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The Dold–Thom theorem via factorization homology

Lauren Bandklayder[1]

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Abstract
We give a new proof of the classical Dold–Thom theorem using factorization homology. Our method is direct and conceptual, avoiding the Eilenberg–Steenrod axioms entirely in favor of a more general geometric argument.

Keywords Dold–Thom theorem · Factorization homology · Algebraic topology

1 Introduction

The Dold–Thom theorem is a classical result giving a beautiful relation between homotopy and homology. It states that for a nice, based topological space M and abelian group A, there are isomorphisms between the homotopy groups of the infinite symmetric product of M with coefficients in A and the reduced homology groups of M itself, again with coefficients in A:

$$\pi_*(\mathrm{Sym}(M; A)) \cong \widetilde{H}_*(M; A).$$

The infinite symmetric product is the space of configurations of points in M labeled by elements of A endowed with a certain topology; see Definition 2 for a precise formulation.

A fundamental result with applications in both algebraic topology and algebraic geometry, this theorem has received much attention since it was first published in 1958. In 1959, Spanier used the equivalence of homology theories exhibited to understand Spanier–Whitehead duality [27]. Later, McCord gave a convenient model for the categorical tensor of a based space and a topological abelian group which gener-

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✉ Lauren Bandklayder
Lbandklayder@mpim-bonn.mpg.de

1 Northwestern University, Evanston, USA

alized Dold and Thom's infinite symmetric product [22]. Inspired in part by McCord, the notes of Floyd and Floyd [13] discuss the infinite symmetric product as a source of models for spectra. Both McCord's model and the notes [13] are treated in the paper of Kuhn, [17], where more historical perspective is also given. Segal, too, gave a generalized model for the infinite symmetric product in [25], viewing it as a labelled configuration space. In 1996, Gajer gave an intersection-homology variant of the Dold–Thom theorem [14]. An equivariant formulation of the theorem was given by dos Santos in [11], which generalizes an equivariant integral-coefficient formulation given by Lima–Filho in [18]. More recently, Suslin and Voevodsky used the theorem to define motivic cohomology, giving a means of translating techniques and results of algebraic topology into algebraic geometry [28]. As giving a complete history of the Dold–Thom theorem would constitute a paper of its own, we have only outlined some of the highlights here, but we hope this convinces the reader of its rich history and utility.

As we now explain, our construction of the infinite symmetric product is an instance of factorization homology, although we shall make no use of the formal machinery of that theory—the interested reader can consult [1–3] and Chapter 5 of [19] for more. Factorization homology acts as a bridge between the algebraic and geometric study of manifolds, taking two pieces of input—geometric data in the form of an n-manifold, M, and algebraic data in the form of an n-disk algebra, $\mathcal{A}$—and returning an invariant denoted $\int_M \mathcal{A}$ and called the factorization homology of M with coefficients in $\mathcal{A}$. Roughly speaking, an n-disk algebra is a functor out of a certain category of disjoint unions of disks which determines a local algebraic structure on n-manifolds, and factorization homology gives a means of gluing together this local data across coordinate patches of a given manifold M by taking a homotopy colimit to obtain a global invariant of M. From this point of view, the Dold–Thom theorem asserts that, when viewing the infinite symmetric product as an n-disk algebra, factorization homology recovers ordinary homology. This perspective is illustrated in Proposition 15.

Remark 1 The reader may have noticed that what we have called the Dold–Thom theorem is not the statement one finds in [10], but a slight variation thereof which considers coefficients in an abelian group. However, one can observe that our definition of the infinite symmetric product will still hold with coefficients in an abelian monoid and in fact there is homeomorphism between our $\mathrm{Sym}(M; \mathbb{N})$ and the classical version of the infinite symmetric product $\mathrm{SP}^\infty(M)$ as defined in [10]. We will focus our attention on proving the variation of the Dold–Thom theorem which considers group coefficients, and in the "Appendix" we will derive the original statement from this variant.

1.1 Motivation for a new approach

The original proof of the Dold–Thom theorem proceeds by verifying that the composition of functors $\pi_*(\mathrm{Sym}(-; A))$ satisfies the Eilenberg–Steenrod axioms for a reduced homology theory [12]. In this note, we will outline a new, direct proof of the Dold–Thom theorem which avoids the Eilenberg–Steenrod axioms entirely. At the heart is a powerful local-to-global argument of Dugger and Isaksen allowing us to recognize the infinite symmetric product as an instance of factorization homology. We imagine the conceptual and geometric nature of this proof lends itself readily to many generalizations and reinterpretations. As the development of factorization homology is relatively modern, another compelling aspect of this approach is in its unification of contemporary and classical constructions in algebraic topology.

