(cf. Theorem 2.7.)

Let $x \in G\backslash N(H)$ such that $HxH \cap Hx^{-1}H = \emptyset$. Then we have the disjoint union $HxH = Hxh_1 \cup \cdots \cup Hxh_n$ for some $h_1, \ldots, h_n \in H$ (assume here that this is a finite union; say that G is finite). Also, then $Hx^{-1}H = h_1^{-1}x^{-1}H \cup \cdots \cup h_n^{-1}x^{-1}H$ is a disjoint union.

Recall that the number of right cosets Hz in HxH is $[H : H \cap x^{-1}Hx] = |H|/|H \cap x^{-1}Hx| (= n)$, and the number of left cosets zH in HxH is $[x^{-1}Hx : H \cap x^{-1}Hx] = |H|/|H \cap x^{-1}Hx|$ ([6, Theorem 1.7.1]). The only left-to-right edges from HxH to $Hx^{-1}H$ are from each vertex in Hxh_i to each vertex in $h_i^{-1}x^{-1}H$ (there are no edges from HxH to HxH, or from $Hx^{-1}H$ to $Hx^{-1}H$). So from HxH to $Hx^{-1}H$, there are $n|H|^2$ such edges. Similarly, we get all the right-to-left edges, i.e., from $Hx^{-1}H$ to HxH. Which of these directed edges is an undirected edge?

Let $hxh_i \to h_i^{-1}x^{-1}k$ be an edge for some $h, k \in H$. Then there is an edge in the reverse direction $\Leftrightarrow (h_i^{-1}x^{-1}k)(hxh_i) \in H \Leftrightarrow x^{-1}khx \in H \Leftrightarrow kh \in xHx^{-1} \Leftrightarrow kh \in xHx^{-1} \cap H$. So let $k \in H$ be fixed. Then $h \in k^{-1}(xHx^{-1} \cap H)$. Thus, there are $|H||xHx^{-1} \cap H|$ two-way edges in Hxh_i to $h_i^{-1}x^{-1}H$, and hence there are $(|H||xHx^{-1} \cap H|)(|H|/|H \cap xHx^{-1}|) = |H|^2$ undirected edges in the connected component $\Gamma_H(HxH \cup Hx^{-1}H)$. Therefore, since there are $n|H|^2$ left-to-right edges and $n|H|^2$ right-to-left edges, the connected component $\Gamma_H(HxH \cup Hx^{-1}H)$ has $2n|H|^2 - |H|^2 = (2|H|^3/|H \cap x^{-1}Hx|) - |H|^2$ edges.

Alternatively, graph-theoretic results can be used to arrive at the same conclusion. If $t \in H_{xH}$, then $x^{-1}t \neq x$ since $x^2 \notin H$. Thus, $x^{-1}t \in N_x$, and it follows that the mapping $p : N_x \to H_{xH}$ defined by $p(a) = xa$ is a bijection. Thus, $|N_x| = |H_{xH}|$.

Suppose that $g \in HxH \cup Hx^{-1}H$; say $g = h_1xh_2$ for some $h_1, h_2 \in H$. Then $HgH \cap Hg^{-1}H = (Hh_1xh_2H) \cap (H(h_1xh_2)^{-1}H) = HxH \cap Hx^{-1}H = \emptyset$. Furthermore, the mapping $q : H_{xH} \to H_{gH}$ defined by $q(a) = h_1ah_1^{-1}$ is a bijection. So $|H_{xH}| = |H_{gH}|$. In particular, $|N_x| = |N_g| = |H_{xH}|$. Similar equalities hold if $g \in Hx^{-1}H$. Hence Theorem 2.8 implies that the underlying undirected graph of $\Gamma_H(HxH \cup Hx^{-1}H)$ is a $(2|H| - |H_{xH}|)$-regular graph. Furthermore, the graph obtained by deleting all of the directed edges in $\Gamma_H(HxH \cup Hx^{-1}H)$ is a $|H_{xH}|$-regular graph.

