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Ascending the divided and going-down properties by absolute flatness

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Abstract This paper aims to show that the “going-down ring” and the “divided ring” properties ascend along flat morphisms whose co-diagonal morphisms are flat, the so-called absolutely flat morphisms introduced by Olivier. But unibranchness hypotheses are necessary as any henselization morphism shows. As a by-product, we get that the “unibranch divided ring” property is preserved by the formation of factor domains and by localization with respect to prime ideals. Moreover, we exhibit some ascent results for the “going-down ring” and “divided ring” properties along integral extensions.

Mathematics Subject Classification 13B24 · 13G05 · 13C15 · 14A05 · 13B40 · 13A15

المُلخَص

تهدف هذه الورقة البحثية إلى إثبات أن خاصتي “حلقة الهبوط إلى أسفل” و “الحلقة المقسومة” تصعدان عبر المُشاكلات المُسطَّحة التي تكون مُشاكلاتها الفُطرية المرافقة مسطحة، أي المُشاكلات المسماة مسطحة بشكل مطلق والتي قدمها أوليفير. ولكن فرضيات التشعب الموحَّد ضرورية كما تبين ذلك أية مُشكلة هنسليّة. كحصيلة جانبية، يتبين لنا أن تكوين المجالات العاملية والمَوْضَعَة بالنسبة للمثاليات الأولية يحافظان على خاصية “الحلقة المقسومة ذات التشعب الموحَّد”. بالإضافة إلى ذلك، نقدم نتائج عن خصائص “حلقة الهبوط إلى أسفل” و “الحلقة المقسومة” تصعد عبر التمديدات الكاملة.

1 Introduction and notation

All rings considered below are commutative with 1. In this paper, a ring is *local* if it has a unique maximal ideal and is *irreducible* if it has a unique minimal prime ideal. A ring is called *primary* provided that its zero-divisors are nilpotent and is called a *weak Baer ring* if the annihilator of each of its elements is generated by an idempotent.

We are primarily interested in the ascent of the “divided ring” and “going-down ring” properties by absolutely flat ring morphisms. We first give some background and definitions that are needed in the paper. Olivier introduced absolutely flat ring morphisms that are flat ring morphisms $R \rightarrow S$ such that $S \otimes_R S \rightarrow S$ is flat (see [26, 27]). Examples are (strict) henselizations and étale morphisms

Dobbs introduced and explored in numerous papers going-down rings and divided integral domains. An integral domain D is called a (*universally*) *going-down domain* if each of its overrings C defines a (*universally*)

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going-down morphism of rings $D \rightarrow C$ [7, 8, 12]. Then an arbitrary ring is called (universally) going-down if each of its factor domains is (universally) going-down. Following Badawi, a prime ideal P of a ring R is called a *divided prime ideal* if P is comparable under inclusion to each (principal) ideal of R [1]. Then a ring R is called *divided* if each of its prime ideals is divided. A divided ring is local and irreducible. Hence the reduced ring R_{red} of a divided ring R is an integral domain. This last property is crucial in the sequel because we will consider (geometrically) unbranched rings whose definitions follow.

A local ring R with integral closure R' is *unbranched* if R_{red} is an integral domain and R' is a local ring. Clearly, a unbranched local ring is irreducible. Moreover, a unbranched local ring R is *geometrically unbranched* if the residual extension at the maximal ideals of R and R' is purely inseparable (see [36, Chapter IX]). Then a local ring R is unbranched (geometrically unbranched), amounts to saying that its henselization R^h (strict henselization R^{sh}) is irreducible [36, Definitions 1 & 2, p. 100].

We present now our main results. Our motivation comes from some papers by Cinquegrani [4–6]. She mainly considers ascent by henselization of the “divided domain” property and, as a corollary, by integral extensions. A recent paper by Badawi and Dobbs gives criteria that characterize a going-down ring R in terms of some divided extension S of R , within the universe of either primary rings or reduced rings [2]. Moreover, we established that the “universally going-down ring” property is ascended by étale morphisms within the context of weak Baer rings [35]. Primary rings and weak Baer rings share a common property; they have a zero-dimensional total quotient ring and their classes contain integral domains. Thus our first task is the generalization of Cinquegrani’s results to absolutely flat morphisms within the above contexts. Our second aim consists in giving ascent results for the “going-down ring” property by using the results of Badawi and Dobbs. As a by-product, we get new results about the ascent of the “going-down ring” property by integral extensions. But all our results are established under either a unbranchedness hypothesis or at least, an irreducibility condition. That the consideration of unbranchedness is necessary is clear, because for instance, if R^h is divided, then R is unbranched and $R \rightarrow R^h$ is absolutely flat.

In Sect. 1, we develop some material about absolutely flat morphisms. Ascent of the primary property by absolutely flat morphisms is considered.

Section 2 begins with a crucial result that is used in the rest of the paper. If $R \rightarrow S$ is an injective ring morphism, $\text{Spec}(S) \rightarrow \text{Spec}(R)$ is a homeomorphism and S is (locally) primary, then S is (locally) divided if R is. It follows that an absolutely flat ring morphism, whose spectral map is a homeomorphism, ascends the “weak Baer locally divided ring” property.

We establish that the “divided unbranched ring” property is sufficiently strong to be preserved by the formation of factor domains and localization with respect to prime ideals. These two results are essential to proving ascent of the “divided ring” property. Our first ascent result is as follows: If $R \rightarrow S$ is a local absolutely flat ring morphism and S is unbranched, then R is a primary divided ring if and only if S is a primary divided ring. In this case, $\text{Spec}(S) \rightarrow \text{Spec}(R)$ is a homeomorphism. This result applies to (strict) henselizations and is globalized. Moreover, in the above result, we can replace the unbranched hypothesis on S by R is geometrically unbranched. As a matter of fact, this property is preserved by absolute flatness. An immediate consequence is that the “locally divided ring” property ascends along absolutely flat morphisms whose domains are normal.

These results combine to yield that if R is a divided unbranched primary ring, then so is its integral closure. Since a weak Baer ring is locally primary, there naturally exists a version for weak Baer rings. We call a ring morphism a *min morphism* if minimal prime ideals of the target contract to minimal prime ideals. Then one of our main results is the following. Let R be a divided unbranched ring and $R \rightarrow S$ a min integral extension where S is unbranched. If S is either primary or $\text{Nil}(S)$ is divided, then S is a divided ring.

Then Sect. 2 ends with some transfer results for properties like the PVD property and the “ i -domain” property.

Section 3 is devoted to the ascent of the “(universally) going-down ring” property along absolutely flat morphisms. We prove two versions in the “weak Baer ring” context. Let R be a universally going-down weak Baer ring and an absolutely flat ring morphism $f : R \rightarrow S$. Then S is a universally going-down weak Baer ring. McAdam proved a result (see [23]) that gives in our context an ascent criterion: if $R \rightarrow S$ is an absolutely flat ring morphism between weak Baer rings and R is a going-down ring, then S is a going-down ring if and only if S is treed. We deduce from this criterion the next result. Let $R \rightarrow S$ be an absolutely flat ring morphism between weak Baer rings. If R is going-down and S is locally unbranched (for instance, if R is locally geometrically unbranched), then S is going-down.

Next we look at the primary context and get a stronger result than Badawi and Dobbs obtained because we add a unbranchedness hypothesis [2, Theorem 3.3]. Let R be a local primary ring, such that $\text{Nil}(R)$ is a

divided prime ideal. Then R is a unibranch going-down ring if and only if its integral closure R' is a divided ring. We deduce from this last theorem the following ascent result: Let R be a local unibranch going-down primary ring, such that $\text{Nil}(R)$ is a divided prime ideal. If $R \rightarrow S$ is a local absolutely flat ring morphism, such that S is unibranch, then S is a going-down primary ring, such that $\text{Nil}(S)$ is a divided prime ideal.