1.2 Overview of this work

For simplicity we will restrict ourselves to the case where M is a smooth manifold. Because any finite CW complex can be approximated up to homotopy by a smooth manifold, the homotopy invariance of the infinite symmetric product immediately implies the theorem for all finite CW complexes, and it is quite possible similar techniques extend this proof even further. Nevertheless, for the purposes of this note we will content ourselves with the case of a manifold.

The rough outline of our proof of the Dold–Thom theorem is as follows. The heart lies in Proposition 15, where we use a local-to-global argument to see that, for $\mathrm{Disk}_{*/M}$ an appropriate category of disks embedded in M, there is a homotopy equivalence between the infinite symmetric product of M and the factorization homology of M:

$$\mathrm{Sym}(M; A) \simeq \operatorname*{hocolim}_{U \in \mathrm{Disk}_{*/M}} \mathrm{Sym}(U; A).$$

In Lemma 18 we will show how to use the Dold–Kan correspondence to pass to chain complexes. It is crucial to this step that we take the homotopy colimit over a *sifted* category, and we wish to emphasize this point to the reader, as it is precisely this fact that allows us to analyze this same homotopy colimit in spaces as one in chain complexes. Once in the context of chain complexes, we finish the proof by showing the homotopy colimit $\mathrm{hocolim}_{U \in \mathrm{Disk}_{*/M}} \mathrm{Sym}(U; A)$ considered as a chain complex is quasi-isomorphic to the reduced singular chain complex, $\widetilde{C}_*(M; A)$, which precisely implies the desired Dold–Thom isomorphisms: $\pi_i(\mathrm{Sym}(M; A)) \cong \widetilde{H}_i(M; A)$. We remark that at this point it may not be clear to the reader why one can consider the expression $\mathrm{hocolim}_{U \in \mathrm{Disk}_{*/M}} \mathrm{Sym}(U; A)$ as a chain complex, this will be elaborated on in Corollary 17 in conjunction with the paragraph directly following it.

In Sect. 2 we will provide the basic definitions we work with, including that of the infinite symmetric product and the relevant categories of disks, before turning to a discussion of the local-to-global argument we will use to make the identification of the infinite symmetric product with factorization homology. In Sect. 3 we will actually prove this identification and derive from it the Dold–Thom theorem with abelian group coefficients. Finally, in "Appendix" we will derive the Dold–Thom theorem as stated in [10] from the case with abelian group coefficients.

2 Preliminaries

2.1 The infinite symmetric product

In this section we briefly recall the definition of the infinite symmetric product associated to a topological space, motivated by the idea of defining the free topological A-module on a space, M.

Definition 2 For M a pointed space with basepoint, $*$, and A an abelian group with identity, e, define the infinite symmetric product of M with coefficients in A to be the space

$$\begin{aligned}&\mathrm{Sym}(M;A)\\ &\quad := \left\{(S,l)\middle|\, * \in S,\ S \subset M,\ |S| < \infty,\ \text{and } l : S \to A \text{ such that } l(*) = e\right\}\big/\sim\end{aligned}$$

where $\sim$ is the equivalence relation generated by $(S,l) \sim (S \cup \{x\}, l')$ when l' is the map which agrees with l on S and sends x to e, topologized with the finest topology making the following maps continuous for any finite set I:

$$\begin{aligned} f_I : M^{I_+} \times A^{I_+} &\to \mathrm{Sym}(M;A) \\ (c : I_+ \to M, l : I_+ \to A) &\mapsto \left[(c(I_+), l' : s \mapsto \sum_{i \in c^{-1}(s)} l(i))\right]. \end{aligned}$$

Here M^{I_+} denotes based maps $I_+ \to M$, A^{I_+} denotes all maps of sets $I_+ \to A$ sending the basepoint to e, and both are endowed with the product topology.

That is, taking the union over all finite I of the maps f_I gives a surjection $M^{I_+} \times A^{I_+} \twoheadrightarrow \mathrm{Sym}(M;A)$, and we say $U \subset \mathrm{Sym}(M;A)$ is open if and only if its inverse image is open under each of these maps. In particular, this topology requires that labels vanish at the basepoint and that points labeled by the identity be forgotten, and allows for points to collide whence their labels add. A first example of this construction is given in the following lemma:

Lemma 3 *For I a finite set, there is a homeomorphism*

$$\mathrm{Sym}(I_+;A) \cong A^I$$

where we consider A as a topological space with the discrete topology and A^I with the product topology.

