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Hopf-cyclic cohomology of the Connes–Moscovici Hopf algebras with infinite dimensional coefficients

B. Rangipour[1] · **S. Sütlü**[2] · **F. Yazdani Aliabadi**[1]

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Abstract We discuss a new strategy for the computation of the Hopf-cyclic cohomology of the Connes–Moscovici Hopf algebra $\mathcal{H}_n$. More precisely, we introduce a multiplicative structure on the Hopf-cyclic complex of $\mathcal{H}_n$, and we show that the van Est type characteristic homomorphism from the Hopf-cyclic complex of $\mathcal{H}_n$ to the Gelfand–Fuks cohomology of the Lie algebra W_n of formal vector fields on $\mathbb{R}^n$ respects this multiplicative structure. We then illustrate the machinery for $n = 1$.

Keywords Hopf-cyclic cohomology · Connes–Moscovici Hopf algebras · Gelfand–Fuks cohomology · Characteristic classes

Contents

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✉ B. Rangipour
bahram@unb.ca

S. Sütlü
serkan.sutlu@isikun.edu.tr

F. Yazdani Aliabadi
fereshteh.yazdani@unb.ca

[1] Department of Mathematics and Statistics, University of New Brunswick, Fredericton, NB, Canada

[2] Department of Mathematics, Işık University, Şile, 34980 Istanbul, Turkey

1 Introduction

The van Est type isomorphism

$$\kappa_n : \bigoplus_{i \equiv * \pmod 2} H^i_{GF}(W_n, \mathbb{C}) \longrightarrow HP^*(\mathcal{H}_n, \delta, 1)$$

of [8, Thm. 11], see also [10, (4.12)], and its relative version

$$\kappa_{n,SO(n)} : \bigoplus_{i \equiv * \pmod 2} H^i_{GF}(W_n, SO(n), \mathbb{C}) \longrightarrow HP^*(\mathcal{H}_n, SO(n), \delta, 1)$$

between the Hopf-cyclic cohomology (with trivial coefficients) of the Connes–Moscovici Hopf algebra $\mathcal{H}_n$, and the Gelfand–Fuks cohomology of the infinite dimensional Lie algebra W_n of formal vector fields over $\mathbb{R}^n$ allowed a link between the characteristic classes of foliations and the total index class of the hypoelliptic signature operator [7]. This way, the scope of the theory of characteristic classes was broadened even further, [9]. As such, a considerable amount of research on the Hopf algebra $\mathcal{H}_n$, and the (periodic) Hopf-cyclic cohomology of Hopf (co)module (co)algebras has been initiated.

The first explicit computations on the Hopf-cyclic cohomology of the Connes–Moscovici Hopf algebras has been carried out by [8,43] for $\mathcal{H}_1$, using the bicrossproduct structure on $\mathcal{H}_n$. Those results were then followed by [51] for $\mathcal{H}_2$, in the presence of a cup product construction with an equivariant extension of the Hopf-cyclic cohomology. Finally, using a van Est type characteristic homomorphism through the Bott complex [4] and the simplicial de Rham complex [13] of Dupont, Moscovici showed in [42] that the elements of the Vey basis for the Gelfand–Fuks cohomology of W_n can be transferred to the Hopf-cyclic cohomology of $\mathcal{H}_n$.

We, on the other hand, introduce in the present paper a multiplicative structure on the Hopf-cyclic cohomology complex of $\mathcal{H}_n$ (and in the presence of a highly non-trivial coefficients), and show that the van Est type characteristic homomorphism of [49] between the Gelfand–Fuks cohomology of W_n and the Hopf-cyclic cohomology of $\mathcal{H}_n$ respects the multiplicative structures on its domain and range. Thus, we can move the characteristic classes to the Hopf-cyclic cohomology by transfering only the mul-

tiplicative generators, and thus obtain a (Vey) basis for the Hopf-cyclic cohomology of $\mathcal{H}_n$.

The Hopf algebra $\mathcal{H}_n$ is introduced in [8], for each $n \in \mathbb{N}$, as an organisational device in the computation of the index of the tranversally elliptic operators on foliations. By its very nature, $\mathcal{H}_n$ is a Hopf algebra of differential operators on the bundle $F^+(M)$ of orientation preserving frames on a flat n-manifold M, and it thus acts naturally on the cross-product algebra $\mathcal{A}_\Gamma := C^\infty(F^+) \rtimes \Gamma$, for any pseudogroup Γ of partial diffeomorphisms on F^+. The structure of $\mathcal{H}_n$ has been investigated extensively through [15,16,26,43,44].