We now describe notational conventions. For a ring R , we denote, respectively, by $Z(R)$, $\text{Tot}(R)$, R_{red} , R' , $\text{Nil}(R)$, $\text{Min}(R)$ and $\text{Spec}(R)$ the set of all its zero-divisors, its total quotient ring, its reduced ring, its integral closure in $\text{Tot}(R)$, its nilradical, its set of all minimal prime ideals and its set of all prime ideals endowed with the Zariski topology. The residue field $R_P/PR_P \simeq \text{Tot}(R/P)$ of R at $P \in \text{Spec}(R)$ is denoted by $k(P)$. Let $f : R \rightarrow S$ be a ring morphism with spectral map ${}^a f : \text{Spec}(S) \rightarrow \text{Spec}(R)$, then $k(P) \otimes_R S$ is the fiber ring at P of f and its spectrum is homeomorphic to ${}^a f^{-1}(\{P\})$.

We now introduce some useful material.

To begin with, the reader is referred to [26, 27] which state the following results that are used in this paper. Absolute flatness is universal (preserved in any base change), stable under composition and preserved by direct limits. Moreover, $f : R \rightarrow S$ is absolutely flat if and only if $R_Q \rightarrow S_Q$ is absolutely flat for each $Q \in \text{Spec}(S)$ and $P := f^{-1}(Q)$.

If R is a local ring, a local-étale R -algebra is a local ring morphism $R \rightarrow S_P$ where $R \rightarrow S$ is étale and P a prime ideal of S . A direct limit of local-étale R -algebras with local transition morphisms is called a local ind-étale algebra. For instance, R^h and R^{sh} are local ind-étales over R (see [36, Chapter VIII]).

(Local) ind-étale morphisms, flat epimorphisms are absolutely flat. Absolutely flat morphisms and ind-étale morphisms are closely related as the next lemma shows. Some results already known are recalled for the reader's convenience.

Lemma 1.1 *Let $R \rightarrow S$ be an absolutely flat ring morphism.*

- (1) *If R is reduced (respectively, absolutely flat), so is S .*
- (2) *$R \rightarrow S$ is universally incomparable and $\text{Nil}(S) = \text{Nil}(R)S$ so that $S_{\text{red}} = S \otimes_R R_{\text{red}}$. It follows that S is zero-dimensional if R is.*
- (3) *If R and S are local with respective maximal ideals M and N and $MS \neq S$, then $N = MS$ and $R \rightarrow S$ is local.*
- (4) *The residual extensions of $R \rightarrow S$ are algebraic and separable.*
- (5) *If $R \rightarrow S$ is local and S is henselian, then $R \rightarrow S$ is local ind-étale. Moreover, there is a factorization $R \rightarrow R^h \rightarrow S$ where $R^h \rightarrow S$ is faithfully flat and local.*
- (6) *If $R \rightarrow S$ is local, there is a factorization $R \rightarrow S \rightarrow R^{sh}$ where $S \rightarrow R^{sh}$ is faithfully flat and local.*
- (7) *If $R \rightarrow S$ is local, then R is geometrically unibranch if and only if S is geometrically unibranch.*

Proof (1) is [27, Corollary 2, p. 51]. (2) Let $Q \subseteq Q'$ be two prime ideals of S both lying over P in R . The fiber $k(P) \otimes_R S$ is absolutely flat because so is $k(P)$ and moreover, $k(P) \rightarrow k(P) \otimes_R S$ is absolutely flat. Therefore, each prime ideal of the fiber is maximal. As ${}^a f^{-1}(P)$ is homeomorphic to $\text{Spec}(k(P) \otimes_R S)$, we get $Q = Q'$. Then $\text{Nil}(S) = \text{Nil}(R)S$ is a consequence of (1) applied to the absolutely flat morphism $R/\text{Nil}(R) \rightarrow S/\text{Nil}(R)S$. To complete the proof, first observe that a ring A is zero-dimensional if and only if A_{red} is absolutely flat and then use (1).

We prove (3). Observe that $R/M \rightarrow S/MS$ is absolutely flat. An appeal to (1) shows that S/MS is absolutely flat. As this ring is local, it is a field and MS is a maximal ideal.

Statement (4) is a consequence of [26, Proposition 3.1 (iii)] because each residual extension of $R \rightarrow S$ identifies with a residual extension of $k(P) \rightarrow k(P) \otimes_R S$ for some $P \in \text{Spec}(R)$.

(5) Assume that $R \rightarrow S$ is local where S is henselian and R and S have respective maximal ideals M and N . There is a factorization $R \rightarrow R^h \rightarrow S$ into local morphisms. In view of (3), S/MS is a field and $N = MS$. Thus $R/M \rightarrow S/MS$ is a separable algebraic extension of fields by (4). Therefore, $R^h/MR^h \rightarrow S/MS$ is separable algebraic. Write the field S/MS as a direct limit of finite separable field extensions k_i of R^h/MR^h . In view of [36, Corollaire, p. 84], we get a direct system of local-étale finite R^h algebras T_i such that $T_i/MT_i = k_i$. Let T be the direct limit of the algebras T_i . Then $R^h \rightarrow T$ is faithfully flat and local and $T/MT \simeq S/MS$. It follows from [36, Proposition 1, p. 81] that $T \simeq S$. Thus $R \rightarrow S$ is local ind-étale.

Now (6) is [27, Corollary, p. 58] and (7) is [26, Corollaire 2.8]. □

Badawi and Dobbs observed that divided rings R verifying the condition $Z(R) = \text{Nil}(R)$ have the same behavior as divided domains [2]. This condition is clearly equivalent to (0) is a primary ideal of R . For this reason, rings verifying this condition are called *primary* by Bowman, O'Carroll and Qureshi in [3, 24] where

it is proved that the tensor product of finitely many fields over a common subfield is locally primary. The reader is warned that this terminology we adopt is different from the usual. A primary ring R is irreducible and if $\text{Min}(R) = \{\mathfrak{m}\}$, then $\text{Tot}(R) = R_{\mathfrak{m}}$, a zero-dimensional ring. Moreover, there is a factorization $R \rightarrow R_P \rightarrow \text{Tot}(R)$ into injective morphisms for each $P \in \text{Spec}(R)$. An overring of a primary ring is primary [2, Lemma 2.6]. Clearly, injective ring morphisms descend the primary property.

Then a primary ring R is divided if and only if $PR_P = P$ for each $P \in \text{Spec}(R)$. This statement makes sense since we can consider that $R_P \subseteq \text{Tot}(R)$. Besides, P is divided if and only if R is the pullback $\mathcal{P}(P)$ defined by $R_P \rightarrow k(P)$ and $R/P \rightarrow k(P)$ [2, Proposition 2.5 (c)].

In this paper, we consider another class of rings whose total quotient rings are zero-dimensional. A ring is called a *weak Baer ring* if the annihilator of each of its elements is generated by an idempotent. Then R is a weak Baer ring amounts to saying that its total quotient ring is absolutely flat and its local rings are integral domains (see for instance, [35, Definition 1.4 ff]). An overring S of a weak Baer ring R is a weak Baer ring. To see this, let $x = a/s$ and $y = b/t$ be elements of S where s and t are regular elements of R and such that $xy = 0$. If e is the idempotent of R such that $0 :_R a = Re$, then $b = be$ so that $y \in Se$. It follows that $0 :_S x = Se$.

Let R be a ring such that $\text{Tot}(R)$ is absolutely flat and let $R \rightarrow S$ be an absolutely flat morphism. Then $\text{Tot}(S)$ is absolutely flat and identifies to $\text{Tot}(R) \otimes_R S$ [33, Lemme 2.5]. This result can be generalized to rings whose total quotient rings are zero-dimensional and thus applies to primary rings and weak Baer rings.

