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RESEARCH ARTICLE

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Prediction intervals for future lifetime of three parameters Weibull observations based on generalized order statistics

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Abstract In this paper, two pivotal statistics are introduced to construct prediction intervals for future lifetime of three parameters Weibull observations based on generalized order statistics, which can be widely applied in reliability theory and lifetime problems. The probability density functions as well as the explicit form of the distribution functions of our pivotal statistics are derived. Monte Carlo simulations are performed to demonstrate the efficiency of the proposed methods and a real data analysis is conducted for illustrative purposes.

Mathematics Subject Classification (2010) 11K45 · 60K10 · 62E15 · 62G30 · 62M20

الملخص

في هذه الورقة، يتم تقديم إحصائين محوريين لبناء فترات تنبؤ لمدى الحياة المستقبلية لثلاثة وسائط من ملاحظات ويبول على أساس إحصاءات مرتبة ومعممة، والتي يمكن تطبيقها على نطاق واسع في نظرية الاعتمادية ومسائل الحياة. وقد اشتقت دالة كثافة الاحتمال، بالإضافة إلى الصيغة الصريحة لدالة إحصاءاتنا المحورية. كما استعملت محاكاة مونتي كارلو لإثبات كفاءة الطرق المقترحة وجرى تحليل بيانات حقيقية لغرض التوضيح.

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1 Introduction

Prediction of unobserved or censored observations is an interesting topic, especially in the viewpoint of actuarial, biological science, physics, medical and engineering sciences. An authoritative review of developments on prediction problems has been prepared by Kaminsky and Nelson [8]. It is well known that quite often the survival data come with a special feature called censoring. Censoring occurs in life testing experiments, when exact survival times are known only for a portion of the individuals or items under study. The experimenter may not always be in a position to observe the life times of all the products (or items) were put on test either intentionally or unintentionally; this may be because of time limitation and/or other restrictions (such as money, mechanical or experimental difficulties, material resources, etc.); see, for example, Nelson [16] and Balakrishnan and Cohen [1].

The Weibull distribution is one of the most widely used distributions in reliability and survival analysis. Because of its various shapes of the probability density function and its convenient representation of the distribution/survival function, the Weibull distribution has been used very effectively for analyzing lifetime data, particularly when the data are censored, which is very common in most life testing experiments. Moreover, Weibull distribution without any doubt is one of the most important models in modern statistics because of its ability to fit data from various fields, ranging from life data to weather data or observations made in economics and business administration, in hydrology, in biology or in the engineering sciences. A commonly used model in reliability theory and lifetime studies is the three-parameter Weibull distribution, which was introduced by the Swedish statistician Waloddi Weibull for the first time in 1939 in connection with his studies on the strength of materials (for more details and applications of Weibull distribution see Rinne [17]).

The prediction intervals for future observations from the exponential distribution have been studied by many authors and among them are Lawless [11,12], Lingappaiah [13–15], Geisser [7], and Barakat et al. [2], while El-Adll [5] studied the same problem for three-parameter Weibull distribution based on ordinary order statistics.

Generalized order statistics (gos) have been introduced as a unified distribution theoretical set-up which contains a variety of models of ordered random variables (rv's) with different interpretations. Since Kamps [9] had introduced the unifying model of gos, the use of such model has been steadily growing along the years. This is due to the fact that such model includes important well-known practical models that had been separately treated in statistical literature. Examples of such models are the ordinary order statistics, sequential order statistics, progressive type II censored order statistics, record values, kth record values and Pfeifer's records. The rv's $X(1, n, \tilde{m}, k), \dots, X(n, n, \tilde{m}, k)$ are called gos based on an absolutely continuous distribution function (df) F with density function (pdf) f, if their joint density function is given by

$$f^{X(1,n,\tilde{m},k),\dots,X(n,n,\tilde{m},k)}(x_1,\dots,x_n) = k\left(\prod_{j=1}^{n-1}\gamma_{j,n}\right)\left(\prod_{i=1}^{n-1}(1-F(x_i))^{m_i} f(x_i)\right)(1-F(x_n))^{k-1} f(x_n), \quad (1)$$

on the cone $F^{-1}(0) < x_1 \le \cdots \le x_n < F^{-1}(1-)$ of $\mathbb{R}^n$, with parameters $n \in \mathbb{N}$, $n \ge 2$, $k > 0$, $\tilde{m} = (m_1, \dots, m_{n-1}) \in \mathbb{R}^{n-1}$, $M_r = \sum_{j=r}^{n-1} m_j$, such that $\gamma_{r,n} = k + n - r + M_r > 0$ for all $r \in \{1, \dots, n-1\}$. Moreover, let $c_{r-1,n} = \prod_{j=1}^{r}\gamma_{j,n}$, $r = 1, \dots, n-1$, and $\gamma_{n,n} = k$. Generalized order statistics based on the standard uniform distribution are denoted by $U(r, n, \tilde{m}, k)$. Choosing the parameters appropriately, models such as ordinary order statistics (oos) ($\gamma_{i,n} = n-i+1$, $i = 1, \dots, n$, i.e. $\tilde{m} = (m_1, \dots, m_{n-1}) = (0, 0, \dots, 0)$ and $k = 1$), sequential order statistics(sos) ($\gamma_{i,n} = (n-i+1)\alpha_i$, $\alpha_1, \dots, \alpha_n > 0$), progressive type II censored order statistics(pos) ($m_i = R_i \in \mathbb{N}_0$, $\tilde{m} = (m_1, \dots, m_{n-1}) \ne (0, 0, \dots, 0)$, $k = m_n + 1$ with $\gamma_{i,n} = n - i + 1 + \sum_{j=i}^{n} R_j$, $1 \le i \le n-1$ and $\gamma_{n,n} = k = R_n + 1$) and Pfeifer's record model ($\gamma_{i,n} = \beta_i$, $\beta_1, \dots, \beta_n > 0$) are particular cases (cf. [4,9]). Barakat et al. [3] studied some bootstrap properties of normalized extreme generalized order statistics.