It is well known that any r-regular graph on v vertices has $vr/2$ edges (cf. [4, p. 4]). Thus, $\Gamma_H(HxH \cup Hx^{-1}H)$ has $|HxH \cup Hx^{-1}H|(2|H| - |H_{xH}|)/2 = (2|H|^2/|H_{xH}|)(2|H| - |H_{xH}|)/2 = (2|H|^3/|H_{xH}|) - |H|^2$ edges, where the first equality follows from Theorem 2.11. Similarly, it follows that there are $|HxH \cup Hx^{-1}H||H_{xH}|/2 = |H|^2$ undirected edges in $\Gamma_H(HxH \cup Hx^{-1}H)$. We record these observations in the following theorem.

Theorem 2.15 *Let H be a subgroup of a finite group G, and let $x \in G\backslash N(H)$ such that $HxH \cap Hx^{-1}H = \emptyset$. If C is the set of vertices of the component of $\Gamma_H(G)$ containing x, then $\Gamma_H(C)$ has $2|H|^3/|H_{xH}| - |H|^2$ edges, and $|H|^2$ undirected edges. In particular, if $|H|$ is prime, then $\Gamma_H(C)$ has $2|H|^3 - |H|^2$ edges.*

Example 2.16 Let $H = \{\iota, (1, 3)(2, 4)\} \leq S_4$. By Theorem 2.11, the order of any component of $\Gamma_H(S_4)$ is either 2, 4, or 8. By Theorem 2.5, it can be seen in Fig. 1 that $N(H) = \{\iota, (1, 3)(2, 4), (1, 2, 3, 4), (1, 4, 3, 2), (1, 2)(3, 4), (1, 4)(2, 3), (1, 3), (2, 4)\}$. Furthermore, it is immediately seen from Theorem 2.9 and Fig. 1 (even if the vertices were not labeled) which vertices would be looped if the definition of $\Gamma_H(G)$ allowed for loops. Note that Theorem 2.15 shows that any component of $\Gamma_H(S_4)$ with eight vertices has $2(2)^3 - (2)^2 = 12$ edges, and $(2)^2 = 4$ undirected edges. As an illustration of Theorem 2.1, the reader can verify that any subgroup of S_4 containing H can be realized as the set of vertices of a union of some components in Fig. 1.

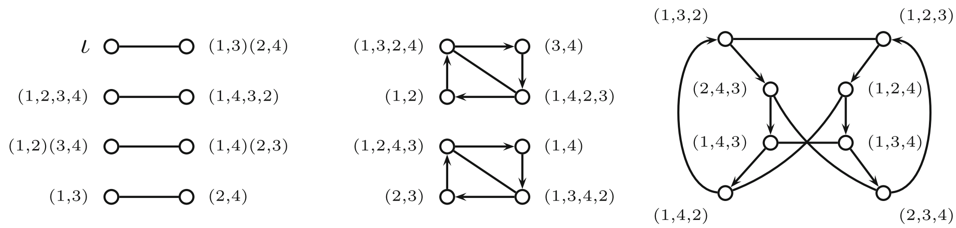


Fig. 1 The graph $\Gamma_{\{\iota,(1,3)(2,4)\}}(S_4)$

3 Applications

Suppose that $t < p$ for some positive integers t and p with p prime, and let G be a group of order $p^n t$ for some positive integer n. It is a straightforward consequence of Sylow's theorem that if H is a subgroup of G with $|H| = p^n$, then H is a normal subgroup G. The above results allow for a generalization of this fact, and the proof below may be of some interest since it is independent of Sylow's theorem.

Theorem 3.1 *Let H be a proper subgroup of a finite group G, and let p be the smallest prime number dividing $|H|$. If $|G|/|H| \leq p$, then H is a normal subgroup of G.*

Proof Suppose that $g \in G$. Let C be the vertex set of the component of $\Gamma_H(G)$ containing g, and set $t = |G|/|H|$. Since $H \neq G$, it follows that $C \neq G$ (e.g., Theorem 2.3 shows that $\Gamma_H(H)$ is a component of $\Gamma_H(G)$). Thus, $|C| \leq |G| - |H| = |H|(t-1)$ since $|H|$ divides the order of any component by Corollary 2.12. Hence

$$|H|/|H_{gH}| \leq |C|/|H| \leq t - 1 < p,$$

where the first inequality holds by Theorem 2.11. But either $|H|/|H_{gH}| = 1$ or there is a prime factor of $|H|$ dividing $|H|/|H_{gH}|$. The minimality of p eliminates the latter case, and thus $H = H_{gH}$. That is, $H = \{h \in H | (hg)H = gH\} = \{h \in H | g^{-1}hg \in H\}$. Thus, $g \in N(H)$. Since g was chosen arbitrarily, H is a normal subgroup of G. □

Example 3.2 Let G be a finite group with $|G| = 700 = 2^2 \cdot 5^2 \cdot 7$, and let H be a subgroup of G with $|H| = 175 = 5^2 \cdot 7$. Then $p = 5$ is the smallest prime number dividing $|H|$ and $|G|/|H| = 4 < 5$. Thus, H is a normal subgroup of G by Theorem 3.1.

