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Representations are adjoint to endomorphisms

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Abstract
The functor that takes a ring to its category of modules has an adjoint if one remembers the forgetful functor to abelian groups: the *endomorphism ring* of linear natural transformations. This uses the self-enrichment of the category of abelian groups. If one considers enrichments into symmetric sequences or even bisymmetric sequences, one can produce an *endomorphism operad* or an *endomorphism properad*. In this note, we show that more generally, given a category $\mathcal{C}$ enriched in a monoidal category $\mathcal{V}$, the functor that associates to a monoid in $\mathcal{V}$ its category of representations in $\mathcal{C}$ is adjoint to the functor that computes the *endomorphism monoid* of any functor with domain $\mathcal{C}$. After describing the first results of the theory we give several examples of applications.

Keywords Endomorphisms of functors · Representations · Enriched categories

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1 Introduction

The functor that takes a ring R to its category of modules has an adjoint, provided that in addition to R-mod, one remembers the forgetful functor R-$\mathsf{mod} \to \mathsf{Ab}$.

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The adjoint sends a functor $F : \mathcal{D} \to \mathsf{Ab}$ to its *endomorphism ring* $\mathcal{E}(F)$ of natural transformations. This fact is familiar to people working on duality results à la Tannaka.

If instead of using the self-enrichment $\langle -, - \rangle : \mathsf{Ab}^{\mathrm{op}} \times \mathsf{Ab} \to \mathsf{Ab}$, one uses an enrichment into symmetric sequences or bisymmetric sequences, then $\mathcal{E}(F)$ can be promoted to an *endomorphism operad* or an *endomorphism properad*. This is summarized in the table:

$\mathcal{E}(F)$	enrichment
endomorphism ring	$\langle X, Y \rangle$
endomorphism operad	$\langle X^{\otimes n}, Y \rangle$
endomorphism properad	$\langle X^{\otimes p}, Y^{\otimes q} \rangle$

In this note we study the general case, replacing Ab by a category $\mathcal{C}$ enriched in a monoidal category $\mathcal{V}$. First we review representations of monoids in the context of an enriched category. Then we describe the *endomorphism monoid* of a functor whose target is an enriched category and show that this construction is adjoint to the representations functor.

After describing the adjunction between monoids in $\mathcal{V}$ and functors with target $\mathcal{C}$, we shall study the basic properties of this adjunction, in particular in the case where the enrichment is also tensored.

In two brief appendices, we provide quick definitions of terms in enriched category theory that we need and give a few examples of contexts in which this setup holds.

The sequel, *Endomorphism operads of functors* [1], contains some explicit computations.

After seeing the definitions of the functors $\mathcal{E}$ and Rep and their adjunction, the reader is encouraged to take a look at the appendix [§ B]. Some of the examples there might be surprising.

2 Monoids and their representations

Let us fix a a bicomplete monoidal category $\mathcal{V}$ and a category $\mathcal{C}$ enriched in $\mathcal{V}$:

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{[-,-]} \mathcal{V}.$$

For convenience, we shall assume given a *locally large* universe enlargement $\mathcal{V} \hookrightarrow \widehat{\mathcal{V}}$ [§ A.1]. Because $\mathcal{V} \hookrightarrow \widehat{\mathcal{V}}$ is fully faithful and monoidal, one has a fully faithful embedding of categories of monoids

$$\mathsf{Mon}(\mathcal{V}) \longhookrightarrow \mathsf{Mon}\big(\widehat{\mathcal{V}}\big).$$

In order to distinguish between the two, we shall say that a monoid in $\widehat{\mathcal{V}}$ is *large*.

Remark 1 (Endomorphism monoid of an object) Thanks to the $\mathcal{V}$-enrichment of $\mathcal{C}$, every object $X \in \mathcal{C}$ has a natural endomorphism monoid $[X, X]$.

Definition 1 (*Representations of monoids*) Let M be a monoid. Its category of representations in $\mathcal{C}$

$$M\text{-}\mathsf{rep}$$

is the large category

- whose objects are (X, α) where X is an object of $\mathcal{C}$ and $\alpha : M \to [X, X]$ is a map of monoids and
- whose morphisms $(X, \alpha) \to (Y, \beta)$ are maps $f : X \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\alpha} & [X,X] \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle f_*} \\ [Y,Y] & \xrightarrow[f^*]{} & [X,Y]. \end{array}$$

The category of representations of M has an evident forgetful functor

$$\mathrm{U}_M : M\text{-}\mathsf{rep} \longrightarrow \mathcal{C}$$

that is both faithful and conservative. The assignment $M \mapsto M\text{-}\mathsf{rep}$ is moreover functorial: given a morphism of monoids $\psi : M \to N$, one has a commutative diagram

$$\begin{array}{ccc} M\text{-}\mathsf{rep} & \xleftarrow{\mathrm{U}_\psi} & N\text{-}\mathsf{rep} \\ & {\scriptstyle \mathrm{U}_M}\searrow \quad \swarrow{\scriptstyle \mathrm{U}_N} & \\ & \mathcal{C} & \end{array}$$