In a wide subclass of gos which contains most of the important practical models when $\gamma_{1,n}, \dots, \gamma_{n,n}$ are assumed to be pairwise different, Kamps and Cramer [10] derived the marginal pdf of the rth gos and the joint pdf of the rth and the sth gos, which are given by

$$f^{X(r,n,\tilde{m},k)}(x_r) = c_{r-1,n} f(x_r)\sum_{i=1}^{r} a_i(r)(\overline{F}(x_r))^{\gamma_{i,n}-1}, \quad (2)$$

$$f^{X(r,n,\tilde{m},k),X(s,n,\tilde{m},k)}(x_r,x_s)=c_{s-1,n}\left(\sum_{i=r+1}^{s}a_i^{(r)}(s)\left(\frac{\overline{F}(x_s)}{\overline{F}(x_r)}\right)^{\gamma_{i,n}}\right)\left(\sum_{i=1}^{r}a_i(r)(\overline{F}(x_r))^{\gamma_{i,n}}\right)$$
$$\times\frac{f(x_r)}{\overline{F}(x_r)}\frac{f(x_s)}{\overline{F}(x_s)},\quad x_r\le x_s,\quad 1\le r<s\le n, \tag{3}$$

where

$$a_i(r)=\prod_{\substack{j=1\\j\ne i}}^{r}\frac{1}{\gamma_{j,n}-\gamma_{i,n}},\ 1\le i\le r\le n,\ a_i^{(r)}(s)=\prod_{\substack{j=r+1\\j\ne i}}^{s}\frac{1}{\gamma_{j,n}-\gamma_{i,n}},\ \ r+1\le i\le s\ \text{ and }\ \overline{F}(x)=1-F(x).$$

A random variable X is said to have three-parameter Weibull distribution, denoted by $W(\eta,\xi,\delta)$, if its probability density function (pdf) is given by

$$f(x)=\begin{cases}\frac{\delta}{\xi}\left(\frac{x-\eta}{\xi}\right)^{\delta-1}\exp\left[-\left(\frac{x-\eta}{\xi}\right)^{\delta}\right], & x>\eta,\\ 0, & x\le\eta,\end{cases} \tag{4}$$

where $\eta\in\mathbb{R}$ is a location parameter, $\xi>0$ is a scale parameter and $\delta>0$ is a shape parameter. The corresponding distribution function (df) is given by

$$F(x)=1-\exp\left[-\left(\frac{x-\eta}{\xi}\right)^{\delta}\right],\quad x\ge\eta. \tag{5}$$

In this paper, we modified two pivotal quantities to construct two exact prediction intervals for future observations from three-parameter Weibull distribution based on generalized order statistics. The rest of the paper is organized as follows: In Sect. 2 we present the main results. Section 3 include Monte Carlo simulation for some important models and an application of real lifetime data is given in Sect. 4.

2 The main results

The following lemma is needed in the proof of Theorem 2.3, which expresses an interesting fact that can be applied for solving other problems.

Lemma 2.1 *Suppose that $X(1,n,\tilde{m},k),\ldots,X(n,n,\tilde{m},k)$ are the first n gos based on Weibull distribution with the pdf (4). Then the rv's*

$$Z_i=\gamma_{i,n}\left[\left(\frac{X(i,n,\tilde{m},k)-\eta}{\xi}\right)^{\delta}-\left(\frac{X(i-1,n,\tilde{m},k)-\eta}{\xi}\right)^{\delta}\right],\ i=1,2,\ldots,n,\ \text{with } X(0,n,\tilde{m},k)\equiv\eta, \tag{6}$$

are independent and identically distributed (iid) according to the standard exponential distribution.

