RESEARCH ARTICLE

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Three frameworks for a general theory of factorization

Abstract We discuss three different frameworks for a general theory of factorization in integral domains: τ-factorization, reduced τ-factorization and Γ-factorization. Let D be an integral domain, $D^{\sharp}$ the non-zero, non-units of D, and τ a symmetric relation on $D^{\sharp}$. For $a \in D^{\sharp}$, $a = \lambda a_1 \cdots a_n$, λ a unit, $a_i \in D^{\sharp}$, $n \geq 1$, and $a_i \tau a_j$ for $i \neq j$, is called a τ-factorization of a and we say a_i is a τ-factor of a. For $a, b \in D^{\sharp}$, $a \mid_{\tau} b$ if a is a τ-factor of b. Then $a \in D^{\sharp}$ is a τ-atom if any τ-factorization of a has $n = 1$ and a is a τ-prime (resp., $\mid_{\tau}$-prime) if $a \mid \lambda a_1 \cdots a_n$ (resp., $a \mid_{\tau} \lambda a_1 \cdots a_n$), $\lambda a_1 \cdots a_n$ a τ-factorization, implies $a \mid a_i$ (resp., $a \mid_{\tau} a_i$) for some i. The theory of reduced τ-factorization is developed similarly, except here we restrict ourselves to reduced τ-factorizations, that is, τ-factorizations $a_1 \cdots a_n$ where the leading unit is omitted (or is 1). The theory of Γ-factorization is as follows. For $a \in D^{\sharp}$, fact(a) (resp., tfact(a)) is the set of (resp., trivial) factorizations of a, $a = \lambda a_1 \cdots a_n$, λ a unit, $n \geq 1$ (resp,. $n = 1$) and fact(D) $= \cup_{a \in D^{\sharp}}$fact($a$), tfact($D$) $= \cup_{a \in D^{\sharp}}$tfact($a$). Let $\Gamma \subseteq$ fact(D) and $\Gamma(a) = \Gamma \cap$ fact(a); the set of Γ-factorizations of a. For $a, b \in D^{\sharp}$, $a \mid_{\Gamma} b$ if some $\lambda a_1 \cdots a_n \in \Gamma(b)$ has $a_i = a$ for some i. We say a is a Γ-atom if $\Gamma(a) \subseteq$ tfact(a) and that a is a Γ-prime (resp., $\mid_{\Gamma}$-prime) if $a \mid \lambda a_1 \cdots a_n$ (resp., $a \mid_{\Gamma} \lambda a_1 \cdots a_n$) where $\lambda a_1 \cdots a_n \in \Gamma$, then $a \mid a_i$ (resp., $a \mid_{\Gamma} a_i$) for some i.

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الملخص:

نناقش ثلاثة أطر مختلفة لنظرية عامة للتحليل في الحلقات الصحيحة: تحليل-τ، وتحليل-τ المُختزَل، وتحليل-Γ. لتكن D حلقة صحيحة، وَ $D^\#$ مجموعة العناصر غير الصفرية التي لا تكون عناصر وحدة في D ، وَ τ علاقة متناظرة على $D^\#$. إذا كان $a \in D^\#$ لـ τ-تحليل $i \neq j$ عندما $a_i \tau a_j$ يحقق وحدة عنصر λ وَ $n \geq 1$ وَ $a_i \in D^\#$ حيث، $\lambda a_1 \cdots a_n$ نسمي فإننا، ونقول أن a_i عامل-τ لـ a. إذا كانت $a, b \in D^\#$، فإننا نكتب $a|_\tau b$ إذا كان a عامل-τ لـ b. يكون $a \in D^\#$ ذرَّة-τ إذا تحقق في أي تحليل-τ لـ a أن $n = 1$، ويكون a τ-أولياً ($|_\tau$-أولياً على التوالي) إذا تحقق ما يلي: إذا كان $a|\lambda a_1 \cdots a_n$ ($a|_\tau \lambda a_1 \cdots a_n$ على التوالي)، حيث $\lambda a_1 \cdots a_n$ تحليل-τ، فإن $a|a_i$ ($a|_\tau a_i$ على التوالي) لإحدى قيم i. يتم تطوير نظرية تحليل-τ المختزل بشكل مماثل باستثناء أننا نحصر أنفسنا في تحليلات τ المختزلة، أي تحليلات-τ $a_1 \cdots a_n$ التي يتم فيها حذف عنصر الوحدة الزائد (أو اعتباره 1). تلخص نظرية تحليلات-Γ كما يلي: لأي a، حيث $fact(a)$ ($tfact(a)$ على التوالي) مجموعة تحليلات a (التافهة على التوالي)، نحصل على $a = \lambda a_1 \cdots a_n$، حيث λ عنصر وحدة، وَ $n \geq 1$ ($n = 1$ بالتوالي)، وَ $fact(D) = \bigcup_{a \in D^\#} fact(a)$ وَ $tfact(D)) = \bigcup_{a \in D^\#} tfact(a)$. لتكن $\Gamma \subseteq fact(D)$ وَ $\Gamma(a) = \Gamma \cap fact(a)$. إذا كانت $a, b \in D^\#$، فإن $a|_\Gamma b$ إذا وجد $\lambda a_1 \cdots a_n \in \Gamma(b)$ بحيث يكون $a_i = a$ لإحدى قيم i. نقول أن a ذرة-Γ إذا كانت $\Gamma(a) \subseteq tfact(a)$، ونقول أن a Γ-أولي ($|_\Gamma$-أولي على التوالي) في حال تحقق ما يلي: إذا كان $a|\lambda a_1 \cdots a_n$ ($a|_\Gamma \lambda a_1 \cdots a_n$ على التوالي)، حيث $\lambda a_1 \cdots a_n \in \Gamma$، فإنه توجد i بحيث $a|a_i$ ($a|_\Gamma a_i$ على التوالي).

1 Introduction

The purpose of this paper is to give three different approaches to a general theory of factorization in integral domains. We isolate four important notions in factorization: a factorization, the related notion of divides, an atom or irreducible element, and a principal prime element. Let D be an integral domain, $U(D)$ the group of units of D, and $D^\sharp = D - (\{0\} \cup U(D))$, the non-zero, non-units of D. For $a \in D^\sharp$, $a = a_1 \cdots a_n$, or more generally, $a = \lambda a_1 \cdots a_n, n \geq 1, \lambda \in U(D), a_i \in D^\sharp$, is a factorization of a and we say a_i is a factor of a. For $a, b \in D^\sharp$, $a \mid b$ if a is a factor of b. Then $a \in D^\sharp$ is irreducible or an atom if for any factorization $a = \lambda a_1 \cdots a_n$ of a, we have $n = 1$. Finally, $a \in D^\sharp$ is prime if whenever $a \mid \lambda a_1 \cdots a_n, \lambda a_1 \cdots a_n$ a factorization, then $a \mid a_i$ for some i.

Now classically, factorization theory considered factorization into atoms or atomic factorization. An integral domain D is atomic if each element of $D^\sharp$ is a product of atoms. One then studies atomic domains with properties weaker than unique factorization. For example, an atomic domain D is a half-factorial domain (HFD) if any two atomic factorizations of $a \in D^\sharp$ have the same length and a (an atomic) domain D is a bounded factorization domain (BFD) if for each $a \in D^\sharp$, there is a natural number $N(a)$ so that for any (atomic) factorization $a_1 \cdots a_n$ of a, we have $n \leq N(a)$. See [2] and [5] for details. However, one can also study non-atomic factorization. Instead of studying factorization into atoms, we could study factorization into primary elements or other distinguished elements. See [1] for details.

Instead of just varying the a_is allowed in a factorization $\lambda a_1 \cdots a_n$, we can restrict the factorizations allowed. McAdam and Swan [10] did this in their study of comaximal factorization. For $a \in D^\sharp$, a comaximal factorization $a = a_1 \cdots a_n$ of a is a factorization where for $i \neq j$, a_i and a_j are comaximal, that is, $(a_i, a_j) = D$. They defined $a \in D^\sharp$ to be pseudo-irreducible if a has no non-trivial comaximal factorization $a = bc$ (or equivalently, any comaximal factorization $a = \lambda a_1 \cdots a_n$ of a has $n = 1$). They also defined $a \in D^\sharp$ to be pseudo-prime if $a \mid bc$ where $(b, c) = D$, then $a \mid b$ or $a \mid c$ (equivalently, if $a \mid \lambda a_1 \cdots a_n$, a comaximal factorization, then $a \mid a_i$ for some i). They then studied domains called comaximal factorization domains (CFDs) with the property that every $a \in D^\sharp$ has a comaximal factorization into pseudo-irreducibles and CFDs called unique comaximal factorization domains (UCFDs) in which comaximal factorization into pseudo-irreducibles is unique up to order and associates.

In [3], the first author and A. Frazier introduced a general theory of factorization involving the notion of a τ-factorization. Let D be an integral domain and τ a symmetric relation on $D^\sharp$. For $a \in D^\sharp$, a τ-factorization of a is a factorization $a = \lambda a_1 \cdots a_n$ where $a_i \tau a_j$ for $i \neq j$. For $a, b \in D^\sharp$, we say that $a \tau$-divides b, denoted $a \mid_\tau b$, if b has a τ-factorization $b = \lambda a_1 \cdots a_n$ where some $a_i = a$. We say that $a \in D^\sharp$ is a τ-atom or τ-irreducible if every τ-factorization of a is trivial: $a = \lambda(\lambda^{-1}a)$. Finally, $a \in D^\sharp$ is τ-prime (resp., $\mid_\tau$-prime) if for any τ-factorization $\lambda a_1 \cdots a_n$ with $a \mid \lambda a_1 \cdots a_n$ (resp., $a \mid_\tau \lambda a_1 \cdots a_n$), then $a \mid a_i$ (resp., $a \mid_\tau a_i$) for

some i. For example, if we take $\tau = D^\sharp \times D^\sharp$, we get the ordinary factorization into non-units, while if we define $a\tau b \Leftrightarrow (a, b) = D$, we get the comaximal factorization of McAdam and Swan. Section 2 reviews the theory of τ-factorization.

In the theory of τ-factorization, the leading unit λ in a τ-factorization $\lambda a_1 \cdots a_n$ turns out to play an important role. However, we could have developed the theory of τ-factorizations by taking a τ-factorization of $a \in D^\sharp$ to be $a = a_1 \cdots a_n$ where $a_i \tau a_j$ for $i \neq j$. We call such a τ-factorization a reduced τ-factorization or a $_r\tau$-factorization. An element $a \in D^\sharp$ is a reduced τ-atom or a $_r\tau$-atom if a has only the trivial $_r\tau$-factorization $a = a$ and a is a reduced τ-prime or $_r\tau$-prime if $a \mid a_1 \cdots a_n$, a $_r\tau$-factorization, then $a \mid a_i$ for some i. The notion of reduced τ-divides is more subtle. For $a, b \in D^\sharp$, a reduced τ-divides b, denoted $a \mid_{r\tau} b$, if $b = a_1 \cdots a_n$, a $_r\tau$-factorization, where some $a_i = a$ and a weakly reduced τ-divides b, denoted $a \mid_{wr\tau} b$, if $a \mid_{r\tau} b$ or $a \sim b$. Finally, $a \in D^\sharp$ is a $\mid_{r\tau}$-prime (resp., $\mid_{wr\tau}$-prime) if $a \mid_{r\tau} a_1 \cdots a_n$ (resp., $a \mid_{wr\tau} a_1 \cdots a_n$), a $_r\tau$-factorization, implies $a \mid_{r\tau} a_i$ (resp., $a \mid_{wr\tau} a_i$) for some i. The theory of reduced τ-factorization is developed in Sect. 3. We believe that the results of Sect. 3 bear out that the choice of allowing a unit in the definition of a τ-factorization is the proper one.

In the final Sect. 4, we introduce the notion of Γ-factorization. Let D be an integral domain and $a \in D^\sharp$. By a (trivial) factorization of a, we mean $a = \lambda a_1 \cdots a_n$ where $n \geq 1 (n = 1)$, $\lambda \in U(D)$, and $a_i \in D^\sharp$ where the order matters (see Remark 2 of Sect. 4 for a more formal definition). Let fact(a) (resp., tfact(a)) be the set of all factorizations (resp., trivial factorizations) of a and let fact$(D) = \cup_{a \in D^\sharp}$fact$(a)$. Let $\Gamma \subseteq$ fact(D). An element of Γ (resp., $\Gamma(a) := \Gamma \cap$ fact(a)) is called a Γ-factorization (resp., Γ-factorization of a). For $a, b \in D^\sharp$, we say that $a \Gamma$-divides b, written $a \mid_\Gamma b$, if there is a $\lambda a_1 \cdots a_n \in \Gamma(b)$ with some $a_i = a$. Then, a is a Γ-atom if $\Gamma(a) \subseteq$ tfact(a) and a is a Γ-prime (resp., $\mid_\Gamma$-prime) if whenever $a \mid \lambda a_1 \cdots a_n$ (resp., $a \mid_\Gamma \lambda a_1 \cdots a_n$), $\lambda a_1 \cdots a_n \in \Gamma$, then $a \mid a_i$ (resp., $a \mid_\Gamma a_i$) for some i. Suppose that τ is a symmetric relation on $D^\sharp$. If we take $\Gamma = \Gamma_\tau$ (resp., $\Gamma = \Gamma_{r\tau}$) to be the set of all τ-factorizations (resp., reduced τ-factorizations), we recover the notion of τ-factorization (resp., reduced τ-factorization).