Denoting by $\widehat{\mathsf{Cat}}$ the very large category of large categories, one gets a *representation functor*

$$\mathsf{Mon}(\mathcal{V}) \xrightarrow{\mathsf{Rep}} \left(\widehat{\mathsf{Cat}}_{/\mathcal{C}}\right)^{\mathrm{op}}.$$

Remark 2 (Representations of large monoids) Since we have required $\widehat{\mathcal{V}}$ to be *locally large*, the definition of the category of representations $M\text{-}\mathsf{rep}$ also makes sense for M a large monoid. Then, the large category $\mathcal{C}$ having been fixed, the representations functor extends to the category of large monoids:

$$\begin{array}{ccc} \mathsf{Mon}(\mathcal{V}) & \xrightarrow{\mathsf{Rep}} & \left(\widehat{\mathsf{Cat}}_{/\mathcal{C}}\right)^{\mathrm{op}}. \\ \big\downarrow & \nearrow & \\ \mathsf{Mon}(\widehat{\mathcal{V}}) & & \end{array}$$

Indeed, let M be a large monoid. The cardinality of the objects of M-rep is bounded by

$$\bigcup_{X\in\mathcal{C}} \mathrm{Hom}_{\widehat{\mathcal{V}}}(M, [X, X]).$$

Since $\mathcal{C}$ is large and $\widehat{\mathcal{V}}$ is locally large, we deduce that M-rep has a large set of objects. Given two representations X and Y of a monoid M, one has

$$\mathrm{Hom}_{M\text{-rep}}(X, Y) \subset \mathrm{Hom}_{\mathcal{C}}(\mathrm{U}_M X, \mathrm{U}_M Y).$$

Hence, since $\mathcal{C}$ has large sets of morphisms, so does M-rep.

3 The endomorphism monoid of a functor

In this section we show that the representation functor $M \mapsto M$-rep has a right adjoint

$$\begin{pmatrix} \mathcal{D} \\ \downarrow F \\ \mathcal{C} \end{pmatrix} \longmapsto \mathcal{E}(F).$$

It takes as inputs large categories $\mathcal{D}$ over $\mathcal{C}$ and outputs the *endomorphism monoid* $\mathcal{E}(F)$ of the functor $F : \mathcal{D} \to \mathcal{C}$.

Remark 3 (Enriched natural transformations) Given a large category $\mathcal{D}$, the category of functors $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$ is naturally enriched in $\widehat{\mathcal{V}}$ as follows. Given two functors $F, G : \mathcal{D} \to \mathcal{C}$, the $\mathcal{V}$-natural transformations from F to G are presented by the object of $\widehat{\mathcal{V}}$ given by

$$\mathrm{Nat}_{\mathcal{V}}(F, G) := \int_{\mathcal{D}}^{*} [F-, G-],$$

where, following Yoneda's original notation [2, § 4], $\int_{\mathcal{D}}^{*}$ denotes the *cointegration* (or end) of a functor $\mathcal{D}^{\mathrm{op}} \times \mathcal{D} \to \mathcal{C}$.

Definition 2 The *endomorphism monoid* of a functor $F : \mathcal{D} \to \mathcal{C}$ is

$$\mathcal{E}(F) := \mathrm{Nat}_{\mathcal{V}}(F, F)$$

the (large) monoid of $\mathcal{V}$-natural transformations of F.

Remark 4 (Functoriality of $\mathcal{E}$) As is the case in any 2-categorical setting, $\mathcal{V}$-natural transformations are compatible with 'horizontal composition' or 'whiskering':

$$\left(\mathcal{D}' \xrightarrow{\Phi} \mathcal{D} \underset{F}{\overset{F}{\underset{\Downarrow}{\rightrightarrows}}} \mathcal{C} \right) \longmapsto \left(\mathcal{D}' \underset{F\circ\Phi}{\overset{F\circ\Phi}{\underset{\Downarrow}{\rightrightarrows}}} \mathcal{C} \right).$$

Thus, the construction $F \mapsto \mathcal{E}(F)$ is functorial in the sense that given

$$\mathcal{D}' \xrightarrow{\Phi} \mathcal{D} \xrightarrow{F} \mathcal{C},$$

one gets a morphism of large monoids

$$\Phi^* : \mathcal{E}(F) \longrightarrow \mathcal{E}(F \circ \Phi).$$

Theorem 1 *The functor* $\mathcal{E}$ *is right adjoint to* Rep

$$\mathsf{Mon}(\widehat{\mathcal{V}}) \underset{\mathcal{E}}{\overset{\mathsf{Rep}}{\rightleftarrows}} \left(\widehat{\mathsf{Cat}}_{/\mathcal{C}}\right)^{\mathrm{op}}.$$

There are a number of examples where this setup gives interesting endomorphism monoids and interesting adjunctions [§ B].

