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Groups up to congruence relation and from categorical groups to c-crossed modules

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Abstract

We introduce a notion of c-group, which is a group up to congruence relation and consider the corresponding category. Extensions, actions and crossed modules (c-crossed modules) are defined in this category and the semi-direct product is constructed. We prove that each categorical group gives rise to a c-group and to a c-crossed module, which is a connected, special and strict c-crossed module in the sense defined by us. The results obtained here will be applied in the proof of an equivalence of the categories of categorical groups and connected, special and strict c-crossed modules.

Keywords Group up to congruence relation · c-crossed module · action · Categorical group

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1 Introduction

Our aim was to obtain for categorical groups an analogous description in terms of certain crossed module type objects as we have for $\mathcal{G}$-groupoids obtained by Brown

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and Spencer [5], which are strict categorical groups, or equivalently, group-groupoids or internal categories in the category of groups. By a categorical group we mean a coherent 2-group in the sense of Baez and Lauda [1]. It is important to note that it is well known that a categorical group is equivalent to a strict categorical group [1,12,19], but we do not have an equivalence between the corresponding categories. This idea brought us to a new notion of group up to congruence relation. In this way we came to the definition of c-group and the corresponding category. Then we defined actions in this category and introduced the notion of c-crossed module. Among this kind of objects we distinguished connected, strict and special c-crossed modules denoted as cssc-crossed modules. We proved that every categorical group gives rise to a cssc-crossed module. The prototypes of all the new concepts introduced in this paper are those obtained from categorical groups. In the sequel to this paper we will prove that there is an equivalence between the category of categorical groups and the category of cssc-crossed modules. We hope that this result will give a chance to consider for categorical groups the problems analogous to those considered and solved in the case of strict categorical groups in terms of group-groupoids and internal categories in [4,6–9].

We would like to thank one of the editors of the journal who pointed out to us the paper of Schommer-Pries, [17], in which that author studies the notion of 2-group in a general bicategory with finite products. The editor's comments suggested that there might be a link between the work in the noted paper and our results in this paper, since an equivalence relation on a set gives a groupoid, and the category of sets with equivalence relations $\widetilde{\mathsf{Sets}}$ carries a bicategory structure. We would note, however, that the paper referred to does not attempt to prove that equivalence that we are seeking, and which will be further investigated in a sequel to this paper. Comparison with the work in [17] does, although, suggest further questions about the links between internal c-crossed modules and categorical groups in bicategories.

In Sect. 2 we recall the definitions of a categorical group, a group-groupoid, and a crossed module in the category of groups. In Sect. 3 we define groups up to congruence relation, shortly c-groups, give examples and consider the corresponding category of c-groups denoted as cGr. We define cKer f, cIm f, for any morphism f in cGr, and normal c-subgroups in any c-group. In Sect. 4 we define split extensions and actions in cGr. After this, we define c-crossed modules and give examples. We introduce the notions of special, strict and connected c-crossed modules and give examples. We prove that every categorical group defines a cssc-crossed module.

2 Preliminaries

Recall the definition of a monoidal category given by Mac Lane [15].

Definition 1 A **monoidal category** is a category $\mathsf{C} = (C_0, C_1, d_0, d_1, i, m)$ equipped with a bifunctor $+ : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ called the monoidal sum, an object 0 called the zero object, and three natural isomorphisms α, λ and ρ. Explicitly,

$$\alpha = \alpha_{x,y,z} : (x + y) + z \xrightarrow{\approx} x + (y + z)$$

is natural for all $x, y, z \in C_0$, and the pentagonal diagram

$$\begin{array}{ccccc} ((x+y)+z)+t & \xrightarrow{\alpha} & (x+y)+(z+t) & \xrightarrow{\alpha} & x+(y+(z+t)) \\ \downarrow{\scriptstyle \alpha+1} & & & & \uparrow{\scriptstyle 1+\alpha} \\ (x+(y+z))+t & & \xrightarrow[\alpha]{\qquad\qquad} & & x+((y+z)+t) \end{array}$$

commutes for all $x, y, z, t \in C_0$. Again, λ and ρ are natural $\lambda_x : 0 + x \overset{\approx}{\to} x$, $\rho_x : x + 0 \overset{\approx}{\to} x$, for all $x \in C_0$, the diagram

$$\begin{array}{ccc} (x+0)+y & \xrightarrow{\alpha} & x+(0+y) \\ \downarrow{\scriptstyle \rho+1} & & \downarrow{\scriptstyle 1+\lambda} \\ x+y & = & x+y \end{array}$$

commutes for all $x, y \in C_0$ and also $\lambda_0 = \rho_0 : 0 + 0 \overset{\approx}{\to} 0$. Moreover "all" diagrams involving α, λ, and ρ must commute. A monoidal category is said to be a monoidal groupoid whenever each morphism is invertible.

In this definition we use the term monoidal sum and denote it as $+$, instead of monoidal product, used in the original definition, and write the operation additively. From the definition it follows that $1_0 + f \approx f + 1_0 \approx f$, for any morphism f. In what follows the isomorphisms α, λ and ρ involved in group-like identities, their inverses, compositions and their monoidal sums will be called *special isomorphisms*. Since $+$ is a bifunctor in a monoidal category we have $d_j(f+g) = d_j(f)+d_j(g)$, $j = 0, 1$, $i(x+y) = i(x) + i(y)$ and the interchange law $(f' + g')(f + g) = f'f + g'g$, whenever the composites $f'f$ and $g'g$ are defined, for any $x, y \in C_0$, $f, f', g, g' \in C_1$.