It was first observed in [26] that $\mathcal{H}_1$ is a bicrossproduct Hopf algebra. Then in [43,44] the authors showed, using its module algebra action on the algebra $\mathcal{A}_\Gamma := C^\infty(F^+) \rtimes \Gamma$, that this is in fact the case for any $n \in \mathbb{N}$.

The domain of the van Est type map, Hopf-cyclic cohomology, is introduced in [8] as a cyclic cohomology theory associated to a Hopf algebra and a pair of elements (called the modular pair in involution, or MPI in short) consisting of a grouplike element in the Hopf algebra, and a character of the Hopf algebra. The theory was then developed through [27,28,32] as a cyclic cohomology theory associated to a (co)algebra, equipped with a Hopf algebra (co)action, and a particular (co)representation of that Hopf algebra as the space of coefficients (called stable-anti-Yetter-Drinfeld modules, or SAYD modules in short), so that [8]'s $HP^*(\mathcal{H}_n, \delta, 1)$ is the (periodic) Hopf-cyclic cohomology with trivial coefficients.

It turned out that the bicrossproduct structure of $\mathcal{H}_n$ was not only helpful in understanding its Hopf algebra structure, but is was also crucial to compute its Hopf-cyclic cohomology. This point of view was taken in [44] to introduce a bicocyclic bicomplex computing the Hopf-cyclic cohomology (with trivial coefficients) of $\mathcal{H}_n$.

On the other hand, nontrivial examples of SAYD modules over bicrossproduct Hopf algebras were developed through [48–50]. More precisely, given a bicrossproduct Hopf algebra associated to a Lie algebra via semi-dualisation [36,37], a SAYD module was associated to any representation of the Lie algebra. In [49], a concrete 4-dimensional SAYD module over the Schwarzian quotient $\mathcal{H}_{1S}$ of $\mathcal{H}_1$ was constructed this way, and the Hopf-cyclic cohomology of $\mathcal{H}_{1S}$ with coefficients in this particular space were computed. Furthermore, it was also observed in [49] that the Connes–Moscovici Hopf algebra $\mathcal{H}_n$ is an example of a semi-dualisation Hopf algebra associated to the Lie algebra W_n of formal vector fields on $\mathbb{R}^n$, and since W_n has no nontrivial finite dimensional representation, $\mathcal{H}_n$ does not admit any nontrivial finite dimensional SAYD module.

It was this last result that prompted us to think about the Hopf-cyclic cohomology of $\mathcal{H}_n$ with infinite dimensional coefficients. In fact, an example of an infinite dimensional SAYD module over a Hopf subalgebra of $\mathcal{H}_1$ was already introduced in [1]. However, there appears to be no attempt in the literature regarding an explicit computation of the Hopf-cyclic cohomology of Connes–Moscovici Hopf algebras with infinite dimensional coefficients.

Now the range of the van Est type homomorphism of [49], the Gelfand–Fuks cohomology of the Lie algebra W_n of formal vector fields on $\mathbb{R}^n$, was the target of a series of attempts [19,20,22–24]. It is known to be finite dimensional [23,25], and provides a universal source for all characteristic classes of foliations [4]. On the other

hand, the cohomology of W_n with nontrivial coefficients has been studied through [18,21,35], see also [17]. In the present paper we shall consider the cohomology of W_n with coefficients in the space of formal differential forms.

In the case of the trivial coefficients, the van Est type characteristic map between the Gelfand–Fuks cohomology of W_n and the Hopf-cyclic cohomology of $\mathcal{H}_n$ has also been considered in [45,46] from the point of view of the integration of invariant forms over simplexes in the spaces of jets of diffeomorphisms. In [49,50], however, the transfer of classes was achieved via a characteristic isomorphism in the opposite direction, *i.e.* from the Hopf-cyclic cohomology to the Gelfand–Fuks cohomology via differentiation. Here we shall adopt this last point of view, introduce a multiplicative structure on the Hopf-cyclic cohomology (with coefficients) bicomplex of $\mathcal{H}_n$, and show that the characteristic homomorphism respects the multiplicative structures on its domain and the range.