Proof Observe that a functor from $\mathcal{D}$ to M-rep over $\mathcal{C}$ consists of:

- at the object level, a monoid map $M \xrightarrow{\psi_X} [F(X), F(X)]$ for each object X of $\mathcal{D}$, and
- at the morphism level, no data, since the value on morphisms is determined by being over $\mathcal{C}$ and the functor from M-rep to $\mathcal{C}$ is faithful.

However, to be a functor, the collection ψ_X must satisfy a condition so that for each map f in $\mathrm{Hom}_{\mathcal{D}}(X, Y)$, the map $F(f)$ is an M-representation map between $F(X)$ and $F(Y)$. This is precisely the condition for the maps ψ_X to assemble to a map $\psi : M \to \mathcal{E}(F)$. Compatibility with the M-representation structures for each object X implies that ψ is a morphism of large monoids. □

Example 1 Let $X : * \to \mathcal{C}$ be an object of $\mathcal{C}$. Then the equalizer formula for the cointegral computing $\mathcal{E}(X)$ collapses to $[X, X]$. So in this case $\mathcal{E}(X)$ recovers the ordinary endomorphism object $[X, X]$.

Example 2 Let $f : \Delta^1 \to \mathcal{C}$ be a morphism of $\mathcal{C}$, with domain X and codomain Y. Again the cointegral has a simple description via the equalizer formula; it is the pullback of $[X, X]$ and $[Y, Y]$ over $[X, Y]$.

$$\mathcal{E}(f) = [X, X] \underset{[X,Y]}{\times} [Y, Y].$$

This is sometimes called the endomorphism monoid of f [3, 13.10].

Remark 5 (Generalized enrichments) We have taken as our fundamental input an enrichment of the category $\mathcal{C}$ in the monoidal category $\mathcal{V}$. A generalization of this framework is to consider instead a lax functor

$$\mathcal{C} \longrightarrow \mathsf{Bimod}_{\bullet}(\mathcal{V})$$

where $\mathsf{Bimod}_{\bullet}(\mathcal{V})$ is the bicategory whose objects are monoids in $\mathcal{V}$, whose morphisms are pointed bimodules, and whose 2-morphisms are maps of bimodules.

Let us present an example of such a generalized enrichment that does not fit directly in our framework. Let $\mathcal{C}$ be a large category, seen as naturally enriched in large sets. There is a lax functor

$$\mathcal{C} \longrightarrow \mathsf{Bimod}_{\bullet}(\widehat{\mathsf{Set}})$$

given on objects by

$$X \longmapsto \mathrm{Aut}(X),$$

which sends a map $f : X \to Y$ to

$$\mathrm{Hom}(X, Y)_f := \mathrm{Hom}(X, Y) \text{ pointed by } f$$

and which sends the composite of two maps f and g to

$$\mathrm{Hom}(X, Y)_f \otimes_{\mathrm{Aut}(Y)} \mathrm{Hom}(Y, Z)_g \longrightarrow \mathrm{Hom}(X, Z)_{fg}.$$

Using the same ideas, one can see how to produce a generalized enrichment out of a $\mathcal{V}$-enriched category $\mathcal{C}$ via

$$X \longmapsto [X, X].$$

The cointegral defining the endomorphism monoid of a functor F has a natural extension to the generalized framework.

The generalized enrichment of our example yields the following adjunction

$$\widehat{\mathsf{Grp}} \underset{\mathcal{A}\mathrm{ut}}{\overset{\mathsf{Rep}}{\rightleftarrows}} \left(\widehat{\mathsf{Cat}}_{/\mathcal{C}}\right)^{\mathrm{op}}.$$

Of course one could—indirectly—obtain the adjunction between representations and automorphism groups by first taking the monoid of endomorphisms and then restricting to groups.

4 Small endomorphism monoids

When the domain category $\mathcal{D}$ of F is small, the endomorphism monoid $\mathcal{E}(F)$ is obviously small. We shall show that this is still the case when $\mathcal{D}$ is large under appropriate accessibility conditions.

Lemma 1 (Accessible reduction) *Assume that the category $\mathcal{C}$ is accessibly enriched* [7], *$\mathcal{D}$ is an accessible category and $F : \mathcal{D} \to \mathcal{C}$ is an accessible functor.*

Let κ be a small cardinal big enough so that $\mathcal{D}$ is κ-accessible and so that both F and $X \mapsto [X, Y]$ commute with κ-filtered colimits. Let us denote by F^κ the restriction of F to the full subcategory $\mathcal{D}^\kappa \subset \mathcal{D}$ of κ-compact objects of $\mathcal{D}$. Then the canonical map

$$\mathcal{E}(F) \longrightarrow \mathcal{E}(F^\kappa)$$

is an isomorphism. In particular $\mathcal{E}(F)$ is a (small) monoid.