The next result is well known (cf. [5, Corollary 4.5]), and it immediately follows from Theorem 3.1 since any prime number dividing $|H|$ is a prime number dividing $|G|$.

Corollary 3.3 *Let H be a proper subgroup of a finite group G, and let p be the smallest prime number dividing $|G|$. If $|G|/|H| = p$, then H is a normal subgroup of G.*

4 The case when $|H| = 2$

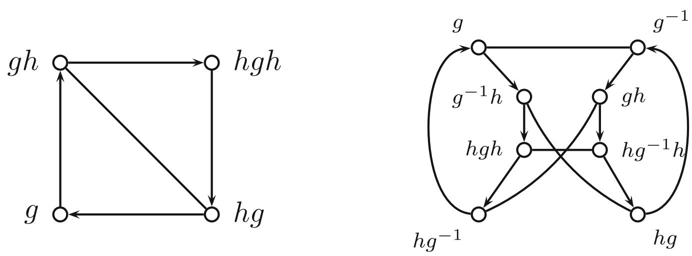
Recall from Sect. 2 that, for a vertex g of the graph $\Gamma_H(G)$, we define the set $N_g = \{v \in G | v \to g$ and $g \to v\}$. That is, N_g is the set of all vertices of $\Gamma_H(G)$ that are adjacent to g via an undirected edge.

Lemma 4.1 *Let H be a subgroup of a finite group G such that $|H| = 2$, and let $g \in G$. Then the following statements hold.*

(1) *The inequality $|N_g| \leq 2$ holds if $g^2 \notin H$, and the inequality $|N_g| \leq 1$ holds if $g^2 \in H$.*
(2) *Suppose that $g \notin N(H)$. Then $|N_g| = 0$ if and only if $g^2 \in H$. In this case, $g = g^{-1}$.*

Proof Note that $\deg(g) \geq 1$ since $|H| > 1$. By Theorem 2.9, it follows that $1 \leq 4 - |N_g| - 2$ if $g^2 \in H$. Also, if $g^2 \notin H$ and $|N_g| \geq 3$, then Theorem 2.9 shows that $\deg(g) = 4 - |N_g| \leq 1$. But clearly $3 \leq |N_g| \leq \deg(g)$. This is a contradiction, and the first claim is proved.

To prove (2), it is clear that $g = g^{-1}$ if $|N_g| = 0$; in particular, $g^2 \in H$ if $|N_g| = 0$. It remains to show that $|N_g| = 0$ if $g^2 \in H$.



(a) Γ_1 **(b)** Γ_2

Fig. 2 The possible components when $|H| = 2$ and $g \notin N(H)$

Suppose that $g \notin N(H)$ and $g^2 \in H$. Then $|N_g| \le 1$ by (1). If $|N_g| = 1$, then $\deg(g) = 4 - 1 - 2 = 1$ by Theorem 2.9. Also, the proof of Theorem 2.8 shows that the component of $\Gamma_H(G)$ containing g is either isomorphic to K^1 or K^2, or g is the vertex of a cycle in $\Gamma_H(G)$. But the first and last of these possibilities are eliminated since $\deg(g) = 1$, and the second possibility fails by Theorem 2.5 because $g \notin N(H)$. Therefore, $|N_g| \neq 1$; that is, $|N_g| = 0$. □

The following theorem characterizes the possible components of $\Gamma_H(G)$ when $|H| = 2$. That the four given components are realizable can be seen from Example 2.16 and the graph $\Gamma_{\{0,3\}}(\mathbb{Z}_6)$.

