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On the topological computation of K_4 of the Gaussian and Eisenstein integers

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Abstract
In this paper we use topological tools to investigate the structure of the algebraic K-groups $K_4(R)$ for $R = Z[i]$ and $R = Z[\rho]$ where $i := \sqrt{-1}$ and $\rho := (1+\sqrt{-3})/2$. We exploit the close connection between homology groups of $\mathrm{GL}_n(R)$ for $n \leq 5$ and those of related classifying spaces, then compute the former using Voronoi's reduction theory of positive definite quadratic and Hermitian forms to produce a very large finite cell complex on which $\mathrm{GL}_n(R)$ acts. Our main result is that $K_4(\mathbb{Z}[i])$ and $K_4(\mathbb{Z}[\rho])$ have no p-torsion for $p \geq 5$.

Keywords Cohomology of arithmetic groups · Voronoi reduction theory · Linear groups over imaginary quadratic fields · K-theory of number rings

Mathematics Subject Classification Primary 19D50; Secondary 11F75

1 Introduction

1.1 Statement of results

Let R be the ring of integers of a number field F. Only very few cases are known where the algebraic K-group $K_4(R)$ has been explicitly computed, the first such $K_4(\mathbb{Z})$ having been determined as recently as 2000 by Rognes [17], building on work

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of Soulé [18]. The goal of this paper is the explicit topological computation of the torsion (away from 2 and 3) in the groups $K_4(R)$ for R one of two special imaginary quadratic examples: the *Gaussian integers* $\mathbb{Z}[i]$ and the *Eisenstein integers* $\mathbb{Z}[\rho]$, where $i := \sqrt{-1}$ and $\rho := (1+\sqrt{-3})/2$. Our work is in the spirit of Lee–Szczarba [12–14], Soulé [19], and Elbaz-Vincent–Gangl–Soulé [7,8] who treated $K_N(\mathbb{Z})$ for small N, and Staffeldt [20] who investigated $K_3(\mathbb{Z}[i])$. As in these works, the first step is to compute the cohomology of $\mathrm{GL}_n(R)$ for $n \leq N+1$; information from this computation is then assembled into information about the K-groups following the program in Sect. 1.2. Using these computations we show the following (Theorem 4.1):

Theorem. *The orders of the groups* $K_4\big(\mathbb{Z}[i]\big)$ *and* $K_4\big(\mathbb{Z}[\rho]\big)$ *are not divisible by any primes* $p \geq 5$.

We remark that this result is not new; in fact, Kolster's work [11] implies the stronger result that $K_4\big(\mathbb{Z}[i]\big)$ and $K_4\big(\mathbb{Z}[\rho]\big)$ vanish. Indeed, if R is the ring of integers of a CM field, then Kolster proved that, assuming the Quillen–Lichtenbaum conjecture, the orders of the groups $K_{4n}(R)$, $n = 1, 2, 3, \ldots$, can be computed in terms of special values of certain L-functions. This deep connection between K-groups and special values of L-functions is now a theorem, thanks to the celebrated work by Voevodsky [21] and Rost, as put into context in [9].

Our work, on the other hand, treats $K_4\big(\mathbb{Z}[i]\big)$ and $K_4\big(\mathbb{Z}[\rho]\big)$ by completely different methods. We only use the definition of the K-groups and explicit results about the cohomology of the relevant arithmetic groups [6], together with Arlettaz's bounds on the kernel of the Hurewicz homomorphism [1], to prove Theorem 4.1. This also explains why our calculations do not allow us to say anything for the primes 2 and 3: both the results of [6] and the injectivity of the Hurewicz map in our cases only hold away from these primes.

1.2 Outline of method

In the rest of this introduction we outline the main steps of our argument. These follow the classical approach for computing algebraic K-groups of number rings due to Quillen [15], which shifts the focus to computing the homology (with nontrivial coefficients) of certain arithmetic groups.

(i) (*Definition*) By definition the algebraic K-group $K_N(R)$ of a ring R is a particular homotopy group of a topological space associated to R: we have $K_N(R) = \pi_{N+1}(BQ(R))$, where $BQ(R)$ is a certain classifying space attached to the infinite general linear group $\mathrm{GL}(R)$. In particular $BQ(R)$ is the classifying space of the category $Q(R)$ of finitely generated R-modules. This is known as Quillen's *Q-construction* of algebraic K-theory [16].