Proof By noting that

$$\left(\frac{X(r,n,\tilde{m},k)-\eta}{\xi}\right)^{\delta}=\sum_{j=1}^{r}\frac{Z_j}{\gamma_{i,n}},\quad r=1,2,\ldots,n,$$

the Jacobian J, can be written in the form

$$J=\frac{1}{c_{n-1}}\left(\frac{\xi}{\delta}\right)^{n}\prod_{j=1}^{n}\left(\frac{x_j-\eta}{\xi}\right)^{1-\delta}.$$

The joint pdf of $X(1,n,\tilde{m},k),\ldots,X(n,n,\tilde{m},k)$ based on Weibull distribution with pdf (4) can be written in the form

$$\begin{aligned} f^{X(1,n,\tilde{m},k),\ldots,X(n,n,\tilde{m},k)}(x_1,\ldots,x_n) &= c_{n-1}\left(\frac{\delta}{\xi}\right)^n \left(\prod_{i=1}^{n} y_i^{\delta-1}\right)\exp\left[-\sum_{i=1}^{n-1}(\gamma_{i,n}-\gamma_{i+1,n})y_i^{\delta}-\gamma_{n,n}y_n^{\delta}\right] \\ &= c_{n-1}\left(\frac{\delta}{\xi}\right)^n \left(\prod_{i=1}^{n} y_i^{\delta-1}\right)\exp\left[-\sum_{i=1}^{n}\gamma_{i,n}(y_i^{\delta}-y_{i-1}^{\delta})\right] \end{aligned}$$

where $y_i=(x_i-\eta)/\xi$. Therefore, we have the following equation:

$$f_{Z_1,\ldots,Z_n}(z_1,\ldots,z_n)=\exp\left[-\sum_{j=1}^{n} z_j\right],$$

which by the Factorization Theorem implies the assertion of the lemma. □

The main goal of this paper is to use the first observed r gos, $X(1,n,\tilde{m},k),\ldots,X(r,n,\tilde{m},k)$, to construct prediction intervals for the sth gos $(1\le r<s\le n)$, through the following two modified statistics:

$$U_{r,s}=\frac{X_s^{\star}-X_r^{\star}}{X_r^{\star}} \tag{7}$$

$$V_{r,s}=\frac{X_s^{\star}-X_r^{\star}}{T_{r:n}^{(\tilde{m},k)}}, \tag{8}$$

where

$$X_j^{\star}=\left(\frac{X(j,n,\tilde{m},k)-\eta}{\xi}\right)^{\delta},\ T_{r:n}^{(\tilde{m},k)}=\sum_{i=1}^{r}\gamma_{i,n}\left[\left(\frac{X(i,n,\tilde{m},k)-\eta}{\xi}\right)^{\delta}-\left(\frac{X(i-1,n,\tilde{m},k)-\eta}{\xi}\right)^{\delta}\right],$$

$i=1,2,\ldots,r$ and $X(0,n,\tilde{m},k)\equiv\eta$.

Theorem 2.2 *Assume that $X(1,n,\tilde{m},k),\ldots,X(r,n,\tilde{m},k)$, are the first observed r gos based on $W(\eta,\xi,\delta)$ with pdf (4). Then the df $F_{U_{r,s:n}}^{(\tilde{m},k)}$ of the statistic $U_{r,s}$ is given by*

$$F_{U_{r,s:n}}^{(\tilde{m},k)}(u)=1-\sum_{i=r+1}^{s}\sum_{j=1}^{r}c_{s-1,n}a_i^{(r)}(s)a_j(r)\left[\gamma_{i,n}(\gamma_{j,n}+\gamma_{i,n}u)\right]^{-1},\quad u\ge 0. \tag{9}$$

Proof By Equations (3), (4) and (5), the joint pdf for the rth and sth gos takes the form

$$\begin{aligned} f^{X(r,n,\tilde{m},k),X(s,n,\tilde{m},k)}(x_r,x_s) &= \left(\frac{\delta}{\xi}\right)^2 c_{s-1,n}\left(\frac{x_r-\eta}{\xi}\right)^{\delta-1}\left(\frac{x_s-\eta}{\xi}\right)^{\delta-1}\sum_{i=r+1}^{s}\sum_{j=1}^{r}a_i^{(r)}(s)a_j(r) \\ &\quad\times\exp\left\{-\left[\gamma_{i,n}\left(\left(\frac{x_s-\eta}{\xi}\right)^{\delta}-\left(\frac{x_r-\eta}{\xi}\right)^{\delta}\right)+\gamma_{j,n}\left(\frac{x_r-\eta}{\xi}\right)^{\delta}\right]\right\}, \\ &\quad\eta<x_r\le x_s,\quad 1\le r<s\le n. \end{aligned} \tag{10}$$

By standard transformation methods, the joint pdf of the subrange

$$W_{r,s}=\left(\frac{X(s,n,\tilde{m},k)-\eta)}{\xi}\right)^{\delta}-\left(\frac{X(r,n,\tilde{m},k)-\eta)}{\xi}\right)^{\delta}\quad\text{and}\quad Y=\left(\frac{X(r,n,\tilde{m},k)-\eta)}{\xi}\right)^{\delta},$$

$f_{W_{r,s},Y}^{(\tilde{m},k)}(w,y)$, is given by

$$f_{W_{r,s},Y}^{(\tilde{m},k)}(w,y) = c_{s-1,n} \sum_{i=r+1}^{s} \sum_{j=1}^{r} a_i^{(r)}(s) a_j(r) \exp\left\{-(\gamma_{i,n} w + \gamma_{j,n} y)\right\}, \quad w > 0,\ y > 0. \tag{11}$$