Remark 4 A pointed map $M \xrightarrow{f} Y$ induces a map $\mathrm{Sym}(M;A) \xrightarrow{\mathrm{Sym}(f;A)} \mathrm{Sym}(Y;A)$ which is given explicitly by the assignment:

$$[(S,l)] \mapsto \left[(f(S), x \mapsto \sum_{s \in f^{-1}(x)} l(s))\right]$$

and, in fact, this allows us to view the infinite symmetric product as a functor

$$\mathrm{Sym}(-; A) : \mathrm{Top}_* \to \mathrm{Top}_*.$$

In the future we will use the notation $\mathrm{Sym}(f; A)\big[(S, l)\big] = \big[f(S), f(l)\big]$. Moreover, this functor is homotopy invariant. Let $H : M \times [0, 1] \to Y$ be a homotopy between two pointed maps $f, g : M \to Y$ and for each element $t \in [0, 1]$, let H_t denote the map $H(-, t) : M \to Y$, then one can define a homotopy between $\mathrm{Sym}(f; A)$ and $\mathrm{Sym}(g; A)$ explicitly as

$$\begin{aligned} H' : \mathrm{Sym}(M; A) \times [0, 1] &\to \mathrm{Sym}(Y; A) \\ ((S, l), t) &\mapsto \mathrm{Sym}(H_t; A)(S, l). \end{aligned}$$

2.2 Categories of disks

From here on we will fix M to be a smooth, pointed n-manifold. The following categories will come up throughout our proof:

Definition 5 Let Mfld_n denote the category whose objects are n-manifolds and with morphisms given by open embeddings.

Definition 6 Let $\mathrm{Disk} \subset \mathrm{Mfld}_n$ denote the full subcategory consisting of objects which are finite disjoint unions of n-dimensional Euclidean spaces.

Of course, the category Disk also depends on n, but since we have fixed a dimension n we will omit that from the notation for ease of reading. We can then consider the over-category $\mathrm{Disk}_{/M}$.

Definition 7 Let $\mathrm{Disk}_{*/M}$ denote the full subcategory of $\mathrm{Disk}_{/M}$ consisting of embeddings $U \hookrightarrow M$ whose image contains the basepoint $* \in M$.

2.3 Local-to-global methods

As mentioned in the introduction, our approach to the Dold–Thom theorem will be to understand the local structure of the infinite symmetric product and leverage this to understand its global structure. We will show that the infinite symmetric product can be constructed from some collection of simpler, more tangible spaces, much in the way that manifolds can be constructed from disks. In particular, we would like to see that there is an equivalence $\mathrm{Sym}(M; A) \simeq \mathrm{hocolim}_{\,U \hookrightarrow M \in \mathrm{Disk}_{*/M}} \mathrm{Sym}(U; A)$.

We will make use of the following definition due to Dugger and Isaksen [7]:

Definition 8 For X a topological space and $\mathcal{U} = \{U_i\}_{i \in I}$ an open cover of X, we say that $\mathcal{U}$ is a *complete cover* if for all finite subsets J of the indexing set I, the intersection $\bigcap_{j \in J} U_j$ admits an open cover by elements of $\mathcal{U}$.

Example 9 For M a based manifold, the over-category $\mathrm{Disk}_{*/M}$ determines a complete cover of M. It is clear that this category determines a cover of M since M is locally

Euclidean, and to see that it is complete one simply notes that given two embeddings of finitely many copies of Euclidean space into M, the intersection of the images of these embeddings is an open submanifold of M, and thus itself admits a cover by disks embedded in M.

Example 10 More generally, for X a topological space, any basis for the topology of X is a complete cover of X.

These complete covers will be our local data, and now we provide a pleasant result of Dugger and Isaksen [7] which states that this local data is sufficient to recover a topological space up to weak equivalence. More rigorously:

Theorem 11 *If $\mathfrak{U} = \{U_i\}_{i\in I}$ is a complete cover, then there is a weak equivalence*

$$\underset{\mathfrak{U}}{hocolim}\ U_i \simeq X$$

between X and the homotopy colimit of the diagram in spaces determined by $\mathfrak{U}$.

Remark 12 We work with complete covers because they are most convenient for our purposes, but one can replace complete covers with many nice covers to obtain analogous results—in [7] one can find versions of Theorem 11 for Cech complexes and hypercovers, generalizing Segal's earlier work in [26], and more recently, in A.3 of [19] Lurie extends these result to the "generalized Seifert–Van Kampen theorem."

We conclude this section with two useful consequences of Theorem 11 which immediately follow from Examples 9 and 10:

Corollary 13 *For M a based manifold, there is a weak equivalence*

$$\underset{U\hookrightarrow M\in \mathrm{Disk}_{*/M}}{\mathrm{hocolim}}\ U \simeq M.$$

Corollary 14 *For X a topological space, and $\mathfrak{U} = \{U_i\}_{i\in I}$ any basis for the topology on X, there is a weak equivalence*

$$\underset{\mathfrak{U}}{\mathrm{hocolim}}\ U_i \simeq X.$$

3 A new proof of the Dold–Thom theorem

We are now ready to prove the Dold–Thom theorem. Our first step, and the core of our proof, will be to appeal to Theorem 11 in order to prove the following:

Proposition 15 *For M a based manifold, there is a weak equivalence*

$$\mathrm{Sym}(M; A) \simeq \underset{U\hookrightarrow M\in \mathrm{Disk}_{*/M}}{\mathrm{hocolim}}\ \mathrm{Sym}(U; A).$$

Proof of Proposition 15 Observe first that the functor

$$\mathrm{Sym}(-;A) : \mathrm{Top}_* \to \mathrm{Top}_*$$

preserves open embeddings.