In order to keep the paper in a reasonable length, and avoid tedious calculations, we shall illustrate the machinery only for $n = 1$. We note, on the other hand, that the whole argument works as well for the Hopf-cyclic cohomology (with any multiplicative coefficients) of any bicrossproduct Hopf algebra associated to a matched pair of Lie algebras, [49,50], as well as those associated to a matched pair of Lie groups, [36,37, 44,57]. We note also that we shall confine ourselves to the Connes–Moscovici Hopf algebras in the present paper, and postpone the application on quantum groups (such as the κ-deformed Poincaré quantum algebra of [39], the quantum Weyl group of [40], and the affine quantum groups of [38]) to a subsequent paper.

The plan of the paper is as follows. In Sect. 2 we consider the space $\Omega^{\leq 1}_n$ of formal differential 0-forms together with 1-forms on $\mathbb{R}^n$. We review the bicrossproduct structure of the Connes–Moscovici Hopf algebra $\mathcal{H}_n$, and then we illustrate the (induced) SAYD module structure of $\Omega^{\leq 1}_n$ over $\mathcal{H}_n$. Section 3 is devoted to the Lie algebra cohomology, with coefficients. In particular, we recall the cohomology of a matched pair Lie algebra, as well as the cohomology of W_n with coefficients in the space $\Omega^{\leq 1}_n$. In Sect. 4 we recall the Hopf-cyclic cohomology, with coefficients, for Hopf algebras. More importantly, it is Sect. 4 in which we introduce a multiplicative structure on the Hopf-cyclic bicomplex. Finally, we show in Sect. 5 that the characteristic isomorphism of [49] respects the multiplicative structures on the Hopf-cyclic complex of $\mathcal{H}_n$ and the Lie algebra cohomology complex of W_n. We illustrate the whole discussion in the case $n = 1$. More explicitly, we transfer the generators of $H^*(W_1, \Omega^{\leq 1}_1)$ to the Hopf-cyclic cohomology $HC^*(\mathcal{H}_1, \Omega^{\leq 1}_{1\delta})$.

2 The space of formal differential forms

2.1 The Connes–Moscovici Hopf algebra $\mathcal{H}_n$

We recall, in this section, the Connes–Moscovici Hopf algebra $\mathcal{H}_n$, and its bicrossproduct structure from [8,44]. Referring the reader to [36,37,55] for a quick review of the bicrossproduct Hopf algebras, as well as the matched pairs of Lie groups and Lie algebras, we begin with the note that we are going to use the Sweedler's notation [56] for the coaction and the comultiplication.

From the group decomposition point of view, the Connes–Moscovici Hopf algebra $\mathcal{H}_n$ is constructed by the Kac decomposition, [31], of the group $\mathrm{Diff}(\mathbb{R}^n)$ of diffeomorphisms of $\mathbb{R}^n$. Accordingly, $\mathrm{Diff}(\mathbb{R}^n) = G \times N$, where $G \cong F\mathbb{R}^n \cong GL_n^{\mathrm{aff}}$ is the group of affine transformations, and

$$N = \{\phi \in \mathrm{Diff}(\mathbb{R}^n) \mid \phi(0) = 0,\ \ \phi'(0) = \mathrm{Id}\}.$$

We thus have the Hopf algebra $\mathcal{U} := U(g\ell_n^{\mathrm{aff}})$, where

$$\begin{aligned} g\ell_n^{\mathrm{aff}} &= \langle \{X_k, Y_i^j \mid 1 \le i, j, k \le n\}\rangle, \\ [Y_i^j, X_k] = \delta_k^j X_i, \quad [X_k, X_\ell] = 0, \quad [Y_i^j, Y_p^q] &= \delta_p^j Y_i^q - \delta_i^q Y_p^j \end{aligned}$$

is the Lie algebra of the group $GL_n^{\mathrm{aff}} := \mathbb{R}^n \rtimes GL_n$, and the Hopf algebra $\mathcal{F} := \mathcal{F}(N)$ of regular functions on N is generated by the functions given by

$$\alpha^i_{jk_1\ldots k_r}(\psi) = \partial_{k_r} \ldots \partial_{k_1} \partial_j (\psi^i(x))|_{x=0}, \qquad 1 \le i, j, k_1, \ldots, k_r \le n,\ \ \psi \in N,$$

or alternatively by the functions

$$\eta^i_{jk\ell_1\ldots\ell_r}(\psi) = \partial_{\ell_r} \ldots \partial_{\ell_1} \left((\psi'(x)^{-1})^i_\nu \partial_j \partial_k \psi^\nu(x) \right) |_{x=0}.$$