(ii) (*Homotopy to homology*) The Hurewicz homomorphism $\pi_{N+1}(BQ(R)) \to H_{N+1}(BQ(R))$ allows one to replace the homotopy group by a homology group without losing too much information; more precisely, what may get lost is information about small torsion primes appearing in its finite kernel.

(iii) (*Stability*) By a stability result of Quillen [15, p. 198] one can pass from $Q(R)$ to the category $Q_{M+1}(R)$ of finitely generated R-modules of rank $\leq M+1$ for sufficiently large M. This amounts to passing from $\mathrm{GL}(R)$ to the finitedimensional general linear group $\mathrm{GL}_{M+1}(R)$. In the cases at hand, a result of Lee and Szczarba allows to reduce to the case $M = N$.

(iv) (*Sandwiching*) The homology groups to be determined are then $H_*(BQ_n(R))$ for $n \leq N + 1$. Rather than computing these directly, one uses the fact that they can be sandwiched between homology groups of $\mathrm{GL}_n(R)$, where the homology is taken with (nontrivial) coefficients in the Steinberg module St_n associated to $\mathrm{GL}_n(R)$.

(v) (*Equivariant homology*) It has been shown for certain number rings R that the homology groups $H_m(\mathrm{GL}_n(R), St_n)$ are isomorphic to the equivariant $\mathrm{GL}_n(R)$-homology of an associated pair (denoted $(X_n^*, \partial X_n^*)$ in Sect. 1.3 below). The standard method to compute the latter uses *Voronoi complexes*. These are relative chain complexes of certain explicit polyhedral reduction domains of a space of positive definite quadratic or Hermitian forms of a given rank, depending respectively on whether $R = \mathbb{Z}$ or R is imaginary quadratic.

(vi) (*Vanishing results*) There are various techniques to show vanishing of homology groups. As a starting point one has vanishing results for $H_n(BQ_1)$ as in Theorem 3.1 below, and for $H_0(GL_n, St_n)$ as in Lee–Szczarba [13], Cor. to Thm 4.1.

For a given N, using (ii) and knowing the results of (iv)–(vi) for all $0 \leq n \leq N + 1$ is often enough to give an upper bound B on the primes p dividing the order of the torsion subgroup $K_{N,\mathrm{tors}}(R)$ of $K_N(R)$.

1.3 Outline of paper

In this paper the sections work backwards through the method outlined in Sect. 1.2 to determine the structure of $K_4(\mathbb{Z}[i])$ and $K_4(\mathbb{Z}[\rho])$. In Sect. 2, we describe the computation of the equivariant homology in question and relate it to the Steinberg homology. In Sect. 3 we use the results on Steinberg homology and some vanishing results to determine the groups $H_m(BQ_n(R))$ (i.e., step (iv) above). A key role here is played by Quillen's stability result (iii) for BQ_n, as refined by Lee–Szczarba in [13], which serves as a stopping criterion. Finally, in Sect. 4 we work out the potential primes entering the kernel of the Hurewicz homomorphism (i.e., step (ii) above), which gives Theorem 4.1.

2 Homology of Voronoi complexes

We first collect the results from [6] concerning the Voronoi complexes attached to $\Gamma = \mathrm{GL}_n(\mathbb{Z}[i])$ or $\Gamma = \mathrm{GL}_n(\mathbb{Z}[\rho])$; this is the necessary information needed for step (v) from Sect. 1.2 above. More details about these computations, including background about how the computations are performed, can be found in [6].

Let F be an imaginary quadratic field with ring of integers R, and let $X_n := \mathrm{GL}_n(\mathbb{C})/\mathrm{U}(n)$ be the symmetric space of $\mathrm{GL}_n(F \otimes_{\mathbb{Q}} \mathbb{R})$. The space X_n can be realized as the quotient of the cone of rank n positive definite Hermitian matrices C_n modulo homotheties (i.e. non-zero scalar multiplication), and a partial Satake compactification

X_n^* of X_n is given by adjoining boundary components to X_n given by the cones of positive semi-definite Hermitian forms with an F-rational nullspace (again taken up to homotheties). We let $\partial X_n^* := X_n^* \smallsetminus X_n$ denote the *boundary* of X_n^*. Then $\Gamma := \mathrm{GL}_n(R)$ acts by left multiplication on both X_n and X_n^*, and the quotient $\Gamma \backslash X_n^*$ is a compact Hausdorff space.