It is not difficult to show that the joint pdf of $U_{r,s} = \dfrac{W_{r,s}}{Y}$ and Y can be written as

$$f_{U_{r,s},Y}^{(\tilde{m},k)}(u,y) = \sum_{i=r+1}^{s} \sum_{j=1}^{r} c_{s-1,n} a_i^{(r)}(s) a_j(r)\, y \exp\left\{-(\gamma_{j,n} + \gamma_{i,n} u) y\right\}, \quad u > 0, \quad y > 0.$$

Thus, we have

$$\begin{aligned} f_{U_{r,s:n}}^{(\tilde{m},k)}(u) &= \int_0^\infty f_{U_{r,s},Y}^{(\tilde{m},k)}(u,y)\,\mathrm{d}y \\ &= c_{s-1,n} \sum_{i=r+1}^{s} \sum_{j=1}^{r} a_i^{(r)}(s) a_j(r) \left(\gamma_{j,n} + \gamma_{i,n} u\right)^{-2}, \quad u > 0. \end{aligned} \tag{12}$$

Integrating (12) form 0 to u and simplifying the result, we obtain (9) which proves the theorem. □

In the following theorem we derive the distribution of the pivotal statistic $V_{r,s}$

Theorem 2.3 *Suppose that $X(1,n,\tilde{m},k),\ldots,X(r,n,\tilde{m},k)$, are the first observed r gos based on three parameters Weibull distribution $W(\eta,\xi,\delta)$. Then the df of the statistic $V_{r,s}$ is given by*

$$F_{V_{r,s:n}}^{(\tilde{m},k)}(v) = 1 - \frac{c_{s-1,n}}{c_{r-1,n}} \sum_{i=r+1}^{s} a_i^{(r)}(s) \left[\gamma_{i,n}(1+\gamma_{i,n} v)^r\right]^{-1}, \quad v \geq 0. \tag{13}$$

Proof In view of Lemma 2.1, the statistic $T_{r:n}^{(\tilde{m},k)}$ has a gamma distribution with the pdf

$$f_{T_{r:n}^{(\tilde{m},k)}}^{(\tilde{m},k)}(t) = \frac{1}{\Gamma(r)} t^{r-1} \mathrm{e}^{-t}, \quad t > 0.$$

By the independence of $W_{r,s}$ and $T_{r:n}^{(\tilde{m},k)}$, the joint pdf of $W_{r,s}$ and $T_{r:n}^{(\tilde{m},k)}$ is given by

$$f_{T_{r:n}^{(\tilde{m},k)},W_{r,s:n}}^{(\tilde{m},k)}(t,w) = f_{T_{r:n}^{(\tilde{m},k)}}^{(\tilde{m},k)}(t) f_{W_{r,s:n}}^{(\tilde{m},k)}(w),$$

and by Eq. (11), with noting that $c_{r-1} \sum_{j=1}^{r} \frac{a_j(r)}{\gamma_{j,n}} = 1$, we get

$$f_{W_{r,s:n}}^{(\tilde{m},k)}(w) = \int_0^\infty f_{W_{r,s},Y}^{(\tilde{m},k)}(w,y)\,\mathrm{d}y = \frac{c_{s-1,n}}{c_{r-1,n}} \sum_{i=r+1}^{s} a_i^{(r)}(s)\, \mathrm{e}^{-\gamma_{i,n} w}, \quad w > 0.$$

Therefore, we obtain

$$f_{T_{r:n}^{(\tilde{m},k)},W_{r,s:n}}^{(\tilde{m},k)}(t,w) = \frac{c_{s-1,n}}{c_{r-1,n}\Gamma(r)} \sum_{i=r+1}^{s} a_i^{(r)}(s)\, t^{r-1} \mathrm{e}^{-(t+\gamma_{i,n} w)}, \quad t > 0,\ w > 0.$$

Putting $V_{r,s} = \dfrac{W_{r,s}}{T_{r:n}^{(\tilde{m},k)}}$, the joint pdf of $T_{r:n}^{(\tilde{m},k)}$ and $V_{r,s}$, after a routine calculations is given by

$$f_{T_{r:n}^{(\tilde{m},k)},V_{r,s:n}}^{(\tilde{m},k)}(t,v) = \frac{c_{s-1,n}}{c_{r-1,n}\Gamma(r)} \sum_{i=r+1}^{s} a_i^{(r)}(s)\, t^{r} \mathrm{e}^{-(1+\gamma_{i,n} v)t}, \quad t > 0,\ v > 0.$$

Hence, the pdf of the pivotal statistic $V_{r,s}$ takes the form

$$f^{(\tilde{m},k)}_{V_{r,s}:n}(v) = \int_0^\infty f^{(\tilde{m},k)}_{T_{r,\tilde{m}},V_{r,s}:n}(t,v)\mathrm{d}t$$
$$= \frac{r c_{s-1,n}}{c_{r-1,n}} \sum_{i=r+1}^{s} a_i^{(r)}(s)\ \left(1+\gamma_{i,n} v\right)^{-(r+1)}, \quad v > 0. \tag{14}$$

Integrating (14) and simplifying the result we obtain (13). This completes the proof of the theorem. □