Any category C with finite products can be considered as a monoidal category where to any given two objects, $+$ assigns their product and 0 is the terminal object. The category of abelian groups Ab is a monoidal category where the tensor product $\otimes$ is the monoidal sum and $\mathbb{Z}$ is the unit object. There are other examples as well [15].

In a monoidal category, if the special isomorphisms α, λ, and ρ are identities, then C is called a **strict monoidal category**.

Let C and C' be two monoidal categories. A *(strict) morphism* of monoidal categories $T : (C, +, 0, \alpha, \lambda, \rho) \to (C', +', 0', \alpha', \lambda', \rho')$ is a functor $T : C \to C'$, such that for all objects x, y, z and morphisms f and g there are equalities $T(x + y) = Tx +' Ty$, $T(f+g) = Tf +' Tg$, $T0 = 0'$, $T\alpha_{x,y,z} = \alpha'_{Tx,Ty,Tz}$, $T\lambda_x = \lambda'_{Tx}$, $T\varrho_x = \rho'_{Tx}$.

Definition 2 [1] If x is an object in a monoidal category, an inverse for x is an object y such that $x + y \approx 0$ and $y + x \approx 0$. If x has an inverse, it is called invertible.

It is well known and easy to show that if any object has a one-sided inverse in a monoidal category, then any object is invertible [1,12].

Definition 3 A **categorical group** $\mathsf{C} = (C_0, C_1, d_0, d_1, i, m)$ is a monoidal groupoid, where all objects are invertible and moreover, for every object $x \in C_0$ there is an object $-x \in C_0$ with a family of natural isomorphisms

$$\varepsilon_x : \ -x + x \approx 0,$$
$$\delta_x : x + (-x) \approx 0$$

such that the following diagrams are commutative:

$$\begin{array}{ccccc}
0 + x & \xrightarrow{\delta_x^{-1}+1} & (x + (-x)) + x & \xrightarrow{a_{x,-x,x}} & x + (-x + x) \\
\downarrow{\scriptstyle \lambda_x} & & & & \downarrow{\scriptstyle 1_x+\varepsilon_x} \\
x & & \xrightarrow[\rho_x^{-1}]{} & & x + 0
\end{array}$$

$$\begin{array}{ccccc}
-x + 0 & \xrightarrow{1+\delta_x^{-1}} & -x + (x + (-x)) & \xrightarrow{a^{-1}_{-x,x,-x}} & (-x + x) + (-x) \\
\downarrow{\scriptstyle \rho_{-x}} & & & & \downarrow{\scriptstyle \varepsilon_x+1_{-x}} \\
-x & & \xrightarrow[\lambda^{-1}_{-x}]{} & & 0 + (-x)
\end{array}$$

It is important and a well-known fact that the definition of a categorical group implies that for any morphism $f : x \to x' \in C_1$ there is a morphism $-f : -x \to -x'$ with natural isomorphisms $-f + f \approx 0$ and $f + (-f) \approx 0$, where the morphism 0 is 1_0 (see e.g. [18]). As in the case of monoidal categories the natural transformations α, λ, ρ, ε, δ, identity transformation $1_{\mathsf{C}} \to 1_{\mathsf{C}}$, their compositions and sums will be called *special isomorphisms*. The categorical group defined above is coherent [1,13], which means that all diagrams involving special isomorphisms commute. For a monoidal category one can find this in [15], Coherence Chapter VII Sect. 2.

A categorical group is called strict if the special isomorphisms α, λ, ρ, ε, and δ are identities. Strict categorical groups are known as group-groupoids (see below for the definition), internal categories in the category of groups or 2-groups in the literature.

The definition of categorical group we gave is Definition 3.1 in Baez and Lauda, [1], where the operation is multiplication and where it is called a coherent 2-group. Sinh [18] calls them "gr-categories", and this name is also used by other authors as well, e.g., Breen [2]. They are called "categories with group structure" in which all morphisms are invertible by Ulbrich [20] and Laplaza [13]. The term categorical group for strict categorical groups is used by Joyal and Street [12], and it is used by Vitale [21,22] and others for non-strict ones.

The functorial properties of addition $+$ implies that in a categorical group we have $-1_x = 1_{-x}$, for any $x \in C_0$. Since an isomorphism between morphisms θ : $f \approx g$ means that there exist isomorphisms $\theta_i : d_i(f) \to d_i(g), i = 0, 1$ with $\theta_1 f = g\theta_0$, the naturality property of special isomorphisms implies that there exist special isomorphisms between the morphisms in C_1. But if $\theta_i, i = 0, 1$ are special isomorphisms, it does not imply that θ is a special isomorphism; in this case we will call θ a *weak special isomorphism*. It is obvious that a special isomorphism between the morphisms in C_1 will be a weak special isomorphism. Note that if $f \approx f'$ is a weak special isomorphism, then the coherence property implies that f' is the unique morphism weakly specially isomorphic to f with the same domain and codomain objects as f'.