A generalization—due to Ash [2, Chapter II] and Koecher [10]—of the polyhedral reduction theory of Voronoi [22] yields a Γ-equivariant explicit decomposition of X_n^* into (Voronoi) cells. Moreover, there are only finitely many cells modulo Γ and we have the following result.

Proposition 2.1 [6, Proposition 3.6] *For* $\Gamma \in \{\mathrm{GL}_n(\mathbb{Z}[i]), \mathrm{GL}_n(\mathbb{Z}[\rho])\}$ *and* $m \in \mathbb{Z}$ *we have* $H_m^\Gamma(X_n^*, \partial X_n^*, \mathbb{Z}) \simeq H_{m-n+1}(\Gamma, \mathit{St}_n)$.

Let $\Sigma_d^* := \Sigma_d(\Gamma)^*$ be a set of representatives of the Γ-inequivalent d-dimensional Voronoi cells that meet the interior X_n, and let $\Sigma_d := \Sigma_d(\Gamma)$ be the subset of representatives of the Γ-inequivalent *orientable* cells in this dimension; here we call a cell *orientable* if all the elements in its stabilizer group preserve its orientation. Note that in our consideration the prime 2 will always be inverted. This entails that only orientable cells can contribute to the homology. One can form a chain complex Vor_*, the *Voronoi complex*, and one can prove that modulo small primes the homology of this complex is the homology $H_*(\Gamma, \mathit{St}_n)$, where St_n is the rank *n Steinberg module* (cf. [4, p. 437]). To keep track of these small primes explicitly, we make the following definition.

Definition 2.2 (*Serre class of small prime power groups*) Given $k \in \mathbb{N}$, we let $\boldsymbol{\mathcal{S}}_{p \le k}$ denote the **Serre class** of finite abelian groups G whose cardinality $|G|$ has all of its prime divisors p satisfying $p \le k$.

For any finitely generated abelian group G, there is a unique maximal subgroup $G_{p \le k}$ of G in the Serre class $\boldsymbol{\mathcal{S}}_{p \le k}$. We say that two finitely generated abelian groups G and G' are **equivalent modulo** $\boldsymbol{\mathcal{S}}_{p \le k}$ and write $G \simeq_{/p \le k} G'$ if the quotients $G/G_{p \le k} \cong G'/G'_{p \le k}$ are isomorphic.

We call the **torsion primes** of a group G those prime numbers p which divide the order of at least one of the finite subgroups of G.

2.1 Voronoi data for $R = \mathbb{Z}[i]$

We now give results for the Voronoi complexes and the equivariant homology of the pairs $(X_n^*, \partial X_n^*)$ in the cases relevant to our paper ($n = 2, 3, 4$). This subsection treats the Gaussian integers; in Sect. 2.2 we treat the Eisenstein integers.

Proposition 2.3 [20]

1. *There is one d-dimensional Voronoi cell for* $\mathrm{GL}_2(\mathbb{Z}[i])$ *for each* $1 \le d \le 3$, *and only the 3-dimensional cell is orientable.*
2. *The number of d-dimensional Voronoi cells for* $\mathrm{GL}_3(\mathbb{Z}[i])$ *is given by*:

d	2	3	4	5	6	7	8
$\lvert\Sigma_d(\mathrm{GL}_3(\mathbb{Z}[i]))^*\rvert$	2	3	4	5	3	1	1
$\lvert\Sigma_d(\mathrm{GL}_3(\mathbb{Z}[i]))\rvert$	0	0	1	4	3	0	1

Proposition 2.4 [6, Table 12] *The number of d-dimensional Voronoi cells for* $\mathrm{GL}_4(\mathbb{Z}[i])$ *is given by*:

d	3	4	5	6	7	8	9	10	11	12	13	14	15
$\lvert\Sigma_d(\mathrm{GL}_4(\mathbb{Z}[i]))^*\rvert$	4	10	33	98	258	501	704	628	369	130	31	7	2
$\lvert\Sigma_d(\mathrm{GL}_4(\mathbb{Z}[i]))\rvert$	0	0	5	48	189	435	639	597	346	120	22	2	2

We remark that for $\mathrm{GL}_3(\mathbb{Z}[i])$ the Voronoi complexes and their homology ranks were originally computed by Staffeldt [20], who even distilled the 3-part for each homology group. After calculating the differentials for this complex one obtains the following homology groups, in agreement with Staffeldt's results:

Proposition 2.5 [20, Theorems IV, 1.3 and 1.4, p.785]

$$H_m(\mathrm{GL}_2(\mathbb{Z}[i]), \mathit{St}_2) \simeq_{/p\leq 3} \begin{cases} \mathbb{Z} & \textit{if } m = 2, \\ 0 & \textit{otherwise}, \end{cases} \tag{1}$$

$$H_m(\mathrm{GL}_3(\mathbb{Z}[i]), \mathit{St}_3) \simeq_{/p\leq 3} \begin{cases} \mathbb{Z} & \textit{if } m = 2, 3, 6, \\ 0 & \textit{otherwise}. \end{cases} \tag{2}$$

In particular, from the above theorem we deduce that the only possible torsion primes for $H_m(\mathrm{GL}_n(\mathbb{Z}[i]), \mathit{St}_n)$ for $n = 2, 3$ are the primes 2 and 3.

While the Voronoi homology of $\mathrm{GL}_4(\mathbb{Z}[i])$ has been determined in all degrees in [6, Theorem 7.2], we will only need the following two special cases.

Proposition 2.6 [6, Theorem 7.2] *For $m = 1, 2$ we have*

$$H_m(\mathrm{GL}_4(\mathbb{Z}[i]), \mathit{St}_4) \simeq_{/p\leq 5} \{0\}. \tag{3}$$

The last column of [6, Table 12] further shows that the elementary divisors of all the differentials in the Voronoi complex for $\mathrm{GL}_4(\mathbb{Z}[i])$ in small degree (in fact for degree ≤ 13) are supported on primes ≤ 3.

We want to show the stronger result that $H_1(\mathrm{GL}_4(\mathbb{Z}[i]), \mathit{St}_4) \simeq_{/p\leq 3} \{0\}$, i.e. we want to show that the prime 5 cannot occur. For this we will need to use spectral sequences. According[1] to [5, VII.7], there is a spectral sequence $E^r_{d,q}$ converging to

[1] More precisely [5, VII.7] constructs a spectral sequence converging to the equivariant homology $H^G_*(X, M)$ of a G-complex X with coefficients in a G-module M; the E^1 page has a form similar to (4). One can formulate an analogous spectral sequence for the equivariant homology of a pair (X, Y) of G-complexes with E^1 page (4), cf. the remarks in [5, VII.7] in the paragraphs preceding equation (7.1).

the equivariant homology groups $H^{\Gamma}_{d+q}(X^*_n, \partial X^*_n; \mathbb{Z})$ of the homology pair $(X^*_n, \partial X^*_n)$, and such that

$$E^1_{d,q} = \bigoplus_{\sigma \in \Sigma^*_d} H_q(\Gamma_\sigma, \mathbb{Z}_\sigma), \tag{4}$$

where $\mathbb{Z}_\sigma$ is the orientation module of the cell σ and Γ_σ the stabilizer of the cell σ. In the remainder of this section we put $n = 4$ and consider $(X^*_4, \partial X^*_4)$.

Proposition 2.7 *Let* $\Gamma = \mathrm{GL}_4(\mathbb{Z}[i])$ *and* $E^1_{d,q}$ *as above.*

(i) *For each* $d = 0, \dots, 4$ *one has* $E^1_{d,4-d} \simeq_{/p\leq 3} \{0\}$.
(ii) *Similarly, for each* $d = 0, \dots, 5$ *one has* $E^1_{d,5-d} \simeq_{/p\leq 3} \{0\}$.

Proof We use the data obtained in [6, Table 12], available at [24].

(i) 1. As there are no cells in Σ^*_d for $d \leq 2$, we have $E^1_{0,4} = E^1_{1,3} = E^1_{2,2} = 0$.

2. Consider now $d = 3$. The stabilizer of each of the four cells in Σ^*_3 lies in $\boldsymbol{\mathcal{S}}_{p\leq 3}$. Thus in particular we have

$$E^1_{3,1} = \bigoplus_{\sigma \in \Sigma^*_3} H_1(\mathrm{Stab}_\sigma, \mathbb{Z}_\sigma) \in \boldsymbol{\mathcal{S}}_{p\leq 3},$$

where $\boldsymbol{\mathcal{S}}_{p\leq 3}$ is as in Definition 2.2.