Theorem 4.2 *Let H be a subgroup of a finite group G such that $|H| = 2$. If C is the set of vertices of a component of $\Gamma_H(G)$, then $\Gamma_H(C)$ is isomorphic to either K^2, $K^{2,2}$, Γ_1, or Γ_2, where Γ_1 and Γ_2 are the graphs given in Fig. 2.*

Proof Suppose that $H = \{e, h\}$, where e denotes the identity element of G, and let $g \in C$. If $g \in N(H)$, then $\Gamma_H(C)$ is isomorphic to either K^2 or $K^{2,2}$ by Theorem 2.7. Henceforth, assume that $g \notin N(H)$. Note that $|C| \in \{4, 8\}$ by Theorem 2.11 and Corollary 2.13.

Case 1 Assume that $|C| = 4$. By Theorem 2.11, it is the case that $HgH \cap Hg^{-1}H \neq \emptyset$, i.e., $HgH = Hg^{-1}H$. By Theorem 2.3, it follows that $C = \{g, gh, hg, hgh\}$. If $g^{-1} = gh$, then $g(hg) = (gh)g = e$ implies that $g^{-1} = hg$, which contradicts that the elements of C are distinct. Thus, $g^{-1} \neq gh$. Similarly, $g^{-1} \neq hg$. If $g^{-1} = hgh$, then $(gh)^2 = e \in H$. Thus, in any case, it can be assumed that C contains an element whose square is the identity element in H. Without loss of generality, let $g^2 = e \in H$. Hence $(hgh)^2 = e \in H$, and $(gh)^2, (hg)^2 \notin H$ since $g \notin N(H)$. Now it is clear that the adjacency relations given in Fig. 2a hold. That these are the only adjacency relations follows immediately from Remark 2.10.

Case 2 Assume that $|C| = 8$. Then Theorem 2.3 implies that $C = \{g, gh, hg, hgh\} \cup \{g^{-1}, g^{-1}h, hg^{-1}, hg^{-1}h\}$. Clearly the adjacency relations given in Fig. 2b hold. By Remark 2.10, these are the only adjacency relations. □

5 The case $|H| = 3$

Before describing the possible components when $|H| = 3$, it will be convenient to introduce the following construction. Given a component $\Gamma_H(C)$ of $\Gamma_H(G)$, let $\Gamma_H(C)^*$ be the graph whose vertices are the elements of $(C \times \{1\}) \cup (\{1\} \times C)$ such that v and w are the initial and terminal vertices of an edge, respectively, if and only if there exist elements $a, b \in C$ with $ab \in H$ such that either $v = (a, 1)$ and $w = (1, b)$, or $v = (1, a)$ and $w = (b, 1)$ (here, a and b need not be distinct). For example, it is easy to check that if $\Gamma_H(C) \cong K^n$ for some $n \in \mathbb{Z}^+$, then $\Gamma_H(C)^* \cong K^{n,n}$. Furthermore, it is straightforward to show that $\Gamma_2 \cong \Gamma_1^*$, where Γ_1 and Γ_2 are the graphs in Fig. 2. It is shown below that these observations extend to the components of $\Gamma_H(G)$ when $|H| = 3$. In particular, there are two possible components (up to isomorphism) corresponding to the case when $|H_{gH}| = 1$; namely, the graphs Γ and Γ^*, where Γ is the graph of order nine in Fig. 3.

Example 5.1 Let $G = A_4 \times \mathbb{Z}_3$ and set $\sigma = (1, 2, 3) \in A_4$. Consider the subgroup $H = \{(\iota, 0), (\sigma, 0), (\sigma^2, 0)\}$ of G. Note that the graph $\Gamma_{\{\iota,\sigma,\sigma^2\}}(A_4)$ is given in Fig. 3, and it is isomorphic to the subgraph of $\Gamma_H(G)$ whose vertices consist of the elements in $\{(\alpha, 0) | \alpha \in A_4\}$.