Proposition 1.2 *Let $R \rightarrow S$ be an absolutely flat ring morphism, where $\text{Tot}(R)$ is zero-dimensional (respectively, absolutely flat). Then $\text{Tot}(S)$ is zero-dimensional (respectively, absolutely flat), $\text{Tot}(S) = S \otimes_R \text{Tot}(R)$, $S \otimes_R B$ is an overring of S for each overring B of R and the integral closures of R and S are related by $S' = S \otimes_R R'$. In particular, we have $\text{Tot}(R_P) = (\text{Tot}(R))_P$ and $(R')_P = (R_P)'$ for each $P \in \text{Spec}(R)$.*

Proof In view of Lemma 1.1(2), $S \otimes_R \text{Tot}(R)$ is zero-dimensional. As $R \rightarrow S$ is flat, there is a ring morphism $\text{Tot}(R) \rightarrow \text{Tot}(S)$ inducing $R \rightarrow S$. Since $S \otimes_R \text{Tot}(R)$ is a pushout, there is a factorization $S \rightarrow S \otimes_R \text{Tot}(R) \rightarrow \text{Tot}(S)$. Observe that $S \otimes_R \text{Tot}(R) \rightarrow \text{Tot}(S)$ is an epimorphism because so is $S \rightarrow \text{Tot}(S)$. As the reduced ring of $S \otimes_R \text{Tot}(R)$ is absolutely flat, $S \otimes_R \text{Tot}(R) \rightarrow \text{Tot}(S)$ is surjective [25, Proposition 2]. Moreover, this map is injective because the flat epimorphism $S \rightarrow S \otimes_R \text{Tot}(R)$ is essential [22, Proposition 2.1, p.111] and the composite $S \rightarrow S \otimes_R \text{Tot}(R) \rightarrow \text{Tot}(S)$ is injective. Hence, we get that $\text{Tot}(S) = S \otimes_R \text{Tot}(R)$. The statement about overrings is clear because $R \rightarrow S$ is flat while the statement about integral closure is a consequence of [27, Theorem 5.1], because $\text{Tot}(S) = \text{Tot}(R) \otimes_R S$. To complete the proof, observe that $R \rightarrow R_P$ is absolutely flat. □

We first look at the ascent of the “primary ring” property by absolutely flat morphisms. Recall that an attached prime ideal P of an R -module M is a prime ideal P of R such that for each finitely generated ideal $I \subseteq P$ there is some $x \in M$ such that $I \subseteq 0 :_R x \subseteq P$. We denote by $\text{Att}(R)$ the set of all attached prime ideals of the R -module R . Then we have $Z(R) = \cup [P \mid P \in \text{Att}(R)]$ while $\text{Nil}(R) = \cap [P \mid P \in \text{Att}(R)]$. For these definitions and further results, see [14,20,32]. Hence, a ring R is primary if and only if $\text{Att}(R)$ has only one element \mathfrak{m} . In this case, \mathfrak{m} is the unique minimal prime ideal of R . If $Q \in \text{Spec}(R)$, then $\text{Att}(R_Q) = \{PR_Q \mid P \subseteq Q \text{ and } P \in \text{Att}(R)\}$ [14, p.407]. It follows that the “primary ring” property localizes (see also [2, Lemma 2.6]).

Proposition 1.3 *Let $f : R \rightarrow S$ be an absolutely flat ring morphism where R is primary with $\text{Min}(R) = \{\mathfrak{m}\}$ and S is irreducible with $\text{Min}(S) = \{\mathfrak{n}\}$.*

- (1) $\text{Tot}(S) = S_{\mathfrak{m}} = S_{\mathfrak{n}}$, $Z(S) = \text{Nil}(S) = \mathfrak{n}$ and $\mathfrak{n} = \mathfrak{m}S$. Hence, S is primary and $\text{Tot}(S) = \text{Tot}(R) \otimes_R S$.
- (2) If A is an overring of R , then $A \otimes_R S$ is an overring of S . Moreover, if R' is the integral closure of R , then $R' \otimes_R S$ is the integral closure of S .
- (3) If \mathfrak{m} is divided, so is \mathfrak{n} .

Proof Since $\text{Tot}(R)$ is zero-dimensional, we can use Proposition 1.2 from which it follows that $\text{Tot}(S) = S_{\mathfrak{m}}$.

Now an element Q of $\text{Att}(S)$ verifies $f^{-1}(Q) \subseteq \mathfrak{m}$ since it can be lifted up to $\text{Tot}(S)$. We thus get $f^{-1}(Q) = \mathfrak{m}$. But \mathfrak{n} is contained in Q and $f^{-1}(\mathfrak{n}) = \mathfrak{m}$ by flatness of f . We deduce from Lemma 1.1(2) that $Q = \mathfrak{n}$. It follows then that $Z(S) = \mathfrak{n} = \text{Nil}(S)$ and $S_{\mathfrak{m}} = S_{\mathfrak{n}}$. Since $\text{Nil}(S) = \text{Nil}(R)S$ by Lemma 1.1(2), we get that $\mathfrak{n} = \mathfrak{m}S$.

Then (2) is a consequence of Proposition 1.2.

Assume that \mathfrak{m} is divided. We can write $nS_{\mathfrak{n}} = nS_{\mathfrak{m}} = \mathfrak{m}R_{\mathfrak{m}}S = \mathfrak{m}S = \mathfrak{n}$. □

Remark 1.4 As in Proposition 1.3, we will have to consider absolutely flat morphisms $f : R \rightarrow S$ where S is irreducible. It would have been pleasant to have a transfer result for irreducibility, at least when R is unbranched and f is local. But this is definitely wrong. It is enough to consider a unbranched ring R which is not geometrically unbranched because $R \rightarrow R^{sh}$ is absolutely flat and R^{sh} is not irreducible. Part (7) of these remarks provides a stronger argument. Here are some positive/negative results.

- (1) When R is unbranched, R^h is irreducible. Moreover, Lemma 1.1(7) shows that S is irreducible when R is geometrically unbranched. Assume that $f : R \rightarrow S$ is a flat epimorphism and that R is irreducible with minimal prime ideal \mathfrak{m} . As f satisfies going-down, each minimal prime of S contracts to \mathfrak{m} . Since $\text{Spec}(S) \rightarrow \text{Spec}(R)$ is injective by [22, Proposition 1.9, p. 109], S is irreducible.
- (2) Suppose that R is irreducible and $R \rightarrow S$ is absolutely flat, then $\text{Nil}(S) = \mathfrak{m}S$ where \mathfrak{m} is the minimal prime ideal of R . Then S is irreducible if and only if $\mathfrak{m}S$ is a prime ideal. As $R \rightarrow S$ is going-down, the preceding condition amounts to the irreducibility of $S_{\mathfrak{m}}$. To see this, use $\text{Nil}(S) = \mathfrak{m}S$ and the fact that $f^{-1}(\mathfrak{n}) = \mathfrak{m}$ for $\mathfrak{n} \in \text{Min}(S)$.
- (3) Now consider an absolutely flat ring morphism $R \rightarrow S$ where R is an integral domain and S is irreducible; then S is an integral domain because S is reduced.
- (4) Let R be a *normal* ring that is, each localization at a prime ideal is an integrally closed domain. If $R \rightarrow S$ is absolutely flat, then S is normal whence locally irreducible (apply [27, Corollary, p. 57] to each local morphism $R_P \rightarrow S_Q$).
- (5) Let $f : R \rightarrow S$ be an absolutely flat morphism such that R is a weak Baer ring and S is locally irreducible; then S is a weak Baer ring. It is enough to remark that $\text{Tot}(S)$ is absolutely flat and that S_Q is an integral domain for each $Q \in \text{Spec}(S)$. This last fact follows from the absolute flatness of $R_P \rightarrow S_Q$ where $P := f^{-1}(Q)$ and (3).
- (6) Let R be a local ring. Then R is (geometrically) unbranched if and only if the integral domain R_{red} is (geometrically) unbranched. Denote by S either R^h or R^{sh} and consider the absolutely flat morphism $R \rightarrow S$. We can read in Lemma 1.1(2) that $S_{\text{red}} = S \otimes_R R_{\text{red}}$. The result follows because S_{red} is either $(R_{\text{red}})^h$ or $(R_{\text{red}})^{sh}$ [17, Section 18].
- (7) It does not hold in general that irreducibility ascends along absolutely flat morphisms. Assume the contrary and consider an absolutely flat integral extension $R \subset S$ where R is an integral domain (for instance, an étale covering of R). In view of (3), S is an integral domain. Because $S \rightarrow S \otimes_R S$ is an absolutely flat integral extension, $S \otimes_R S$ is also an integral domain. Since this integral extension is essential by [35, Proposition 1.1] and $S \rightarrow S \otimes_R S \rightarrow S$ is injective, we find that $S \otimes_R S \rightarrow S$ is bijective. This last property is a characterization of an epimorphism [22, Lemme 1.0, p. 108]. It follows then that $R \rightarrow S$ is a faithfully flat epimorphism whence an isomorphism [22, Lemme 1.2, p. 109], a contradiction.