3. For $d = 4$, we note that none of the ten cells in Σ^*_4 has its orientation preserved under the action of its stabilizer, so $E^1_{4,0} = 0 \bmod \boldsymbol{\mathcal{S}}_{p\leq 2}$.

(ii) 1. As there are no cells in Σ^*_d for $d \leq 2$, we have $E^1_{0,5} = E^1_{1,4} = E^1_{2,3} = 0$.

2. Consider now $d = 3$ and $d = 5$. The stabilizer of each of the four cells in Σ^*_3 and each of the 33 cells in Σ^*_5 lies in $\boldsymbol{\mathcal{S}}_{p\leq 3}$. Thus in particular we have

$$E^1_{3,2} \in \boldsymbol{\mathcal{S}}_{p\leq 3}, \quad E^1_{5,0} \in \boldsymbol{\mathcal{S}}_{p\leq 3}.$$

3. Finally, for $d = 4$, there is only one cell (out of ten) in Σ^*_4, denoted by σ^1_4, that contains a subgroup of order 5. We must therefore show that there is no 5-torsion in the group $H_1(\mathrm{Stab}(\sigma^1_4), \tilde{\mathbb{Z}})$ (where $\tilde{\mathbb{Z}}$ is the orientation module $\mathbb{Z}_{\sigma^1_4}$). Indeed, the subgroup K_1 of $\mathrm{Stab}(\sigma^1_4)$ preserving the orientation of σ^1_4 is isomorphic to $\mathbb{Z}/4\mathbb{Z} \times A_5$, where A_5 is the alternating group on five letters, with abelianization $H_1(\mathrm{Stab}(\sigma^1_4), \tilde{\mathbb{Z}}) \simeq H_1(K_1, \mathbb{Z}) \simeq \mathbb{Z}/4\mathbb{Z}$ (for the first equality, which holds mod $\boldsymbol{\mathcal{S}}_{p\leq 2}$, we make use of Lemmas 8.2 and 8.3 in [8]) lies in $\boldsymbol{\mathcal{S}}_{p\leq 3}$. Thus there can be no 5-torsion from here, which completes the proof. □

Corollary 2.8 *For* $\Gamma = \mathrm{GL}_4(\mathbb{Z}[i])$ *one has* $H_1(\Gamma, St_4) \simeq H^{\Gamma}_4(X^*_4, \partial X^*_4, \mathbb{Z}) \simeq_{/p\leq 3} \{0\}$ *and* $H_2(\Gamma, St_4) \simeq H^{\Gamma}_5(X^*_4, \partial X^*_4, \mathbb{Z}) \simeq_{/p\leq 3} \{0\}$.

2.2 Voronoi homology data for $R = \mathbb{Z}[\rho]$

Now we turn to the Eisenstein case.

Proposition 2.9 [6, Tables 1 and 11]

1. *There is one* d*-dimensional Voronoi cell for* $\mathrm{GL}_2(\mathbb{Z}[\rho])$ *for each* $1 \le d \le 3$, *and only the* 3*-dimensional cell is orientable.*
2. *The number of* d*-dimensional Voronoi cells for* $\mathrm{GL}_3(\mathbb{Z}[\rho])$ *is given by*:

d	2	3	4	5	6	7	8
$\lvert\Sigma_d(\mathrm{GL}_3(\mathbb{Z}[\rho]))^*\rvert$	1	2	3	4	3	2	2
$\lvert\Sigma_d(\mathrm{GL}_3(\mathbb{Z}[\rho]))\rvert$	0	0	1	2	1	1	2

3. *The number of* d*-dimensional Voronoi cells for* $\mathrm{GL}_4(\mathbb{Z}[\rho])$ *is given by*:

d	3	4	5	6	7	8	9	10	11	12	13	14	15
$\lvert\Sigma_d(\mathrm{GL}_4(\mathbb{Z}[\rho]))^*\rvert$	2	5	12	34	82	166	277	324	259	142	48	15	5
$\lvert\Sigma_d(\mathrm{GL}_4(\mathbb{Z}[\rho]))\rvert$	0	0	0	8	50	129	228	286	237	122	36	10	5

After calculating the differentials we find the same results as for the homology of $\mathbb{Z}[i]$ above:

Proposition 2.10 [6, Theorems 7.1 and 7.2 with Propositions 3.2 and 3.6]

$$H_m(\mathrm{GL}_2(\mathbb{Z}[\rho]), St_2) \simeq_{/p\le 3} \begin{cases} \mathbb{Z} & \text{if } m = 2, \\ 0 & \text{otherwise}, \end{cases} \tag{5}$$

$$H_m(\mathrm{GL}_3(\mathbb{Z}[\rho]), St_3) \simeq_{/p\le 3} \begin{cases} \mathbb{Z} & \text{if } m = 2, 3, 6, \\ 0 & \text{otherwise}, \end{cases} \tag{6}$$

For $m = 1, 2$ *we have*

$$H_m(\mathrm{GL}_4(\mathbb{Z}[\rho]), St_4) \simeq_{/p\le 5} \{0\}. \tag{7}$$

As with $\mathbb{Z}[i]$, a more refined analysis of the $\Gamma = GL_4(\mathbb{Z}[\rho])$ case shows that $H_m^{\Gamma}(X_4^*, \partial X_4^*, \mathbb{Z})$ contains no 5-torsion for $m = 4, 5$:

Proposition 2.11 *Let* $\Gamma = \mathrm{GL}_4(\mathbb{Z}[\rho])$ *and* $E^1_{d,q}$ *as above.*

(i) *For each* $d = 0, \ldots, 4$ *one has* $E^1_{d,4-d} \simeq_{/p\le 3} \{0\}$.
(ii) *Similarly, for each* $d = 0, \ldots, 5$ *one has* $E^1_{d,5-d} \simeq_{/p\le 3} \{0\}$.

Proof The argument is very similar to that of the proof of Proposition 2.7. We use the data obtained in [6, Table 11], available at [24].

(i) 1. As there are no cells in Σ_d^* for $d \le 2$, we have $E^1_{0,4} = E^1_{1,3} = E^1_{2,2} = 0$.
2. For $d = 3$, there are two cells in Σ_3^*, with stabilizer in $\boldsymbol{\mathcal{S}}_{p\le 3}$, and hence

$$E^1_{3,1} = \bigoplus_{\sigma \in \Sigma_3^*} H_1(\mathrm{Stab}(\sigma), \mathbb{Z}_\sigma) \in \boldsymbol{\mathcal{S}}_{p\le 3}.$$

3. For $d = 4$, we note that none of the five cells in Σ_4^* has its orientation preserved under the action of its stabilizer, so $E^1_{4,0} = 0 \bmod \boldsymbol{\mathcal{S}}_{p\leq 2}$.

(ii) 1. As there are no cells in Σ_d^* for $d \leq 2$, we have $E^1_{0,5} = E^1_{1,4} = E^1_{2,3} = 0$.

2. Consider now $d = 3$ and $d = 5$. The stabilizer of each of the two cells in Σ_3^* and each of the 12 cells in Σ_5^* lies in $\boldsymbol{\mathcal{S}}_{p\leq 3}$. Thus in particular we have

$$E^1_{3,2} \in \boldsymbol{\mathcal{S}}_{p\leq 3}, \qquad E^1_{5,0} \in \boldsymbol{\mathcal{S}}_{p\leq 3}.$$

3. Finally, for $d = 4$, there is only one cell (out of five) in Σ_4^*, denoted by σ_4^1, that contains a subgroup of order 5. We must therefore show that there is no 5-torsion in the group $H_1(\mathrm{Stab}(\sigma_4^1), \tilde{\mathbb{Z}})$ (where $\tilde{\mathbb{Z}}$ is the orientation module $\mathbb{Z}_{\sigma_4^1}$). Indeed, the subgroup K_1 of $\mathrm{Stab}(\sigma_4^1)$ preserving the orientation of σ_4^1 is isomorphic to $\mathbb{Z}/6\mathbb{Z} \times A_5$, where A_5 is the alternating group on five letters, with abelianization $H_1(\mathrm{Stab}(\sigma_4^1), \tilde{\mathbb{Z}}) = H_1(K_1, \mathbb{Z}) \simeq \mathbb{Z}/6\mathbb{Z}$, which lies in $\boldsymbol{\mathcal{S}}_{p\leq 3}$. Thus there can be no 5-torsion from here, which completes the proof. $\square$

Corollary 2.12 *For* $\Gamma = \mathrm{GL}_4(\mathbb{Z}[\rho])$ *one has* $H_1(\Gamma, \mathit{St}_4) \simeq H_4^\Gamma(X_4^*, \partial X_4^*, \mathbb{Z}) \simeq_{/p\leq 3} \{0\}$ *and* $H_2(\Gamma, \mathit{St}_4) \simeq H_5^\Gamma(X_4^*, \partial X_4^*, \mathbb{Z}) \simeq_{/p\leq 3} \{0\}$.