2 τ-Factorizations

In this section, we give a brief review of the theory of τ-factorizations. Proofs and examples for claims may be found in Sect. 2 of [3] unless otherwise noted. Much of the material from [3] comes from [4]. The theory of τ-factorizations is further investigated in the dissertations [7,9,11] and [12].

Throughout D will be an integral domain with quotient field K. Let $D^* = D - \{0\}$, $U(D)$ the group of units of D, and $D^\sharp = D^* - U(D)$, the non-zero, non-units of D. Also throughout this section (except in the following definitions), τ will be a symmetric relation on $D^\sharp$. However, in Sect. 4, we will consider ordered τ-factorizations, the case where τ need not be symmetric. We next define three important properties that τ may have. As usual, $a \sim b$ means that a and b are associates.

Definition 2.1 Let D be an integral domain and τ a relation on $D^\sharp$. We call τ *multiplicative* (resp., *divisive*) if for $a, b, c \in D^\sharp$ (resp., $a, a', b, b' \in D^\sharp$), $a\tau b$ and $a\tau c$ imply $a\tau bc$ and $b\tau a$ and $c\tau a$ imply $bc\tau a$ (resp., $a\tau b$, $a'|a$ and $b'|b$ imply $a'\tau b'$). We say that τ is *associate preserving* if for $a, b, b' \in D^\sharp$ with $b \sim b'$, $a\tau b$ implies $a\tau b'$ and $b\tau a$ implies $b'\tau a$.

We next give the fundamental definition of a τ-factorization.

Definition 2.2 For $a \in D^\sharp$, D be an integral domain and τ a relation on $D^\sharp$, we define $a = \lambda a_1 \cdots a_n, n \geq 1, \lambda \in U(D), a_i \in D^\sharp$, to be a *$\tau$-factorization of a* if $a_i \tau a_j$ for each $i \neq j$. We say that a is a *τ-product of the a_i* and that a_i is a *τ-factor of a*. For $a, b \in D^\sharp$, we say that $a\tau$-divides b, written $a|_\tau b$, if a is a τ-factor of b. We call $a = \lambda(\lambda^{-1}a)$ a *trivial τ-factorization of a*.

Note that if $a = \lambda a_1 \cdots a_n$ is a τ-factorization, then so is each rearrangement $a = \lambda a_{\sigma(1)} \cdots a_{\sigma(n)}$, $\sigma \in S_n$. Also, observe that for $a, b \in D^\sharp$, the following conditions are equivalent: (1) $a\tau b$, (2) ab is a τ-factorization, (3) λab is a τ-factorization for all $\lambda \in U(D)$, (4) there is a τ-factorization $\lambda a_1 \cdots a_n$ where some $a_i = a$ and $a_j = b$ for $i \neq j$.

We pause to give several examples.

Example 2.3 Throughout D will be an integral domain.

(1) $\tau = D^\sharp \times D^\sharp$. This gives the usual notions of a factorization and divides. Of course, τ is both multiplicative and divisive.

(2) $\tau = \varnothing$. Here $a \in D^\sharp$ has only the trivial τ-factorization and $a|_\tau b \Leftrightarrow a \sim b$. Vacuously, τ is both multiplicative and divisive.

(3) Let S be a non-empty subset of $D^\sharp$ and take $\tau = S \times S$, so $a\tau b \Leftrightarrow a, b \in S$. Here τ is multiplicative (divisive) if and only if S is multiplicatively closed (closed under non-unit factors). A non-trivial τ-factorization is up to unit factors just a factorization into elements from S. Thus, if we take S to be the set of atoms of D, we get the usual factorization of an element into irreducible factors. In this case, every element of $D^\sharp$ is an atom or has a non-trivial τ-factorization if and only if D is atomic. We could also take S to be the set of prime elements, prime power elements, primary elements, or other distinguished elements such as rigid elements or t-pure elements. These last examples are examples of non-atomic factorizations. See [1] for details. We could also replace S by a subset S' where for each $s \in S$ there exists exactly one $s' \in S'$ with $s' \sim s$. For example, for $D = \mathbb{Z}$, take S to be the set of prime elements and take $S' = \{n \in \mathbb{N} | n$ is prime$\}$. Here, τ is not associate preserving. Examples of this type are one of the reasons we chose to include a unit factor in the definition of a τ-factorization. Sometimes, it is of interest to replace $S \times S$ by $S \times S - \Delta = \{(s,t) \in S \times S | s \neq t\}$. For example, if $P = \{p_\alpha\}$ is a set of non-associate primes, take $S = \{p_\alpha^k | p_\alpha \in P, k \geq 1\}$. Then for $\tau = S \times S - \Delta$, a non-trivial τ-factorization is just a product $\lambda p_{\alpha_1}^{k_1} \cdots p_{\alpha_n}^{k_n}$ where $p_{\alpha_1}, \ldots, p_{\alpha_n}$ are distinct elements of P and each $k_i \geq 1$.

(4) Let I be an ideal of D and define $a\tau b \Leftrightarrow a - b \in I$. A special case is $D = \mathbb{Z}$ and $I = (n)$, so $a\tau_n b \Leftrightarrow a \equiv b \mod n$. Here, τ_n is multiplicative or associate preserving only for $n = 2$ and is never divisive. The relation τ_n is investigated in [3,4,7] and [11].

(5) Let $\star$ be a star-operation on D and define $a\tau_\star b \Leftrightarrow (a,b)^\star = D$, that is, a and b are $\star$-coprime or $\star$-comaximal. (Recall that a star operation $\star$ on D is a closure operation on the set of non-zero fractional ideals of D that satisfies $(xA)^\star = xA^\star$ and $D^\star = D$, see [6]). It is easily checked that $\tau_\star$ is both multiplicative and divisive. In the case where $\star = d$ (the d-operation $A \longrightarrow A_d = A$), we have the comaximal factorization of McAdam and Swan [10]. Also of interest is the case where $\star$ is the v-operation $(A \to A_v = (A^{-1})^{-1})$.

(6) Related to factorizations into v-coprime elements, we have factorizations into relatively prime elements. Define for $a, b \in D^\sharp a\tau_{[\ ]} b \Leftrightarrow [a,b] = 1$, that is, a and b have no common non-unit factor. While divisive, $\tau_{[\ ]}$ need not be multiplicative. The relation $\tau_{[\ ]}$ is investigated in [11] and [12].

The notion of a τ-factorization on D like that of a topology on a set is very general. But part of the power of the definition is its generality. Of course, in topology we usually have other axioms such as the separation axioms. The analogy for τ-factorization is the conditions that τ is associated-preserving, divisive, or multiplicative. We have found the divisive condition to be the most useful and is powerful enough to obtain interesting results such as Theorem 2.12.

Given a factorization, we often want to further factor certain terms or want to combine terms. In general, neither action preserves τ-factorizations. Our first proposition states that if τ is divisive (resp., multiplicative), then the refinement of a τ-factorization obtained by τ-factoring one or more terms (resp., combining terms in a τ-factorization) again gives a τ-factorization. This good behavior was the main reason for introducing the notions of multiplicative and divisive relations.

Proposition 2.4 *Let D be an integral domain and let τ be a relation on $D^\sharp$.*

(1) *Suppose that τ is divisive. Let $a, b, b' \in D^\sharp$ where $b \sim b'$. Then $a\tau b \Leftrightarrow a\tau b'$. So τ is associate preserving. Thus $a = \lambda a_1 \cdots a_n$ is a τ-factorization of a if and only if $a_1 \cdots (\lambda a_i) \cdots a_n$ is a τ-factorization of a. Hence, when τ is divisive, or more generally associate preserving, we can dispense with the unit λ.*

(2) *Suppose that τ is divisive. Let $a = a_1 \cdots a_n$ be a τ-factorization of a and let $a_i = b_{i1} \cdots b_{im_i}$ be a τ-factorization of a_i (possibly the trivial factorization $a_i = b_{i1}$). Then $a = b_{11} \cdots b_{1m_1} b_{21} \cdots b_{2m_2} \cdots b_{n1} \cdots b_{nm_n}$ is a τ-factorization of a, called a τ-refinement of a. Thus when τ is divisive, a τ-refinement of a τ-factorization is a τ-factorization.*

(3) *Suppose that τ is multiplicative. If $\{1, 2, \ldots, n\} = A_1 \dot\cup \cdots \dot\cup A_s$ (disjoint union) with each A_i non-empty and $b_i = \Pi\{a_j | j \in A_i\}$, then $a = \lambda b_1 \cdots b_s$ is a τ-factorization of a.*

We next discuss the relation $|_\tau$ in more detail. Let D be an integral domain and τ a relation on $D^\sharp$. Let $a, a', b, b', c \in D^\sharp$. Certainly, $a|_\tau b \Rightarrow a|b$, but the converse is false. We have (1) $a|_\tau a$ and (2) $a|_\tau b$ and $b|_\tau a \Leftrightarrow a \sim b$. If $b \sim b'$, then $a|_\tau b \Leftrightarrow a|_\tau b'$. If $a \sim a'$ and τ is associate preserving, then $a|_\tau b \Leftrightarrow a'|_\tau b$. However, in general $a \sim a'$ and $a|_\tau b \nRightarrow a'|_\tau b$. If τ is divisive, then (3) $a|_\tau b$ and $b|_\tau c \Rightarrow a|_\tau c$. However, in general this is also false. If τ is both multiplicative and divisive, then $a|_\tau b$ and $b\tau c \Rightarrow ac|_\tau bc$.

Definition 2.5 For $a \in D^\sharp$, D an integral domain and τ a relation on $D^\sharp$, a is τ-*irreducible* or a τ-*atom* if the only τ-factorizations of a are the trivial ones. Let τ' be another relation on $D^\sharp$. By a τ-*atomic* τ'-*factorization for* $a \in D^\sharp$, we mean a τ'-factorization of a into τ-atoms. We say that D is τ-τ' *atomic* if each $a \in D^\sharp$ has a τ-atomic τ'-factorization. When $\tau = \tau'$, we simply say a τ-*atomic factorization* or that D is τ-*atomic*. We say that a τ-factorization $a = \lambda a_1 \cdots a_n$ is τ-*unrefineable* or τ-*complete* if it has no proper τ-refinements. And D is τ-*complete* if every $a \in D^\sharp$ has a τ-complete factorization.

Note that an associate of a τ-atom is again a τ-atom. Of course, a τ-atomic factorization is τ-complete. By Proposition 2.4(2) for τ divisive, a τ-complete factorization is the same thing as a τ-atomic factorization. This is not true in general. Hence for τ divisive, D is τ-complete if and only if it is τ-atomic.

Definition 2.6 Let τ be a relation on $D^\sharp$, D an integral domain. Then $a \in D^\sharp$ is τ-*prime* if whenever $a | \lambda a_1 \cdots a_n$ where $\lambda a_1 \cdots a_n$ is a τ-factorization, then $a | a_i$ for some i. We call $a \in D^\sharp$ a $|_\tau$-*prime* if whenever $a |_\tau \lambda a_1 \cdots a_n$ where $\lambda a_1 \cdots a_n$ is a τ-factorization, then $a |_\tau a_i$ for some i.

An associate of a τ-prime element is again τ-prime. If τ is associate preserving, then an associate of a $|_\tau$-prime element is again $|_\tau$-prime. However, in general an associate of a $|_\tau$-prime element need not be $|_\tau$-prime.

Remark 1 Note that τ-primes and $|_\tau$-primes are a special case of what we might call a τ_1-τ_2-τ_3-*prime* (where τ_1, τ_2 and τ_3 are relations on $D^\sharp$): whenever $a |_{\tau_2} \lambda a_1 \cdots a_n$ where $\lambda a_1 \cdots a_n$ is a τ_1-factorization, then $a |_{\tau_3} a_i$ for some i. For example, a τ-prime is a τ-τ'-τ'-prime where $\tau' = D^\sharp \times D^\sharp$ [Example 2.3(1)] and a $|_\tau$-prime is a τ-τ-τ-prime.

Let D be an integral domain, τ be a relation on $D^\sharp$, and $a \in D^\sharp$. Clearly if a is irreducible (resp., prime), then a is τ-irreducible (resp., τ-prime) and if a is τ-prime or $|_\tau$-prime, then a is τ-irreducible. If τ is multiplicative and divisive, then $a \tau$-prime implies a is $|_\tau$-prime (see Proposition 2.7 below). But in general, a prime or τ-prime element need not be $|_\tau$-prime. Note that in the definitions of τ-irreducible, τ-prime, and $|_\tau$-prime, we did not restrict ourselves to the case of τ-factorizations $\lambda a_1 \cdots a_n$ of length $n = 2$ as is usual. We next note that if τ is multiplicative, we can restrict ourselves to the case $n = 2$. In general this is not the case.