The Hopf algebra $\mathcal{F}$ is a $\mathcal{U}$-module algebra by the action

$$(Z \triangleright f)(\psi) := \left.\frac{d}{dt}\right|_{t=0} f(\psi \triangleleft \exp(tZ)), \qquad f \in \mathcal{F},\ \ Z \in g\ell_n^{\mathrm{aff}},$$

and $\mathcal{U}$ is a $\mathcal{F}$-comodule coalgebra by the coaction

$$\begin{aligned} &\blacktriangledown : g\ell_n^{\mathrm{aff}} \longrightarrow g\ell_n^{\mathrm{aff}} \otimes \mathcal{F}, \\ &\blacktriangledown(X_k) = X_k \otimes 1 + Y_i^j \otimes \eta^i_{jk}, \qquad \blacktriangledown(Y_i^j) = Y_i^j \otimes 1 \end{aligned} \tag{2.1}$$

which is extended to a coaction $\blacktriangledown : \mathcal{U} \to \mathcal{U} \otimes \mathcal{F}$.

Remark 2.1 In view of the non-degenerate pairing [49, (3.50)], see also [8, Prop. 3] or [6, Prop. 3], $\mathcal{F}$ is isomorphic with the Hopf algebra $R(\mathfrak{n})$ of representative functions on $U(\mathfrak{n})$, where $\mathfrak{n}$ is the Lie algebra of the group N, and the coaction (2.1) dualizes the left $\mathfrak{n}$-action on $g\ell_n^{\mathrm{aff}}$.

Finally, it follows from [44, Prop. 2.14] that $(\mathcal{F}, \mathcal{U})$ is a matched pair of Hopf algebras, and from [44, Thm. 2.15] that ${\mathcal{H}_n}^{\mathrm{cop}} \cong \mathcal{F} \blacktriangleright\!\!\blacktriangleleft \mathcal{U}$.
To review the bicrossproduct structure of $\mathcal{H}_n$, from the Lie algebra decomposition point of view, we consider the Lie algebra W_n of formal vector fields on $\mathbb{R}^n$. Elements of W_n are expressed as $\sum_{i=1}^n f^i(x^1, \ldots, x^n)\partial_i$, where $f^i(x^1, \ldots, x^n)$ is a formal

power series in the indeterminates $x^1, \dots, x^n$, for any $i = 1, \dots, n$. It is an infinite dimensional vector space

$$W_n = \Big\langle \{e_i := \partial_i,\ e_i^j := x^j \partial_i,\ e_i^{jk_1\dots k_r} := x^j x^{k_1} \dots x^{k_r} \partial_i \mid 1 \le i, j, k_1 \dots k_r \le n\} \Big\rangle,$$

with the Lie bracket given by

$$\begin{aligned}
&[e_i, e_j] = 0, \qquad [e_k, e_i^j] = \delta_k^j e_i, \qquad [e_\ell, e_i^{jk_1\dots k_r}] = \delta_\ell^j e_i^{k_1\dots k_r},\\
&[e_i^j, e_p^{q\ell_1\dots\ell_r}] = \delta_i^q e_p^{j\ell_1\dots\ell_r} + \sum_{m=1}^{n} \delta_i^{\ell_m} e_p^{jq\ell_1\dots\widehat{\ell_m}\dots\ell_r} - \delta_p^j e_i^{q\ell_1\dots\ell_r},\\
&[e_i^{jk_1\dots k_r}, e_p^{q\ell_1\dots\ell_s}]\\
&\quad = \delta_i^q e_p^{jk_1\dots k_r\ell_1\dots\ell_s} + \sum_{m=1}^{s} \delta_i^{\ell_m} e_p^{jqk_1\dots k_r\ell_1\dots\widehat{\ell_m}\dots\ell_s} - \delta_p^j e_i^{q\ell_1\dots\ell_s k_1\dots k_r}\\
&\qquad - \sum_{m=1}^{r} \delta_p^{k_m} e_i^{jq\ell_1\dots\ell_s k_1\dots\widehat{k_m}\dots k_r}.
\end{aligned}$$