Proof Using the universal property of the cointegrals, it is enough to show the existence of compatible maps

$$\mathcal{E}(F^\kappa) \xrightarrow{\varphi_X} [F(X), F(X)]$$

for every $X \in \mathcal{D}$, such that for every κ-compact X^κ, the map φ_{X^κ} is equal to the projection map $\pi_{X^\kappa} : \mathcal{E}(F^\kappa) \to [F(X^\kappa), F(X^\kappa)]$.

Since every $X \in \mathcal{D}$ is canonically the κ-filtered colimit $X = \operatorname{colim}_{X^\kappa \to X} X^\kappa$ of the κ-compact objects over it,

$$[F(X), F(X)] = \lim_{X^\kappa \to X} [F(X^\kappa), F(X)].$$

Every map $g : X^\kappa \to X$ induces a morphism

$$\mathcal{E}(F^\kappa) \xrightarrow{\pi_{X^\kappa}} [F(X^\kappa), F(X^\kappa)] \xrightarrow{g_*} [F(X^\kappa), F(X)]$$

and given $h : \underline{X}^\kappa \to X^\kappa$, one can draw a commutative diagram

$$\begin{array}{ccccc}
\mathcal{E}(F^\kappa) & \xrightarrow{\pi_{X^\kappa}} & [F(X^\kappa), F(X^\kappa)] & \xrightarrow{g_*} & [F(X^\kappa), F(X)] \\
{\scriptstyle \pi_{\underline{X}^\kappa}}\downarrow & & \downarrow{\scriptstyle h^*} & & \downarrow{\scriptstyle h^*} \\
[F(\underline{X}^\kappa), F(\underline{X}^\kappa)] & \xrightarrow[h_*]{} & [F(\underline{X}^\kappa), F(X^\kappa)] & \xrightarrow[g_*]{} & [F(\underline{X}^\kappa), F(X)]
\end{array}$$

where the commutation of the first square is guaranteed by the universal property of $\mathcal{E}(F^\kappa)$. This shows that we get a well-defined morphism φ_X for every $X \in \mathcal{D}$.

By construction of φ_X, the following diagram commutes

$$\begin{array}{ccc}
\mathcal{E}(F^\kappa) & \xrightarrow{\varphi_X} & [F(X), F(X)] \\
{\scriptstyle \pi_{X^\kappa}}\downarrow & & \downarrow{\scriptstyle g^*} \\
[F(X^\kappa), F(X^\kappa)] & \xrightarrow[g_*]{} & [F(X^\kappa), F(X)],
\end{array}$$

hence when g is the identity of a κ-compact object X^κ, we get $\pi_{X^\kappa} = \varphi_{X^\kappa}$ as promised.

Let $f : X \to Y$ be a morphism in $\mathcal{D}$. We need to check the commutativity of the induced square

$$\begin{array}{ccc} \mathcal{E}(F^\kappa) & \xrightarrow{\varphi_X} & [F(X), F(X)] \\ {\scriptstyle \varphi_Y}\downarrow & & \downarrow{\scriptstyle f_*} \\ [F(Y), F(Y)] & \xrightarrow[f^*]{} & [F(X), F(Y)]. \end{array}$$

By accessibility again, one may check the equality $f^*\varphi_Y = f_*\varphi_X$ after projection $g^* : [F(X), F(Y)] \to [F(X^\kappa), F(Y)]$ for every $g : X^\kappa \to X$. Then by the commutativity of the diagrams

$$\begin{array}{ccc} \mathcal{E}(F^\kappa) & \xrightarrow{\varphi_X} & [F(X), F(X)] \\ {\scriptstyle \pi_{X^\kappa}}\downarrow & & \downarrow{\scriptstyle f_* g^*} \\ [F(X^\kappa), F(X^\kappa)] & \xrightarrow[f_* g_*]{} & [F(X^\kappa), F(Y)] \end{array}$$

and

$$\begin{array}{ccc} \mathcal{E}(F^\kappa) & \xrightarrow{\varphi_Y} & [F(Y), F(Y)] \\ {\scriptstyle \pi_{X^\kappa}}\downarrow & & \downarrow{\scriptstyle (fg)^*} \\ [F(X^\kappa), F(X^\kappa)] & \xrightarrow[(fg)_*]{} & [F(X^\kappa), F(Y)], \end{array}$$

we may conclude the desired result.

Remark 6 (Accessibility of the category of representations) In view of the previous reduction lemma, one may wonder whether $\mathrm{U}_M : M\text{-}\mathsf{rep} \to \mathcal{C}$ is an accessible functor between accessible categories whenever $\mathcal{C}$ is accessibly enriched.

This appears to be an intricate question in general: it is still unknown whether the category of bigebras over some well-known props are actually accessible. In the particular case where $\mathcal{C}$ is accessibly tensored (or cotensored), this question receives a positive answer. We shall give more details about this case in the next section.