Example 1 Let X be a topological space and $x \in X$ be a point in X. Consider the category $\Pi_2(X, x)$, whose objects are paths $x \to x$, and whose morphisms are homotopy classes of paths between paths, where $f, g : x \to x$. This category is a categorical group, for the proof see [1] and the paper of Hardie, Kamps and Kieboom [10,11].

One can see more examples in [1], and also we will give them in the sequel to this paper, where we will construct a categorical group for any cssc-crossed module as defined below in Sect. 5.

We define (strict) morphisms between categorical groups, which satisfy conditions of (strict) morphisms of monoidal categories. Note that this definition implies: $T(-x) = -T(x)$ and $T(-f) = -T(f)$, for any object x and arrow f in a categorical group. Categorical groups form a category with (strict) morphisms between them. For any categorical group $\mathsf{C} = (C_0, C_1, d_0, d_1, i, m)$ denote $\operatorname{Ker} d_0 = \{f \in C_1 \mid d_0(f) \approx 0\}$ and $\operatorname{Ker} d_1 = \{f \in C_1 \mid d_1(f) \approx 0\}$.

Lemma 1 *Let $\mathsf{C} = (C_0, C_1, d_0, d_1, i, m)$ be a categorical group. For any $f \in \operatorname{Ker} d_1$ and $g \in \operatorname{Ker} d_0$ we have a weak special isomorphism $f + g \approx g + f$.*

Proof Suppose $d_0(g) = 0'$ and $d_1(f) = 0''$, where $0' \approx 0 \approx 0''$. The interchange law implies $(1_{0''} + g)(f + 1_{0'}) = f + g$ and $(g + 1_{0''})(1_{0'} + f) = g + f$. Let $\gamma : 0'' \approx 0'$ be a special isomorphism. Applying the coherence property of a categorical group, we easily obtain that the left sides of both noted equalities are isomorphic to $g\gamma f$, and both are weak special isomorphisms. This implies that there is a weak special isomorphism $f + g \approx g + f$. □

The analogous statement is well known for group-groupoids, where instead of the isomorphisms we have equalities in the definitions of $\operatorname{Ker} d_0$ and $\operatorname{Ker} d_1$ and in the final result [5].

Below we recall the definition of crossed module introduced by Whitehead in [23]. A *crossed module* (A, B, μ) consists of a group homomorphism $\mu\colon A \to B$ together with an action $(b, a) \mapsto b \cdot a$ of B on A such that for $a, a_1 \in A$ and $b \in B$

CM1. $\mu(b \cdot a) = b + \mu(a) - b$, and
CM2. $\mu(a) \cdot a_1 = a + a_1 - a$.

For an extensive treatment of crossed modules, see [3, Part I].

Here are some examples of crossed modules.

- The inclusion of a normal subgroup $N \to G$ is a crossed module with the action by conjugation of G on N. In particular, any group G can be regarded as a crossed module $1_G\colon G \to G$.
- For any group G, modules over the group ring of G are crossed modules with $\mu = 0$.

- For any group G the object $\mu\colon G \to \operatorname{Aut} G$ is a crossed module, where $\mu(g)\cdot g' = \mu(g)(g')$ for any $g, g' \in G$.

A *morphism* $(f, g)\colon (A, B, \mu) \to (A', B', \mu')$ of crossed modules is a pair $f\colon A \to A'$, $g\colon B \to B'$ of morphisms of groups such that $g\mu = \mu' f$ and f is an operator morphism over g, i.e., $f(b \cdot a) = g(b) \cdot f(a)$ for $a \in A$, $b \in B$. So crossed modules and morphisms of them, with the obvious composition of morphisms $(f', g')(f, g) = (f' f, g' g)$ form a category.

Definition 4 A *group-groupoid* G is a *group object* in the category of groupoids, which means that it is a groupoid G equipped with functors

(i) $+\colon G \times G \to G$, $(a, b) \mapsto a + b$;
(ii) $u\colon G \to G$, $a \mapsto -a$;
(iii) $0\colon \{\star\} \to G$, where $\{\star\}$ is a singleton, □

which are called respectively sum, inverse and zero, satisfying the usual axioms for a group.

The definition we gave was introduced by Brown and Spencer in [5] under the name $\mathcal{G}$-*groupoid*, where the group operation is multiplication. The term group-groupoid was used later in [4]. It is interesting that a group object in the category of small categories called $\mathcal{G}$-*category* is a group-groupoid. As it is noted by the authors of [5] this fact was known to Duskin.

Example 2 If X is a topological group, then the fundamental groupoid $\pi_1 X$ of the space X is a group-groupoid [5].

Example 3 For a group X, the direct product $G = X \times X$ is a group-groupoid. Here the domain and codomain homomorphisms are the projections; the object inclusion homomorphism is defined by the diagonal homomorphism $i(x) = (x, x)$, for any $x \in X$ and the composition of arrows is defined by $(x, y) \circ (z, t) = (z, y)$ whenever $x = t$, for any $x, y, z, t \in X$.

Theorem 1 [5] *The categories of crossed modules and of group-groupoids are equivalent.*

According to the authors of [5], this result was known to Verdier. It was then used by Duskin and was later discovered independently by Brown and Spencer. It was proved by Porter that the analogous statement is true in the more general setting of a category of groups with operations [16].