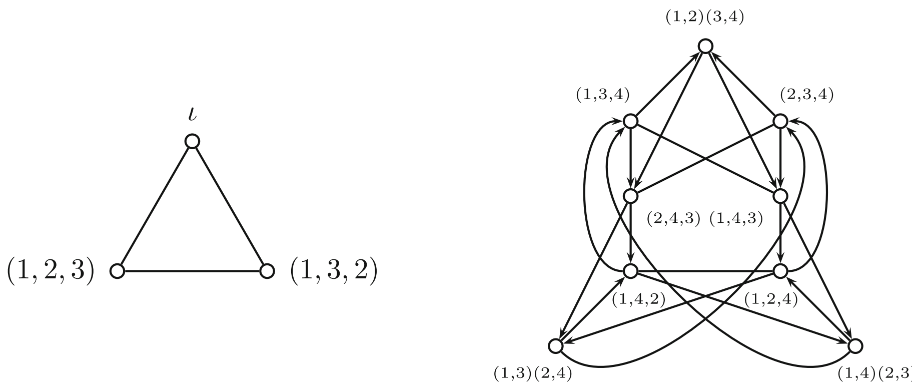


Fig. 3 The graph $\Gamma_{\{\iota,(1,2,3),(1,3,2)\}}(A_4)$

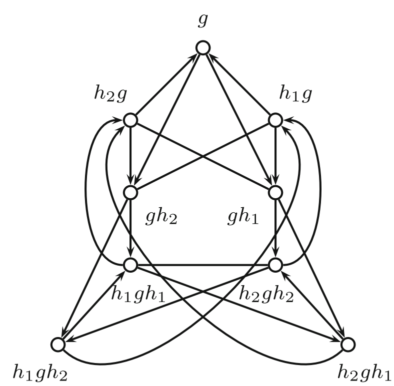


Fig. 4 The only possible component having nine vertices when $|H| = 3$

Let Γ be the component of order nine in Fig. 3. Define a bijection ψ from the set $C = (A_4\backslash\{\iota, \sigma, \sigma^2\}) \times (\mathbb{Z}_3\backslash\{0\})$ onto the vertices of Γ^* by $\psi(\alpha, 1) = (\alpha, 1)$ and $\psi(\alpha, 2) = (1, \alpha)$ for each $\alpha \in A_4\backslash\{\iota, \sigma, \sigma^2\}$. If $(\alpha_1, z_1), (\alpha_2, z_2) \in C$, then clearly $(\alpha_1, z_1) \to (\alpha_2, z_2)$ if and only if $\psi(\alpha_1, z_1)$ and $\psi(\alpha_2, z_2)$ are the initial and terminal vertices, respectively, of an edge in Γ^*. Thus, $\Gamma_H(C) \cong \Gamma^*$. Similarly, it is easy to check that $(\{\iota, \sigma, \sigma^2\} \times \{1\}) \cup (\{\iota, \sigma, \sigma^2\} \times \{2\})$ are the vertices of a component of $\Gamma_H(G)$ that is isomorphic to $\Gamma_{\{\iota,\sigma,\sigma^2\}}(\{\iota, \sigma, \sigma^2\})^* \cong K^{3,3}$. Therefore, $\Gamma_H(G)$ is isomorphic to the union of the four graphs K^3, $K^{3,3}$, Γ, and Γ^*.

The following theorem characterizes the possible components of $\Gamma_H(G)$ when $|H| = 3$. That the four given components are realizable can be seen from Example 5.1.

Theorem 5.2 *Let H be a subgroup of a finite group G such that $|H| = 3$. If C is the set of vertices of a component of $\Gamma_H(G)$, then $\Gamma_H(C)$ is isomorphic to either K^3, $K^{3,3}$, Γ, or Γ^*, where Γ is the graph given in Fig. 4.*

Proof Suppose that $H = \{e, h_1, h_2\}$, where e denotes the identity element of G, and let $g \in C$. If $g \in N(H)$, then $\Gamma_H(C)$ is isomorphic to either K^3 or $K^{3,3}$ by Theorem 2.7. Henceforth, assume that $g \notin N(H)$. Note that $|C| \in \{9, 18\}$ by Theorem 2.11 and Corollary 2.13.

Case 1 Suppose that $|C| = 9$. By Theorem 2.11, it is the case that $HgH \cap Hg^{-1}H \neq \emptyset$, i.e., $HgH = Hg^{-1}H$. By Theorem 2.3, it follows that $C = \{g, gh_1, h_1g, h_1gh_1, gh_2, h_2g, h_2gh_2, h_1gh_2, h_2gh_1\}$. The inverse of any element in C is again an element in C, and $|C|$ is odd. This implies that C contains an element that is its own inverse. Therefore, without loss of generality, assume that $g^2 = e$. Hence $(h_1gh_2)^2 = (h_2gh_1)^2 = e$. Now it is clear that the adjacency relations given in Fig. 4 hold. That these are the only adjacency relations follows by Remark 2.10.