Cogebras over a dg-operad over a field give an example of an accessibly enriched context [B.1.3] that is neither tensored, nor cotensored, in which P-cog is accessible for any dg-operad P [4].

5 The case of tensored enrichment

In the case where $\mathcal{C}$ is tensored over $\mathcal{V}$, the additional structure allows one to say more about the adjunction between representations and endomorphisms, particularly when the tensor structure is well-behaved.

5.1 The adjunction in the accessibly tensored case

In the case where forgetful functors are accessible, we no longer need to have jumps in sizes and we get a refined adjunction with the category of *small* monoids.

Proposition 1 (Accessibly tensored case) *Assume that* $\mathcal{C}$ *is accessibly tensored over* $\mathcal{V}$*. Then there is an adjunction*

$$\mathsf{Mon}(\mathcal{V}) \underset{\mathcal{E}}{\overset{\mathsf{Rep}}{\rightleftarrows}} (\mathsf{Acc}_{/\mathcal{C}})^{\mathrm{op}}$$

in which Acc *is the very large category of large accessible categories and accessible functors.*

For this one restricts the adjunction $\mathsf{Rep} \dashv \mathcal{E}$ using accessible reduction [1] and the following lemmas.

Lemma 2 *If* $\mathcal{C}$ *is accessibly tensored, then every monoid* M *induces an accessible monad* $\widetilde{M}$ *on* $\mathcal{C}$ *whose category of modules is canonically equivalent to* M-rep *as an object of* $\widehat{\mathsf{Cat}}_{/\mathcal{C}}$

$$M\text{-}\mathsf{rep} = \widetilde{M}\text{-}\mathsf{mod}.$$

As a consequence, the category of representations M-rep *is accessible and the forgetful functor*

$$M\text{-}\mathsf{rep} \xrightarrow{\mathrm{U}_M} \mathcal{C}$$

is accessible.

Proof Because the functor $M \mapsto (M \otimes -)$ is monoidal [4], each monoid M induces an accessible monad $\widetilde{M}$ with underlying functor $X \mapsto M \otimes X$. As a consequence its category of modules is accessible and the forgetful functor

$$\widetilde{M}\text{-}\mathsf{mod} \xrightarrow{\mathrm{U}_{\widetilde{M}}} \mathcal{C}$$

is accessible.

We now claim that there is a canonical equivalence of categories above $\mathcal{C}$

$$M\text{-}\mathsf{rep} = \widetilde{M}\text{-}\mathsf{mod},$$

compatible with the forgetful functors. Let (X, α) be a representation of M. Then the monoid morphism $\alpha : M \to [X, X]$ is equivalent by adjunction to an $\widetilde{M}$-module structure $\widetilde{\alpha} : M \otimes X \to X$. Let (Y, β) be another representation of M, then $f : X \to Y$ is a morphism of representations if

$$\begin{array}{ccc} M & \xrightarrow{\alpha} & [X, X] \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle f_*} \\ [Y, Y] & \xrightarrow[f^*]{} & [X, Y]. \end{array}$$

commutes. By adjunction the top right part of the diagram is equivalent to $M \otimes X \to X \to Y$ and the bottom left is equivalent to $M \otimes X \to M \otimes Y \to Y$ so that the commutativity of the above square is equivalent to the commutativity of

$$\begin{array}{ccc} M \otimes X & \xrightarrow{M \otimes f} & M \otimes Y \\ {\scriptstyle\widetilde{\alpha}}\downarrow & & \downarrow{\scriptstyle\widetilde{\beta}} \\ X & \xrightarrow[f]{} & Y. \end{array}$$

Hence $f : X \to Y$ is a morphism of M-representations if and only if it is a morphism of $\widetilde{M}$-modules. □

Lemma 3 *Let $F : \mathcal{D} \to \mathcal{C}$ be an accessible functor with accessible domain. Then the counit of the adjunction* Rep $\dashv \mathcal{E}$ *applied to F*

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & \mathcal{E}(F)\text{-rep} \\ & {\scriptstyle F}\searrow \quad \swarrow{\scriptstyle \mathrm{U}_{\mathcal{E}(F)}} & \\ & \mathcal{C} & \end{array}$$

is given by an accessible functor.