3 Groups up to congruence relation

Let X be a non-empty set with an equivalence relation R on X. Denote such a pair by X_R. Define a category whose objects are the pairs X_R and morphisms are functions $\underset{\sim}{f\colon X_R} \to Y_S$, such that $f(x) \sim_S f(y)$, whenever $x \sim_R y$. Denote this category by $\widetilde{\mathsf{Sets}}$.

Note that for $X_R, Y_S \in Ob\left(\widetilde{\mathsf{Sets}}\right)$, the product $X_R \times Y_S$ is a product object in Sets with the equivalence relation $R \times S$ defined by

$$(x, y) \sim_{R\times S} (x_1, y_1) \Leftrightarrow x \sim_R x_1 \quad \text{and} \quad y \sim_S y_1$$

We now define a *group up to congruence relation* or briefly *c-group* as follows.

Definition 5 Let G_R be an object in $\widetilde{\mathsf{Sets}}$ and

$$\begin{aligned} m\colon G \times G &\longrightarrow G \\ (a, b) &\longmapsto a + b \end{aligned}$$

a morphism in $\widetilde{\mathsf{Sets}}$, i.e., $m \in \widetilde{\mathsf{Sets}}((G \times G)_{R\times R}, G_R)$. G_R is called a *c-group* if the following axioms are satisfied.

(i) $a + (b + c) \sim_R (a + b) + c$ for all $a, b, c \in G$;
(ii) there exists an element $0 \in G$ such that $a + 0 \sim_R a \sim_R 0 + a$, for all $a \in G$;
(iii) for each $a \in G$ there exists an element $-a$ such that $a + (-a) \sim_R 0$ and $-a + a \sim_R 0$.

In a c-group G_R, the given element $0 \in G$ is called *zero element* and for any $a \in G$ the given element $-a \in G$ is called the *inverse* of a. The congruences involved in the conditions (i)–(iii) of the definition, their compositions and sums will be called *special congruences*.

Remark 1 Let G_R be a c-group. Then we have the following:

(a) if $a \sim_R b$ and $c \sim_R d$ for $a, b, c, d \in G$ then $a + c \sim_R b + d$;
(b) if an element $0' \in G$ different from 0 satisfies the congruence (ii) in Definition 5, then $0 \sim_R 0'$;
(c) if an element $a' \in G$ different from -a satisfies the congruences involved in condition (iii) in Definition 5, then $a' \sim_R -a$;
(d) if $a \sim_R b$ then $-a \sim_R -b$.

Example 4 Every group G is a c-group where the congruence relation is equality.

So the concept of c-group generalizes that of group.

Example 5 This example comes from Mac Lane's paper [14], where the author regards the quotient group as a group with congruence relation. Let G be a group and H a normal subgroup in G. The quotient group G/H can be regarded as a group with the same elements as the group G and with the congruence relation $g \sim g'$ if and only if $g - g' \in H$. The operations are defined in the same way as in G and they preserve congruences. Such a group is a c-group, where group identities are in fact satisfied up to equality.

Example 6 Let X be a topological space and $x \in X$. The set $P(X, x)$ of all closed paths at x is a c-group with the composition of paths. Here the congruence relation is homotopy of paths.

Example 7 Let $\mathbb{Z}^* = \mathbb{Z}\backslash\{0\}$. Define an equivalence relation on $\mathbb{Z}^*$ by $x \sim_R y \Leftrightarrow xy > 0$. Then $\mathbb{Z}^*$ becomes a c-group with respect to the multiplication. The unit is the number 1 and the inverse for any number is this number itself.

Example 8 In a categorical group C the set C_1 of morphisms and the set C_0 of objects are both c-groups. The congruence relations are isomorphisms between arrows and between objects, respectively.

Example 9 Any group can be endowed with a c-group structure. To show this recall that every group G can be regarded as a part of a certain crossed module in the category of groups, for example $G \to \operatorname{Aut} G$. According to Theorem 1 there exists a group-groupoid $C = (C_0, C_1, d_o, d_1, i, m)$, for which $d_1|_{\operatorname{Ker} d_0} \colon \operatorname{Ker} d_0 \to C_0$, is a crossed module and is isomorphic to $G \to \operatorname{Aut} G$. $\operatorname{Ker} d_0$ is a c-group, the congruence relation on it is induced by the congruence relation on C_1, which is the relation being isomorphic between the morphisms. From this follows that G has also a c-group structure, group identities are satisfied up to equality, and naturally all special isomorphisms are equalities.

Note that in this case the congruence relation on G is trivial, i.e., $G_R = G \times G$, and any group can be considered as a c-group with trivial congruence relation on it.

The following example is due to the referee.

Example 10 Any magma can be considered as a c-group with trivial congruence relation.

Definition 6 Let G_R be a c-group. If $a + b \sim_R b + a$ for all $a, b \in G$, then G_R is called a *c-abelian (or c-commutative)* c-group.

Definition 7 Let G_R and H_S be c-groups. A morphism $f \in \widetilde{\mathsf{Sets}}(G_R, H_S)$ such that $f(a + b) = f(a) + f(b)$ for any $a, b \in G$ is called a *c-group morphism* from G_R to H_S.

From the definition it follows that a morphism between c-groups preserves congruences between elements; moreover we obtain that $f(0) \sim 0$ and $f(-a) \sim -f(a)$, for any $a \in G$. As a result we obtain that a morphism between c-groups carries special congruences to special congruences between pairs of elements.