To see it this, suppose $g : X \to Y$ is an open embedding. We'll first show $\mathrm{Sym}(g) := \mathrm{Sym}(g; A) : \mathrm{Sym}(X; A) \to \mathrm{Sym}(Y; A)$ is open. Note that for any finite set I, we have the commutative diagram

$$\begin{array}{ccc} X^{I_+} \times A^{I_+} & \xrightarrow{f_{I,X}} & \mathrm{Sym}(X;A) \\ {\scriptstyle g^I \times \mathrm{id}^I}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Sym}(g)} \\ Y^{I_+} \times A^{I_+} & \xrightarrow[f_{I,Y}]{} & \mathrm{Sym}(Y;A) \end{array}$$

where $f_{I,X}$ and $f_{I,Y}$ are the maps topologizing X and Y, respectively. For any open $U \subset \mathrm{Sym}(X; A)$, we want to see that $\mathrm{Sym}(g)(U)$ is open in $\mathrm{Sym}(Y; A)$. We know by definition that $f^{-1}_{I,X}(U)$ is open, and because g is an open embedding, $\big(g^I \times \mathrm{id}^I\big) \circ \big(f^{-1}_{I,X}(U)\big)$ is also open. Now

$$\mathrm{Sym}(g)(U) = \bigcup_{I\text{-finite}} f_{I,Y}\big((g^I \times \mathrm{id}^I) \circ (f^{-1}_{I,X}(U))\big)$$

so that $\mathrm{Sym}(g)(U)$ is open as desired.

Now to see $\mathrm{Sym}(g)$ is an embedding, it suffices to see it is an injection. For this, assume the equivalence class $\big[(g(S), g(l))\big]$ is the same as $\big[(g(T), g(m))\big]$, then we will see it follows that $\big[(S,l)\big]$ is the same equivalence class as $\big[(T, m)\big]$.

From our assumption of the equivalence $\big[(g(S), g(l))\big] = \big[(g(T), g(m))\big]$, we can write without loss of generality the following equalities:

$$\begin{aligned} &(1)\ g(S) = g(T) \cup g(S'),\\ &(2)\ g(l)|_{g(T)} = g(m),\\ &(3)\ g(l)|_{g(S')} \equiv e. \end{aligned}$$

The first line, with the fact that g is an embedding, implies that T is a subset of S, so we have the equality $S = T \amalg (S - T)$. Then we will be done if we show that $l|_T = m$ and that $l|_{S-T} \equiv e$, because then the equalities $\big[(S,l)\big] = \big[(T \amalg (S - T), m \amalg e)\big] = \big[(T, m)\big]$ will immediately follow. Let us first consider $l|_T$, the case of $l|_{S-T}$ will be similar. We know by assumption (2) that $g(l)|_{g(T)} = g(m)$ which implies that that for t an element of T, we have the string of equalities

$$
\begin{aligned}
m(t) &= \sum_{x \in g^{-1}(g(t))} m(x) \\
&= g(m)(g(t)) \\
&= g(l)(g(t)) \\
&= \sum_{x \in g^{-1}(g(t))} l(x) \\
&= l(t)
\end{aligned}
$$

where the first and last line come from the fact that g is an embedding, the second and fourth lines are by definition, and the third line is our assumption (2). This shows $l|_T = m$ as desired. To see the equivalence $l|_{S-T} \equiv e$, note by our assumption (3), as above we have the following equalities for s an element of $S - T$,

$$
\begin{aligned}
e &= g(l)(g(s)) \\
&= \sum_{x \in g^{-1}(g(s))} l(x) \\
&= l(s)
\end{aligned}
$$

which proves the claim, and further concludes the proof that $\mathrm{Sym}(-; A)$ preserves open embeddings.

If we let $\mathrm{Open}(X)$ denote the poset of open subsets of a topological space X ordered by inclusion, the above discussion then allows us to consider the functor

$$
\mathrm{Sym}(-; A) : \mathrm{Disk}_{*/M} \to \mathrm{Open}(\mathrm{Sym}(M; A))
$$

which sends an object $\iota : U \hookrightarrow M$ to $\mathrm{im}(\mathrm{Sym}(\iota; A)) = \mathrm{Sym}(U; A)$ viewed as a subspace of $\mathrm{Sym}(M; A)$. To apply Theorem 11, we now just need to see that the collection $\mathcal{S} = \{\mathrm{Sym}(U; A)\}_{\mathrm{Disk}_{*/M}}$ gives a complete cover of $\mathrm{Sym}(M; A)$ (Figs. 1, 2).