Proof The top map of the diagram if accessible because the two other maps are accessible [2] and the forgetful functor $\mathcal{E}(F)$-rep $\to \mathcal{C}$ is conservative. □

Remark 7 In the accessibly tensored case, the representation functor factors through the category of accessible monads on $\mathcal{C}$. Using an adapted version of a result of Janelidze and Kelly [5], one can show that the adjunction Rep $\dashv \mathcal{E}$ factors as a composite of adjunctions

$$\mathsf{Mon}(\mathcal{V}) \underset{\longleftarrow}{\xrightarrow{M \longmapsto \widetilde{M}}} \mathsf{Monads}_{\mathrm{acc}}(\mathcal{C}) \underset{\longleftarrow}{\xrightarrow{\mathsf{Mod}}} \left(\mathsf{Acc}_{/\mathcal{C}}\right)^{\mathrm{op}}.$$

5.2 Faithfulness of Rep

The question of reconstructing a monoid M out of its category M-rep of representations is an old one, in the Tannakian context for example [B.2.1]. Such a result cannot be obtained in general without additional hypotheses. Instead one can look at the opportunity of recovering M as a *submonoid* of $\mathcal{E}(\mathrm{U}_M)$.

This is the question of faithfulness of the Rep functor which is of independent interest. As an example, one can view Joyal's results on analytic monads [6] as saying in particular that the representation functor is faithful in the case where $\mathcal{C}$ is the category of sets operadically enriched in symmetric sequences.

The representation functor $M \mapsto M\text{-rep}$ is a priori not faithful. A trivial example of this takes $\mathcal{C}$ to be the empty category. A nontrivial example of independent interest is given by looking at the functor $P \mapsto P\text{-cog}$ mapping a dg-operad to its category of cogebras. Indeed, one can show that there exists a non-zero dg-operad without nontrivial cogebras [7]:

$$\exists\, P \neq 0, \quad P\text{-cog} = 0.$$

However, when $\mathcal{C}$ is tensored, we get a criterion to check whether the representation functor is faithful.

Proposition 2 (Faithfulness of representations) *Assume that $\mathcal{C}$ is faithfully tensored over $\mathcal{V}$, then the representations functor*

$$\mathsf{Mon}(\mathcal{V}) \xrightarrow{\mathsf{Rep}} \left(\widehat{\mathsf{Cat}}_{/\mathcal{C}}\right)^{\mathrm{op}}$$

is faithful. Equivalently, for every monoid M, the unit map

$$M \longrightarrow \mathcal{E}(\mathrm{U}_M)$$

is a monomorphism.

Proof Let $\phi, \psi : M \rightrightarrows N$ be two morphisms of monoids such that

$$N\text{-rep} \xrightarrow{\mathrm{U}_\phi = \mathrm{U}_\psi} M\text{-rep}.$$

If $\phi_!$ denotes the (partially defined) left adjoint to U_ϕ and $\psi_!$ the (partially defined) left adjoint to U_ψ, then one has $\phi_! = \psi_!$. Let X be an object of $\mathcal{C}$, because $\mathcal{C}$ is tensored over $\mathcal{V}$, the monoid M acts on $M \otimes X$ and $M \otimes X$ is then the free representation of M induced on X. The same goes for $N \otimes X$. As a consequence, one has

$$\mathrm{U}_\phi \circ \phi_!(M \otimes X) = \mathrm{U}_\psi \circ \psi_!(M \otimes X) = N \otimes X.$$

Using the units of the adjunctions, one then gets that

$$M \otimes X \xrightarrow{\phi \otimes X = \psi \otimes X} N \otimes X.$$

Since this is true for every X, we get $\phi = \psi$. □

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A Terminology of enriched categories

We let the reader turn to Kelly [8] for a detailed exposition on categories enriched in a monoidal category $(\mathcal{V}, \otimes, \mathbf{1})$. In order to not be bothered by size issues, we fix once and for all three infinite inaccessible cardinals $\mathrm{L} < \mathrm{XL} < \mathrm{XXL}$ and use the dictionary

$$\text{small} := \text{L-small}; \quad \text{large} := \text{XL-small}; \quad \text{very large} := \text{XXL-small}.$$

We now assume that $\mathcal{V}$ is large (has large sets of objects and morphisms) and has all small limits and colimits. In what follows we consider a large $\mathcal{V}$-enriched category

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{[-,-]} \mathcal{V}$$

and assume that $\mathcal{C}$ is large.

A.1 Enlargement of the universe

For convenience (when computing over large diagrams), we shall enlarge $\mathcal{V}$: we choose a very large monoidal category $(\widehat{\mathcal{V}}, \otimes, \mathbf{1})$ with a full monoidal embedding

$$(\mathcal{V}, \otimes, \mathbf{1}) \hookrightarrow \left(\widehat{\mathcal{V}}, \otimes, \mathbf{1}\right).$$

The enlarged universe can be chosen to be locally large, have all large limits and colimits and the embedding can be assumed to commute with small limits and colimits. This is discussed for example by Kelly [8, §2.6] (albeit in the closed symmetric setting).

The $\mathcal{V}$-category $\mathcal{C}$ can now without effort be seen as a $\widehat{\mathcal{V}}$-category

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{[-,-]} \mathcal{V} \hookrightarrow \widehat{\mathcal{V}}.$$

A.2 Properties of enrichments

Definition 3 (*Closed monoidal category*) One says that $\mathcal{V}$ is *closed* when the functor $Y \mapsto Y \otimes X$ has a right adjoint $Z \mapsto X^Z$ for each object X in $\mathcal{V}$.