Remark 1.5 Recall that a ring morphism $R \rightarrow S$ is called *submersive* if $\text{Spec}(S) \rightarrow \text{Spec}(R)$ is a surjective map and the Zariski topology on $\text{Spec}(R)$ is the quotient topology. It is easy to show that if $R \rightarrow S$ is submersive with an injective spectral map, then $\text{Spec}(S) \rightarrow \text{Spec}(R)$ is actually a homeomorphism. This will be used in next sections for either injective integral morphisms or going-down morphisms whose spectral maps are surjective (for instance, faithfully flat morphisms). Indeed, such morphisms are submersive (see for instance [31]).

The following remark will also be useful. Let $f : R \rightarrow S$ be a ring morphism. If ${}^af : \text{Spec}(S) \rightarrow \text{Spec}(R)$ is a homeomorphism, then $Q = \sqrt{PS}$ for each $Q \in \text{Spec}(S)$ and $P := f^{-1}(Q)$. Indeed, the relations ${}^af^{-1}(V(P)) = V(PS)$ and ${}^af(V(Q)) = V(P)$ hold because af is a continuous closed map (see [18, I, Section 1.2]).

2 Ascent of the divided property

Our goal in this section is the extension of some Cinquegrani’s results to rings which may not be integral domains and absolutely flat morphisms. Moreover, we will exhibit new results. In this paper a ring morphism $R \rightarrow S$ is called an *i-morphism* (respectively, *unbranched*, *h-morphism*) if the map $\text{Spec}(S) \rightarrow \text{Spec}(R)$ is injective (respectively, bijective, a homeomorphism). Recall that an *h-morphism* $R \rightarrow S$ induces an order

isomorphism $\text{Spec}(S) \rightarrow \text{Spec}(R)$. Also, an overring B of a ring A is called a unibranch overring if $A \rightarrow B$ is unibranch (this does not mean that the overring is a unibranch ring!)

We give here a useful result which is outside of the absolutely flat context.

Proposition 2.1 *Let $f : R \rightarrow S$ be an injective h -morphism of rings (for instance, if $R \rightarrow S$ is unibranch and either has the going-down property or is integral).*

- (1) $S_Q = S_P$ for each $Q \in \text{Spec}(S)$ and $P := Q \cap R$ so that $R_P \rightarrow S_Q$ is injective.
- (2) If S is (locally) primary and R is (locally) divided, then S is (locally) divided.
- (3) If S is a weak Baer ring and R is locally divided, then S is locally divided.

Proof We prove (1). As $\text{Spec}(S) \rightarrow \text{Spec}(R)$ is a homeomorphism, we get that $Q = \sqrt{PS}$ by Remark 1.5. Consider the multiplicative subset $f(R \setminus P)$ and its saturated associated multiplicative subset T . Then $S \setminus T$ is the union of all prime ideals \mathfrak{q} of S such that $f^{-1}(\mathfrak{q}) \subseteq P$. But $f^{-1}(\mathfrak{q}) \subseteq P$ implies that \mathfrak{q} is contained in Q because $\mathfrak{q} = \sqrt{f^{-1}(\mathfrak{q})S}$ in view of Remark 1.5. From $Q \subseteq S \setminus T$, we deduce $T = S \setminus Q$ and $S_Q = S_P$.

Assume that R is divided and S is primary. Let $Q \in \text{Spec}(S)$ lying over $P \in \text{Spec}(R)$. It follows that $PR_P = P$ because R is primary. Let $x \in QS_Q$. From $S_Q = S_P$ and $Q = \sqrt{PS}$, we deduce that there is some integer n such that $x^n \in PR_P S = PS$ so that $x \in Q$. Therefore, $QS_Q = Q$ for each Q implies that S is divided.

Now the local statement follows easily because $R_P \rightarrow S_Q$ is injective.

To get (3), we can reduce to the local case of (2) because S_Q is an integral domain for $Q \in \text{Spec}(S)$. \square

Corollary 2.2 *Let $f : R \rightarrow S$ be an h -morphism. If R is a weak Baer locally divided ring and f is absolutely flat, then S is a weak Baer locally divided ring.*

Proof From Remark 1.4(5), we deduce that S is a weak Baer ring because the homeomorphism $\text{Spec}(S) \rightarrow \text{Spec}(R)$ forces S to be locally irreducible. Then we can use Proposition 2.1(2), because each local morphism $R_P \rightarrow S_Q$, associated with $Q \in \text{Spec}(S)$ and $P := f^{-1}(Q)$, is faithfully flat whence injective and S_Q is an integral domain. \square

We next outline a generalization to arbitrary rings of [6, Proposizione 2.1] because it exhibits an interesting morphism and gives Proposition 2.3 when $S = R^h$. Actually this result is crucial to proving [5, Proposizione 1.1] in the integral domain case used in Proposition 2.4. Moreover, this result has its own interest.

Proposition 2.3 *If R is a divided (geometrically) unibranch ring, then so is each factor domain of R . Moreover, for each $P \in \text{Spec}(R)$, there is a factorization of the natural map $R/P \rightarrow (R/P)' \rightarrow R'/PR'$ into injective integral morphisms.*

Proof We mimic with suitable changes [6, Proposizione 2.1], using Badawi's definition of divided rings. Let R be a unibranch divided ring. The integral closure R' of R is a local ring and so is R'/PR' for $P \in \text{Spec}(R)$. Then we can exhibit an injective ring morphism $\varphi : (R/P)' \rightarrow R'/PR'$ by setting $\varphi(\bar{r}/\bar{s}) = r/s$, where \bar{r}/\bar{s} is an element of the quotient field $k(P)$ of R/P and r/s is the class of r/s in R'/PR' . To build the ring morphism φ , copy the proof of [6, Proposizione 2.1], taking into account that if an element $s \in R$ does not belong to P then $P \subseteq Rs^n$ for each integer n . Then φ is injective because $R \rightarrow R'$ is injective integral from which it follows that $PR' \cap R = P$ since $R \rightarrow R'$ has lying-over. Next observe that $R/P \rightarrow (R/P)' \rightarrow R'/PR' \simeq (R/P) \otimes_R R'$ is the natural map and is therefore integral and injective. It follows that $(R/P)' \rightarrow R'/PR'$ is integral and hence $(R/P)'$ is a local integral domain. Thus R/P is divided and unibranch. Assume that R is geometrically unibranch and denote by M and M' the respective maximal ideals of R and R' . Then M'/PR' is above the unique maximal ideal \mathfrak{M} of $(R/P)'$ lying over M/P . As $R/M \rightarrow R'/M'$ is purely inseparable, so is $(R/P)/(M/P) \rightarrow (R/P)'/\mathfrak{M}$, showing that R/P is geometrically unibranch. \square

Proposition 2.4 *Let $f : R \rightarrow S$ be a local absolutely flat ring morphism where R is a divided ring and S is a henselian irreducible ring. Then R/P is unibranch for each $P \in \text{Spec}(R)$ and the following statements hold:*

- (1) PS is a prime ideal of S for each $P \in \text{Spec}(R)$.
- (2) $R \rightarrow S$ is an h -morphism whose set-theoretic inverse is given by $P \mapsto PS$ for $P \in \text{Spec}(R)$. In particular, $\text{Spec}(S)$ is linearly ordered.
- (3) $S_Q = S_P$ for each $Q \in \text{Spec}(S)$ and $P := f^{-1}(Q)$.