Case 2 Suppose that $|C| = 18$ and let $a \in C$. By Theorem 2.3, it follows that $C = \{a, ah_1, h_1a, h_1ah_1, ah_2, h_2a, h_2ah_2, h_1ah_2, h_2ah_1\} \cup \{a^{-1}, a^{-1}h_1, h_1a^{-1}, h_1a^{-1}h_1, a^{-1}h_2, h_2a^{-1}, h_2a^{-1}h_2, h_1a^{-1}h_2, h_2a^{-1}h_1\}$.

Let Γ be the graph given in Fig. 4, and define a mapping ψ from the vertices of Γ^* onto C such that $\psi(hgh', 1) = hah'$ and $\psi(1, hgh') = ha^{-1}h'$ for all vertices $(hgh', 1)$ and $(1, hgh')$ of Γ^*. Clearly ψ is a bijection. Also, it is straightforward to check that if v and w are the initial and terminal vertices, respectively, of an edge in Γ^*, then $\psi(v) \to \psi(w)$. But $\deg^+(v) = \deg^-(v) = 3$ for every vertex v of Γ^*, and thus the converse of the previous statement is valid by Remark 2.10. Hence $\Gamma_H(C) \cong \Gamma^*$. □

6 The case when G/H is a finite abelian group

Let H be a normal subgroup of a group G such that G/H is a finite abelian group, and let $K = \{x \in G \mid x^2 \in H\}$. It is easily verified that K is a subgroup of G containing H.

We have already observed that $\Gamma_H(G)$ has two types of connected components: (i) complete graph components of the form xH for $x \in K$, and (ii) complete bipartite graph components of the form $xH \cup x^{-1}H$ for $x \in G\backslash K$ (see Theorem 2.7). Let $\alpha(G, H)$ be the number of complete graph components of $\Gamma_H(G)$ and $\beta(G, H)$ be the number of complete bipartite graph components of $\Gamma_H(G)$. Then $\alpha = \alpha(G, H) = |K/H| \geq 1$, $\beta = \beta(G, H) = (|G/H| - |K/H|)/2 \geq 0$, and $|G| = |H|(\alpha + 2\beta)$.

In Theorem 6.2, we give necessary conditions on $\alpha(G, H)$ and $\beta(G, H)$; in Theorem 6.3, we show that these conditions are also sufficient to realize such a graph as $\Gamma_H(G)$. However, we first give an example where $\Gamma_H(G) \cong \Gamma_{H'}(G')$, but $G \ncong G'$, $H \ncong H'$, and $K \ncong K'$.

Example 6.1 Let $G = \mathbb{Z}_2 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_{25}$ with $H = \{0\} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_9 \oplus \{0\}$, and let $G' = \mathbb{Z}_2^4 \oplus \mathbb{Z}_3^2 \oplus \mathbb{Z}_5^2$ with $H' = \mathbb{Z}_2^2 \oplus \{0\}^2 \oplus \mathbb{Z}_3^2 \oplus \{0\}^2$. Then $K = \{g \in G | 2g \in H\} = \mathbb{Z}_2 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_9 \oplus \{0\}$ and $K' = \{g \in G' | 2g \in H'\} = \mathbb{Z}_2^4 \oplus \mathbb{Z}_3^2 \oplus \{0\}^2$. Also, $|G| = |G'| = 3600$, $|H| = |H'| = 36$, and $|K| = |K'| = 144$. Therefore, by Theorem 2.7, it is straightforward to check that $\Gamma_H(G) \cong \Gamma_{H'}(G')$. In fact, each graph consists of four complete graphs K^{36} and 48 complete bipartite graphs $K^{36,36}$. Note that none of the groups are isomorphic.