Remark 2 If $f \colon G_R \to H_S$ and $g \colon H_S \to N_T$ are two c-group morphisms, then $gf \colon G_R \to N_T$ is also a c-group morphism. Further for each c-group G_R there is a unit morphism $1_G \colon G_R \to G_R$ such that 1_G is the identity function on G_R. Therefore, we have a category of c-groups with c-group morphisms; denote this category by cGr.

Let G_R and H_S be c-groups, and $f \colon G_R \to H_S$ a morphism of c-groups.

Definition 8 The subset $\operatorname{cKer} f = \{a \in G_R \mid f(a) \sim_S 0_H\}$ is said to be the *c-kernel* of the c-group morphism f.

Note that $\operatorname{cKer} f$ is a c-group with the congruence relation induced by G_R. In particular, $\operatorname{cKer} d_0$, for a categorical group $\mathsf{C} = (C_0, C_1, d_0, d_1, i, m)$, is a c-group with the congruence relation on $\operatorname{cKer} d_0$ induced by the isomorphisms in C_1.

Definition 9 The subset $\mathrm{cIm}\, f = \{b \in H_S \mid \exists a \in G_R, f(a) \sim_S b\}$ is said to be the *c-image* of the morphism f.

Lemma 2 *Let G be a c-group with congruence relation R. Then the quotient set G/R becomes a group with the operation defined by the induced map*

$$\begin{aligned} m^*\colon G/R \times G/R &\longrightarrow G/R, \\ ([a],[b]) &\longmapsto [a]+[b] = [a+b]. \end{aligned}$$

Example 4 gives a full embedding of categories $\mathcal{E}\colon \mathsf{Gr} \to \mathsf{cGr}$, and Lemma 2 gives a functor

$$\mathcal{Q}\colon \mathsf{cGr} \to \mathsf{Gr}.$$

It is easy to see that the functor $\mathcal{Q}$ is left adjoint to the functor $\mathcal{E}$.

It is well known that the zero group $\mathbf{0}$ is an initial and terminal object in the category of groups Gr. It is a terminal object in the category cGr as well, but note that $\mathbf{0}$ is not an initial object in cGr.

Definition 10 Let G_R be a c-group and let H be a subset of the underlying set of G. H is called a c-subgroup in G_R if H_S is a c-group with the addition and congruence relation S induced by G_R.

Let G_R be a c-group and let H be a subset of G. If for an element $a \in G$ there exists an element $b \in H$ such that $a \sim_R b$ then we write $a \mathrel{\widetilde{\in}} H$. If H and H' are two subsets of G_R, then we write $H \mathrel{\widetilde{\subset}} H'$ if for any $h \in H$ we have $h \mathrel{\widetilde{\in}} H'$. If $H \mathrel{\widetilde{\subset}} H'$ and $H' \mathrel{\widetilde{\subset}} H$, then we write $H \sim H'$.

Definition 11 Let G_R be a c-group and let $H_S \subseteq G_R$ a c-subgroup in G_R. Then H_S is called a *normal c-subgroup* if $g + h - g \mathrel{\widetilde{\in}} H_S$ for any $h \in H_S$ and $g \in G$.

The condition given in the definition is equivalent to the condition $g + H_S - g \mathrel{\widetilde{\subset}} H_S$, and it is equivalent itself to the condition $g + H_S \sim H_S + g$ for any $g \in G$.

Definition 12 Let G_R be a c-group and let $H_S \subseteq G_R$ be a c-subgroup in G_R. Then H_S is called a *perfect c-subgroup* if $g \mathrel{\widetilde{\in}} H$ implies $g \in H$, for any $g \in G$.

Definition 13 A c-group G_R is called *connected* if $g \sim g'$ for any $g, g' \in G$.

Lemma 3 *Let G_R and H_S be c-groups and let $f\colon G_R \to H_S$ be a morphism of c-groups. Then*

(i) $\mathrm{cKer}\, f$ *is a perfect and normal c-subgroup in G_R;*
(ii) $\mathrm{cIm}\, f$ *is a perfect c-subgroup in H_S.*

Proof This follows from the definitions. □

Now we shall construct the quotient object G/H, where H is a normal c-subgroup of a c-group G. Consider the classes $\{g + H \mid g \in G\}$. If $g + H \cap g' + H \neq \emptyset$, then we obtain $-g + g' \widetilde{\in} H$, which implies that $g + H \sim g' + H$. Now consider the set of these classes $\{\text{cl}(g + H) \mid g \in G\}$, where $\text{cl}(g + H) = \cup\{x \in G \mid x \widetilde{\in} g + H\}$. We define $G/H = \{\text{cl}(g + H) \mid g \in G\}$. An addition operation in this set is defined by $\text{cl}(g + H) + \text{cl}(g' + H) = \text{cl}((g + g') + H)$, for any $g, g' \in G$. This operation is well defined, it is associative and we have the unit element $\text{cl}(0 + H)$. Actually the constructed object is a group, the congruence relation on G/H is the equality "$=$". We have an obvious surjective morphism $p : G \to G/H$.