Remark 2.4 The $(1-\alpha)100\%$ predictive conference intervals for the future unobserved value of $X(s,n,\tilde{m},k)$, based on the pivotal statistics $U_{r,s}$ and $V_{r,s}$, respectively, are given by

$$\left(x_r,\,(1+u_\alpha)^{1/\delta}(x_r-\eta)+\eta\right), \tag{15}$$

and

$$\left(x_r\ ,\ \left[\xi^\delta t_r v_\alpha + (x_r-\eta)^\delta\right]^{1/\delta} + \eta\right), \tag{16}$$

where x_r is an observed value of $X(r,n,\tilde{m},k)$, u_α can be obtained from Eq. (9) by solving the nonlinear equation $F^{(\tilde{m},k)}_{U_{r,s}:n}(u_\alpha) = 1-\alpha$, t_r is an observed value of $T^{(\tilde{m},k)}_{r:n}$ and v_α can be obtained from Eq. (13) by solving the nonlinear equation $F^{(\tilde{m},k)}_{V_{r,s}:n}(v_\alpha) = 1-\alpha$.

3 Simulation study

In this section, Monte Carlo simulations are conducted to investigate the efficiency of the obtained results in the preceding section. For this purpose an algorithm is constructed. In the simulation study, we generate 100,000 ordered random samples, for any value of s, each sample of size n from three-parameter Weibull distribution $W(\eta,\xi,\delta)$ for some values of $\eta,\ \xi$ and δ. The coverage probability and the average interval width based on the two statistics $U_{r,s}$ and $V_{r,s}$ are computed for three special cases from gos.

Algorithm

Step 1 choose the values of r, s and n,

Step 2 solve the nonlinear equations $F^{(\tilde{m},k)}_{U_{r,s}:n}(u_\alpha) = 1-\alpha$ and $F^{(\tilde{m},k)}_{V_{r,s}:n}(v_\alpha) = 1-\alpha$, numerically, to obtain the values of u_α and v_α at $\alpha = 0.05,\ 0.1$, where $F^{(\tilde{m},k)}_{U_{r,s}:n}(u)$ and $F^{(\tilde{m},k)}_{V_{r,s}:n}(v)$ are given by Eqs. (9) and (13), respectively,

Step 3 generate n generalized order statistics $X(1,n,\tilde{m},k),\ldots,X(r,n,\tilde{m},k)$, based on $W(\eta,\xi,\delta)$ for a given values of $\eta,\ \xi,\ \delta$ using the following algorithm which is due to El-Adll [5] (see also Barakat et al. [2]):

(a) generate r independent Uniform (0, 1) observations $W_1,\ldots,W_r$,

(b) set $V_i = W_i^{\frac{1}{\gamma_{i,n}}}$ for $i = 1,2,\ldots r$,

(c) set $U(r,n,\tilde{m},k) = 1-\prod_{i=1}^r V_i$; thus, in view of Cramer [4], definition 3.1.5, $U(r,n,\tilde{m},k)$ is the rth uniform gos,

(d) set $X(r,n,\tilde{m},k) = F^{-1}(U(r,n,\tilde{m},k))$; then $X(r,n,\tilde{m},k)$ for $r = 1,2,\ldots n$, is the rth gos based on the df F,

Step 4 determine the lower and the upper bounds of the predictive intervals using steps 2, 3 and relations Eqs. (15) and (16),

Step 5 define a counter, c, as follows: $c = c+1$, if $X(s,n,\tilde{m},k)$ lies within the predictive interval; otherwise, set $c = c$,

Step 6 repeat steps 3, 4 and 5, 100,000 times,

Step 7 compute the coverage probability $(c/100{,}000)$ and the average interval width of the predictive confidence interval (PCI) of $X(s,n,\tilde{m},k)$.

Finally, all the computations are prepared by Mathematica 8 (Tables 1, 2, 3).

Table 1 Coverage probability and average width when $n = 20$ for $W(3.7, 1.1, 2.2)$ oos model ($m_1 = \cdots = m_{n-1} = 0$ and $k = 1$)