Proposition 2.7 *Let D be an integral domain and let τ be a relation on $D^\sharp$.*

(1) *Suppose that τ is multiplicative and let $a \in D^\sharp$. Then a is τ-irreducible (resp., τ-prime, $|_\tau$-prime) if and only if a has no τ-factorization $a = \lambda a_1 a_2$ (resp., for a τ-factorization $\lambda a_1 a_2$, $a | \lambda a_1 a_2 \Rightarrow a | a_1$ or $a | a_2$, $a |_\tau \lambda a_1 a_2 \Rightarrow a |_\tau a_1$ or $a | a_2$).*

(2) *If τ is both multiplicative and divisive, then a τ-prime element is $|_\tau$-prime.*

Let D be an integral domain. For relations τ_1, τ_2 on $D^\sharp$, define $\tau_1 \le \tau_2 \Leftrightarrow \tau_1 \subseteq \tau_2$, that is, $a \tau_1 b \Rightarrow a \tau_2 b$. Observe that $\tau_1 \le \tau_2$ if and only if each τ_1-factorization is a τ_2-factorization. Let $\mathcal{R}$ be the set of relations on $D^\sharp$. So $\mathcal{R}$ is partially ordered by $\le$. Note that $\emptyset$ [Example 2.3(2)] is the least element of $\mathcal{R}$ and the usual factorization is given by $\tau = D^\sharp \times D^\sharp$ [Example 2.3(1)], the greatest element. Suppose that $\star_1$ and $\star_2$ are two star-operations on D with $\star_1 \le \star_2$, that is, $A^{\star_1} \subseteq A^{\star_2}$ for all $A \in F(D)$. Then $\tau_{\star_1} \le \tau_{\star_2}$. Suppose that τ_1, τ_2 are relations on $D^\sharp$ with $\tau_1 \le \tau_2$. Then a τ_1-factorization of $a \in D^\sharp$ is also a τ_2-factorization of a. Thus if a is a τ_2-atom (resp., τ_2-prime), then a is a τ_1-atom (resp., τ_1-prime). Hence, we have the previously mentioned fact that an atom (resp., prime) of D is a τ-atom (resp., τ-prime). Observe that if $\tau_1 \le \tau_2$, then $a |_{\tau_1} b \Rightarrow a |_{\tau_2} b$ for any $a, b \in D^\sharp$ and the converse is true if τ_2 is both multiplicative and associate preserving, but not in general (define τ_1, τ_2 on $\mathbb{Z}^\sharp$ by $a \tau_1 b \Leftrightarrow |a| = |b| = 2$ and $2 \tau_2 2$, $2 \tau_2 - 2$, $-2 \tau_2 2$). However, a $|_{\tau_2}$-prime need not be a $|_{\tau_1}$-prime.

There is a natural extension of the notion of a UFD to τ-factorizations. Let D be an integral domain and τ a relation on $D^\sharp$. We say that D is a τ-*UFD* if (1) D is τ-atomic and (2) if $\lambda a_1 \cdots a_n = \mu b_1 \cdots b_m$ are two τ-atomic factorizations, then $n = m$ and after re-ordering, if necessary, $a_i \sim b_i$ for each i. We leave it to the reader to define a τ-τ'-UFD using τ-atomic τ'-factorizations. The following lemma used to prove Theorem 2.9 shows the importance of $|_\tau$-primes.

Proposition 2.8 *Let D be an integral domain and τ a relation on $D^\sharp$. Suppose that $\lambda p_1 \cdots p_n = \mu q_1 \cdots q_m$ are two τ-factorizations where the p_i are $|_\tau$-prime and the q_i are τ-atoms. Then $n = m$ and after re-ordering, if necessary, $p_i \sim q_i$.*

Theorem 2.9 *Let D be an integral domain and τ a relation on $D^\sharp$.*

(1) *Suppose that every element of $D^\sharp$ has a τ-factorization into $|_\tau$-primes. Then D is a τ-UFD. Moreover, $a \in D^\sharp$ is τ-irreducible if and only if a is an associate of a $|_\tau$-prime.*

(2) *Suppose that τ is divisive and that D is a τ-UFD. Then a τ-irreducible element of D is $|_\tau$-prime (and of course the converse always holds).*

(3) *For τ divisive, the following are equivalent:*
   (a) D is a τ-UFD,
   (b) every element of $D^\sharp$ has a τ-factorization into $|_\tau$-primes, and
   (c) D is τ-atomic and every τ-irreducible element of D is $|_\tau$-prime.

In Theorem 2.9(3) if we replace $|_\tau$-prime by τ-prime, we have (a)$\Leftarrow$(b)$\Leftrightarrow$(c), but (a)$\not\Rightarrow$(b). The counterexample for the implication (a)$\Rightarrow$(b) is given in [9] which contains a thorough investigation of τ-UFDs. However, a domain D in which every element is a τ-product of associates of $|_\tau$-primes need not be a τ-UFD. In a τ-UFD every element of D need not be a τ-product of τ-primes (resp., $|_\tau$-primes).

We can also define the following τ-factorization properties weaker than τ-unique factorization. Let D be an integral domain and τ a relation on $D^\sharp$. We have already defined τ-atomic. We say that D satisfies τ-ACCP if for each infinite sequence $\{a_n\}_{n=1}^\infty$ of elements of $D^\sharp$ with $a_{n+1}|_\tau a_n$ for each $n \geq 1$, there is an N (depending on the sequence) with $a_{k+1} \sim a_k$ for each $k \geq N$. The domain D is a τ-*half-factorial domain* (τ-HFD) if D is τ-atomic and whenever $\lambda a_1 \cdots a_n = \mu b_1 \cdots b_m$ are two τ-atomic factorizations, then $n = m$.

We say that D is a τ-*bounded factorization domain* (τ-BFD) if D is τ-atomic and for each $a \in D^\sharp$, there is a natural number $N(a)$ so that if $a = \lambda a_1 \cdots a_n$ is a τ-atomic factorization of a, then $n \leq N(a)$. Note that for τ divisive, D is a τ-BFD if and only if for each $a \in D^\sharp$, there is a natural number $N(a)$ so that for any τ-factorization $a = \lambda a_1 \cdots a_n$, $n \leq N(a)$. This follows from Proposition 2.4(2) which gives that for τ divisive a maximal length τ-factorization is a τ-atomic factorization. Thus for τ divisive, a BFD is a τ-BFD.

We say that D is a τ-*idf-domain* if each $a \in D^\sharp$ has at most finitely many non-associate τ-factors that are τ-atoms. D is a τ-*finite-factorization domain* (τ-FFD) if D is τ-atomic and each $a \in D^\sharp$ has only finitely many τ-factorizations (up to order and associates) into τ-irreducibles. Clearly, a τ-FFD is a τ-BFD. Suppose that τ is divisive. Then a modification of the proof of [2, Theorem 5] gives that the following are equivalent: (1) D is a τ-FFD, (2) D is a τ-atomic τ-idf-domain, (3) each $a \in D^\sharp$ has only finitely many τ-factorizations up to order and associates.

Theorem 2.10 *Let D be an integral domain and τ a relation on $D^\sharp$. If D is a τ-UFD, then D is a τ-HFD and a τ-FFD and either of these conditions implies that D is a τ-BFD. If further τ is divisive, then D a τ-BFD implies D satisfies τ-ACCP and if D satisfies τ-ACCP, then D is τ-atomic.*

Let D be an integral domain and τ a relation on $D^\sharp$. Note that if D satisfies ACCP, then D satisfies τ-ACCP. Thus for τ divisive, ACCP $\Rightarrow$ τ-atomic. So for τ divisive, a UFD, FFD, HFD, and BFD are τ-atomic. Thus a FFD (resp., BFD) is a τ-FFD (resp., τ-BFD) for τ divisive. However, an atomic domain need not be τ-atomic, even if τ is both multiplicative and divisive. For [8] gives an example of an atomic domain that is not a CFD.

We next note that for τ divisive, a UFD is a τ-UFD. Thus a UFD is a UCFD. The proof uses the following lemma which states that for τ divisive, a τ-atomic factorization is a mix of atomic factorizations and coprime factorizations.

Lemma 2.11 *Let D be an integral domain and let τ be a divisive relation on $D^\sharp$. Let $a_1 \cdots a_n$ be a τ-atomic factorization. Then for $i \neq j$, either $[a_i, a_j] = 1$ or $a_i \sim a_j$ are atoms.*

Theorem 2.12 *Let D be a UFD and τ a divisive relation on $D^\sharp$. Then D is a τ-UFD.*

As the following example shows, D can be a τ-UFD without τ being divisive. This example also shows that even in a τ-UFD, the τ atoms do not determine $|_\tau$.

Example 2.13 Let D be a UFD and define τ on $D^\sharp$ by $a\tau b \Leftrightarrow a$ and b are non-zero principal primes. So τ is divisive. Now clearly D is a τ-UFD with the sets of τ-atoms, τ-primes, $|_\tau$-primes, and non-zero principal primes coinciding. Observe that $a |_\tau b \Leftrightarrow a$ is a prime with $a \mid b$ or $a \sim b$. Suppose that $\tau \leq \tau'$; so τ' need not be divisive. Then D is still a τ'-UFD with the set of τ'-atoms, τ'-primes, $|_{\tau'}$-primes, and non-zero principal primes coinciding. However, $|_{\tau'}$ may change. For if p and q are primes with $p \nmid_{\tau'} \lambda q$ for any unit λ, then $p |_\tau pq$, but $p \nmid_{\tau'} pq$.

We have based our factorization theory on τ-atomic factorizations. We could have instead used τ-complete factorizations. Let D be an integral domain. Define D to be a τ-*complete HFD* (resp., τ-*complete UFD*) if (1) D is τ-complete and (2) if $a = \lambda a_1 \cdots a_n = \mu b_1 \cdots b_m$ are two τ-complete factorizations of $a \in D^\sharp$, then $n = m$ (resp., and after re-ordering, if necessary, $a_i \sim b_i$ for $i = 1, \ldots, n$). We say that D is a τ-*complete FFD* (resp., τ-*complete BFD*) if for each $a \in D^\sharp$ there are only finitely many τ-complete factorizations for a up to units, order, and associates (resp., there is a natural number $N(a)$ so that for each τ-complete factorization $a = \lambda a_1 \cdots a_n, n \leq N(a)$).

Now even in a τ-UFD, a τ-factorization cannot necessarily be τ-refined to a τ-atomic factorization. Let us say that an integral domain D is τ-*atomizable* (resp., τ-*completeable*) if each τ-factorization of D can be τ-refined to a τ-atomic (resp., τ-complete) factorization. We have the following implications:

$$\tau\text{-atomizable} \Rightarrow \tau\text{-atomic}$$
$$\Downarrow \qquad\qquad\qquad \Downarrow$$
$$\tau\text{-ACCP} \Rightarrow \tau\text{-completeable} \Rightarrow \tau\text{-complete}.$$

Note that for τ divisive, τ-complete $\Rightarrow$ τ-atomizable, but τ-complete need not imply τ-ACCP since an atomic domain need not satisfy ACCP. In general, none of the implications can be reversed. The topics in the previous two paragraphs are discussed in more detail in [9] in the context of Γ-factorization.

3 Reduced τ-factorization

We maintain the notation from Sect. 2: D is an integral domain and τ a symmetric relation on $D^\sharp$. In a τ-factorization $a = \lambda a_1 \cdots a_n$ for a we allowed a leading unit λ. In this section we consider τ-factorizations without the unit λ, or equivalently, where $\lambda = 1$.

Definition 3.1 Let D be an integral domain and τ, τ_1 and τ_2 be relations on $D^\sharp$. For $a \in D^\sharp$, a *reduced τ-factorization* ($_r\tau$-*factorization*) of a is $a = a_1 \cdots a_n$ where $a_i \in D^\sharp$ and $a_i \tau a_j$ for $i \neq j$. The $_r\tau$-factorization $a = a$ is called *the trivial $_r\tau$-factorization of a*. So $a \in D^\sharp$ is a *reduced τ-atom or $_r\tau$-atom* if a has only the trivial $_r\tau$-factorization. A *reduced τ_1-atomic reduced τ_2-factorization* (a $_r\tau_1$-*atomic* $_r\tau_2$-*factorization*) of a is a $_r\tau_2$-factorization of a into $_r\tau_1$-atoms. Likewise, we define a $_r\tau_1$-*atomic τ_2-factorization* and τ_1-*atomic $_r\tau_2$-factorization*. A $_r\tau_1$-atomic $_r\tau_1$-factorization is a called a $_r\tau_1$-*atomic factorization*. We say that D is $_r\tau_1$-$_r\tau_2$-*atomic* if each $a \in D^\sharp$ has a $_r\tau_1$-atomic $_r\tau_2$-factorization. Likewise, we define $_r\tau_1$-τ_2-*atomic* and τ_1-$_r\tau_2$-*atomic*. We say that D is $_r\tau$-*atomic* if D is $_r\tau$-$_r\tau$-atomic.

The notion of a reduced τ-factorization was introduced in [7] and investigated more fully in [11]. Much of this material comes from [11]. The following proposition states some simple facts about reduced τ-factorizations and reduced τ-atoms.