Definition 4 (*Tensored*) One says that $\mathcal{C}$ is *tensored* over $\mathcal{V}$ whenever $\mathcal{V}$ is closed and for every $X \in \mathcal{C}$ and $M \in \mathcal{V}$, the functor

$$Y \longmapsto [X, Y]^M$$

is $\mathcal{V}$-representable by an object denoted $M \otimes X \in \mathcal{C}$. In that case, since $\mathcal{V}$ is closed the induced functor

$$(\mathcal{V}, \otimes, \mathbf{1}) \xrightarrow{M \longmapsto (M \otimes -)} (\mathrm{Fun}(\mathcal{C}, \mathcal{C}), \circ, \mathrm{id}_{\mathcal{C}})$$

is naturally endowed with a monoidal structure.

Definition 5 (*Faithfully tensored*) We shall say that $\mathcal{C}$ is *faithfully* tensored over $\mathcal{V}$ if it is tensored and the functor

$$\mathcal{V} \xrightarrow{M \longmapsto (M \otimes -)} \mathrm{Fun}(\mathcal{C}, \mathcal{C})$$

is faithful.

Definition 6 (*Accessibly tensored*) We shall say that $\mathcal{C}$ is *accessibly* tensored over $\mathcal{V}$ if it is tensored, both $\mathcal{V}$ and $\mathcal{C}$ are accessible and for every $M \in \mathcal{V}$, the functor

$$\mathcal{C} \xrightarrow{X \longmapsto M \otimes X} \mathcal{C}$$

is accessible.

Definition 7 (*Accessibly enriched*) When $\mathcal{V}$ and $\mathcal{C}$ are both accessible, we shall say that $\mathcal{C}$ is *accessibly enriched* if there exists a small cardinal κ such that for every $Y \in \mathcal{C}$, the functor

$$\mathcal{C}^{\mathrm{op}} \xrightarrow{X \longmapsto [X, Y]} \mathcal{V}$$

commutes with κ-cofiltered limits.

Remark 8 One can check that if $\mathcal{C}$ is accessibly tensored, it is then accessibly enriched.

B Examples of contexts of application

In this appendix, we give several application contexts for the adjunction

$$\mathsf{Mon}(\widehat{\mathcal{V}}) \underset{\mathcal{E}}{\overset{\mathsf{Rep}}{\rightleftarrows}} \left(\widehat{\mathsf{Cat}}_{/\mathcal{C}}\right)^{\mathrm{op}}.$$

In each context, the terminology is specific, both for monoids and for their categories of representations.

B.1 Using a closed symmetric monoidal category

In the next examples, we fix a presentable closed symmetric monoidal category $(\mathcal{C}, \otimes, \mathbf{1})$ and denote its internal hom by $\langle -, - \rangle$. We then consider several enrichments for $\mathcal{C}$.

Potential examples of such closed symmetric monoidal categories include the category of sets, vector spaces or coassociative cogebras (more generally cogebras over Hopf operads). It also includes the categories of sheaves valued in those categories.

B.1.1 Self enrichment

This one is the most obvious, since the monoidal structure of $\mathcal{C}$ is closed, it is self-enriched via

$$[X, Y] := \langle X, Y \rangle.$$

In this context, the general idea of the adjunction $\mathsf{Rep} \dashv \mathcal{E}$ was well-known to people doing reconstruction theorems à la Tannaka. It appears for example in Street's *Quantum groups: a path to current algebra* [9, Ch. 16].

B.1.2 Operadic enrichment

Let us denote by $\mathcal{C}^{\mathbf{S}^{\mathrm{op}}}$ the category of symmetric sequences: sequences of objects $M(n)$ of $\mathcal{C}$ endowed with right $\mathbf{S}_n$-actions for every natural n. The category $\mathcal{C}$ is accessibly tensored over the category of symmetric sequences via the formula

$$M \triangleleft X := \coprod_{n \in \mathbf{N}} M(n) \otimes_{\mathbf{S}_n} X^{\otimes n}.$$

This induces a monoidal structure on symmetric sequences

$$M \triangleleft N := \coprod_{n \in \mathbf{N}} M(n) \otimes_{\mathbf{S}_n} N^{\circledast n}.$$

Where $\circledast$ denotes the convolution of symmetric sequences. The associated enrichment is given by

$$[X, Y](n) := \langle X^{\otimes n}, Y \rangle.$$

Monoids in symmetric sequences are called operads

$$\mathsf{Op}(\mathcal{C}) := \mathsf{Mon}\left(\mathcal{C}^{\mathbf{S}^{\mathrm{op}}}, \triangleleft, \mathbf{1}_{\triangleleft}\right)$$