r	s	90%	95%	$AWU_{90\%}$	$AWU_{95\%}$	90%	95%	$AWV_{90\%}$	$AWV_{95\%}$
6	7	0.89904	0.95011	0.1537	0.2044	0.89905	0.95049	0.1536	0.2041
6	8	0.90182	0.95151	0.2597	0.3252	0.90175	0.95122	0.2595	0.3248
6	9	0.89953	0.95022	0.3551	0.4325	0.89961	0.95050	0.3547	0.4319
6	10	0.89924	0.95024	0.4469	0.5351	0.89940	0.94993	0.4464	0.5344
6	11	0.89940	0.95016	0.5378	0.6365	0.90031	0.95013	0.5372	0.6356
6	12	0.90024	0.95026	0.6293	0.7381	0.90026	0.95032	0.6285	0.7370
6	13	0.90072	0.94961	0.7235	0.8426	0.90024	0.94964	0.7226	0.8414
6	14	0.89972	0.94923	0.8226	0.9525	0.90025	0.94917	0.8215	0.9511
6	15	0.90027	0.94992	0.9300	1.0717	0.89971	0.94986	0.9288	1.0701
6	16	0.90028	0.94991	1.0478	1.2023	0.89984	0.94994	1.0464	1.2004
9	10	0.90015	0.95015	0.1441	0.1897	0.90033	0.94981	0.1439	0.1893
9	11	0.90024	0.94914	0.2471	0.3057	0.89990	0.94954	0.2466	0.3047
9	12	0.90126	0.95141	0.3439	0.4133	0.90115	0.95151	0.3430	0.4119
9	13	0.89962	0.94990	0.4400	0.5194	0.89952	0.95005	0.4389	0.5177
9	14	0.90033	0.95052	0.5397	0.6292	0.90054	0.95019	0.5381	0.6269
9	15	0.89913	0.94999	0.6462	0.7462	0.89910	0.94976	0.6445	0.7438
9	16	0.89894	0.94895	0.7624	0.8739	0.89859	0.94870	0.7603	0.8709
12	13	0.90064	0.95018	0.1552	0.2030	0.90043	0.95039	0.1547	0.2022
12	14	0.89869	0.94946	0.2706	0.3323	0.89925	0.94951	0.2695	0.3305
12	15	0.89887	0.94935	0.3848	0.4587	0.89897	0.94856	0.3830	0.4559
12	16	0.90092	0.95061	0.5065	0.5928	0.90070	0.95090	0.5041	0.5892
15	16	0.90022	0.95121	0.1958	0.2548	0.89929	0.95063	0.1946	0.2526

Table 2 Coverage probability and average width when $n = 20$ for $W(3.7, 1.1, 2.2)$ pos model with censoring scheme $(50, 20, 11, 19^*1)$ that is $N = 50$, $n = 20$, $R_1 = 11, R_2 = R_3 = \cdots = R_{20} = 1$ (where N is the total items put on a life test, n is the purposed observed failures and r is the actual observed failures)

r	s	90%	95%	$AWU_{90\%}$	$AWU_{95\%}$	90%	95%	$AWV_{90\%}$	$AWV_{95\%}$
6	7	0.89756	0.94795	0.1137	0.1512	0.89798	0.94784	0.1135	0.1507
6	8	0.90052	0.95012	0.1920	0.2403	0.90071	0.95046	0.1915	0.2395
6	9	0.89917	0.95002	0.2622	0.3192	0.89899	0.94964	0.2614	0.3180
6	10	0.89995	0.94947	0.3297	0.3947	0.89968	0.94948	0.3287	0.3932
6	11	0.89850	0.94864	0.3961	0.4686	0.89833	0.94862	0.3949	0.4669
6	12	0.90045	0.94994	0.4638	0.5439	0.90013	0.94985	0.4625	0.5420
6	13	0.90090	0.95125	0.5334	0.6211	0.90060	0.95078	0.5319	0.6189
6	14	0.90021	0.95095	0.6061	0.7017	0.90037	0.95058	0.6044	0.6993
6	15	0.89895	0.94942	0.6838	0.7877	0.89886	0.94901	0.6818	0.7850
6	16	0.89995	0.95061	0.7706	0.8841	0.90022	0.95016	0.7683	0.8810
9	10	0.90161	0.95132	0.1062	0.1397	0.90204	0.95106	0.1059	0.1393
9	11	0.90003	0.95024	0.1820	0.2251	0.90014	0.94984	0.1814	0.2241
9	12	0.89808	0.94863	0.2529	0.3039	0.89831	0.94878	0.2520	0.3025
9	13	0.89961	0.94997	0.3237	0.3821	0.89975	0.94996	0.3225	0.3802
9	14	0.89977	0.95035	0.3967	0.4624	0.90006	0.94996	0.3952	0.4602
9	15	0.90018	0.95082	0.4743	0.5477	0.90044	0.95101	0.4724	0.5450
9	16	0.89818	0.94908	0.5596	0.6414	0.89834	0.94924	0.5574	0.6383
12	13	0.89924	0.94928	0.1138	0.1489	0.89918	0.94967	0.1134	0.1481
12	14	0.89904	0.94973	0.1985	0.2438	0.89929	0.95012	0.1975	0.2422
12	15	0.90018	0.95045	0.2823	0.3365	0.90034	0.95017	0.2808	0.3342
12	16	0.90065	0.95076	0.3713	0.4347	0.90039	0.95068	0.3691	0.4314
15	16	0.90022	0.95076	0.1435	0.1868	0.90026	0.95017	0.1426	0.1851

4 An illustrative example

The order random variables play an important role for the lifetime prediction methods because if m items are put simultaneously in a life test, the weakest component will fail first, followed by the second weakest and so on until all have failed. For example, in manufacture we are interested in the time to failure after n units are put in a life test. In such cases, the observations arrive in ascending order of magnitude and do not have to be ordered after collection of the data. The practical importance of such experiments is evident. Moreover, the possibility is now open of terminating the experiment before its conclusion by stopping after a given time (Type I censoring) or after a given number of failures (Type II censoring). It may be of interest to predict