3 Vanishing and sandwiching

In this section, we carry out the sandwiching argument (step (iv) of Sect. 1.2). As a first step we invoke a vanishing result for homology groups for BQ_1 due to Quillen [15, p. 212]. In our cases this result boils down to the following statement:

Proposition 3.1 *For the rings* $R = \mathbb{Z}[i]$ *and* $\mathbb{Z}[\rho]$, *we have*

$$H_n(BQ_1(R)) = 0 \quad \textit{whenever } n \geqslant 3.$$

For $R = \mathbb{Z}[i]$ a slightly stronger result is proved in [20, Lemma I.1.2]. However, we will not need this stronger result for $\mathbb{Z}[i]$, or its analogue for $\mathbb{Z}[\rho]$.

Using our homology data from Sect. 2 and Proposition 3.1, we can get for both rings $R = \mathbb{Z}[i]$ and $R = \mathbb{Z}[\rho]$ the following result:

Proposition 3.2 $H_5\big(BQ(R)\big) \simeq_{/p\leq 3} \mathbb{Z}$.

Proof For brevity we will drop R from the notation, as the argument is the same for both cases. We will successively determine $H_5(BQ_j)$ for $j = 1, \ldots, 5$ and then identify the last group via stability with $H_5(BQ)$. For this, we will combine results from Sect. 2 with Quillen's long exact sequence for different j, given by

$$\cdots \longrightarrow H_n(BQ_{j-1}) \longrightarrow H_n(BQ_j) \longrightarrow H_{n-j}(\mathrm{GL}_j, \mathit{St}_j) \longrightarrow H_{n-1}(BQ_{j-1}) \longrightarrow \cdots. \tag{8}$$

The case $j = 1$. By Proposition 3.1 we have $H_n(BQ_1) = 0$ for $n \geq 3$.

The case $j=2$. From the above sequence (8) for $j=2$, we get

$$\underbrace{H_5(BQ_1)}_{=0} \longrightarrow H_5(BQ_2) \longrightarrow H_3(\mathrm{GL}_2, St_2) \longrightarrow \underbrace{H_4(BQ_1)}_{=0},$$

whence $H_5(BQ_2)=0 \mod \boldsymbol{\mathcal{S}}_{p\leq 3}$ by (1) and (5).

The case $j=3$. Now we invoke another result of Staffeldt, who showed (see [20, proof of Theorem I.1.1] that

$$H_4(BQ_2) = H_4(BQ_3) = \mathbb{Z} \mod \boldsymbol{\mathcal{S}}_{p\leq 3}. \tag{9}$$

From (8) for $j=3$ we get the exact sequence, working mod $\boldsymbol{\mathcal{S}}_{p\leq 3}$,

$$H_5(BQ_2) \longrightarrow H_5(BQ_3) \longrightarrow \underbrace{H_2(\mathrm{GL}_3, St_3)}_{=\mathbb{Z}\text{ (by (2), (6))}} \longrightarrow \underbrace{H_4(BQ_2)}_{=\mathbb{Z}\text{ (by (9))}} \longrightarrow \underbrace{H_4(BQ_3)}_{=\mathbb{Z}\text{ (by (9))}} \longrightarrow \underbrace{H_1(\mathrm{GL}_3, St_3)}_{=0\text{ (by (2), (6))}}.$$

Since the leftmost group $H_5(BQ_2)$ vanishes modulo $\boldsymbol{\mathcal{S}}_{p\leq 3}$ by the case $j=2$, this sequence implies that $H_5(BQ_3)=\mathbb{Z} \mod \boldsymbol{\mathcal{S}}_{p\leq 3}$.