Proposition 3.2 *Let D be an integral domain and τ, τ_1 and τ_2 be relations on $D^\sharp$.*

(1) *A $_r\tau$-factorization is a τ-factorization.*
(2) *A τ-atom is a $_r\tau$-atom. So $D\tau_1$-τ_2-atomic implies D is $_r\tau_1$-τ_2-atomic.*
(3) *Suppose that τ is associate preserving. Then a $_r\tau$-atom is a τ-atom. So the following are equivalent: (a) D is τ-atomic, (b) D is $_r\tau$-τ-atomic, (c) D is τ-$_r\tau$-atomic, and (d) D is $_r\tau$-atomic.*
(4) *For $a \in D^\sharp$, a is a τ-atom if and only if every associate of a is a $_r\tau$-atom.*

Proof (1) Clear. (2) Suppose that a is a τ-atom. Then a has only trivial τ-factorizations and hence only the trivial $_r\tau$-factorization. So a is a $_r\tau$-atom. The second statement is clear. (3) Let a be a reduced τ-atom. If a is not a τ-atom, we have a τ-factorization $a = \lambda a_1 \cdots a_n$ where $n > 1$. But then $a = (\lambda a_1) \cdot a_2 \cdots a_n$ is a non-trivial $_r\tau$-factorization for a, a contradiction. So a is a τ-atom if and only if it is a $_r\tau$-atom. For $a \in D^\sharp$, if a has a τ-atomic factorization $a = \lambda a_1 \cdots a_n$, then $a = (\lambda a_1) \cdot a_2 \cdots a_n$ is a τ-atomic $_r\tau$-factorization and hence a $_r\tau$-atomic $_r\tau$-factorization. The equivalence of (a)–(d) easily follows. (4) ($\Rightarrow$) If a is a τ-atom, then each associate of a is a τ-atom and hence a $_r\tau$-atom. ($\Leftarrow$) Suppose that each associate of a is a $_r\tau$-atom. Suppose that is not a τ-atom: so $a = \lambda a_1 \cdots a_n$, a τ-factorization with $n > 1$. But then $\lambda^{-1} a = a_1 \cdots a_n$ is a not a $_r\tau$-atom; a contradiction. $\qquad\square$

We next give some examples.

Example 3.3 Throughout, D will be an integral domain.

(1) (An example of a $_r\tau$-atom with an associate that is not a $_r\tau$-atom and hence not a τ-atom and a τ-atomic domain that is not $_r\tau$-atomic.) Define τ on $\mathbb{Z}^\sharp$ by $a\tau b \Leftrightarrow a, b \geq 2, a = 4, b = -4$, or $a = -4, b = 4$. Now each positive prime of $\mathbb{Z}$ is a τ-atom; so $\mathbb{Z}$ is τ-atomic. Here -4 is a $_r\tau$-atom, but not a τ-atom since $-4 = (-1) \cdot 2 \cdot 2$ and the associate 4 of -4 is not a $_r\tau$-atom. Now $-16 = -4 \cdot 4$; so -16 is not a $_r\tau$-atom. But there is no $_r\tau$-atomic factorization for -16 (as $4 = 2 \cdot 2$). Thus $\mathbb{Z}$ is not $_r\tau$-atomic and hence not $\tau -_r\tau$-atomic.
(2) Define τ on $\mathbb{Z}^\sharp$ by $\pm p\, \tau \pm q$ where p, q are any positive primes and $4\tau 4$. So τ is not associate preserving. The τ-atoms and $_r\tau$-atoms are $\pm p$ where p is a prime. Thus $\mathbb{Z}$ is τ-atomic and $_r\tau$-atomic and each associate of a $_r\tau$-atom is a $_r\tau$-atom; but τ is not associate preserving. Note that $\mathbb{Z}$ is a τ-UFD and a $_r\tau$-UFD (defined using $_r\tau$-factorizations).
(3) Define τ on $\mathbb{Z}^\sharp$ by $p\tau q$ and $-p\tau - q$ where p, q are any positive primes of $\mathbb{Z}$. The τ-atoms are $\pm p$, p a non-zero prime. Now $\mathbb{Z}$ is a τ-UFD. The $_r\tau$-atoms of $\mathbb{Z}$ are $\pm p$ and $-p_1 \cdots p_{2n}$ where p and p_i are positive primes. Note that $\mathbb{Z}$ is still a $_r\tau$-UFD. If we add $-4\tau - 4$, then $-4 \cdot -4 = 2 \cdot 2 \cdot 2 \cdot 2$, so $\mathbb{Z}$ is no longer a $_r\tau$-UFD, but is still a τ-UFD.

Recall that $a \in D^\sharp$ is τ-prime if whenever $a \mid \lambda a_1 \cdots a_n$, a τ-factorization, then $a \mid a_i$ for some i. We define $a \in D^\sharp$ to be $_r\tau$-prime if $a \mid a_1 \cdots a_n$, a $_r\tau$-factorization, then $a \mid a_i$ for some i. Unlike the case for τ-atoms and $_r\tau$-atoms, the notions of τ-prime and $_r\tau$-prime coincide.

Proposition 3.4 *Let D be an integral domain and τ a symmetric relation on $D^\sharp$. Then $a \in D^\sharp$ is $_r\tau$-prime if and only if it is τ-prime. Hence a $_r\tau$-prime is a $_r\tau$-atom.*

Proof ($\Leftarrow$) This is immediate since a $_r\tau$-factorization is a τ-factorization. ($\Rightarrow$) Suppose that a is a $_r\tau$-prime. Suppose that $a \mid \lambda a_1 \cdots a_n$, a τ-factorization. Then $a \mid a_1 \cdots a_n$ where $a_1 \cdots a_n$ is now a $_r\tau$-factorization. So $a \mid a_i$ for some i. Thus a is a τ-prime. □

We next want to consider "reduced $\mid_\tau$-primes". To do this, we need to define "reduced τ-divides". Now for $a, b \in D^\sharp$, the "natural" way to do this is to define $a \mid_{r\tau} b$ if $b = a_1 \cdots a_{i-1} \cdot a \cdot a_{i+1} \cdots a_n$ where $a_1 \cdots a_{i-1} \cdot a \cdot a_{i+1} \cdots a_n$ is a $_r\tau$-factorization. However, as we shall see, there are several problems with this definition. So we make the following two definitions.

Definition 3.5 Let D be an integral domain and τ a relation on $D^\sharp$. We say that *a reduced τ-divides b*, denoted $a \mid_{r\tau} b$, if $b = a_1 \cdots a_{i-1} \cdot a \cdot a_{i+1} \cdots a_n$ where $a_1 \cdots a_{i-1} \cdot a \cdot a_{i+1} \cdots a_n$ is a $_r\tau$-factorization of b. We say that *a weakly reduced τ-divides b*, denoted $a \mid_{wr\tau} b$, if either (1)$a \mid_{r\tau} b$ or (2) $a \sim b$.

Our next proposition concerns the relations $\mid_\tau$, $\mid_{r\tau}$ and $\mid_{wr\tau}$.

Proposition 3.6 *Let D be an integral domain and τ a symmetric relation on $D^\sharp$. Let $a, b \in D^\sharp$.*

(1) $a \mid_{r\tau} b \Rightarrow a \mid_{wr\tau} b \Rightarrow a \mid_\tau b$.
(2) *The following are equivalent.*
  (a) $a \mid_{r\tau} b \Leftrightarrow a \mid_\tau b$ *for all $a, b \in D^\sharp$,*
  (b) $a \mid_{r\tau} b \Leftrightarrow a \mid_{wr\tau} b$ *for all $a, b \in D^\sharp$, and*
  (c) D *is a field or $U(D) = \{1\}$.*
(3) *Suppose that τ is associate preserving and that $a \mid_\tau b$, so $b = \lambda a_1 \cdots a_{n-1}a = (\lambda a_1) \cdot a_2 \cdots a_{n-1} \cdot a$, a $_r\tau$-factorization. If $n > 1$, $a \mid_{r\tau} b$ and hence $a \mid_{wr\tau} b$. If $n = 1$, $b = \lambda a$, so $a \mid_{wr\tau} b$, but $a \mid_{r\tau} b \Leftrightarrow \lambda = 1$.*
(4) $a \mid_\tau b \Leftrightarrow a \mid_{wr\tau} b$ *for all $a, b \in D^\sharp$ if τ is associate preserving. If τ is multiplicative, the converse is true.*

Proof (1) Clear. (2) (c) $\Rightarrow$ (a), (b) Clear. (a) $\Rightarrow$ (b), (c) Suppose that D is not a field, so there is an $a \in D^\sharp$, and let $\lambda \in U(D) - \{1\}$. Thus $a \mid_\tau \lambda a$ and $a \mid_{wr\tau} \lambda a$, but $a \nmid_{r\tau} \lambda a$. (3) Clear. (4) The first statement follows from (3). Suppose that τ is multiplicative but not associate preserving. So there exist $a, b \in D^\sharp$ and $\lambda \in U(D)$ with $a\tau b$ but $a \not\tau (\lambda b)$. Now certainly $a \mid_\tau (\lambda ab)$. But $a \nmid_{wr\tau} (\lambda ab)$. For $a \mid_{wr\tau} (\lambda ab)$ implies $a \mid_{r\tau} (\lambda ab)$ or $a \sim (\lambda ab)$. Now if $a \mid_{r\tau} (\lambda ab)$, then $\lambda ab = a \cdot c$ where $a\tau c$ or $\lambda ab = a$. The first case $\lambda ab = ac$ gives $c = \lambda b$, so $a\tau\lambda b$, a contradiction. The second case, $\lambda ab = a$ gives that b is a unit, also a contradiction. Likewise $a \sim \lambda ab$ leads to a contradiction. □

If τ is not multiplicative the converse of the first statement of (4) does not hold in general. Define τ on $\mathbb{Z}^\sharp$ by $\pm 2\tau \pm 2$, $\pm 2\tau \pm 4$, $\pm 2\tau 8$, $\pm 4\tau 8$, and $b\tau a$ whenever $a\tau b$. Note that τ is neither associate preserving nor multiplicative, since $(2, -8), (2, 16) \notin \tau$. However $a \mid_\tau b \Leftrightarrow a \mid_{wr\tau} b$ for all $a, b \in \mathbb{Z}^\sharp$.

Corollary 3.7 *Let D be an integral domain and τ a symmetric relation on $D^\sharp$.*

(1) *The $_r\tau$-factorization theory on D with $|_{r\tau}$ coincides with the τ-factorization theory on D if and only if D is a field or $U(D) = \{1\}$.*
(2) *If D is associate preserving, then the $_r\tau$-factorization theory on D with $|_{wr\tau}$ coincides with the τ-factorization theory on D. If τ is multiplicative, the converse is true.*

Recall that $a \in D^\sharp$ is a $|_\tau$-prime if whenever $a \mid_\tau \lambda a_1 \cdots a_n$, $\lambda a_1 \cdots a_n$ a τ-factorization, then $a|_\tau a_i$ for some i.

Definition 3.8 Let D be an integral domain and τ a relation on $D^\sharp$. We define $a \in D^\sharp$ to be a *(weakly) reduced τ-divides prime*, $|_{r\tau}$-prime ($|_{wr\tau}$-prime), if whenever $a \mid_{r\tau} a_1 \cdots a_n$ ($a \mid_{wr\tau} a_1 \cdots a_n$), $a_1 \cdots a_n$ a $_r\tau$-factorization, then $a \mid_{r\tau} a_i$ ($a \mid_{wr\tau} a_i$) for some i.

It is easily checked that a $|_{wr\tau}$-prime or $|_{r\tau}$-prime is a $_r\tau$-atom. In fact, a $|_{wr\tau}$-prime is even a τ-atom. However, a $|_\tau$-prime need not be a $|_{wr\tau}$-prime nor a $|_{r\tau}$-prime. For with the relation τ_3 on $\mathbb{Z}^\sharp$ defined by $a\tau_3 b \Leftrightarrow a \equiv b$ mod 3 (see Example 2.3(4)), 5 is a $|_{\tau_3}$-prime (Example 3.9), but $5 \mid_{wr\tau_3} 5 \cdot (5 \cdot 7^2) = (5 \cdot 7) \cdot (5 \cdot 7)$, while $5 \nmid_{wr\tau_3} 5 \cdot 7$ with similar relations holding for $|_{r\tau_3}$. Also, 7 is a $|_{wr\tau_3}$-prime, but not a $|_{r\tau_3}$-prime and its associate -7 is not a $|_{wr\tau_3}$-prime. A $|_{r\tau}$-prime need not be a $|_{wr\tau}$-prime. For example, define τ on $\mathbb{Z}^\sharp$ by $a\tau b \Leftrightarrow a, b \geq 2$. Now -4 is vacuously a $|_{r\tau}$-prime, but -4 is not a $|_{wr\tau}$-prime since $-4 \mid_{wr\tau} 4 = 2 \cdot 2$, but $-4 \nmid_{wr\tau} 2$. The same example also shows that a $|_{r\tau}$-prime need not be a $|_\tau$-prime nor a τ-atom and that an associate of a $|_{r\tau}$-prime need not be a $|_{r\tau}$-prime. Finally, a $|_{wr\tau}$-prime need not be a $|_\tau$-prime. Define τ on $\mathbb{Z}^\sharp$ by $2\tau 16$, $16\tau 2$, $-4\tau 8$ and $8\tau - 4$. Then 16 is a $|_{wr\tau}$-prime but not a $|_\tau$-prime as $16 \mid_\tau 32 = (-1)(-4)8$ but $16 \nmid_\tau 8, -4$. The following example illustrates the differences between τ-factorizations and $_r\tau$-factorizations.