Table 3 Coverage probability and average width when $n = 20$, for $W(3.7, 1.1, 2.2)$ sos model with $m_1 = \cdots = m_{n-1} = \alpha - 1,\ k = \alpha$ that is $\gamma_{i,n} = (n - i + 1)\alpha_i,\ \alpha_i = \alpha = 2, \forall i = 1, 2, \ldots, n$

r	s	90%	95%	$AWU_{90\%}$	$AWU_{95\%}$	90%	95%	$AWV_{90\%}$	$AWV_{95\%}$
6	7	0.90124	0.95111	0.1122	0.1492	0.90156	0.95098	0.1121	0.1490
6	8	0.89947	0.94921	0.1895	0.2372	0.89928	0.94946	0.1893	0.2369
6	9	0.90066	0.95088	0.2591	0.3156	0.90070	0.95070	0.2588	0.3151
6	10	0.89834	0.94906	0.3257	0.3900	0.89817	0.94892	0.3253	0.3895
6	11	0.89964	0.94936	0.3927	0.4648	0.89986	0.94918	0.3922	0.4641
6	12	0.90157	0.95060	0.4595	0.5389	0.90179	0.95026	0.4588	0.5380
6	13	0.89945	0.94939	0.5276	0.6145	0.89945	0.94951	0.5269	0.6136
6	14	0.89980	0.94980	0.6001	0.6949	0.90011	0.94972	0.5993	0.6939
6	15	0.90022	0.95025	0.6786	0.7820	0.90021	0.95013	0.6779	0.7809
6	16	0.89930	0.94940	0.7653	0.8782	0.89910	0.94960	0.7645	0.8770
9	10	0.90116	0.94972	0.1052	0.1385	0.90148	0.94982	0.1050	0.1381
9	11	0.90005	0.95019	0.1803	0.2230	0.89999	0.95018	0.1799	0.2224
9	12	0.90012	0.94941	0.2509	0.3015	0.90048	0.94946	0.2503	0.3006
9	13	0.89975	0.95012	0.3212	0.3791	0.89961	0.95059	0.3203	0.3779
9	14	0.89928	0.94995	0.3941	0.4594	0.89916	0.94990	0.3930	0.4578
9	15	0.90028	0.95059	0.4711	0.5440	0.90009	0.95082	0.4699	0.5422
9	16	0.89901	0.94874	0.5567	0.6381	0.89907	0.94904	0.5551	0.6359
12	13	0.89858	0.94973	0.1131	0.1480	0.89826	0.94967	0.1127	0.1473
12	14	0.90081	0.95081	0.1975	0.2426	0.90004	0.95059	0.1968	0.2413
12	15	0.90082	0.95018	0.2807	0.3346	0.90046	0.94956	0.2793	0.3326
12	16	0.90020	0.95023	0.3696	0.4327	0.90041	0.94989	0.3678	0.4299
15	16	0.90153	0.95164	0.1429	0.1860	0.90143	0.95150	0.1421	0.1844

Table 4 The failure voltages in kilovolts per millimeter for 20 specimens

32.0	35.4	36.2	39.8	41.2	43.3	45.5	46.0	46.2	46.4
46.5	46.8	47.3	47.3	47.6	49.2	50.4	50.9	52.4	56.3

the time at which all the components will have failed or to predict the mean failure time of the unobserved lifetimes. In these cases, the interval or point predict are of interest.

In this section, an example for real data is presented to demonstrate the importance of results obtained in Sect. 2. The data were given by Lawless [12, p. 189]. It consists of voltage levels at which failures occurred in a certain type of electrical cable insulation (Type 1 insulation) when specimens were subjected to an increasing voltage stress in a laboratory experiment. The test involved 20 specimens and the failure voltages in kilovolts per millimeter are given in Table 4.

For the purposed data, Gini statistic, (see [6]), as well as the max p value method, are applied to get the best fitting to Weibull distribution. Moreover, prediction intervals for the unobserved failures, $x_s, s = r + 1, r + 2, \ldots, n$ are obtained.

We use Gini statistic and the max p value method to show that $c = 9.1973$ is very close to optimum value and the maximum p value $= 0.999996$, and in this case the maximum Likelihood estimate of b is $b = 47.7383$ which gives a good fitting to two-parameter Weibull distribution ($a = 0.0$) see Table 5.

Assume that the first r failures, ($r = 9, 12, 15, 18$) are observed. Prediction intervals for the unobserved failures $x_s,\ s = r + 1, r + 2, \ldots, 20$, based on oos, with $\alpha = 0.1,\ 0.05$ are obtained. The results are presented in Table 6.

5 Discussion and concluding remarks

Two pivotal quantities are modified to predict future observations from three-parameter Weibull distribution. Numerical results of these pivotal quantities for three different models are presented through simulation studies. Finally, an example has been given to illustrate the results discussed in this paper.

From the simulation studies (Tables 1, 2, 3) it is clear that

1. In all cases the coverage probability is close to $1 - \alpha,\ \alpha = 0.1,\ 0.5$.
2. When s is fixed, the AIW of the PCI of $X(s, n, \tilde{m}, k)$ decreases, with increasing r as expected, since more available data improved prediction results.