Proof There is a factorization $R \rightarrow R^h \rightarrow S$ where the second morphism is faithfully flat by Lemma 1.1(5). It follows that R^h is irreducible and R is therefore unbranched. We can assume that $R \rightarrow S$ is local ind-étale by Lemma 1.1(5). Let $P \in \text{Spec}(R)$, then R/P is unbranched by Proposition 2.3. Let \mathfrak{m} be the minimal prime ideal of R . In view of Lemma 1.1(2), $\mathfrak{m}S$ is the nilradical of S so that $\mathfrak{m}S = \mathfrak{n}$, the minimal prime ideal of S . Then $A := R/\mathfrak{m}$ is a divided domain and $B := S/\mathfrak{m}S$ is local ind-étale over A and an integral domain. As S is henselian, so is B because \mathfrak{n} is the nilradical of S [36, Remarque p. 5]. We are now in position to apply a result of Cinquegrani: S/PS is an integral domain for each $P \in \text{Spec}(R)$ [5, Proposizione 1.1]. Then (1) follows easily.

Next suppose that $Q, Q' \in \text{Spec}(S)$ both contract to P in R ; then by incomparability we get that $Q = PS = Q'$. Hence $\text{Spec}(S) \rightarrow \text{Spec}(R)$ is injective and surjective, because $R \rightarrow S$ is faithfully flat. Now $R \rightarrow S$ has the going-down property. It follows from Remark 1.5 that $\text{Spec}(R) \rightarrow \text{Spec}(S)$ is a homeomorphism because $R \rightarrow S$ is a submersive morphism.

Then (3) is a consequence of (2) and Proposition 2.1. □

Next Lemma clarifies Lemma 1.3 under the conditions of Proposition 2.4.

Lemma 2.5 *There is a bijective map $\text{Att}(S) \rightarrow \text{Att}(R)$ under the hypotheses of Proposition 2.4. It follows that S is primary if R is primary.*

Proof The following fact is known: $f^*(\text{Att}(B)) = \text{Att}(A)$ for a faithfully flat ring morphism $f : A \rightarrow B$ [20, Theorem 2.2]. Now if \mathfrak{m} is the minimal prime ideal of R , then $\mathfrak{m}S = \mathfrak{n}$, the unique minimal prime ideal of S . Then observe that R is primary if and only if $P = \mathfrak{m}$ for each $P \in \text{Att}(R)$ and use Proposition 2.4. □

We are now ready for our first transfer result.

Theorem 2.6 *Let $f : R \rightarrow S$ be an absolutely flat local ring morphism where S is unbranched. Then R is a primary divided ring if and only if S is a primary divided ring. In this case, $R \rightarrow S$ is an h -morphism.*

Proof A faithfully flat ring morphism $f : A \rightarrow B$ descends the “divided ring” property because $f^{-1}(IB) = I$ for each ideal I of A . Moreover, absolute flatness is stable under composition. Thus to establish the proof, we can as well assume that S is an irreducible henselian ring by considering $R \rightarrow S \rightarrow S^h$. The “primary ring” property is also descended by faithful flatness (actually, by injectivity).

First assume that R is divided and primary. Then S is primary by Lemma 2.5. In order to check that S is divided, it is enough to show that S is the pullback $\mathcal{P}(Q)$ for each $Q \in \text{Spec}(S)$. Now set $P := f^{-1}(Q)$; we know that $S_Q = S_P$ by Proposition 2.4(3) while $S/Q = S/PS$ by Proposition 2.4(2). Then tensor the pullback $\mathcal{P}(P)$ over R by S . As $R \rightarrow S$ is flat, we get a new pullback defined by the ring morphisms $S_P = S_Q \rightarrow k(P) \otimes_R S$ and $S/PS = S/Q \rightarrow k(P) \otimes_R S$. Since $R \rightarrow S$ is absolutely flat, so is $k(P) \rightarrow k(P) \otimes_R S$. It follows that $k(P) \otimes_R S$ is an absolutely flat ring by [27, Theorem 2.2]. But $k(P) \otimes_R S$ is the fiber ring at P and is therefore a field because $\text{Spec}(S) \rightarrow \text{Spec}(R)$ is injective. There is a factorization $S \rightarrow k(P) \otimes_R S \rightarrow k(Q)$ by definition of a pushout. The composite being an epimorphism of rings so is the last morphism. As its range is a field, $k(P) \otimes_R S \rightarrow k(Q)$ is actually bijective [22, Corollaire 1.3 p.109]. We have thus proved that S is the pullback $\mathcal{P}(Q)$ so that Q is divided.

The converse is straightforward because $f : R \rightarrow S$ is faithfully flat. □

Remark We may also use Proposition 2.1 to prove the previous theorem once we know that $\text{Spec}(S) \rightarrow \text{Spec}(R)$ is a homeomorphism and S is primary.

Corollary 2.7 *Let $R \rightarrow S$ be an absolutely flat ring morphism. If R is locally primary (for instance, if R is a weak Baer ring), S is locally unbranched and R is locally divided, then S is locally divided.*

Proof Let Q be a prime ideal of S lying over $P \in \text{Spec}(R)$. Then $R_P \rightarrow S_Q$ is absolutely flat and local. □

Corollary 2.8 *Let R be a ring. If R is unbranched (respectively, geometrically unbranched), then R^h (respectively, R^{sh}) is a primary divided ring if and only if R is a primary divided ring. In this case, $R \rightarrow R^h$ (respectively, $R \rightarrow R^{sh}$) is an h -morphism.*

Theorem 2.9 *Let $R \rightarrow S$ be a local absolutely flat morphism. Then R is divided, geometrically unbranched and primary if and only if S is divided, geometrically unbranched and primary. In this case $R \rightarrow S$ is an h -morphism.*

Proof First assume that the conditions hold for R and let $R \rightarrow T$ be a strict henselization of R . There is a factorization $R \rightarrow S \rightarrow T$ where the last morphism is local and faithfully flat by Lemma 1.1(6). Then T and S are geometrically unbranched by [26, Corollaire 2.8]. Moreover, T is divided and primary thanks to Theorem 2.6. These two properties are descended by local faithfully flat morphisms. This has been shown at the beginning of the proof of Theorem 2.6. Hence, the conditions hold for S . Now $R \rightarrow T$ is unbranched by Theorem 2.6. As $S \rightarrow T$ has lying-over, $R \rightarrow S$ is unbranched.

Conversely, assume that the conditions hold for S ; then R is primary and divided by faithful flatness of $R \rightarrow S$. It follows from the proof of a theorem by Paxia [30, Theorem 3, p. 404], that R and S have isomorphic strict henselizations. Therefore, R is geometrically unbranched (we can also use [26, Corollaire 2.8]). \square

A ring is called *normal* if each of its localizations at a prime ideal is an integrally closed domain. Such a ring is evidently locally geometrically unbranched.

Proposition 2.10 *Let $R \rightarrow S$ be an absolutely flat morphism where R is a normal ring. If R is locally divided, so is S .*

Proof We can assume that $R \rightarrow S$ is local. In this case, R is geometrically unbranched and primary and then Theorem 2.9 is available. \square

Our previous results enable us to get ascent results for integral morphisms. As a first result, we show that seminormalization ascends the locally divided property, at least in the case where the base ring R is a weak Baer ring (see Sect. 1). As the total quotient ring of a weak Baer ring R is absolutely flat, its seminormalization R^s is the seminormalization of R in its total quotient ring (see [37]). Let P be a prime ideal of R ; then Proposition 1.2 ensures that $\text{Tot}(R_P) = \text{Tot}(R)_P$ so that $(R_P)^s = (R^s)_P$. Moreover, R^s is a weak Baer ring because an overring of R .

Proposition 2.11 *Let R be a weak Baer ring. If R is locally divided, so is R^s .*

Proof It is well-known that $\text{Spec}(R^s) \rightarrow \text{Spec}(R)$ is a homeomorphism. Then we can use Proposition 2.1(3) since R^s is a weak Baer ring. \square

We now consider ascent of the divided property through integral extensions. We need a property of henselian pullbacks.

Proposition 2.12 *Let R be a divided ring and $P \in \text{Spec}(R)$. The following statements are equivalent:*

- (1) R is henselian.
- (2) R_P and R/P are henselian.