Lemma 4 (i) *If G is a c-group and H is a normal c-subgroup in G, then for any group G' and c-group morphism $f : G \to G'$, if $f(h) = 0$ for any $h \in H$, there exists a unique morphism $\theta : G/H \to G'$, in* cGr *such that $\theta p = f$.*

(ii) *If H is a perfect normal c-subgroup in G, then* $H = \text{cKer}\, p$.

Proof This follows by easy checking. □

4 Actions and crossed modules in cGr

An extension in the category cGr is defined in a similar way to that in the category of groups.

Definition 14 Let $A, B \in$ cGr. An *extension* of B by A is a sequence

$$0 \longrightarrow A \xrightarrow{\;i\;} E \xrightarrow{\;p\;} B \longrightarrow 0 \qquad (1)$$

in which p is surjective and i is the c-kernel of p in cGr. We say that an extension is *split* if there is a morphism $s\colon B \to E$, such that $ps = 1_B$.

We shall identify $a \in A$ with its image $i(a)$. We shall use the notation $b \cdot a = s(b) + (a - s(b))$. Then a split extension induces an action (on the left) of B on A. We have the following conditions for this action:

(i) $b \cdot (a + a_1) \sim (b \cdot a) + (b \cdot a_1)$,
(ii) $(b + b_1) \cdot a \sim b \cdot (b_1 \cdot a)$,
(iii) $0 \cdot a \sim a$,
(iv) If $a \sim a_1$ and $b \sim b_1$ then $b \cdot a \sim b_1 \cdot a_1$,

for $a, a_1 \in A$ and $b, b_1 \in B$.

Here and in what follows we omit congruence relation symbols for A and B.

Let $A, B \in$ cGr and suppose that B acts on A satisfying the conditions (i)–(iv). In this case we will say, that we have *an action in* cGr. Consider the product $B \times A$ in cGr. We have the operation in $B \times A$, defined in an analogous way to that in the case of groups:

$(b', a') + (b, a) = (b' + b, a' + b' \cdot a)$ for any $b, b' \in B, a, a' \in A$.

This operation is associative up to the relation defined by $(b, a) \sim (b', a')$ if and only if $b \sim b'$ and $a \sim a'$, which is a congruence relation. Obviously, we have a zero element $(0, 0)$ in $B \times A$ and the opposite element for any pair $(b, a) \in B \times A$ is $(-b, -b \cdot (-a))$. Therefore, we have a semidirect product $B \ltimes A$ in cGr.

Definition 15 Let $f : D \to D'$ be a morphism in cGr. f is called an isomorphism up to congruence relation or c-isomorphism if there is a morphism $f' : D' \to D$, such that $ff' \sim 1_{D'}$ and $f'f \sim 1_D$.

We will denote such an isomorphism by $\stackrel{\sim}{\approx}$.

We have a natural projection $p' : B \ltimes A \to B$. The c-kernel of p' is not isomorphic to A as it would be in the case of groups, but we have an isomorphism up to congruence relation cKer $p' \stackrel{\sim}{\approx} A$.

Let $0 \to A \to E \to B \to 0$ be a split extension of B by A in cGr. Then we have an action of B on A and the corresponding semidirect product $B \ltimes A$. In this case we obtain a c-isomorphism $E \stackrel{\sim}{\approx} B \ltimes A$ given by the correspondences analogous to the group case.

Definition 16 Let G and H be two c-groups, let $\partial : G \to H$ be a morphism of c-groups and let H act on G. We call (G, H, ∂) a *c-crossed module* if the following conditions are satisfied:

(i) $\partial(b \cdot a) = b + (\partial(a) - b)$,
(ii) $\partial(a) \cdot a_1 \sim a + (a_1 - a)$.

for $a, a_1 \in G$ and $b \in H$.

Let (G, H, ∂) and (G', H', ∂') be two c-crossed modules. A *c-crossed module morphism* is a pair of morphisms $\langle f, g \rangle : (G, H, \partial) \to (G', H', \partial')$ such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\partial} & H \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ G' & \xrightarrow{\partial'} & H' \end{array}$$

is commutative, and for all $b \in H$ and $a \in G$, we have $f(b \cdot a) = g(b) \cdot f(a)$, where f and g are morphisms of c-groups.

c-crossed modules and morphisms of c-crossed modules form a category.

Example 11 Any crossed module in the category of groups can be endowed with the structure of a c-crossed module by applying Example 4 or 9.

For other examples see Sect. 5.

Let $G \in \mathsf{cGr}$ and H be a normal c-subgroup in G. It is easy to see that in general we do not have a usual action by conjugation of G on H.

Lemma 5 *If H is a perfect normal c-subgroup of a c-group G, then we have an action of G on H in the category* cGr *and the inclusion morphism defines a c-crossed module.*

Proof This is by easy checking. □

For a categorical group $\mathsf{C} = (C_0, C_1, d_0, d_1, i, m)$, we have a split extension

$$0 \longrightarrow \text{cKer}\, d_0 \xrightarrow{j} C_1 \xrightarrow{d_0} C_0 \longrightarrow 0 \tag{2}$$

where $\mathrm{cKer}\, d_0$ is a c-group, a congruence relation on it is defined naturally as an isomorphism between the arrows in $\mathrm{cKer}\, d_0$. Now we define an action of C_0 on $\mathrm{cKer}\, d_0$ by

$$\begin{aligned} C_0 \times \mathrm{cKer}\, d_0 &\longrightarrow \mathrm{cKer}\, d_0, \\ (r, c) &\longmapsto r \cdot c = i(r) + (j(c) - i(r)). \end{aligned}$$