Theorem 6.2 *Let H be a normal subgroup of a group G such that G/H is a finite abelian group. Then $\alpha(G, H) = 2^k$ for some integer $k \geq 0$ and $2^j | \beta(G, H)$, where $j = max\{k - 1, 0\}$.*

Proof Let $\alpha = \alpha(G, H)$ and $\beta = \beta(G, H)$. By the Fundamental Theorem of Finite Abelian Groups, $G/H \cong \mathbb{Z}_{2^{m_1}} \oplus \cdots \oplus \mathbb{Z}_{2^{m_k}} \oplus L$, where $k \geq 0$, each $m_i \geq 1$, and $|L| \geq 1$ is odd. Thus, $K/H \cong (\mathbb{Z}_2)^k$; so $\alpha = |K/H| = 2^k$. Now $|G/H| = (2^{m_1 + \cdots + m_k})|L| = \alpha + 2\beta = 2^k + 2\beta$ with $m_1 + \cdots + m_k \geq k$. Hence $2^k | 2\beta$, and thus $2^j | \beta$, where $j = \max\{k - 1, 0\}$. □

Theorem 6.3 *Let $k \geq 0$ be an integer, $j = max\{k - 1, 0\}$, $n \geq 0$ an integer such that $2^j | n$, and $h \geq 1$ an integer. Then there is a finite abelian group G with subgroup H such that $|H| = h$, $\alpha(G, H) = 2^k$, and $\beta(G, H) = n$. Moreover, $|G| = 2^k |H| + n|H| = (2^k + n)h$.*

Proof We prove this in several cases. Let $m = 2^k$, $\alpha = \alpha(G, H)$, and $\beta = \beta(G, H)$.

(a) Let $m = 2^k$ with $k \geq 0$, and $n = 0$.
Define $G = (\mathbb{Z}_2)^k \oplus \mathbb{Z}_h$ and $H = \{0\} \oplus \mathbb{Z}_h$. Then $K/H = G/H \cong (\mathbb{Z}_2)^k$. Thus, $\alpha = |K/H| = 2^k = m$ and $\beta = (|G/H| - |K/H|)/2 = 0 = n$.

(b) Let $m = 2^0 = 1$ and $n \geq 1$.
Define $G = \mathbb{Z}_{2n+1} \oplus \mathbb{Z}_h$ and $H = \{0\} \oplus \mathbb{Z}_h$. Then $G/H \cong \mathbb{Z}_{2n+1}$, and $K/H = \{0\}$ since $2n + 1$ is odd. Thus, $\alpha = |K/H| = 1$ and $\beta = (|G/H| - |K/H|)/2 = (2n + 1 - 1)/2 = n$.

(c) Let $m = 2^k$ with $k \geq 1$, and $n \geq 1$. Write $n = 2^r q$ with $r \geq j$ and $q \geq 1$ odd. We consider two subcases.

  (i) Let $r > j \geq k - 1 \geq 0$.
  Define $G = (\mathbb{Z}_2)^k \oplus \mathbb{Z}_d \oplus \mathbb{Z}_h$, where $d = 2^{r-k+1}q + 1$ and $H = \{0\}^{k+1} \oplus \mathbb{Z}_h$. Then $G/H \cong (\mathbb{Z}_2)^k \oplus \mathbb{Z}_d$, and $K/H \cong (\mathbb{Z}_2)^k$ since d is odd. Thus, $\alpha = |K/H| = 2^k$ and $\beta = (|G/H| - |K/H|)/2 = (2^k(2^{r-k+1}q + 1) - 2^k)/2 = 2^{r+1}q/2 = 2^r q = n$.

  (ii) Let $r = j = k - 1 \geq 0$.
  Since q is odd, $q + 1 = 2^{b-k}d$ for integers b and d with $b > k$ and $d \geq 1$ odd. Define $G = (\mathbb{Z}_2)^{k-1} \oplus \mathbb{Z}_{2^{b-k+1}} \oplus \mathbb{Z}_d \oplus \mathbb{Z}_h$ and $H = \{0\}^{k+1} \oplus \mathbb{Z}_h$. Then $G/H \cong (\mathbb{Z}_2)^{k-1} \oplus \mathbb{Z}_{2^{b-k+1}} \oplus \mathbb{Z}_d$ and $K/H \cong (\mathbb{Z}_2)^k$ since d is odd. Thus, $\alpha = |K/H| = 2^k$ and $\beta = (|G/H| - |K/H|)/2 = (2^{k-1}2^{b-k+1}d - 2^k)/2 = 2^k(2^{b-k}d - 1)/2 = 2^k(q + 1 - 1)/2 = 2^k q/2 = 2^{k-1}q = 2^r q = n$. □

Example 6.4 We illustrate each case in the above proof. Let $\alpha = \alpha(G, H)$, $\beta = \beta(G, H)$, and $h = |H| = 6$. The reader may verify that the given group G and subgroup $H \cong \mathbb{Z}_6$ yield the desired α and β.