Proof We may mimic the proof of Cinquegrani [4, Teorema 2.6]. We can also use the fact that a local ring is henselian if and only if its reduced ring is henselian [36, Remarque, p. 5]. As R is divided, this ring has only one minimal prime ideal \mathfrak{n} and the reduced rings of R , R_P , R/P are, respectively, R/\mathfrak{n} , $(R/\mathfrak{n})_{P/\mathfrak{n}}$ and $(R/\mathfrak{n})/(P/\mathfrak{n})$. Moreover, R/\mathfrak{n} is a divided domain. Therefore, we can assume that R is an integral domain and then apply the above-quoted result. \square

Lemma 2.13 *Let R be a unbranched divided primary ring and let $P \in \text{Spec}(R)$. Then $(R_P)^h$ is isomorphic to $(R^h)_P$. It follows that R_P is a divided unbranched primary ring.*

Proof The first remark is that R^h is a divided ring and that $R \rightarrow R^h$ is an h -morphism by Corollary 2.8. In view of Proposition 2.4, $(R^h)_P = (R^h)_Q$ where Q is the unique prime ideal of R^h such that $P := R \cap Q$. Therefore, $(R^h)_P$ is henselian by Proposition 2.12. To complete the proof, it is enough to use the universal properties of henselization and localization (details can be seen in the proof of [5, Lemma 2.2]).

In view of Corollary 2.8, R^h is a divided primary ring. Hence, R^h is irreducible and so is $(R^h)_P$ for each $P \in \text{Spec}(R)$. From the first part of the proof, we infer that $(R_P)^h$ has a unique minimal prime ideal. It follows that R_P is unbranched. \square

Proposition 2.14 *Let R be a unbranched divided ring. Then the rings R_P are unbranched and divided for each $P \in \text{Spec}(R)$. If in addition $\text{Tot}(R)$ is zero-dimensional, then $R \rightarrow R'$ is an h -morphism.*



Proof Let P be a prime ideal of R , then $\text{Nil}(R_P)$ is the unique minimal prime ideal of R_P and contracts to the unique minimal prime ideal \mathfrak{m} of R . We have therefore $\text{Nil}(R_P) = \mathfrak{m}_P$. Now observe that R/\mathfrak{m} is a divided unibranch integral domain by Proposition 2.3 if R is divided and unibranch. Then Lemma 2.13 asserts that $T := (R/\mathfrak{m})_{(P/\mathfrak{m})}$ is divided and unibranch. But T is isomorphic to $R_P/\mathfrak{m}_P = (R_P)_{\text{red}}$. Then observe that for a local ring A , we have that $(A^h)_{\text{red}} = (A_{\text{red}})^h$ so that A is unibranch when A_{red} is. It follows that R_P is unibranch and evidently, divided.

Assume that $\text{Tot}(R)$ is zero-dimensional. Then $(R')_P = (R_P)'$ is a consequence of Proposition 1.2. Because R_P is unibranch, R'_P is a local ring. It follows easily that $R \rightarrow R'$ is unibranch whence an h -morphism. \square

Theorem 2.15 *Let R be a unibranch divided primary ring. Then its integral closure R' is a divided ring and $R \rightarrow R'$ is an h -morphism.*

Proof Since R is primary, so is its overring R' . We deduce from Proposition 2.1 that R' is divided because $R \rightarrow R'$ is an h -morphism by Proposition 2.14. \square

Theorem 2.16 *Let R be a locally unibranch and divided weak Baer ring. Then its integral closure R' is a locally divided ring.*

Proof We know that R' is a weak Baer ring. Let $Q \in \text{Spec}(R')$ and $P := Q \cap R$. Since R_P is primary, we get that $(R_P)'$ is a divided ring and that $R_P \rightarrow (R_P)'$ is an h -morphism by Theorem 2.15. As $\text{Tot}(R)$ is zero-dimensional, we have $(R_P)' = (R')_P$. These facts combine to yield that $(R')_P$ is local and in fact is equal to R'_Q . Therefore, R' is locally divided. \square

We studied 1-split rings in [35]. A ring morphism f is called *essential* if for any ring morphism g such that $g \circ f$ exists and is injective, then g is injective. Then a ring R is called *1-split* if each integral essential extension of R is unibranch.

Recall that a ring morphism $f : R \rightarrow S$ is called a *min morphism* if ${}^a f(\text{Min}(S)) \subseteq \text{Min}(R)$ [13]. If $R \rightarrow S$ is an essential injective morphism where $\text{Tot}(R)$ is absolutely flat, then $R \rightarrow S$ is a min morphism. It is enough to reconsider the proof of [35, Lemma 1.6].

Lemma 2.17 *Let R be a divided henselian ring with minimal prime ideal \mathfrak{m} .*

- (1) R/\mathfrak{m} is a 1-split ring.
- (2) Let $R \rightarrow S$ be a min integral extension where S is irreducible, then $R \rightarrow S$ is an h -morphism.

Proof Proposition 2.12 states that the divided domain R/\mathfrak{m} is locally henselian. To conclude for (1), apply [35, Theorem 4.4].

Now let $R \rightarrow S$ be a min integral extension such that $\text{Min}(S) = \{\mathfrak{n}\}$, the unique minimal prime ideal of S . Then \mathfrak{n} contracts to \mathfrak{m} in R and $R/\mathfrak{m} \rightarrow S/\mathfrak{n}$ is an integral extension between two integral domains. Such an extension is essential by [35, Proposition 1.1]. Since R/\mathfrak{m} is 1-split [35, Theorem 4.4], $R/\mathfrak{m} \rightarrow S/\mathfrak{n}$ is unibranch so that $\text{Spec}(S) \rightarrow \text{Spec}(R)$ is a homeomorphism. \square

Next theorem generalizes a result of Cinquegrani [5, Teorema 2.5].

Theorem 2.18 *Let R be a divided unibranch ring and $R \rightarrow S$ a min integral extension where S is unibranch. If S is either primary or $\text{Nil}(S)$ is divided, then $R \rightarrow S$ is an h -morphism and S is a divided ring.*

Proof First assume that S is primary so that R is also primary. Consider a min integral extension $R \rightarrow S$ where S is unibranch. Set $A := R^h$ and $B := A \otimes_R S$. The natural map $A \rightarrow B$ is an integral extension because $R \rightarrow A$ is flat. This extension is also a min morphism. Indeed, a minimal prime ideal \mathfrak{n} of B contracts to \mathfrak{m} , the unique minimal prime ideal of R because $S \rightarrow B$ is flat and $R \rightarrow S$ is a min morphism. As $R \rightarrow A$ has incomparability, \mathfrak{n} contracts to an element of $\text{Min}(A)$. Then observe that B is the henselization of S [17, Proposition 18.6.8]. It follows that B has a unique minimal prime ideal. Moreover, A is divided by Corollary 2.8. Therefore, $R \rightarrow S$ is unibranch thanks to Lemma 2.17(2) because injectivity of spectral maps is descended by surjective spectral maps. To conclude, use Proposition 2.1 since S is primary. Now in the general case, R_{red} and S_{red} are integral domains since R and S are unibranch. Moreover, $\text{Nil}(R)$ and $\text{Nil}(S)$ are the unique minimal prime ideals of R and S so that $\text{Nil}(S)$ contracts to $\text{Nil}(R)$. To conclude, it is enough to use

the primary case since $R/\text{Nil}(R)$ is a divided unbranched integral domain by Proposition 1.3 and $S/\text{Nil}(S)$ is unbranched. \square

Now we intend to establish transfer results for some classes of (locally) divided domains. They concern (global) pseudo-valuation integral domains. We will get other transfer results which are an introduction for the next section about going-down rings.

Next results are consequences of two ascent properties. A ring is called *semi-hereditary* if each of its finitely generated ideals is projective. Then the “semi-hereditary ring” property is ascended by absolutely flat morphisms [27, Definition, p. 54 ff]. It follows that the transfer of the “valuation domain” property holds for local absolutely flat morphisms [27, Consequence, p. 55; 29, Teorema 1].