The case $j=4$. Moreover, since $H_2(\mathrm{GL}_4, St_4) = H_1(\mathrm{GL}_4, St_4) = 0 \mod \boldsymbol{\mathcal{S}}_{p\leq 3}$ by Propositions 2.6, 2.7 and 2.11, the sequence (8) for $j=4$ gives in a similar way that

$$H_5(BQ_4) = H_5(BQ_3) = \mathbb{Z} \mod \boldsymbol{\mathcal{S}}_{p\leq 3}. \tag{10}$$

The case $j=5$. This is the most complicated of all the cases to handle. Note that BQ is an H-space which implies that $H_*(BQ)\otimes\mathbb{Q}$ is the enveloping algebra of $\pi_*(BQ)\otimes\mathbb{Q}$. It is well-known that $K_0(\mathbb{Z}[i])=\mathbb{Z}$, $K_1(\mathbb{Z}[i])=\mathbb{Z}/2$ and $K_2(\mathbb{Z}[i])=0$ [3, Appendix] as well as $K_3(\mathbb{Z}[i])=\mathbb{Z}\oplus\mathbb{Z}/24$ (given by Merkurjev–Suslin, cf. e.g. Weibel [23], Theorem 73 in combination with Example 28), so modulo $\boldsymbol{\mathcal{S}}_{p\leq 3}$ we have

$$\pi_1(BQ)\otimes\mathbb{Q} = K_0(\mathbb{Z}[i])\otimes\mathbb{Q} = \mathbb{Q},$$

as well as $\pi_2(BQ)\otimes\mathbb{Q} = \pi_3(BQ)\otimes\mathbb{Q} = 0$, and

$$\pi_4(BQ)\otimes\mathbb{Q} = K_3(\mathbb{Z}[i])\otimes\mathbb{Q} = \mathbb{Q}.$$

A very similar argument works for $\mathbb{Z}[\rho]$.

Hence $H_5(BQ)\otimes\mathbb{Q}$ contains the product of $\pi_1(BQ)\otimes\mathbb{Q}$ by $\pi_4(BQ)\otimes\mathbb{Q}$ and so its dimension is at least 1.

The stability result foreshadowed in step (iii) of Sect. 1.2 (resulting for a Euclidean domain Λ from $H_0(\mathrm{GL}_n(\Lambda), St_n)=0$ for $n\geq 3$ [13, Corollary to Theorem 4.1]), now implies that one has $H_5(BQ)=H_5(BQ_5)$. By the above we get that the rank of $H_5(BQ_5)=H_5(BQ)$ is at least 1.

Therefore, invoking yet again Quillen's exact sequence (8), this time for $j=5$, and using the above result that $H_5(BQ_4)$ is equal to $\mathbb{Z}$ modulo $\boldsymbol{\mathcal{S}}_{p\leq 3}$, we deduce from

$$\underbrace{H_5(BQ_4)}_{=\mathbb{Z}\text{ by (10)}} \longrightarrow H_5(BQ_5) \longrightarrow \underbrace{H_0(\mathrm{GL}_5, St_5)}_{=0}$$

that $H_5(BQ) = H_5(BQ_5)$ must be equal to $\mathbb{Z}$ modulo $\mathcal{S}_{p\leq 3}$ as well. Thus $H_5(BQ)$ cannot contain any p-torsion with $p > 3$. □

4 Relating $K_4(R)$ and $H_5(BQ(R))$ via the Hurewicz homomorphism

It is well known that for a number ring R the space $BQ(R)$ is an infinite loop space. Hence a theorem due to Arlettaz [1, Theorem 1.5] shows that the kernel of the corresponding Hurewicz homomorphism $K_4(R) = \pi_5(BQ) \to H_5(BQ)$ is certainly annihilated by 144 (cf. Definition 1.3 in loc.cit., where this number is denoted R_5). Thus that kernel lies in $\mathcal{S}_{p\leq 3}$ (Definition 2.2).

Therefore this Hurewicz homomorphism is injective modulo $\mathcal{S}_{p\leq 3}$. For $R = \mathbb{Z}[i]$ or $\mathbb{Z}[\rho]$, Proposition 3.2 implies that $H_5(BQ)$ contains no p-torsion for $p > 3$. After invoking Quillen's result that $K_{2n}(R)$ is finitely generated and Borel's result that the rank of $K_{2n}(R)$ is zero for any number ring R and $n > 0$, we obtain the following theorem:

Theorem 4.1 *The groups $K_4(\mathbb{Z}[i])$ and $K_4(\mathbb{Z}[\rho])$ lie in $\mathcal{S}_{p\leq 3}$.*

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