Table 5 Values of p value for successive values of c

c	p value	c	p value	c	p value
9.1965	0.999611	9.1971	0.99990	9.1977	0.999811
9.1966	0.999659	9.1972	0.999948	9.1978	0.999763
9.1967	0.999708	9.1973	0.999996	9.1979	0.999715
9.1968	0.999756	9.1974	0.999956	9.1980	0.999667
9.1969	0.999804	9.1975	0.999907	9.1981	0.999619
9.1970	0.999852	9.1976	0.999859	9.1982	0.999571

Table 6 Lower and upper bounds for $x_s,\ s = r+1, r+2, \ldots, 20$, for the data of Table 4 based on oos

r	s	x_s	$U_{90\%}$ PCI	$U_{95\%}$ PCI	$V_{90\%}$ PCI	$V_{95\%}$ PCI
9	10	46.4	(46.2, 47.9737)	(46.2, 48.4933)	(46.2, 47.9331)	(46.2, 48.4415)
9	11	46.5	(46.2, 49.1232)	(46.2, 49.7406)	(46.2, 49.0606)	(46.2, 49.6658)
9	12	46.8	(46.2, 50.1292)	(46.2, 50.8134)	(46.2, 50.0498)	(46.2, 50.7215)
9	13	47.3	(46.2, 51.0731)	(46.2, 51.8108)	(46.2, 50.9798)	(46.2, 51.7051)
9	14	47.3	(46.2, 51.9937)	(46.2, 52.7782)	(46.2, 51.8885)	(46.2, 52.6605)
9	15	47.6	(46.2, 52.9202)	(46.2, 53.7483)	(46.2, 52.8043)	(46.2, 53.6201)
9	16	49.2	(46.2, 53.8833)	(46.2, 54.7546)	(46.2, 53.7574)	(46.2, 54.6167)
9	17	50.4	(46.2, 54.9245)	(46.2, 55.8423)	(46.2, 54.7891)	(46.2, 55.6949)
9	18	50.9	(46.2, 56.1158)	(46.2, 57.0885)	(46.2, 55.9709)	(46.2, 56.9318)
9	19	52.4	(46.2, 57.6205)	(46.2, 58.6708)	(46.2, 57.4653)	(46.2, 58.5039)
9	20	56.3	(46.2, 60.0267)	(46.2, 61.2385)	(46.2, 59.8585)	(46.2, 61.0587)
12	13	47.3	(46.8, 48.4013)	(46.8, 48.8627)	(46.8, 48.4915)	(46.8, 48.9718)
12	14	47.3	(46.8, 49.4937)	(46.8, 50.0471)	(46.8, 49.6299)	(46.8, 50.2001)
12	15	47.6	(46.8, 50.5031)	(46.8, 51.124)	(46.8, 50.6745)	(46.8, 51.3097)
12	16	49.2	(46.8, 51.5121)	(46.8, 52.1918)	(46.8, 51.7133)	(46.8, 52.405)
12	17	50.4	(46.8, 52.5786)	(46.8, 53.3162)	(46.8, 52.807)	(46.8, 53.5544)
12	18	50.9	(46.8, 53.7812)	(46.8, 54.5835)	(46.8, 54.0362)	(46.8, 54.8463)
12	19	52.4	(46.8, 55.2851)	(46.8, 56.1749)	(46.8, 55.5691)	(46.8, 56.4649)
12	20	56.3	(46.8, 57.674)	(46.8, 58.7389)	(46.8, 57.9975)	(46.8, 59.0675)
15	16	49.2	(47.6, 49.3069)	(47.6, 49.7861)	(47.6, 49.5596)	(47.6, 50.0908)
15	17	50.4	(47.6, 50.5675)	(47.6, 51.152)	(47.6, 50.9568)	(47.6, 51.5877)
15	18	50.9	(47.6, 51.8717)	(47.6, 52.5477)	(47.6, 52.3741)	(47.6, 53.09)
15	19	52.4	(47.6, 53.4379)	(47.6, 54.2213)	(47.6, 54.0505)	(47.6, 54.8673)
15	20	56.3	(47.6, 55.8635)	(47.6, 56.8408)	(47.6, 56.6112)	(47.6, 57.6168)
18	19	52.4	(50.9, 53.5676)	(50.9, 54.2618)	(50.9, 53.6822)	(50.9, 54.3822)
18	20	56.3	(50.9, 56.4137)	(50.9, 57.372)	(50.9, 56.5927)	(50.9, 57.542)

3. In most cases, the AIW of the PCI of $X(s, n, \tilde{m}, k)$ based on statistic $U_{r,s}$ is closer to the AIW of $X(s, n, \tilde{m}, k)$ based on the statistic $V_{r,s}$.
4. The AIW of the PCI of $X(s, n, \tilde{m}, k)$ decreases, when the sample size increases.
5. The AIW of the PCI of $X(s, n, \tilde{m}, k)$ increases, when α decreases.

From Sect. 4, it is noted that the accuracy of prediction intervals of x_s, $s > r$ for real data depends on the size of the actual observations $x_1, x_2, \ldots, x_r$, the difference $s - r$ and the goodness of fitting data to Weibull model (see Table 6).

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