Proposition 2.19 *Let $R \rightarrow S$ be an absolutely flat ring morphism between integral domains. If R is a Prüfer domain, so is S .*

Proof For an integral domain R , the following properties are identical: R is semi-hereditary, R is a Prüfer domain [21, Corollary 3]. To conclude, it is enough to use the ascent of the “semi-hereditary ring” property. \square

If we consider an absolutely flat morphism $R \rightarrow S$ such that R is a Prüfer domain and S is connected, then S is an integral domain because R is an integrally closed domain [27, Corollary, p. 57]. Hence, S is a Prüfer domain by the previous proposition.

Global pseudo-valuation domains (GPVD) were defined by Dobbs and Fontana [11]. An integral domain R is called a GPVD if R is a subring of a Prüfer domain T such that $R \rightarrow T$ is unbranched and there is a common nonzero radical ideal I of R and T , such that R/I and T/I are zero-dimensional. Hypotheses on I are equivalent to I is a common ideal of R and T and R/I and T/I are absolutely flat.

Recall that a ring morphism $R \rightarrow S$ is radiciel if $\text{Spec}(S) \rightarrow \text{Spec}(R)$ is injective and its residual extensions are purely inseparable, or equivalently, $R \rightarrow S$ is universally an i -morphism [18, I, Section 3.7]. Hence, in view of Lemma 1.1(4), the residual extensions of an absolutely flat radiciel ring morphism are isomorphic. In the two following results we have not been able to show that the radiciel hypothesis is superfluous, but we suspect that these results are not valid without a kindred hypothesis. Indeed, since the beginning, spectral injectivity under various forms is a necessary condition.

Theorem 2.20 *Let $R \rightarrow S$ be an absolutely flat radiciel ring morphism (for instance, a flat epimorphism) between integral domains. If R is a GPVD, so is S .*

Proof Using the above notation for R , we set $U := T \otimes_R S$. Then U is a Prüfer domain and $S \rightarrow U$ is injective by flatness of $R \rightarrow S$. We also set $J := IS$. Then J is nonzero because a flat morphism whose domain is an integral domain is injective [29, Proposizione 3, p. 109]. If K is the quotient field of T , then $L := U \otimes_T K$ is the quotient field of U by absolute flatness of $T \rightarrow U$ (see Proposition 1.2). Then U can be identified with its image $W := TS$ in L . It follows that $JW = IST = ITS = J$. Hence J is a common ideal of S and U . As $R/I \rightarrow S/IS$ and $T/I \rightarrow U/IS$ are absolutely flat, S/J and W/J are absolutely flat rings.

Now let Q_1 and Q_2 be two prime ideals of U both contracting to the same prime ideal in S . As $\text{Spec}(T) \rightarrow \text{Spec}(R)$ is injective and $\text{Spec}(U) \rightarrow \text{Spec}(T)$ is injective, we get $Q_1 = Q_2$ and $S \rightarrow U$ is an i -morphism. Finally, because spectral surjectivity is universal, we get that $\text{Spec}(U) \rightarrow \text{Spec}(S)$ is bijective. This completes the proof. \square

We use the following definition of a *pseudo-valuation domain* (PVD): a local domain (R, M) is a PVD if there is a (unique) valuation overring V of R whose maximal ideal is M [19, Theorem 2.7]. This implies that $\text{Spec}(R) = \text{Spec}(V)$. Next corollary can be considered as a companion result of [19, Theorem 1.7]: if R is a PVD and T is an overring such that $R \rightarrow T$ satisfies incomparability, then T is also a PVD. Indeed, absolutely flat morphisms have the incomparability property.

Corollary 2.21 *Let $R \rightarrow S$ be an absolutely flat radiciel local ring morphism between integral domains. If R is a PVD, then S is a PVD.*

Proof As observed in [11], a PVD is nothing but a local GPVD. \square

Remark 2.22 It follows easily that the locally pseudo-valuation domain (LPVD) property is transferred by absolutely flat radiciel morphisms between integral domains since for such a ring morphism $R \rightarrow S$, we have that $R_P \rightarrow S_Q$ is absolutely flat and radiciel for each $Q \in \text{Spec}(S)$ lying over $P \in \text{Spec}(R)$.



Recall that an integral domain R is called *quasi-Prüfer* if for each $P \in \text{Spec}(R)$ and each $Q \in \text{Spec}(R[X])$ such that $Q \subseteq P[X]$, then $Q = (Q \cap R)[X]$. An equivalent condition is that the integral closure of R is a Prüfer domain [16, Corollary 6.5.14]. As integral closure is preserved by absolutely flat morphisms between integral domains (see Proposition 1.3), the “quasi-Prüfer domain” property is ascended by absolutely flat ring morphisms between integral domains.

In view of Corollary 2.21, it may be asked whether a geometrically unbranched divided domain is a PVD. The answer is negative since there exists an integrally closed divided domain which is not a PVD [9, Remark 4.10 (b)]

An i -domain is an integral domain R such that $R \rightarrow T$ is an i -morphism for each of its overrings T [28]. Then R is an i -domain amounts to saying that its integral closure R' is Prüfer and $R \rightarrow R'$ is an i -morphism [28, Proposition 2.14].

Proposition 2.23 *Let $R \rightarrow S$ be a local absolutely flat morphism between integral domains. If R is an i -domain and S is unbranched, then S is an i -domain.*

Proof Since R is quasi-Prüfer, so is S . As S is unbranched, its integral closure $S' = S \otimes_R R'$ is local whence a valuation domain. Since $\text{Spec}(S')$ is linearly ordered and $R' \rightarrow S'$ is incomparable, $\text{Spec}(S') \rightarrow \text{Spec}(S)$ is injective because so is $\text{Spec}(R') \rightarrow \text{Spec}(R)$. \square

3 Ascent of the going-down property

Ascent of the “going-down ring” property by absolutely flat morphisms has only been considered in our paper [35] where we proved that if $R \rightarrow S$ is an étale morphism and R is a universally going-down weak Baer ring, then S is a universally going-down weak Baer ring [35, Theorem 3.3]. Moreover, the local rings of a universally going-down weak Baer ring are geometrically unbranched [35, Theorem 3.4]. In this section we extend the above result to absolutely flat morphisms. Besides, it was natural to look at the ascent of the “going-down ring” property by absolutely flat morphisms. An additional motivation is provided by the following result of McAdam. Let $R \subseteq S$ be an extension of domains satisfying incomparability and such that R is going-down. Then S is going-down if and only if S is treed [23, Lemma A]. Since absolutely flat morphisms have the incomparability property, McAdam’s result applies to absolutely flat extension of integral domains.

In a joint paper with Dobbs, we studied weak Baer (universally) going-down rings [13]. If R is a weak Baer ring, then R is (universally) going-down amounts to saying that each overring S (in $\text{Tot}(R)$) defines a (universally) going-down ring morphism $R \rightarrow S$. In the primary rings context this statement holds also for the going-down property [10, Corollary 2.6]. Hence, the “going-down ring” property admits in both contexts the same overrings characterization as in the integral domain context.

Our main tools are the two following results of Badawi and Dobbs:

Proposition 3.1 *Let R be a ring.*

- (1) [2, Theorem 3.3] *If R is primary and $\text{Nil}(R)$ is a divided prime ideal, then R is a local going-down ring if and only if R has a divided integral (unbranched) overring.*
- (2) [2, Theorem 3.4] *If R is reduced, then R is a treed going-down ring if and only if its seminormalization R^s is a locally divided ring.*

Note that the parenthetical statement (unbranched) in (1) is superfluous because the spectrum of a divided ring S is linearly ordered. Hence, an integral extension $R \rightarrow S$ is unbranched by incomparability.

We begin our study within the weak Baer rings context.