(a) For $\alpha = 8 = 2^3$ and $\beta = 0$, let $G = (\mathbb{Z}_2)^3 \oplus \mathbb{Z}_6$ and $H = \{0\}^3 \oplus \mathbb{Z}_6$.
(b) For $\alpha = 1 = 2^0$ and $\beta = 7$, let $G = \mathbb{Z}_{15} \oplus \mathbb{Z}_6$ and $H = \{0\} \oplus \mathbb{Z}_6$.
(c) (i) For $\alpha = 4 = 2^2$ and $\beta = 40 = 2^3 \cdot 5$, let $G = (\mathbb{Z}_2)^2 \oplus \mathbb{Z}_{21} \oplus \mathbb{Z}_6$ and $H = \{0\}^3 \oplus \mathbb{Z}_6$.

(ii) For $\alpha = 4 = 2^2$ and $\beta = 18 = 2 \cdot 3^2$, let $G = \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_6$ and $H = \{0\}^3 \oplus \mathbb{Z}_6$.

Note that the “G/H is abelian” hypothesis in Theorem 6.2 is necessary. For example, let G be the dihedral group of order 16, and let $H = \{e, r^4\}$, where r is a rotation of $\pi/4$ radians. If R is the set of eight reflections in G, then $K = \{e, r^2, r^4, r^6\} \cup R$. So $|K| = 12$, and hence $\alpha(G, H) = 6$.

If, on the other hand, K happens to be a group, then every element of K/H has order at most two. In this case, K/H is an abelian group. If K/H is also finite, then Theorem 6.2 implies that $\alpha(G, H) = 2^k$ for some integer $k \geq 0$. This yields the following corollary.

Corollary 6.5 *Let H be a normal subgroup of a group G such that G/H is a finite group. If $K = \{x \in G | x^2 \in H\}$ is a group, then $\alpha(G, H) = 2^k$ for some integer $k \geq 0$.*

Example 6.6 Let H' be any subgroup of a finite abelian group G', and let $K' = \{x \in G' | x^2 \in H'\}$. Let $G = A_4 \times G'$ and $H = \{(1)\} \times H'$. Then H is a normal subgroup of G and $G/H \cong A_4 \times G'/H'$ is a finite nonabelian group. Moreover, $K = \{x \in G | x^2 \in H\} = \{(1), (1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3)\} \times K'$ is a group. By Corollary 6.5, $\alpha(G, H) = 2^k$ for some integer $k \geq 0$. (Alternatively, note that $\alpha(G, H) = |K|/|H| = 4|K'|/|H'| = 4 \cdot \alpha(G', H')$, which is a power of two by Theorem 6.2.)

Acknowledgments This research was begun while J. Fasteen participated in an NSF sponsored REU program at the University of Tennessee during the summer of 2002. We would like to thank the referees for carefully reviewing this article. Their suggestions improved the quality of this paper.

References

1. Anderson, D.F.; Axtell, M.C.; Stickles, J.A. Jr.: Zero-divisor graphs in commutative rings. In: Fontana, M.; Kabbaj, S.-E.; Olberding, B.; Swanson, I. (eds.) Commutative Algebra, Noetherian and Non-Noetherian Perspectives, pp. 23–45. Springer, New York (2011)
2. Anderson, D.F.; Badawi, A.: The total graph of a commutative ring. J. Algebra **320**, 2706–2719 (2008)
3. Anderson, D.F.; Livingston, P.S.: The zero-divisor graph of a commutative ring. J. Algebra **217**, 434–447 (1999)
4. Bollobás, B.: Modern Graph Theory. Springer, New York (1998)
5. Dummit, D.S.; Foote, R.M.: Abstract Algebra, 3rd edn. Wiley, New York (2004)
6. Hall, M. Jr.: The Theory of Groups. The Macmillan Company, NY (1959)