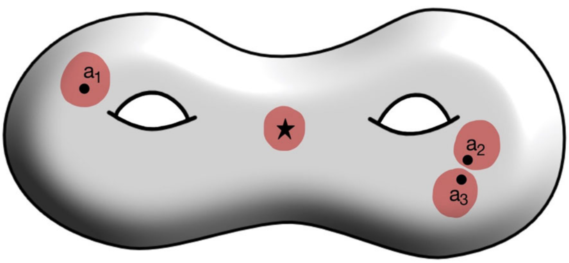

Fig. 1 An illustration that $\mathcal{S}$ covers $\mathrm{Sym}(M; A)$

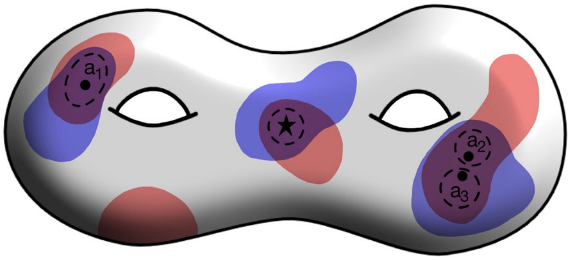

Fig. 2 An illustration that $\mathcal{S}$ is complete

- *To see that* $\mathcal{S}$ *covers* $\mathrm{Sym}(M; A)$
  Because M is a manifold, and in particular is Hausdorff, for any finite labelled configuration $(x, l) \in \mathrm{Sym}(M; A)$ one can find a finite disjoint union of disks in M, each of which contains a unique point of x. Such a disjoint union gives an element of $\mathrm{Disk}_{*/M}$ whose image under $\mathrm{Sym}(-; A)$ contains (x, l), and because (x, l) was arbitrary, this shows that $\mathcal{S}$ gives an open cover of $\mathrm{Sym}(M; A)$.
- *To see that* $\mathcal{S}$ *is complete*
  We must see that any finite intersection $\mathrm{Sym}(U_1; A) \cap \cdots \cap \mathrm{Sym}(U_n; A)$ of elements in $\mathcal{S}$ admits a cover by elements of $\mathcal{S}$. Let's again consider an arbitrary configuration $(x, l) \in \mathrm{Sym}(U_1; A) \cap \cdots \cap \mathrm{Sym}(U_n; A)$. Since each U_i is open in M, the finite intersection $U_1 \cap \cdots \cap U_n$ is also in M, so one can repeat the above argument and find a finite disjoint union of disks $D_x \cong \amalg_1^n \mathbb{R}_i^n$ inside of $U_1 \cap \cdots \cap U_n \subset M$, each of which contains a unique point of x, and whose image under $\mathrm{Sym}(-; A)$ contains (x, l). Doing this for each configuration $x \in \mathrm{Sym}(U_1; A) \cap \cdots \cap \mathrm{Sym}(U_n; A)$ determines a cover of this intersection by elements in $\mathcal{S}$, as desired.

We can now apply Theorem 11 to see there is an equivalence

$$\underset{U \hookrightarrow M \in \mathrm{Disk}_{*/M}}{\mathrm{hocolim}} \mathrm{Sym}(U; A) \xrightarrow{\simeq} \mathrm{Sym}(M; A).$$

□

Remark 16 As described in the introduction, the homotopy colimit above is equivalent to the factorization homology of M with coefficients in the n-disk algebra in spaces determined by the functor $\mathrm{Sym}(-; A)$, motivating the title of this note. To be slightly more precise, one should view M as a zero-pointed manifold, pointed by the basepoint, and use the theory of factorization homology for zero-pointed manifolds, see 2.2 of [2].

Corollary 17 *There are homotopy equivalences of spaces*

$$\begin{aligned}\mathrm{Sym}(M; A) &\simeq \mathrm{hocolim}\left(\mathrm{Disk}_{*/M} \xrightarrow{\mathrm{Sym}(-;A)} \mathrm{Top}\right)\\ &\simeq \mathrm{hocolim}\left(\mathrm{Disk}_{*/M} \xrightarrow{\widetilde{H}_0(-;A)} \mathrm{Top}\right)\end{aligned}$$

where, for $U \hookrightarrow M$ *in* $\mathrm{Disk}_{*/M}$*, we consider* $\widetilde{H}_0(U; A) \cong A^{\pi_0(U)\setminus[*]}$ *as a discretely topologized space.*

Proof For I a finite set, we have seen in Lemma 3 the homeomorphism $\mathrm{Sym}(I_+; A) \cong A^I$. Thus, since for any $U \hookrightarrow M$ in $\mathrm{Disk}_{*/M}$ there is a homotopy equivalence $U \simeq \pi_0(U)$, the corollary follows from the above proposition combined with the homotopy invariance [Remark 4] of $\mathrm{Sym}(-; A)$. □