Given an operad P, its category of representations is called the category of P-algebras. One thus gets an adjunction

$$\mathsf{Op}(\mathcal{C}) \underset{\mathcal{E}}{\overset{\mathsf{Alg}}{\rightleftarrows}} \left(\mathsf{Acc}_{/\mathcal{C}}\right)^{\mathrm{op}}.$$

B.1.3 The other (cogebraic) operadic enrichment

This time we let $\mathcal{C}^{\mathbf{S}}$ be the category of symmetric sequences with left actions of the symmetric groups. It admits a monoidal structure given by

$$M \triangleright N := \coprod_{n\in\mathbf{N}} M^{\circledast n} \otimes_{\mathbf{S}_n} N(n)$$

and the associated enrichment is

$$[X, Y](n) := \langle X, Y^{\otimes n}\rangle.$$

Since left and right actions of symmetric groups are equivalent, one has an equivalence of categories

$$\mathsf{Mon}\Big(\mathcal{C}^{\mathbf{S}}, \triangleright, \mathbf{1}_{\triangleright}\Big) = \mathsf{Mon}\Big(\mathcal{C}^{\mathbf{S}^{\mathrm{op}}}, \triangleleft, \mathbf{1}_{\triangleleft}\Big) = \mathsf{Op}(\mathcal{C}).$$

In this case, the category of representations of an operad P is its category of cogebras. Conversely, the functor $\mathcal{E}$ associates to a functor F, seen as an object of the functor category, its *coendomorphism operad*.

In general, the category of P-cogebras may not be presentable, although (for example) it is presentable if the ground category is dg-vector spaces [4]. Thus, one has the adjunction

$$\mathsf{Op}(\mathsf{dgVect}) \underset{\mathcal{E}}{\overset{\mathsf{Cog}}{\rightleftarrows}} \left(\mathsf{Acc}_{/\mathsf{dgVect}}\right)^{\mathrm{op}}.$$

This example arises naturally in applications and appeared, for example, in unpublished work by May, who considered it well-known.

In one application, the singular chains functor from topological spaces to chain complexes factors through the category of E_∞-cogebras in chain complexes.

The following stable improvement of this example was pointed out to us by Arone: the coendomorphism operad of the suspension functor from pointed spaces to spectra can be shown to be weakly equivalent to the commutative operad [10].

B.1.4 Propic enrichments

Going further, one can enrich $\mathcal{C}$ in the category of bisymmetric sequences $\mathcal{C}^{\mathbf{S}^{\mathrm{op}}\times\mathbf{S}}$ using

$$[X, Y](p, q) := \langle X^{\otimes p}, Y^{\otimes q}\rangle.$$

There are several monoidal structures on bisymmetric sequences compatible with these enrichment objects, depending on the classes of graphs involved in the definition of the monoidal structure. One can allow connected graphs, in which case the monoids are properads [11, 2.1], or allow only simply connected graphs, in which case the monoids are dioperads [12, 4.2]. Similar but more exotic examples are also possible [13].

B.2 Examples with exogenous enrichments

B.2.1 Representations of topological monoids

The following example is taken from the duality between topological groups and their categories of representations due to Tannaka [14]. The category of finite dimensional vector spaces is canonically enriched in topological spaces. Since this category is small, one gets an adjunction

$$\mathsf{Mon}(\mathsf{Top}) \underset{\mathcal{E}}{\overset{\mathsf{Rep}_{\mathrm{fd}}}{\rightleftarrows}} (\mathsf{Cat}_{/\mathsf{Vect}_{\mathrm{fd}}})^{\mathrm{op}}$$

where $\mathcal{E}$ associates to any functor $F : \mathcal{D} \to \mathsf{Vect}_{\mathrm{fd}}$ its topological monoid of endomorphisms.

B.2.2 Bigebras

Let $\boldsymbol{K}$ be a field. The category of associative $\boldsymbol{K}$-algebras is naturally cotensored over $\boldsymbol{K}$-cogebras: given a cogebra V and an algebra Λ, convolution gives $\mathrm{Hom}_{\boldsymbol{K}}(V, \Lambda)$ a structure of associative algebra. This cotensorization comes with an enrichment and a tensorization [15].

Monoid objects in cogebras are bigebras. Given a bigebra H, it is an exercise to verify that the category of representations H-rep is naturally isomorphic to the category of H-module algebras studied by Hopf theorists [16, 4.1.1] equipped with the functor to algebras forgetting the H-module structure. We thus obtain an adjunction

$$\mathsf{Bigebras} \underset{\mathcal{E}}{\overset{\mathsf{Mod}}{\rightleftarrows}} (\mathsf{Acc}_{/\mathsf{Alg}})^{\mathrm{op}}$$

where for an accessible functor $F : \mathcal{D} \to \mathsf{Alg}$, the endomorphism bigebra $\mathcal{E}(F)$ is universal among bigebras acting compatibly on the objects of $\mathcal{D}$.

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