Example 3.9 Let τ_3 be the relation on $\mathbb{Z}^\sharp$ defined by $a\tau_3 b \Leftrightarrow a \equiv b$ mod 3. So τ_3 is not associate preserving.

(1) τ_3-factorizations
    (a) The τ_3-atoms are $\pm p$, p a prime and $\pm 3n$, $n \geq 2$, $3 \nmid n$.
    (b) The τ_3-primes are $\pm p$, p a prime and $\pm 3p$, p a prime, $p \neq 3$.
    (c) The $|_{\tau_3}$-primes are $\pm p$, p a prime, $p \neq 3$.
    (d) $\mathbb{Z}$ is τ_3-atomic, even a τ_3-HFD, but not a τ_3-UFD.
(2) $_r\tau_3$-factorizations
    (a) The $_r\tau_3$-atoms are $\pm p$, p a prime; $\pm 3n$, $n \geq 2$, $3 \nmid n$; pq, p, q primes, $p \not\equiv q$ mod 3, $p, q \neq 3$; $-pq$, p, q primes, $p \equiv q$ mod 3, $p, q \neq 3$.
    (b) The $_r\tau_3$-primes are $\pm p$, p a prime and $\pm 3p$, $p \neq 3$ a prime.
    (c) The $|_{r\tau_3}$-primes: none.
    The $|_{wr\tau_3}$-primes are p, a prime, $p \equiv 1$ mod 3 and $-p$, p a prime, $p \equiv 2$ mod 3.
    (d) $\mathbb{Z}$ is $_r\tau_3$-atomic, but it is not a $_r\tau_3$-HFD as $(7 \cdot 2) \cdot (7 \cdot 2) \cdot (7 \cdot 2) = (-49) \cdot 2 \cdot 2 \cdot 2 \cdot (-7)$ are $_r\tau$-atomic factorizations of length 3 and 5.

The fact that there are no $|_{r\tau_3}$-primes is a special case of the following result.

Proposition 3.10 *Let D be an integral domain with $char(D) \neq 2$ and let τ be a reflexive, symmetric relation on $D^\sharp$. Then D has no $|_{r\tau}$-primes. Thus $\mathbb{Z}$ has no $|_{r\tau_n}$-primes for any $n \geq 0$.*

Proof Let $a \in D^\sharp$. Note that $a \neq -a$ since $char(D) \neq 2$. Now $a \mid_{r\tau} a \cdot a = (-a) \cdot (-a)$, but $a \nmid_{r\tau} (-a)$. Hence a cannot be a $|_{r\tau}$-prime. $\square$

All of the material in Sect. 2 concerning τ-factorization has an analog for $_r\tau$-factorization (in fact, two analogs in some cases since we can use either $|_{r\tau}$ or $|_{wr\tau}$). We have seen (Corollary 3.7) that for τ associate preserving, we can omit the leading unit in a τ-factorization (that is, use $_r\tau$-factorization) once we replace $|_{r\tau}$ by $|_{wr\tau}$. This, along with Proposition 3.10, suggests that when working with $_r\tau$-factorizations we should replace $|_{r\tau}$ by $|_{wr\tau}$. But this in turn suggests that τ-factorization is preferable over $_r\tau$-factorization since we can use the natural relation $|_\tau$. As illustrated in Example 3.9, the theory of τ-factorization is often "cleaner" than the corresponding theory of $_r\tau$-factorization. For example, having an associate of a τ-atom remain a τ-atom is certainly a desirable property. Finally, the leading unit in a τ-factorization allows the flexibility in constructing some natural factorizations as pointed out in Example 2.3(3).

4 Γ-Factorizations

In this section we give an introduction to Γ-factorizations. Γ-factorizations were first studied in [11] and this investigation is continued in [9]. Many results not stated here are given in [9] and [11] which reveal the power, generality, and usefulness of the Γ-factorization framework. Let D be an integral domain with quotient field K. We maintain the notation from Sects. 2 and 3 with the exception that τ need not be symmetric. For $a \in D^\sharp$, we call $a = \lambda a_1 \cdots a_n, n \geq 1, \lambda \in U(D), a_i \in D^\sharp$ *a factorization of a* and a *reduced factorization of a* is $a = a_1 \cdots a_n, n \geq 1, a_i \in D^\sharp$. Note that the order of the factors matters. In the case $n = 1$ we call the (reduced) factorization $a = \lambda(\lambda^{-1}a)$ $(a = a)$ a (reduced) trivial factorization. We will consider a reduced factorization to be a factorization where $\lambda = 1$, and conversely. For $a \in D^\sharp$ let fact(a) (resp., rfact(a)) be the set of factorizations of a (resp., reduced factorizations of a). And let tfact(a) (resp., trfact(a)) be the set of trivial factorizations of a (resp., trivial reduced factorizations of a). Finally, fact$(D) = \cup_{a \in D^\sharp}$fact$(a)$, rfact$(D) = \cup_{a \in D^\sharp}$rfact$(a)$, tfact$(D) = \cup_{a \in D^\sharp}$tfact$(a)$ and trfact$(D) = \cup_{a \in D^\sharp}$trfact$(a)$.

Remark 2 We could have more rigorously defined *a factorization in D* to be the "formal word" $(\lambda, a_1, \ldots, a_n, 1, 1, \ldots)$ where $\lambda \in U(D), n \geq 1$ and $a_i \in D^\sharp$. A *reduced factorization in D* is a factorization $(1, a_1, \ldots, a_n, 1, 1, \ldots)$. Let fact$(D)$ be the set of these factorizations in D. Define $\pi : fact(D) \longrightarrow D^\sharp$ by $\pi((\lambda, a_1, \ldots, a_n, 1, 1, \ldots)) = \lambda a_1 \cdots a_n$. If $(\lambda, a_1, \ldots, a_n, 1, 1, \ldots) \in$ fact(D) with $\pi((\lambda, a_1, \ldots, a_n, 1, 1, \ldots)) = a$ we call $(\lambda, a_1, \ldots, a_n, 1, 1, \ldots)$ *a factorization of a* and write $a = \lambda a_1 \cdots a_n$.

Definition 4.1 Let $\Gamma \subseteq fact(D)$. An element of Γ (resp., $\Gamma(a) := \Gamma \cap fact(a)$) is called a Γ-*factorization* (resp., Γ-*factorization of a*). Let $a, b \in D^\sharp$. We say that *a (weakly)* Γ-*divides b*, written $a \mid_\Gamma b$ ($a \mid_{w\Gamma} b$), if there exists a Γ-factorization of b, $b = \lambda a_1 \cdots a_n$, where some $a_i = a$ (or $a \sim b$). We also say that a_i is a Γ-*factor of b*. An element $a \in D^\sharp$ is Γ-*irreducible* or a Γ-*atom* if $\Gamma \cap$ fact$(a) \subseteq$ tfact(a). We denote the set of Γ-atoms of D by atom(Γ). A Γ-*atomic factorization of a* is a Γ-factorization $a = \lambda a_1 \cdots a_n$ where each a_i is a Γ-atom and D is said to be Γ-*atomic* if each $a \in D^\sharp$ with $\Gamma(a) \neq \emptyset$ has a Γ-atomic factorization. An element $a \in D^\sharp$ is a Γ-*prime* (resp., $\mid_\Gamma$-*prime*, $\mid_{w\Gamma}$-*prime*) if whenever $a \mid \lambda a_1 \cdots a_n$ (resp., $a \mid_\Gamma \lambda a_1 \cdots a_n, a \mid_{w\Gamma} \lambda a_1 \cdots a_n$), a Γ-factorization, then $a \mid a_i$ (resp., $a \mid_\Gamma a_i, a \mid_{w\Gamma} a_i$) for some i. A proper ideal I of D is called a Γ-*prime ideal* if whenever $\lambda a_1 \cdots a_n \in I, \lambda a_1 \cdots a_n \in \Gamma$, we have $a_i \in I$ for some i. We denote the set of Γ-prime ideals by Spec$_\Gamma(D)$ and the set of Γ-prime (resp., $\mid_\Gamma$-prime, $\mid_{w\Gamma}$-prime) elements of D by pSpec$_\Gamma(D)$ (resp., pSpec$_{\mid_\Gamma}(D)$, pSpec$_{\mid_{w\Gamma}}(D)$).

Note that according to our definition of a Γ-atomic domain D, an element of $D^\sharp$ need not to be a Γ-product of Γ-atoms. In fact, if $\Gamma = \emptyset$, every element of $D^\sharp$ is a Γ-atom, but no element of $D^\sharp$ has a Γ-atomic factorization. We will give some examples of Γ-factorizations later. For the moment we content ourselves with two examples. Let τ be a symmetric relation on $D^\sharp$, D an integral domain. Let Γ_τ (resp., $\Gamma_{r\tau}$) be the set of all τ-factorizations (resp., reduced τ-factorizations) on D. So a Γ_τ-factorization (resp., $\Gamma_{r\tau}$-factorization) is just a τ-factorization (resp., reduced τ-factorization). We have $a \mid_{\Gamma_\tau} b \Leftrightarrow a \mid_\tau b, a \mid_{\Gamma_{r\tau}} b \Leftrightarrow a \mid_{r\tau} b$ and $a \mid_{w\Gamma_{r\tau}} b \Leftrightarrow a \mid_{wr\tau} b$. Similar statements hold for τ-prime, $r\tau$-prime, $\mid_\tau$-prime, $\mid_{r\tau}$-prime and $\mid_{wr\tau}$-prime.

Observe that $a \in D^\sharp$ is a Γ-prime if and only if (a) is a Γ-prime ideal. Thus an associate of a Γ-prime is again a Γ-prime. This is not true for Γ-atoms or $\mid_\Gamma$-primes as seen by taking Γ to be $\Gamma_{r\tau}$. The notions of Γ-atom, Γ-prime and $\mid_\Gamma$-prime are all special cases of a Γ_1-Γ_2-Γ_3-prime.

Definition 4.2 Let D be an integral domain. Let $\Gamma_1, \Gamma_2, \Gamma_3 \subseteq$ fact(D). We call $a \in D^\sharp$ a Γ_1-Γ_2-Γ_3-*prime* if for each Γ_1-factorization $\lambda a_1 \cdots a_n, a \mid_{\Gamma_2} \lambda a_1 \cdots a_n$ implies $a \mid_{\Gamma_3} a_i$ for some i. We denote the set of Γ_1-Γ_2-Γ_3-*primes* by pSpec$_{\Gamma_1\text{-}\Gamma_2\text{-}\Gamma_3}(D)$.

Thus a Γ-fact(D)-fact(D)-prime is just a Γ-prime and a Γ-Γ-Γ-prime is a $\mid_\Gamma$-prime. We next note that a Γ-atom is just a Γ-trfact(D)-tfact(D)-prime or Γ-trfact(D)-fact(D)-prime.

Proposition 4.3 *Let D be an integral domain and $\Gamma \subseteq fact(D)$.*

(0) *For $a, b \in D^\sharp, a \mid_\Gamma b \Rightarrow a \mid_{w\Gamma} b \Rightarrow a \mid b$.*
(1) *tfact$(D) \subseteq \Gamma \Leftrightarrow a, b \in D^\sharp$ with $a \sim b$ implies $a \mid_\Gamma b$.*
(1r) *trfact$(D) \subseteq \Gamma \Leftrightarrow a \mid_\Gamma a$ for all $a \in D^\sharp$.*
(2) *$\Gamma \subseteq$ tfact$(D) \Leftrightarrow$ for $a, b \in D^\sharp, a \mid_\Gamma b$ implies $a \sim b$.*
(2r) *$\Gamma \subseteq$ trfact$(D) \Leftrightarrow$ for $a, b \in D^\sharp, a \mid_\Gamma b$ implies $a = b$.*
(3) *$\Gamma =$ tfact$(D) \Leftrightarrow$ for $a, b \in D^\sharp, a \sim b \Leftrightarrow a \mid_\Gamma b$.*

(3r) $\Gamma = trfact(D) \Leftrightarrow$ for $a, b \in D^\sharp$, $a = b \Leftrightarrow a \mid_\Gamma b$.
(4) Let $a \in D^\sharp$. The following are equivalent:
    (a) a is Γ-irreducible,
    (b) a is Γ-trfact(D)-Γ_3-prime for some $\Gamma_3 \subseteq fact(D)$,
    (c) a is Γ-trfact(D)-tfact(D)-prime, and
    (d) a is Γ-trfact(D)-Γ_3-prime for every $\Gamma_3 \supseteq tfact(D)$.
(5) The following are equivalent:
    (a) $\Gamma \subseteq tfact(D)$,
    (b) every element of $D^\sharp$ is a Γ-atom, and
    (c) every element of $D^\sharp$ is a Γ-prime.

Proof (0) Clear. (1)($\Rightarrow$) Suppose $a \sim b$; so $b = \lambda a$ for some $\lambda \in U(D)$. But $b = \lambda a$ is a trivial factorization of b and hence a Γ-factorization of b. So $a \mid_\Gamma b$. ($\Leftarrow$) Let $b = \lambda a$ be a trivial factorization of b. We need that λa is in Γ. Now $a \sim b$, so $a \mid_\Gamma b$. So $b = \mu a_1 \cdots a_n$ is a Γ-factorization where say $a_i = a$. So $\lambda a = b = \mu a_1 \cdots a_{i-1} \cdot a \cdot a_{i+1} \cdots a_n$. If $n > 1$, cancelling a gives $\lambda = \mu a_1 \cdots a_{i-1} a_{i+1} \cdots a_n$, a contradiction. So $n = 1$ and hence $\lambda = \mu$, so $b = \lambda a$ is in Γ. The proof of (1r) is similar.