Now we wish to compute this homotopy colimit in the category of connective chain complexes. Observe that the functor $\widetilde{H}_0(-; A) : \mathrm{Disk}_{*/M} \to \mathrm{Top}_*$ factors through connective chain complexes in the following fashion:

$$\begin{array}{ccc} \mathrm{Disk}_{*/M} & \xrightarrow{\widetilde{H}_0(-;A)} & \mathrm{Top} \\ & {\scriptstyle \widetilde{H}_0(-;A)[0];\ d=0} \searrow \qquad \nearrow {\scriptstyle |U(-)|} & \\ & \mathrm{Ch}_{\geq 0} \underset{\mathrm{DK}}{\simeq} \mathrm{sAb} & \end{array}$$

where $\mathrm{DK} : \mathrm{Ch}_{\geq 0} \to \mathrm{sAb}$ is part of the Dold–Kan correspondence taking chain complexes to simplicial abelian groups, and $|U(-)| : \mathrm{sAb} \to \mathrm{Top}$ is the composition of the forgetful functor U from simplicial abelian groups to simplicial sets and geometric realization $|-|$ from simplicial sets to topological spaces. We will henceforth simply use $\widetilde{H}_0(-; A)$ to denote the functor $\widetilde{H}_0(-; A)[0] : \mathrm{Disk}_{*/M} \to \mathrm{Ch}_{\geq 0}$.

Lemma 18 *The natural map*

$$\left| U \circ \mathrm{DK}\left(\operatorname{hocolim}\left(\mathrm{Disk}_{*/M} \xrightarrow{\widetilde{H}_0(-;A)} \mathrm{Ch}_{\geq 0}\right)\right)\right| \xrightarrow{\simeq} \operatorname{hocolim}\left(\mathrm{Disk}_{*/M} \xrightarrow{\widetilde{H}_0(-;A)} \mathrm{Top}\right)$$

is an equivalence of spaces.

Proof To see this equivalence, it suffices to see that the composition of functors $|U \circ DK(-)|$ commutes with the homotopy colimit on the left. Since both geometric realization and the Dold–Kan functor commute with all homotopy colimits, our task is reduced to seeing that the forgetful functor U commutes with this homotopy colimit. A general fact about forgetful functors is that they commute with homotopy sifted homotopy colimits, so we will attempt to exploit this fact. The category $\mathrm{Disk}_{*/M}$ need not be sifted, however, if one inverts isotopy equivalences $\mathrm{Disk}_{*/M}$, one obtains an ∞-category which we'll denote by $\mathcal{D}\mathrm{isk}_{*/M}$ that *is* homotopy sifted. Analogously, one can invert isotopy equivalences in $\mathrm{Disk}_{/M}$ to obtain a homotopy sifted ∞-category we will denote by $\mathcal{D}\mathrm{isk}_{/M}$. For more rigorous details on the construction of these ∞-categories, see [1] and [3]. These four categories fit in to the following diagram:

$$\begin{array}{ccc} \mathcal{D}\mathrm{isk}_{*/M} & \longrightarrow & \mathcal{D}\mathrm{isk}_{/M} \\ \uparrow & & \uparrow \\ \mathrm{Disk}_{*/M} & \longrightarrow & \mathrm{Disk}_{/M} \end{array}$$

where the top horizontal functor is full and final and the right vertical functor is a localization in the sense of 5.2.7 of [20] and hence final, 2.21 of [3]. In this particular case, that is sufficient to imply the left vertical functor is also a localization and hence final, so there is an equivalence $\text{hocolim}_{\text{Disk}_{*/M}} \widetilde{H}_0(-;A) \simeq \text{hocolim}_{\mathcal{D}\text{isk}_{*/M}} \widetilde{H}_0(-;A)$ which, with the siftedness of $\mathcal{D}\text{isk}_{*/M}$ 3.3.2 of [1], proves the lemma. □

To summarize, we have shown the equivalences

$$\begin{aligned}\pi_i(\text{Sym}(M;A)) &\cong \pi_i\left(\text{hocolim}\left(\text{Disk}_{*/M} \xrightarrow{\widetilde{H}_0(-;A)} \text{Top}\right)\right)\\ &\cong H_i\left(\text{hocolim}\left(\text{Disk}_{*/M} \xrightarrow{\widetilde{H}_0(-;A)} \text{Ch}_{\geq 0}\right)\right)\end{aligned}$$

where the first equivalence follows from Proposition 15 and Corollary 17, and the second from Lemma 18. Now all that remains is to see that this does, in fact, agree with the singular reduced homology of M. This is roughly a special case of the fact that factorization homology recovers ordinary homology; nevertheless, for the reader unfamiliar with the language of factorization homology we will include a proof of this special case.

Lemma 19 *The canonical map of chain complexes*

$$\text{hocolim}\left(\text{Disk}_{*/M} \xrightarrow{\widetilde{H}_0(-;A)} \text{Ch}_{\geq 0}\right) \xrightarrow{\simeq} \widetilde{C}_*(M;A)$$

is a quasi-isomorphism.