Proposition 3.2 *Let there be a going-down weak Baer ring R and let $f : R \rightarrow S$ be an absolutely flat ring morphism, Q a prime ideal of S and $P := f^{-1}(Q)$. If the integral domain R_P is geometrically unbranched, then S_Q is going-down and geometrically unbranched, $R_P \rightarrow S_Q$ is an h -morphism and S_Q is an integral domain.*

Proof The ring morphism $R_P \rightarrow S_Q$ is absolutely flat and local. It follows from Proposition 1.2 that $\text{Tot}(S_Q) = \text{Tot}(R_P) \otimes_{R_P} S_Q$ and $\text{Tot}(S_Q)$ is absolutely flat. Now observe that the seminormalization of a ring A , whose total quotient ring is absolutely flat, is the seminormalization of A in $\text{Tot}(A)$ [37, Proposition



4.2] because an absolutely flat ring is seminormal [34, Lemme 2.2]. Then the seminormalization of S_Q identifies to $(R_P)^s \otimes_{R_P} S_Q$ by [34, Corollaire 2.8] so that $(R_P)^s \rightarrow (S_Q)^s$ is absolutely flat. This morphism is also local because $R_P \rightarrow (R_P)^s$ and $S_Q \rightarrow (S_Q)^s$ are h -morphisms. Since R_P is a treed going-down domain, $(R_P)^s$ is a divided integral domain by Proposition 3.1(2). Observe that $(R_P)^s$ is geometrically unbranched because so is R_P . It follows that $(S_Q)^s$ is geometrically unbranched and divided by Theorem 2.9 whence locally divided. Moreover, $(R_P)^s \rightarrow (S_Q)^s$ is an h -morphism. Using again Proposition 3.1(2), we find that S_Q is going-down and $R_P \rightarrow S_Q$ is an h -morphism. It follows that S_Q is irreducible, whence an integral domain by Remark 1.4(3). \square

Theorem 3.3 *Let R be a locally geometrically unbranched going-down weak Baer ring and $f : R \rightarrow S$ an absolutely flat ring morphism. Then S is a locally geometrically unbranched going-down weak Baer ring and $R \rightarrow S$ is locally an h -morphism.*

Proof The “going-down ring” property is local [10, Proposition 2.1]. Hence, taking into account Proposition 3.2, it is enough to show that S is a weak Baer ring. We know that $\text{Tot}(S)$ is absolutely flat. Besides, each S_Q is an integral domain for $Q \in \text{Spec}(S)$. It follows that S is a weak Baer ring. \square

Now we can complete our result [35, Theorem 3.3].

Theorem 3.4 *Let R be a universally going-down weak Baer ring and an absolutely flat ring morphism $f : R \rightarrow S$. Then S is a universally going-down weak Baer ring.*

Proof Because a universally going-down weak Baer ring is locally geometrically unbranched [35, Theorem 3.4], S is a weak Baer ring by Theorem 3.3. Let (P) be the universally going-down integral domain property. Then (P) is stable under direct limits [13, Proposition 3.13] and hence is ascended by strict henselizations by [35, Theorem 3.3] because a strict henselization is local ind-étale. Moreover, (P) is descended by faithful flatness [13, Remark 3.14 (b)]. Now consider $Q \in \text{Spec}(S)$ and $P := f^{-1}(Q)$. We get an absolutely flat local ring morphism $R_P \rightarrow S_Q$ and a factorization $R_P \rightarrow S_Q \rightarrow (R_P)^{sh}$ where the second morphism is faithfully flat by Lemma 1.1. Hence, S_Q is a universally going-down integral domain for each $Q \in \text{Spec}(S)$. Therefore, S is universally going-down by [13, Proposition 3.2]. \square

Next we give a version of McAdam’s result quoted at the beginning of the section.

Proposition 3.5 *Let $R \rightarrow S$ be an absolutely flat ring morphism between weak Baer rings. If R is a going-down ring, then S is a going-down ring if and only if S is treed.*

Proof We can reduce to the case of a local morphism between integral domains. Then it is enough to apply [23, Lemma A] to the incomparable injective morphism $R \rightarrow S$. \square

Fontana proved that for an integral domain R , the statement “ $\text{Spec}(R)$ is linearly ordered” is logically equivalent to “there exists a unbranched (overring) extension $R \subseteq S$ where S is a divided integral domain” [15, Théorème 2.14]. This allows us to give a slightly different version of Theorem 3.3.

Theorem 3.6 *Let $R \rightarrow S$ be an absolutely flat ring morphism between weak Baer rings. If R is going-down and S is locally unbranched (for instance, if R is locally geometrically unbranched), then S is going-down.*

Proof We can assume that $R \rightarrow S$ is an absolutely flat local extension between integral domains and that S is unbranched. We first show that S has a unbranched extension $S \rightarrow U$ where U is divided. There is a unbranched integral overring $R \rightarrow T$ where T is divided by Proposition 3.1(1). Then $U := S \otimes_R T$ is an integral overring of S by Proposition 1.3 and $T \rightarrow U$ is absolutely flat. Since $S \rightarrow U$ is integral, we have $S' = U'$ so that U is unbranched. It follows that $T \rightarrow U$ is local. We infer from Theorem 2.6 that U is a divided domain and $T \rightarrow U$ is unbranched. Hence, $R \rightarrow S \rightarrow U$ is unbranched. As $S \rightarrow U$ satisfies lying-over, $S \rightarrow U$ is unbranched. By the above-quoted Fontana’s result, $\text{Spec}(S)$ is linearly ordered. In short, S is treed and the proof is complete by Proposition 3.5. \square

The preceding results are deduced from the going-down criterion of Proposition 3.1(2) working in the universe of reduced rings. Our goal is now to use the criterion of Proposition 3.1(1) on primary rings.

Theorem 3.7 *Let R be a local going-down unbranched primary ring, such that $\text{Nil}(R)$ is divided. Then R^h is a local going-down primary ring, such that $\text{Nil}(R^h)$ is divided.*



Proof There exists a divided integral overring S of R by Proposition 3.1(1). As R and S have the same integral closure, S is unbranched. Moreover, S is primary by [2, Lemma 2.6]. From [17, Proposition 18.6.8], we deduce that $S^h = R^h \otimes_R S$ because $R \subseteq S$ is an integral extension and S is local. Hence $R^h \rightarrow S^h$ is an integral extension. By Corollary 2.7, S^h is divided and primary and $R^h \rightarrow S^h$ is unbranched. Now $S^h = S \otimes_R R^h$ is an overring of R^h by Proposition 1.3(2) because S is an overring of R . Moreover, $Z(R^h) = \text{Nil}(R^h)$ is a divided prime ideal by Proposition 1.3(3) since R^h is irreducible. Another application of Proposition 3.1(1) shows that R^h is a going-down local ring. \square

Theorem 3.8 *Let R be a local primary ring, such that $\text{Nil}(R)$ is a divided prime ideal. Then R is a unbranched going-down ring if and only if its integral closure R' is a divided ring.*

Proof Assume that R is a unbranched going-down primary ring. We deduce from Proposition 3.1(1) that R has a divided integral overring S such that $R \rightarrow S$ is unbranched. As $R' = S'$, we find that S is unbranched. Besides, S is divided and primary. Thus $R' = S'$ is divided by Theorem 2.15. The converse is clear in view of Proposition 3.1(1). \square

Theorem 3.9 *Let R be a local unbranched going-down primary ring, such that $\text{Nil}(R)$ is a divided prime ideal. If $R \rightarrow S$ is a local absolutely flat ring morphism such that S is unbranched, then S is a going-down primary ring, such that $\text{Nil}(S)$ is a divided prime ideal.*

Proof Thanks to Proposition 1.3(1) and (3), S is primary and $\text{Nil}(S)$ is a divided prime ideal since S is irreducible. In view of Proposition 1.3(2), the integral closure of S is $S' = R' \otimes_R S$ so that $R' \rightarrow S'$ is absolutely flat. Moreover, S' is local and hence, $R' \rightarrow S'$ is local. Then Theorem 2.9 applies since R' is geometrically unbranched, primary as an overring of a primary ring and is divided by Theorem 3.8. Hence S' is divided and the conclusion follows from Theorem 3.8. \square

Corollary 3.10 *Let R be a local geometrically unbranched going-down primary ring, such that $\text{Nil}(R)$ is a divided prime ideal. If $R \rightarrow S$ is a local absolutely flat ring morphism, then S is a geometrically unbranched going-down primary ring, such that $\text{Nil}(S)$ is a divided prime ideal.*

Proof Use Theorem 3.9 and [26, Corollaire 2.8] which states that S is geometrically unbranched. \square

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