Proposition 1 *The action of C_0 on $\mathrm{cKer}\, d_0$ satisfies the conditions for an action in cGr.*

Proof First we shall show the congruence $r \cdot (c + c') \sim r \cdot c + r \cdot c'$.

$$\begin{aligned} r \cdot (c + c') &= i(r) + ((c + c') - i(r)) \\ &\sim (i(r) + ((c - i(r)) + i(r)) + (c' - i(r))) \\ &\sim (i(r) + (c - i(r))) + (i(r) + (c' - i(r))) \\ &= r \cdot c + r \cdot c'. \end{aligned}$$

Next we shall show that $(r + r') \cdot c \sim r \cdot (r' \cdot c)$.
We have

$$\begin{aligned} (r + r') \cdot c &= i(r + r') + (c - i(r + r')) \\ &\sim i(r) + ((i(r') + c - i(r')) - i(r)) \\ &= r \cdot (r' \cdot c). \end{aligned}$$

It is trivial that $0 \cdot c \sim c$ and $r \cdot 0 \sim 0$. Now we shall show that if $r \sim r'$ and $c \sim c'$ then $r \cdot c \sim r' \cdot c'$. We have

$$\begin{aligned} r \cdot c &= i(r) + (c - i(r)) \\ &\sim i(r') + (c' - i(r')) \\ &= r' \cdot c'. \end{aligned}$$

□

5 cssc-crossed modules and the main theorem

Definition 17 A c-crossed module (G, H, ∂) will be called connected if G is a connected c-group.

For a categorical group $\mathsf{C} = (C_0, C_1, d_0, d_1, i, m)$ denote $d_1|_{\mathrm{cKer}\, d_0}$ by d.

Proposition 2 *Let $\mathsf{C} = (C_0, C_1, d_0, d_1, i, m)$ be a categorical group. Then $(\mathrm{cKer}\, d_0, C_0, d)$ is a connected c-crossed module.*

Proof cKer d_0 is a connected c-group, which follows from the fact that any two arrows in cKer d_0 have domains isomorphic to 0 and that C is a groupoid. Note that the congruence relation on C_0 is generated by the isomorphisms between the objects in C_0. Therefore d preserves the congruence relation on cKer d_0 since $f \approx f'$ implies that $d_1 f \approx d_1 f'$. For the first condition for a crossed module we have $d(r \cdot c) = d(i(r) + (c - i(r))) = r + (dc - r)$, for any $c \in$ cKer d_0 and $r \in C_0$.

For the second condition we have to prove that $(dc) \cdot c' \sim c + (c' - c)$ for $c, c' \in$ cKer d_0, which follows from the fact that cKer d_0 is a connected c-group. □

Now we shall introduce another sort of c-group denoted $\mathbf{Star}_{\mathsf{C}}0$ for any categorical group $\mathsf{C} = (C_0, C_1, d_0, d_1, i, m)$. By definition $\mathbf{Star}_{\mathsf{C}}0 = \{f \in C_1 \mid d_0(f) = 0\}$. An addition operation is defined by $f + f' = (f + f')\gamma$, where $f + f' : 0 + 0 \to d_1(f) + d_1(f')$ is a sum in C_1, i.e., the same as the sum in cKer d_0 and $\gamma : 0 \to 0 + 0$ is the unique special isomorphism in C_1. The $\sim$-relation on $\mathbf{Star}_{\mathsf{C}}0$ is induced by the relation on C_1, which is the relation of being isomorphic in C_1, and it is a congruence relation on $\mathbf{Star}_{\mathsf{C}}0$. It is obvious that $d\mid_{\mathbf{Star}_{\mathsf{C}}0}$ preserves the congruences. The operation in $\mathbf{Star}_{\mathsf{C}}0$ is associative up to congruence. The zero element in $\mathbf{Star}_{\mathsf{C}}0$ is the zero arrow 0; we have $f + 0 \sim f, 0 + f \sim f$, for any $f \in \mathbf{Star}_{\mathsf{C}}0$. The opposite morphism of f in C_1 is $-f : -0 \to -d_1 f$. There is a unique special isomorphism $\kappa : 0 \approx -0$. Define the opposite morphism $-f$ in $\mathbf{Star}_{\mathsf{C}}0$ as $-f\kappa$. One can easily see that $f + (-f) \approx 0$ in $\mathbf{Star}_{\mathsf{C}}0$ and the $\sim$-relation is a congruence. Therefore $\mathbf{Star}_{\mathsf{C}}0$ is a c-group. C_0 is also a c-group, where the congruence relation is given by isomorphisms between the objects. Now we will define an action of C_0 on $\mathbf{Star}_{\mathsf{C}}0$. By definition $r \cdot c = (i(r) + (c - i(r)))\gamma$ for any $r \in C_0$, $c \in \mathbf{Star}_{\mathsf{C}}0$, where γ is a special isomorphism $0 \approx r + (0 - r)$, which is unique as we know already. Here we check the action identities. We have

$$\begin{aligned} r \cdot (c + c') &= (i(r) + ((c + c') - i(r))\gamma \\ &\sim (i(r) + (c - i(r)))\gamma_1 + i(r) + (c' - i(r)))\gamma_2 \\ &= r \cdot c + r \cdot c' \end{aligned}$$

for any $r \in C_0$, $c, c' \in \mathbf{Star}_{\mathsf{C}}0$. The other three conditions of action for c-groups are checked similarly.