(2)($\Rightarrow$) Suppose $a \mid_\Gamma b$; so $b = \lambda a_1 \cdots a_n$ a Γ-factorization. But $\Gamma \subseteq tfact(D)$ gives $n = 1$ so $b \sim a$. ($\Leftarrow$) Let $b = \lambda a_1 \cdots a_n$ be a Γ-factorization. Now $a_1 \mid_\Gamma b$, so $a_1 \sim b$. Hence $a_1 = \mu b$ where $\mu \in U(D)$. So $b = \lambda(\mu b) \cdot a_2 \cdots a_n$. So $n > 1$ forces $a_2, \ldots, a_n$ to be units, a contradiction. Hence $n = 1$, so $b = \lambda a_1 \in tfact(D)$. The proof of (2r) is similar.

(3) This follows from (1) and (2). The proof of (3r) is similar.

(4) $(b) \Rightarrow (a)$ Suppose that a is Γ-trfact(D)-Γ_3-prime. Let $a = \lambda a_1 \cdots a_n$ be a Γ-factorization of a. Then $a \mid_{\text{trfact}(D)} \lambda a_1 \cdots a_n$. So $a \mid_{\Gamma_3} a_i$ for some i. Hence $a|a_i$, say $a_i = ba$, $b \in D^*$. So $a = \lambda a_1 \cdots a_n = \lambda a_1 \cdots a_{i-1} \cdot (ba) \cdot a_{i+1} \cdots a_n$. Hence $n = 1$; so a is Γ-irreducible. $(a) \Rightarrow (d)$ Suppose that a is Γ-irreducible. Suppose that $a \mid_{\text{trfact}(D)} \lambda a_1 \cdots a_n$ where $\lambda a_1 \cdots a_n$ is a Γ-factorization. By (3r) $a \mid_{\text{trfact}(D)} \lambda a_1 \cdots a_n$ implies $a = \lambda a_1 \cdots a_n$. Now a Γ-irreducible gives $n = 1$, so $a = \lambda a_1$. Hence $a \sim a_1$, so $a \mid_{\Gamma_3} a_1$. So a is a Γ-trfact(D)-Γ_3-prime. $(d) \Rightarrow (c) \Rightarrow (b)$ Clear.

(5) This follows from the definitions. $\qquad\square$

Our next proposition gives some properties of $\cup \Gamma_\alpha$, $\cap \Gamma_\alpha$ and $\Gamma_2 \backslash \Gamma_1$ for $\Gamma_\alpha, \Gamma_1, \Gamma_2 \subseteq fact(D)$. Its simple proof is left to the reader.

Proposition 4.4 *Let D be an integral domain, $\Gamma_1, \Gamma_2, \Gamma_\alpha \subseteq fact(D)$ and $a, b \in D^\sharp$.*

(1) $(\underset{\alpha}{\cup}\Gamma_\alpha)(a) = \underset{\alpha}{\cup}\Gamma_\alpha(a)$.
(2) $(\underset{\alpha}{\cap}\Gamma_\alpha)(a) = \underset{\alpha}{\cap}\Gamma_\alpha(a)$.
(3) *If $\Gamma_1 \subseteq \Gamma_2$, then $a \mid_{\Gamma_1} b \Rightarrow a \mid_{\Gamma_2} b$.*
(4) $(\Gamma_2 \backslash \Gamma_1)(a) = (\Gamma_2(a)) \backslash (\Gamma_1(a))$.
(5) $a \mid_{(\underset{\alpha}{\cup}\Gamma_\alpha)} b \Leftrightarrow a \mid_{\Gamma_\alpha} b$ *for some α.*
(6) $a \mid_{(\underset{\alpha}{\cap}\Gamma_\alpha)} b \Rightarrow a \mid_{\Gamma_\alpha} b$ *for all α.*
(7) *If $a \mid_{\Gamma_2} b$ and $a \nmid_{\Gamma_1} b$, then $a \mid_{(\Gamma_2 \backslash \Gamma_1)} b$.*

Example 4.5 Let D be an integral domain and $\{I_\alpha\}$ a family of ideals of D. Define $a\tau_{I_\alpha}b \Leftrightarrow a - b \in I_\alpha$ (Example 2.3(4)) and let $\Gamma_{\tau_{I_\alpha}}$ be the set of τ_{I_α}-factorizations. Then $\underset{\alpha}{\cap}\Gamma_{\tau_{I_\alpha}} = \Gamma_{\tau_{\cap_\alpha I_\alpha}}$ In particular, for $D = \mathbb{Z}$ and natural numbers $n_1, \ldots, n_s$, we have $\Gamma_{\tau_{n_1}} \cap \cdots \cap \Gamma_{\tau_{n_s}} = \Gamma_{\tau_m}$ (or $\tau_{n_1} \cap \cdots \cap \tau_{n_s} = \tau_m$) where m is the LCM of $n_1, \ldots, n_s$.

Example 4.6 Let Γ_{τ_n} be the set of all τ_n-factorizations (Example 2.3(4)). Note that $2 \mid_{\Gamma_{\tau_3}} 14$ and $2 \mid_{\Gamma_{\tau_5}} 14$, but $2 \nmid_{\Gamma_{\tau_3} \cap \Gamma_{\tau_5}} 14$, or equivalently $2 \nmid_{\Gamma_{\tau_{15}}} 14$, hence the converse of (6) fails in general. The converse of (7) need not to hold. For example, let $\Gamma_1 = \{2 \cdot 2 \cdot 3\}$ and $\Gamma_2 = \{4 \cdot 3\}$ and note that $3 \mid_{\Gamma_1} 12$, $3 \mid_{\Gamma_2} 12$ and $3 \mid_{\Gamma_2 \backslash \Gamma_1} 12$.

We next show how the various forms of "prime" elements behave with respect to $\cup_\alpha \Gamma_\alpha$ and $\cap_\alpha \Gamma_\alpha$.

Theorem 4.7 *Let D be an integral domain and let $\Gamma_1, \Gamma_1{}', \Gamma_2, \Gamma_2{}', \Gamma_3, \Gamma_3{}', \Gamma_\alpha$ be subsets of fact(D). Then the following hold.*

(1) *If $\Gamma_1 \subseteq \Gamma_2$, then $Spec_{\Gamma_2}(D) \subseteq Spec_{\Gamma_1}(D)$.*

(2) $Spec_{\underset{\alpha}{\cup}\Gamma_\alpha}(D) = \cap Spec_{\Gamma_\alpha}(D).$

(3) *If $\Gamma_1 \subseteq \Gamma_1'$, then $pSpec_{\Gamma_1'\text{-}\Gamma_2\text{-}\Gamma_3}(D) \subseteq pSpec_{\Gamma_1\text{-}\Gamma_2\text{-}\Gamma_3}(D).$*

(4) $pSpec_{(\cup\Gamma_\alpha)\text{-}\Gamma_2\text{-}\Gamma_3}(D) = \underset{\alpha}{\cap}pSpec_{\Gamma_\alpha\text{-}\Gamma_2\text{-}\Gamma_3}(D).$

(5) $\underset{\alpha}{\cup}pSpec_{\Gamma_\alpha\text{-}\Gamma_2\text{-}\Gamma_3}(D) \subseteq pSpec_{(\cap\Gamma_\alpha)\text{-}\Gamma_2\text{-}\Gamma_3}(D).$

(6) *If $\Gamma_2 \subseteq \Gamma_2'$, $pSpec_{\Gamma_1\text{-}\Gamma_2'\text{-}\Gamma_3}(D) \subseteq pSpec_{\Gamma_1\text{-}\Gamma_2\text{-}\Gamma_3}(D).$*

(7) $pSpec_{\Gamma_1\text{-}(\cup\Gamma_\alpha)\text{-}\Gamma_3}(D) = \underset{\alpha}{\cap}pSpec_{\Gamma_1\text{-}\Gamma_\alpha\text{-}\Gamma_3}(D).$

(8) $\underset{\alpha}{\cup}pSpec_{\Gamma_1\text{-}\Gamma_\alpha\text{-}\Gamma_3}(D) \subseteq pSpec_{\Gamma_1\text{-}(\cap\Gamma_\alpha)\text{-}\Gamma_3}(D).$

(9) *If $\Gamma_3 \subseteq \Gamma_3'$, $pSpec_{\Gamma_1\text{-}\Gamma_2\text{-}\Gamma_3}(D) \subseteq pSpec_{\Gamma_1\text{-}\Gamma_2\text{-}\Gamma_3'}(D).$*

(10) $\underset{\alpha}{\cup}pSpec_{\Gamma_1\text{-}\Gamma_2\text{-}\Gamma_\alpha}(D) \subseteq pSpec_{\Gamma_1\text{-}\Gamma_2\text{-}(\cup\Gamma_\alpha)}(D).$

(11) $pSpec_{\Gamma_1\text{-}\Gamma_2\text{-}(\cap\Gamma_\alpha)}(D) \subseteq \underset{\alpha}{\cap}pSpec_{\Gamma_1\text{-}\Gamma_2\text{-}\Gamma_\alpha}(D).$

(12) $pSpec_{\Gamma_2}(D)\backslash pSpec_{\Gamma_1}(D) \subseteq pSpec_{\Gamma_2\backslash\Gamma_1}(D).$

(13) $atom(\underset{\alpha}{\cup}\Gamma_\alpha) = \underset{\alpha}{\cap}atom(\Gamma_\alpha).$

(14) $\underset{\alpha}{\cup}atom(\Gamma_\alpha) \subseteq atom(\underset{\alpha}{\cap}\Gamma_\alpha).$

(15) $atom(\Gamma_2)\backslash atom(\Gamma_1) \subseteq atom(\Gamma_2\backslash\Gamma_1).$

Proof (1) *Let $I \in Spec_{\Gamma_2}(D)$ and $\lambda a_1 \cdots a_n \in I \cap \Gamma_1 \subseteq I \cap \Gamma_2$, then $a_i \in I$ for some i. Hence $I \in Spec_{\Gamma_1}(D)$.*
   (2) ($\subseteq$) *This follows from (1).* ($\supseteq$) *Let $I \in \cap Spec_{\Gamma_\alpha}(D)$. Let $\lambda a_1 \cdots a_n \in I \cap (\underset{\alpha}{\cup}\Gamma_\alpha)$; so $\lambda a_1 \cdots a_n \in I \cap \Gamma_\alpha$ for some α. Since I is a Γ_α-prime, some $a_i \in I$. So I is a $\underset{\alpha}{\cup}\Gamma_\alpha$-prime.*

   (3) *The proof is similar to that of* (1).

   (4) ($\subseteq$) *This follows from (3).* ($\supseteq$) *Suppose $a \in \underset{\alpha}{\cap}pSpec_{\Gamma_\alpha\text{-}\Gamma_2\text{-}\Gamma_3}(D)$, so $a \in pSpec_{\Gamma_\alpha\text{-}\Gamma_2\text{-}\Gamma_3}(D)$ for each α. Let $a \mid_{\Gamma_2} \lambda a_1 \cdots a_n$, a $\underset{\alpha}{\cup}\Gamma_\alpha$-factorization, that is, $\lambda a_1 \cdots a_n \in \Gamma_\alpha$ for some α. Hence $a \mid_{\Gamma_3} a_i$ for some i.*

   (5) *and* (12) *follow from* (3). *The proof of* (6) *and* (9) *are similar to that of* (3).

   (7) ($\subseteq$) *Follows from* (6). ($\supseteq$) *Suppose $a \in \underset{\alpha}{\cap}pSpec_{\Gamma_1\text{-}\Gamma_\alpha\text{-}\Gamma_3}(D)$, then $a \in pSpec_{\Gamma_1\text{-}\Gamma_\alpha\text{-}\Gamma_3}(D)$ for each α. Let $a \mid_{\underset{\alpha}{\cup}\Gamma_\alpha} \lambda a_1 \cdots a_n$, a Γ_1-factorization, which implies that $a \mid_{\Gamma_{\alpha_0}} \lambda a_1 \cdots a_n$ for some α_0. Hence $a \mid_{\Gamma_3} a_i$ for some i.*

   (8) *Follows from* (6).
   (10) *and* (11) *follow from* (9).
   (13), (14) *and* (15) *follow from* (3)–(5) *and Proposition 4.3(4).*   □

Example 4.8 Let $\Gamma_1 = \{(-1)2 \cdot 3\}$ and $\Gamma_2 = \{-2 \cdot 3\}$, both subsets of fact($\mathbb{Z}$). Then $\Gamma_1 \cap \Gamma_2 = \emptyset$, so atom($\Gamma_1 \cap \Gamma_2$) $= \mathbb{Z}^\sharp$. On the other hand, atom(Γ_1)$\cup$atom(Γ_2) $= \mathbb{Z}^\sharp - \{-6\}$. So the reverse inclusion of (14) (and hence of (5)) does not hold in general. Now let $\Gamma_1 = \Gamma_2$, then atom($\Gamma_2\backslash\Gamma_1$) $= D^\sharp$, but atom(Γ_2)\atom(Γ_1) $= \emptyset$. So the reverse inclusion of (15) [and hence of (12)] does not hold.