Proof From Corollary 14 we know there is an equivalence $M \simeq \text{hocolim}_{U \hookrightarrow M \in \text{Disk}_{*/M}} U$, and thus we have the following quasi-isomorphisms:

$$\begin{aligned}\widetilde{C}_*(M;A) &= \widetilde{C}_*\left(\underset{U \hookrightarrow M \in \text{Disk}_{*/M}}{\text{hocolim}} U;A\right)\\ &\simeq \underset{U \hookrightarrow M \in \text{Disk}_{*/M}}{\text{hocolim}} \widetilde{C}_*(U;A)\\ &\simeq \underset{U \hookrightarrow M \in \text{Disk}_{*/M}}{\text{hocolim}} \widetilde{C}_*(\pi_0(U);A)\end{aligned}$$

where the last equivalence comes from our assumption that U is homotopy equivalent to $\pi_0(U)$.

Now there is an obvious map

$$\underset{U \hookrightarrow M \in \text{Disk}_{*/M}}{\text{hocolim}} \widetilde{H}_0(U;A) \to \underset{U \hookrightarrow M \in \text{Disk}_{*/M}}{\text{hocolim}} \widetilde{C}_*(\pi_0(U);A)$$

which induces an isomorphism on homology, and the lemma follows. □

Stringing the above together, we arrive immediately at the Dold–Thom theorem:

Theorem 20 (Dold–Thom). *For M a smooth pointed manifold and for every integer $i \in \mathbb{Z}$, there is an isomorphism of groups*

$$\pi_i(\mathrm{Sym}(M; A)) \cong \widetilde{H}_i(M; A).$$

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Appendix: The original statement of the Dold–Thom theorem

In its original form [10], the Dold–Thom theorem for smooth manifolds is the following:

Theorem 21 (Dold–Thom). *For M a connected, based manifold and for every integer $i \in \mathbb{Z}$, there is an isomorphism of groups*

$$\pi_i(\mathrm{SP}^\infty(M)) \cong \widetilde{H}_i(M; \mathbb{Z}).$$

Here, $\mathrm{SP}^\infty(M)$ is defined as the colimit of the finite symmetric powers of M; ie.

$$\mathrm{SP}^\infty M := \operatorname*{colim}_i X^{\times i}/\Sigma_i.$$

In this "Appendix" we will show how this statement follows easily from the statement in Theorem 20. To arrive at the classical statement one must first observe that Definition 2 only uses the monoid structure of A and not its full group structure. With this observation, it is clear that $\mathrm{Sym}(M; \mathbb{N})$ is defined and homeomorphic to $\mathrm{SP}^\infty(M)$.

We remark that while in [10] Dold and Thom only proved a statement about infinite symmetric product with coefficients in $\mathbb{N}$ and integral homology, their proof can be imitated nearly verbatim to obtain the analogous results in the case of abelian group coefficients—that is to say, one can simply check directly that the composition of functors $\pi_*(\mathrm{Sym}(-; A))$ satisfies the Eilenberg–Steenrod axioms and agrees with singular homology, and this argument will run through just as it does in the case where $A = \mathbb{N}$. In fact, considering only abelian group coefficients actually simplifies the original proof. In particular, the subtlety in the axiomatic approach to Theorem 21 is in checking the exactness axiom for a homology theory—namely, one needs a homotopy fiber sequence:

$$\mathrm{Sym}(N; \mathbb{N}) \to \mathrm{Sym}(M; \mathbb{N}) \xrightarrow{p} \mathrm{Sym}(M/N; \mathbb{N}),$$

for $N \subset M$ a based subspace and p the map induced by the quotient map $M \to M/N$. The map p is not a fibration when considering coefficients in $\mathbb{N}$, so in [9] Dold and Thom developed the theory of quasifibrations in order obtain this fiber sequence. However, later in [22], McCord showed that when one considers the infinite symmetric product with coefficients in an abelian group A, the map p does become a fibration, immediately giving the desired fiber sequence. It is for this reason we feel justified in referring to both of Theorems 20 and 21 "the Dold–Thom theorem." Let us now derive Theorem 21 from Theorem 20.