Definition 18 A c-crossed module (G, H, ∂) will be called strict if it satisfies the c-crossed module conditions, but where the $\sim$-relation in the second condition is replaced by equality, i.e.,

(i) $\partial(b \cdot a) = b + (\partial(a) - b)$,
(ii) $\partial(a) \cdot a_1 = a + (a_1 - a)$,

for $a, a_1 \in G$ and $b \in H$.

Definition 19 In a c-crossed module (G, H, ∂) a congruence $g \sim g'$ in G will be called a weak special congruence if $\partial(g) \sim \partial(g')$ is a special congruence in H.

Since in a c-crossed module (G, H, ∂) the morphism ∂ carries any special congruence to a special congruence between pairs of elements, in a crossed module every special congruence in G is a weak special congruence.

Definition 20 A c-crossed module (G, H, ∂) will be called *special* if for any congruence $\gamma : \partial c \sim r$, there exists $c' \sim c$, such that $\partial c' = r$, where $c, c' \in G$ and $r \in H$. If γ is a special congruence, then c' is the unique element in G which is weakly equivalent to c.

If a c-crossed module is connected, strict and special we will say that it is a cssc-crossed module. This kind of crossed module is exactly that we were looking for for the description of categorical groups up to equivalence of the corresponding categories, which will be proved in the sequel to this paper.

Theorem 2 *For a categorical group* $\mathsf{C} = (C_0, C_1, d_0, d_1, i, m)$ *the triple* $(\mathbf{Star}_{\mathsf{C}}0, C_0, d)$ *is a cssc-crossed module.*

Proof First we shall show that we have the equality in the first condition of the crossed module $(\mathbf{Star}_{\mathsf{C}}0, C_0, d)$. We have

$$\begin{aligned} d(r \cdot c) &= d((i(r) + (c - i(r))\varepsilon) \\ &= d(i(r) + (c - i(r))) \\ &= r + (dc - r) \end{aligned}$$

where $\varepsilon : 0 \to r + (0 - r)$ is a special isomorphism. Now we shall show that we have the equality in the second condition of a crossed module. First we compute the left side of the condition. We have

$$dc \cdot c' = (i(dc) + (c' - i(dc)))\gamma$$

where $\gamma : 0 \to dc + (0 - dc)$ is a special isomorphism and $i(dc) + (c' - i(dc))$ is a morphism $dc + (0 - dc) \to dc + (dc' - dc)$. We have $-c + i(dc) \in \mathrm{cKer}\, d_1$ and $c' \in \mathrm{cKer}\, d_0$; by Lemma 1 we obtain that there is a weak special isomorphism

$$(-c + i(dc)) + c' \approx c' + (-c + i(dc));$$

this implies that $i(dc) + c' \approx c + c' - c + i(dc)$, which implies $id(c) + (c' - i(dc)) \approx c + (c' - c)$; this gives a weakly special isomorphism

$$id(c) \cdot c' \approx c + (c' - c).$$

By the definition of a sum in $\mathbf{Star}_{\mathsf{C}}0$ for the right side we have $c + (c' - c) = (c + (c' - c)\varphi)\psi$, where $\varphi : 0 \to 0 - 0$ and $\psi : 0 \to 0 + 0$ are special isomorphisms. Here we have in mind that $d(-c) = -dc$ and $i(-dc) = -i(dc)$. Obviously, we have a weak special isomorphism $c + (c' - c)\varphi \approx i(dc) + (c' - i(dc))$, then there is a special isomorphism between the domains of these morphisms $\theta : 0 + 0 \to dc + (0 - dc)$, such that $(i(dc) + (c' - i(dc)))\theta = c + (c' - c)\varphi$. Here we applied that the codomains of these morphisms are equal. Since ψ, θ and γ are special isomorphisms, we obtain that $\theta\psi = \gamma$. Then $(c + (c' - c)\varphi)\psi = (i(dc) + (c' - i(dc)))\gamma$, which means that for the c-crossed module $(\mathbf{Star}_{\mathsf{C}}0, C_0, d)$ we have an equality in the second condition for c-crossed modules.

The crossed module is connected by the definition of $\mathbf{Star}_{\mathsf{C}}0$. Now we shall prove that this crossed module is a special c-crossed module. Let $c \in \mathbf{Star}_{\mathsf{C}}0$, and there is a congruence $\gamma : dc \sim r$, which means that γ is an isomorphism in C_1. Take $c' = \gamma c$, then we will have $c' \approx c$ in C_1, which means that $c' \sim c$ in $\mathbf{Star}_{\mathsf{C}}0$. Suppose γ is a special congruence, then it is a special isomorphism in C_1. From the coherence property of C we have that γ is the unique special isomorphism from dc to r and therefore there is a unique morphism $d_0c \to r$, which is weakly specially isomorphic to c and it is a composition γc. Therefore, c' is unique with this property and $(\mathbf{Star}_{\mathsf{C}}0, C_0, d)$ is a special c-crossed module. □

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