Proposition 4.9 *Let D be an integral domain. Let Γ_i, $\Gamma_i' \subseteq$ fact(D) for $i = 1, 2, 3$ with $\Gamma_1' \subseteq \Gamma_1$, $\Gamma_2' \subseteq \Gamma_2$ and $\Gamma_3' \supseteq \Gamma_3$. Then a Γ_1-Γ_2-Γ_3-prime is a Γ_1'-Γ_2'-Γ_3'-prime.*

Proof Suppose a is Γ_1-Γ_2-Γ_3-prime. Suppose that $a \mid_{\Gamma_2'} \lambda a_1 \cdots a_n \in \Gamma_1'$ Now $\lambda a_1 \cdots a_n \in \Gamma_1' \subseteq \Gamma_1$ and $a \mid_{\Gamma_2'} \lambda a_1 \cdots a_n$ and $\Gamma_2' \subseteq \Gamma_2 \Rightarrow a \mid_{\Gamma_2} \lambda a_1 \cdots a_n$. So $a\Gamma_1$-Γ_2-Γ_3-prime $\Rightarrow a \mid_{\Gamma_3} a_i$ for some i. But then $a \mid_{\Gamma_3'} a_i$ since $\Gamma_3' \supseteq \Gamma_3$.   □

Corollary 4.10 *Let D be an integral domain and let τ_i, τ_i', $i = 1, 2, 3$, be symmetric relations on $D^\sharp$. Suppose that $\tau_1 \geq \tau_1'$, $\tau_2 \geq \tau_2'$ and $\tau_3 \leq \tau_3'$. If a is a τ_1-τ_2-τ_3-prime, then a is a τ_1'-τ_2'-τ_3'-prime.*

Corollary 4.11 *Let D be an integral domain and let $\Gamma_1, \Gamma_2 \subseteq$ fact(D) with $\Gamma_1 \subseteq \Gamma_2$.*

(1) *A Γ_2-atom is a Γ_1-atom.*
(2) *A Γ_2-prime is a Γ_1-prime.*
(3) *A Γ_1-prime is a Γ_1-atom.*

Proof (1), (2) *Clear.* (3) *Let a be a Γ_1-prime. Now a is Γ_1-prime if and only if a is Γ_1-fact(D)-fact(D)-prime. So a is Γ_1-trfact(D)-fact(D)-prime, or equivalently, a is a Γ_1-atom, by Proposition 4.3.4.*   □

We can define "the" Γ-gcd $[a,b]_\Gamma$ for $a,b \in D^\sharp$. We say $[a,b]_\Gamma = d$ if (1) $d \mid_\Gamma a$ and $d \mid_\Gamma b$ and (2) if $c \mid_\Gamma a$ and $c \mid_\Gamma b$, then $c \mid_\Gamma d$. If there does not exist $d \in D^\sharp$ with $d \mid_\Gamma a$ and $d \mid_\Gamma b$, we write $[a,b]_\Gamma = 1$. This will be pursued in a future paper.

For τ-factorization, we defined three additional conditions that the relation τ may satisfy: associate preserving, divisive, and multiplicative. These conditions on τ lead to desirable factorization properties. We next give several conditions that $\Gamma \subseteq \text{fact}(D)$ may satisfy. These properties (along with others) were introduced in [11] and these properties (along with others) are further studied in [9].

Definition 4.12 Let D be an integral domain and $\Gamma \subseteq \text{fact}(D)$. Then we say Γ is

(1) *symmetric* if for any $\lambda a_1 \cdots a_n \in \Gamma$ and $\sigma \in S_n$, we have $\lambda a_{\sigma(1)} \cdots a_{\sigma(n)} \in \Gamma$,
(2) *reflexive* if for any $a \in D^\sharp$ and $n \geq 1$, we have the n-fold product $a \cdots a \in \Gamma$,
(3) *associate preserving* if for any $\lambda a_1 \cdots a_n \in \Gamma$, then $\mu a_1 \cdots (\mu_i a_i) \cdots a_n$ is also a Γ-factorization for any $\mu, \mu_i \in U(D)$,
(4) *divisive* if for any $\lambda a_1 \cdots a_n \in \Gamma$ and $a \mid a_i$ ($a \in D^\sharp$) for $i \in \{1,\ldots,n\}$, we have $\lambda a_1 \cdots a_{i-1} \cdot a \cdot a_{i+1} \cdots a_n \in \Gamma$,
(5) *refinable* if for any $\lambda a_1 \cdots a_n \in \Gamma$ and $b_{i1} \cdots b_{im_i}$ a Γ-factorization of a_i, we have $\lambda b_{11} \cdots b_{1m_1} \cdot b_{21} \cdots b_{2m_2} \cdots b_{n1} \cdots b_{nm_n} \in \Gamma$,
(6) *combinable* if for any $\lambda a_1 \cdots a_n \in \Gamma$ and $k \in \{1,2,\ldots,n-1\}$, then $\lambda a_1 \cdots a_{k-1} \cdot (a_k a_{k+1}) \cdot a_{k+2} \cdots a_n \in \Gamma$,
(7) *unital* if whenever $\lambda a_1 \cdots a_n \in \Gamma$, then $\mu a_1 \cdots a_n \in \Gamma$ for any unit μ,
(8) *divisible* if for any $\lambda a_1 \cdots a_n \in \Gamma$, the subproduct $\lambda a_{i_1} a_{i_2} \cdots a_{i_k} \in \Gamma$ for each $1 \leq i_1 < i_2 < \cdots < i_{k-1} < i_k \leq n$, and
(9) *(reduced) normal* if $(\text{trfact}(D) \subseteq \Gamma)$ $\text{tfact}(D) \subseteq \Gamma$.

Let τ be a (symmetric) relation on $D^\sharp$. Then Γ_τ is symmetric, unital, divisible, reduced, and normal. If τ is reflexive (resp., associate preserving, divisive), then Γ_τ is reflexive (resp., associate preserving, divisive). If τ is divisive (resp., multiplicative), then Γ_τ is refinable (resp., combinable).

Suppose that Γ is combinable. Then any $a \in D^\sharp$ with $\Gamma(a) \neq \emptyset$ which is not a Γ-atom has a Γ-factorization of length 2. If Γ is divisive and combinable, then it is divisible; and if Γ is divisive and unital, then it is associate preserving.

Proposition 4.13 *Let D be an integral domain and let Γ be a unital subset of $\text{fact}(D)$. For $a \in D^\sharp$, the following are equivalent:*

(1) *a is a Γ-atom,*
(2) *λa is a Γ-atom for any $\lambda \in U(D)$,*
(3) *a is a Γ-$\text{tfact}(D)$-Γ_3-prime for some $\Gamma_3 \subseteq \text{fact}(D)$, and*
(4) *λa is a Γ-$\text{tfact}(D)$-Γ_3-prime for some $\Gamma_3 \subseteq \text{fact}(D)$, for any $\lambda \in U(D)$.*

Proposition 4.14 *Let D be an integral domain and $\Gamma \subseteq \text{fact}(D)$. Then the following statements hold.*

(1) *If Γ is normal, then $\mu a \mid_\Gamma \lambda a$ for any $\mu, \lambda \in U(D)$.*
(2) *If $a \mid_\Gamma b$ and $b \mid_\Gamma a$, then $a \sim b$.*
(3) *Let Γ be unital. If $\lambda \in U(D)$, $a \mid_\Gamma b \Leftrightarrow a \mid_\Gamma \lambda b$.*
(4) *Suppose Γ is associate preserving. If $\lambda, \mu \in U(D)$, then $a \mid_\Gamma b \Leftrightarrow \mu a \mid_\Gamma \lambda b$.*
(5) *Let Γ be refinable and associate preserving. If $a \mid_\Gamma b$ and $b \mid_\Gamma c$, then $a \mid_\Gamma c$.*

Proposition 4.15 *Let D be an integral domain and $\Gamma_1, \Gamma_2, \Gamma_\alpha \subseteq \text{fact}(D)$. The following hold.*

(1) *If $\Gamma_1 \subseteq \Gamma_2$ and Γ_1 is reflexive (resp., normal, reduced), then Γ_2 is reflexive (resp., normal, reduced).*
(2) *If each Γ_α is reflexive (resp., symmetric, associate preserving, combinable, divisive, divisible, normal, reduced, unital), so are $\underset{\alpha}{\cup}\Gamma_\alpha$ and $\underset{\alpha}{\cap}\Gamma_\alpha$.*
(3) *If each Γ_α is refinable, then $\underset{\alpha}{\cap}\Gamma_\alpha$ is refinable.*

Proof (1) Follows from the definition. (2) If each Γ_α is reflexive, (1) gives $\underset{\alpha}{\cup}\Gamma_\alpha$ is reflexive. Now, for any $a \in D^\sharp$, the n-fold product $a \cdots a \in \Gamma_\alpha$. Hence $\underset{\alpha}{\cap}\Gamma_\alpha$ is reflexive. The argument for normal or reduced is similar.

Suppose Γ_α is symmetric for each α and let $\lambda a_1 \cdots a_n \in \underset{\alpha}{\cap}\Gamma_\alpha$ (resp., in $\underset{\alpha}{\cup}\Gamma_\alpha$), then $\lambda a_1 \cdots a_n \in \Gamma_\alpha$ for all α (resp., for some α). So $\lambda a_{\sigma(1)} \cdots a_{\sigma(n)} \in \Gamma_\alpha$ for all α (resp., for some α) and $\lambda a_{\sigma(1)} \cdots a_{\sigma(n)} \in \underset{\alpha}{\cap}\Gamma_\alpha$ (resp., $\lambda a_{\sigma(1)} \cdots a_{\sigma(n)} \in \Gamma_\alpha \subseteq \underset{\alpha}{\cup}\Gamma_\alpha$). The arguments for the other properties are similar.

(3) Suppose Γ_α is refinable for all α. Let $\lambda a_1 \cdots a_n \in \underset{\alpha}{\cap}\Gamma_\alpha$ and $a_i = b_{i1} \cdots b_{im_i} \in \underset{\alpha}{\cap}\Gamma_\alpha$, then both $\underset{\alpha}{\cap}\Gamma_\alpha$-factorizations are Γ_α-factorizations for each α. Hence $\lambda b_{11} \cdots b_{1m_1} \cdot b_{21} \cdots b_{2m_2} \cdots b_{n1} \cdots b_{nm_n}$ is a Γ_α-factorization for each α, that is, $\lambda b_{11} \cdots b_{1m_1} \cdot b_{21} \cdots b_{2m_2} \cdots b_{n1} \cdots b_{nm_n} \in \underset{\alpha}{\cap}\Gamma_\alpha$. $\square$

The converse of (2) fails in general. For example consider $\Gamma_1 = \{(-1)(-6), 2\cdot 3, (-1)\cdot(-2)\cdot(-3), (-1)\cdot 2\cdot 3\}$ and $\Gamma_2 = \{(-6), 2\cdot(-3), (-2)\cdot 3, (-1)\cdot 2\cdot(-3), (-1)\cdot(-2)\cdot 3\}$. Now, $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $\Gamma_1 \cup \Gamma_2 = \{(-6), (-1)(-6), 2\cdot 3, 2\cdot(-3), (-2)\cdot 3(-1)\cdot 2\cdot 3, (-1)\cdot(-2)\cdot(-3), (-1)\cdot 2\cdot(-3), (-1)\cdot(-2)\cdot 3\}$ are both associate preserving (resp., divisive, unital), but neither Γ_1 nor Γ_2 is associate preserving (resp., divisive, unital). We can construct similar examples in which the sets are pairwise disjoint by splitting the property among the sets so the union has the desired property. We next give some examples of Γ-factorizations.

Example 4.16 Let D be an integral domain.

(1) Let τ be a symmetric relation on $D^\sharp$. $\Gamma_\tau = \{\lambda a_1 \cdots a_n \mid a_i \tau a_j \text{ for } i \neq j\}$ is the set of τ-factorizations and $\Gamma_{r\tau} = \{a_1 \cdots a_n \mid a_i \tau a_j \text{ for } i \neq j\}$ is the set of reduced τ-factorizations. So $|_{\Gamma_\tau} = |_\tau$ and $|_{\Gamma_{r\tau}} = |_{r\tau}$.

(2) $\Gamma = \text{fact}(D)$. We have the usual factorization, $|_\Gamma = |$, Γ-atom = atom, and $|_\Gamma$-prime = Γ-prime = principal prime. Note that $\Gamma = \Gamma_\tau$ where $\tau = D^\sharp \times D^\sharp$. Of course Γ satisfies all of the conditions given in Definition 4.12.

(3) $\Gamma = \emptyset$. Here every element of $D^\sharp$ is a Γ-atom and a Γ-prime. We never have $a \mid_\Gamma b$; so each element is a $|_\Gamma$-prime. Also, $[a, b]_\Gamma = 1$. Note that $\Gamma \neq \Gamma_\tau$ for any relation τ. Here Γ vacuously satisfies all of the condition given in Definition 4.12 except (2) and (9).