We will actually derive a direct generalization of Theorem 21:

Theorem 22 *Let A be a discrete monoid and denote by $GG(A)$ the Grothendieck group completion of A. For M a connected, based manifold and for every integer $i \in \mathbb{Z}$, there is an isomorphism of groups*

$$\pi_i(\mathrm{Sym}(M; A)) \cong \widetilde{H}_i(M; GG(A)).$$

We will need the following classical result:

Proposition 23 *For X a grouplike topological monoid, there is a weak equivalence*

$$X \simeq \Omega \mathrm{B} X.$$

Proof One can construct a quasifibration $\mathrm{E}X \to \mathrm{B}X$ whose total space $\mathrm{E}X$ is contractible and which has geometric fiber X and homotopy fiber $\Omega\mathrm{B}X$, for details see, for example, Proposition D.2 of [16], Theorem 7.6 of [21], or [23]. □

In particular, because $\mathrm{Sym}(M; -)$ is connected whenever M is connected, this gives the following corollary:

Corollary 24 *For M a connected topological space and A a discrete monoid, there is a weak equivalence*

$$\mathrm{Sym}(M; A) \simeq \Omega\mathrm{B}\,\mathrm{Sym}(M; A).$$

In light of this corollary, it suffices to see that there is an equivalence $\Omega\mathrm{B}\,\mathrm{Sym}(M; A) \simeq \mathrm{Sym}(M; GG(A))$, as then we would have the equivalences:

$$\begin{aligned}\pi_i(\mathrm{Sym}(M; A)) &\cong \pi_i(\mathrm{Sym}(M; GG(A)) \\ &\cong \widetilde{H}_i(M; GG(A)),\end{aligned}$$

recovering Theorem 22, and the classical statement of the Dold–Thom theorem (21) as special case when $A = \mathbb{N}$ and $GG(A) = \mathbb{Z}$.

Proposition 25 *For M a based n-manifold, there is a weak equivalence*

$$\Omega\mathrm{B}\,\mathrm{Sym}(M; A) \simeq \mathrm{Sym}(M; GG(A)).$$

Proof Using complete covers precisely as in the proof of Proposition 15, we see there is a weak equivalence $\mathrm{Sym}(M; A) \simeq \mathrm{hocolim}_{\mathrm{Disk}_{*/M}} \mathrm{Sym}(-, A)$. Upon taking topological group completions, this induces an equivalence $\Omega\mathrm{B}\,\mathrm{Sym}(M; A) \simeq \Omega\mathrm{B}\ \mathrm{hocolim}_{\mathrm{Disk}_{*/M}} \mathrm{Sym}(-, A)$. We would like to commute group completion with this homotopy colimit. Because $\Omega\mathrm{B}$ is the total left derived functor of group completion [8], it preserves all homotopy colimits of A_∞ spaces. It thus suffices to see that ours is a homotopy colimit of A_∞ spaces. Any homotopy sifted homotopy colimit of spaces in which all the spaces and maps are A_∞ is itself a homotopy colimit in A_∞ spaces, so analogously to the argument in Lemma 18 we will appeal to the fact that $\mathrm{hocolim}_{\mathrm{Disk}_{*/M}} \mathrm{Sym}(-, A)$ is equivalent to the homotopy sifted homotopy colimit $\mathrm{hocolim}_{\mathcal{D}\mathrm{isk}_{*/M}} \mathrm{Sym}(-, A)$ to arrive at the equivalence:

$$\Omega\mathrm{B}\,\mathrm{Sym}(M; A) \simeq \underset{\mathrm{Disk}_{*/M}}{\mathrm{hocolim}}\, \Omega\mathrm{B}\,\mathrm{Sym}(-; A).$$

Using homotopy invariance, $\mathrm{Sym}(U; A)$ is equivalent to $\mathrm{Sym}(\pi_0(U); A)$ for U a finite disjount union of disks embedded in M. Then just as in Lemma 3, one can easily see $\mathrm{Sym}(\pi_0(U); A) \cong A^{\pi_0(U)\backslash[*]}$ where $\pi_0(U)\backslash\,[*]$ denotes the fundamental group of U without the homotopy class of the basepoint. This leads us to the equivalences:

$$\begin{aligned}\Omega\,\mathrm{B}\,\mathrm{Sym}(M; A) &\simeq \underset{U \hookrightarrow M \in \mathrm{Disk}_{*/M}}{\mathrm{hocolim}} \Omega\mathrm{B}(A^{\pi_0(U)\backslash[*]}) \\ &\simeq \underset{U \hookrightarrow M \in \mathrm{Disk}_{*/M}}{\mathrm{hocolim}} GG(A)^{\pi_0(U)\backslash[*]} \\ &\simeq \mathrm{Sym}(M; GG(A)) \text{ by Proposition 15.}\end{aligned}$$

□

Remark 26 While Proposition 23 assumes X is connected, we note that this condition is unnecessary for Proposition 25. On the other hand, our proof of Proposition 25 relies on M being a manifold, whereas Proposition 23 holds for any connected topological monoid.

The Dold–Thom theorem as stated in [10] and Theorems 21 and 22 now follow immediately for any smooth manifold. As remarked in Sect. 1.2, one can again simply appeal to homotopy invariance of $\mathrm{SP}^\infty(-)$ combined with approximation of finite CW complexes by smooth manifolds to arrive at a more general theorem for finite CW complexes, and it is the author's hope to generalize these methods further still in future work.

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