(4) $\Gamma = \text{tfact}(D)$. Here every element of $D^\sharp$ is a Γ-atom and a Γ-prime. Also, $a \mid_\Gamma b \Leftrightarrow a \sim b$. So each $a \in D^\sharp$ is also a $|_\Gamma$-prime. We have $[a, b]_\Gamma = 1 \Leftrightarrow a \not\sim b$. Note that $\Gamma = \Gamma_\tau$ where $\tau = \emptyset$. Here Γ satisfies all the conditions given in Definition 4.12 except (2).

(5) Let τ be a not necessarily symmetric relation on $D^\sharp$. Then $\lambda a_1 \cdots a_n \in \text{fact}(D)$ is an *ordered τ-factorization* if $a_i \tau a_j$ for $i < j$ and $a \mid_\tau^{\text{ord}} b$ if $b = \lambda a_1 \cdots a_n$ is an ordered τ-factorization where some $a_i = a$. Take Γ_τ^{ord} to be the set of all ordered τ-factorizations. So $a \mid_\tau^{\text{ord}} b \Leftrightarrow a \mid_{\Gamma_\tau^{\text{ord}}} b$. In a similar manner we can define a reduced ordered τ-factorization, $|_{r\tau}^{\text{ord}}$ and $|_{wr\tau}^{\text{ord}}$. If we let $\Gamma_{r\tau}^{\text{ord}}$ be the set of reduced ordered τ-factorizations, then $|_{r\tau}^{\text{ord}} = |_{\Gamma_{r\tau}^{\text{ord}}}$ and $|_{wr\tau}^{\text{ord}} = |_{w\Gamma_{r\tau}^{\text{ord}}}$. As a concrete example, define τ on $D[x]^\sharp$ by $f \tau g \Leftrightarrow \deg(f) \leq \deg(g)$. Then an ordered τ-factorization is a factorization $\lambda f_1 \cdots f_n$ where λ is a unit and $f_i \in D[x]^\sharp$ with $\deg(f_1) \leq \deg(f_2) \leq \cdots \leq \deg(f_n)$. Note that Γ_τ is not symmetric, divisive, refinable, nor combinable, but satisfies the other conditions given in Definition 4.12. As a second example we could consider factorizations in $\mathbb{Z}$ of the form $(\pm 1)2^{n_1} \cdot 3^{n_2} \cdots p_k^{n_k}$ where p_i is the ith prime.

(6) Let S be a non-empty subset of $D^\sharp$ and take $\tau = S \times S$, so $a \tau b \Leftrightarrow a, b \in S$ (Example 2.3(3)). Take $\Gamma_S = \{\lambda a_1 \cdots a_n \mid a_i \in S, n \geq 1\}$. Note that for any unit λ and $a \in D^\sharp$, $\lambda a \in \Gamma_\tau$, but $\lambda a \in \Gamma_S \Leftrightarrow a \in S$. In fact, $\Gamma_\tau = \Gamma_S \cup \text{tfact}(D)$. Suppose $a \sim b$. Then $a \mid_\tau b$, but $a \mid_{\Gamma_S} \Leftrightarrow a \in S$. Here we have $c \mid_\tau d \Leftrightarrow c \mid_{w\Gamma_S} d$. Note that $\text{Spec}_{\Gamma_\tau}(D) = \text{Spec}_{\Gamma_S}(D)$, $\text{atom}(\Gamma_\tau) = \text{atom}(\Gamma_S)$, $\text{pSpec}_{\Gamma_\tau}(D) = \text{pSpec}_{\Gamma_S}(D)$, and $\text{pSpec}_{|_{\Gamma_\tau}}(D) = \text{pSpec}_{|_{w\Gamma_S}}(D)$.

We next define the Γ analogs of τ-UFD, τ-HFD, τ-BFD, τ-FFD and τ-ACCP. Note that these definitions agree with the τ-factorization definitions in the case where $\Gamma = \Gamma_\tau$.

Definition 4.17 An integral domain D is called a Γ-*unique factorization domain* (Γ-*UFD*) if D is Γ-atomic and whenever $\lambda a_1 \cdots a_n = \mu b_1 \cdots b_m$ are two Γ-factorizations into Γ-atoms, then $n = m$ and after reordering (if necessary) $a_i \sim b_i$. We say D is a Γ-*half-factorial domain* (Γ-*HFD*) if D is Γ-atomic and for any two Γ-atomic factorizations $\lambda a_1 \cdots a_n = \mu b_1 \cdots b_m$ we have $m = n$. The domain D is called a Γ-*bounded factorization domain* (Γ-*BFD*) if D is Γ-atomic and for each $a \in D^\sharp$ there is a positive integer $N_\Gamma(a)$ so that for any Γ-factorization $\lambda a_1 \cdots a_n$ of a, $n \leq N_\Gamma(a)$. We say that D is a Γ-*finite factorization domain* (Γ-*FFD*) if D is Γ-atomic and each $a \in D^\sharp$ has only finitely many Γ-factorizations (up to order and associates) into Γ-atoms. We say that D satisfies the Γ-*ascending chain condition on principal ideals* (Γ-*ACCP*) if for any infinite sequence $\{a_i\}_{i=1}^\infty$ such that $a_{i+1} \mid_\Gamma a_i$ there is a positive integer N (depending on the sequence) with $a_{k+1} \sim a_k$ for each $k > N$.

Note that Γ-UFD ⇒ Γ-HFD and Γ-FFD, and that Γ-HFD or Γ-FFD ⇒ Γ-BFD. If Γ is divisive, unital and refinable, we have Γ-BFD ⇒ Γ-ACCP ⇒ Γ-atomic (see Theorem 4.19). We next give a Γ analog of Proposition 2.8 that will be used in proving Theorem 4.19(4).

Theorem 4.18 *Let D be an integral domain and $\Gamma \subseteq fact(D)$ be divisible, divisive and unital. Suppose $p_1 \cdots p_n = q_1 \cdots q_m$ are two Γ-factorizations so that each p_i is a Γ-atom and each q_i is a Γ-prime (resp. $|_\Gamma$-prime). Then $n = m$ and after reordering (if necessary) $p_i \sim q_i$. Moreover, if each $a \in D^\sharp$ has a Γ-factorization into Γ-primes (resp. $|_\Gamma$-primes), D is a Γ-UFD.*

Proof First, let us assume that each q_i is a $|_\Gamma$-prime. Without loss of generality, say $q_m \mid_\Gamma p_n$. So $p_n = \alpha a_1 \cdots q_m \cdots a_k \in \Gamma$. Since p_n is a Γ-atom, $k = 0$ and $p_n = \alpha q_m$. Cancelling gives two Γ-factorizations $\lambda q_1 \cdots q_{m-1} = (\mu \cdot \alpha) p_1 \cdots p_{n-1}$ (here we use divisible and unital). By induction, $m - 1 = n - 1$ and after reordering (if necessary) $q_i \sim p_i$ for each $1 \le i < n$. Therefore, $n = m$ and after reordering (if necessary) each $p_i \sim q_i$.

Now suppose that each q_i is a Γ-prime. Then $q_1 \mid p_i$ for some i, say $p_i = a_i q_1$. So $p_1 \cdots p_{i-1} \cdot (a_i q_1) \cdot p_{i+1} \cdots p_n = q_1 \cdots q_m$. Cancelling q_1 gives $p_1 \cdots p_{i-1} \cdot a_i \cdot p_{i+1} \cdots p_n = q_2 \cdots q_m$ where both sides are again Γ-factorizations since Γ is divisive and divisible. Now either $q_2 \mid p_j$ for some $j \ne i$ or $q_2 \mid a_i$. In the first case $p_j = a_j q_2$ and as before $p_1 \cdots a_i \cdots a_j \cdots p_n = q_3 \cdots q_m$ where both are Γ-factorizations. If $q_2 \mid a_i$, then $a_i = a_i' q_2$ for some a_i'. So $p_i = a_i' q_1 q_2$. In this case $p_1 \cdots p_{i-1} \cdot a_i' \cdot p_{i+1} \cdots p_n = q_3 \cdots q_m$ where both are again Γ factorizations. Continuing in this manner, we can partition $q_1, q_2, \ldots, q_m$ (as an ordered set) into $\{q_{ij}\}$ so that $p_i = a_i q_{i1} \cdots q_{ik_i}$ (need not be a Γ-factorization). Hence $\prod q_{ij} = q_1 \cdots q_m$. Thus $q_1 \cdots q_m = p_1 \cdots p_n = (a_1 q_{11} \cdots q_{1k_1}) \cdots (a_n q_{n1} \cdots q_{nk_n}) = a_1 \cdots a_n q_1 \cdots q_m$. Hence each a_i is a unit. Since Γ is divisive and unital $p_i = a_i q_{i1} \cdots q_{ik_i}$ is a Γ-factorization of the Γ-atom p_i. Hence $k_i = 1$. Therefore $n = m$ and after reordering (if necessary) $p_i \sim q_i$. □

We end with the following theorem. Note that Theorem 4.19(4) gives a proof for Theorem 2.12 (which is much simpler than the proof given in [3]).

Theorem 4.19 *Let D be an integral domain and $\Gamma \subseteq fact(D)$.*

(1) *If D has ACCP, then D has Γ-ACCP.*
(2) *Suppose Γ is divisive, unital and refinable. If D satisfies Γ-ACCP, then D is Γ-atomic.*
(3) *Suppose Γ is divisive, unital and refinable. If D is a Γ-BFD, then D satisfies Γ-ACCP.*
(4) *Suppose Γ is divisive, divisible, unital and refinable. If D is a UFD, then D is a Γ-UFD.*

Proof (1) If there is an infinite sequence $\{a_i\}_{i=1}^\infty$ with $a_{i+1} \mid_\Gamma a_i$ that does not stabilize, then the same sequence contradicts the fact that D has ACCP. Hence, there is a positive integer N such that for each $k \ge N$, $a_k \sim a_{k+1}$ and D has Γ-ACCP. (2) Let $a \in D^\sharp$, and suppose that it is not a Γ-atom and has no Γ-atomic factorization. So there is a Γ-factorization, say $a = b_1 \cdots b_n$, which is not a Γ-atomic factorization. So some b_i, say b_1, is neither a Γ-atom nor has a Γ-atomic factorization. Say that $b_1 = b_{11} \cdots b_{1m_1}$ is neither a Γ-atom nor a Γ-atomic factorization of b_1. So $a = b_{11} \cdots b_{1m_1} b_2 \cdots b_n$ is not a Γ-atomic factorization of a. Again, we may suppose that $b_{11} = b_{21} \cdots b_{2m_2}$ is neither a Γ-atom nor a Γ-atomic factorization of b_{11}. So $a = b_{21} \cdots b_{2m_2} \cdot b_{12} \cdots b_{1m_1} b_2 \cdots b_n$ is not a Γ-atomic factorization of a. Continuing in this fashion, and we obtain the infinite sequence $\{b_1, b_{11}, b_{21}, b_{31}, \ldots\}$ with $b_{11} \mid_\Gamma b_1$ and $b_{(i+1)1} \mid_\Gamma b_{i1}$ that does not stabilize. This contradiction shows that every element is either a Γ-atom or has a Γ-atomic factorization. Therefore D is Γ-atomic. (3) Let D be a Γ-BFD. Suppose that there is an infinite sequence $\{a_i\}_{i=1}^\infty$ with $a_{i+1} \mid_\Gamma a_i$ that does not stabilize. Now $a_2 \mid_\Gamma a_1$, so $a_1 = \lambda b_1 \cdots b_{i-1} \cdot a_2 \cdot b_{i+1} \cdots b_n \in \Gamma$ where $n > 1$ since $a_1 \not\sim a_2$. Since $a_3 \mid_\Gamma a_2$, we have $a_2 = \mu c_1 \cdots c_{j-1} \cdot a_3 \cdot c_{j+1} \cdots c_m \in \Gamma$ where $m > 1$ since $a_2 \not\sim a_3$. Since Γ is divisive and unital, $(\mu c_1) \cdot c_2 \cdots c_{j-1} \cdot a_3 \cdot c_{j+1} \cdots c_m \in \Gamma$. Thus Γ refinable gives $a_1 = \lambda b_1 \cdots b_{i-1} \cdot (\mu c_1) \cdot c_2 \cdots c_{j-1} \cdot a_3 \cdot c_{j+1} \cdots c_m \cdot b_{i+1} \cdots b_n \in \Gamma$. Continuing this process, we get arbitrarily long Γ-factorizations for a_1. This contradiction gives that D satisfies Γ-ACCP. (4) Since a UFD has ACCP, D is Γ-atomic by (2). Now, we show that each Γ-atom is a Γ-prime and by Theorem 4.18 D will be a Γ-UFD. Suppose that $a \in D^\sharp$ is a Γ-atom and $a \mid b_1 \cdots b_n \in \Gamma$. Since D is a UFD, it is a GCD domain and so $a = a_1 \cdots a_n$, where each $a_i \in D^*$ and $a_i \mid b_i$. Now collect all those a_is that are non-units. Say $a_{i1}, a_{i2}, \ldots, a_{ik}$ are non-units and say that the product of the rest of them is α. Then $a = \alpha a_{i1} a_{i2} \cdots a_{ik}$ is a Γ-factorization of a, which implies that $k = 1$. Hence $a \mid b_{i1}$ and a is a Γ-prime. □

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