Setting

$$\mathfrak{s} := \Big\langle \{e_i := \partial_i,\ e_i^j := x^j \partial_i \mid 1 \le i, j \le n\} \Big\rangle \cong g\ell_n^{\mathrm{aff}},$$

and

$$\mathfrak{n} := \Big\langle \{e_i^{jk_1\dots k_r} := x^j x^{k_1} \dots x^{k_r} \partial_i \mid 1 \le i, j, k_1 \dots k_r \le n\} \Big\rangle,$$

we obtain at once the matched pair decomposition $W_n = \mathfrak{s} \bowtie \mathfrak{n}$. The mutual actions are, via [37, Prop. 8.3.2],

$$e_i^{jk_1\dots k_r} \triangleright e_\ell = \begin{cases} -\delta_\ell^j e_i^{k_1}, & \text{if } r = 1,\\ 0, & \text{if } r \ge 2, \end{cases} \qquad e_i^{jk_1\dots k_r} \triangleright e_p^q = 0,$$

and

$$\begin{aligned}
e_i^{jk_1\dots k_r} \triangleleft e_\ell &= \begin{cases} 0, & \text{if } r = 1,\\ -\delta_\ell^j e_i^{k_1\dots k_r}, & \text{if } r \ge 2, \end{cases}\\
e_i^{jk_1\dots k_r} \triangleleft e_p^q &= \delta_i^q e_p^{jk_1\dots k_r} - \delta_p^j e_i^{qk_1\dots k_r} - \sum_{m=1}^{n} \delta_p^{k_m} e_i^{jqk_1\dots\widehat{k_m}\dots k_r}.
\end{aligned}$$

We next recall the concept of a Lie-Hopf algebra, [50] and see also [53].

Definition 2.2 Let a Lie algebra $\mathfrak{g}$ act on a commutative Hopf algebra $\mathcal{F}$ by derivations, and $\mathcal{F}$ coacts on $\mathfrak{g}$. Then $\mathcal{F}$ is said to be a $\mathfrak{g}$-Hopf algebra if

1. the coaction $\blacktriangledown : \mathfrak{g} \to \mathfrak{g} \otimes \mathcal{F}$ of $\mathcal{F}$ on $\mathfrak{g}$ is a map of Lie algebras, where the bracket on $\mathfrak{g} \otimes \mathcal{F}$ is given by

$$[X \otimes f, Y \otimes g] := [X, Y] \otimes fg + Y \otimes \varepsilon(f) X \triangleright g - X \otimes \varepsilon(g) Y \triangleright f, \quad (2.2)$$

for any $X, Y \in \mathfrak{g}$, and any $f, g \in \mathcal{F}$,
2. the comultiplication and counit of $\mathcal{F}$ are $\mathfrak{g}$-linear, *i.e.* $\Delta(X \triangleright f) = X \bullet \Delta(f)$, and $\varepsilon(X \triangleright f) = 0$, where the action "$\bullet$" is given by

$$\begin{aligned} X \bullet (f^1 \otimes \cdots \otimes f^q) \\ := X_{(1)_{<0>}} \triangleright f^1 \otimes X_{(1)_{<1>}} X_{(2)_{<0>}} \triangleright f^2 \otimes \cdots \\ \cdots \otimes X_{(1)_{<q-1>}} \ldots X_{(q-1)_{<1>}} X_{(q)} \triangleright f^q, \end{aligned} \quad (2.3)$$

for any $X \in \mathfrak{g}$, and any $f^1, \ldots, f^q \in \mathcal{F}$.

The proof of the following proposition is similar to that of [52, Prop. 2.10], and hence is omitted.

Proposition 2.3 *The commutative Hopf algebra $\mathcal{F} = \mathcal{F}(N)$ is an $\mathfrak{s}$-Hopf algebra.*

As a result, it follows from [52, Thm. 2.6], see also [53, Thm. 2.14], that $(\mathcal{F}(N), U(\mathfrak{s}))$ is a matched pair of Hopf-algebras, and the bicrossproduct Hopf algebra $\mathcal{F}(N) \bowtie U(\mathfrak{s}) = \mathcal{F}(N) \bowtie U(g\ell_n^{\mathrm{aff}})$ is isomorphic (as Hopf algebras) with ${\mathcal{H}_n}^{\mathrm{cop}}$.

2.2 SAYD structure over $\mathcal{H}_n$

It was observed in [49] that the only finite dimensional AYD module over the Connes–Moscovici Hopf algebra $\mathcal{H}_n$ is the trivial one, $\mathbb{C}_\delta$. On the other hand, the dual of the space of formal exterior differential 1-forms was considered in [1] as an infinite dimensional nontrivial example, over a Hopf subalgebra of $\mathcal{H}_1$. In this section we study the space of formal differential $\leq$ 1-forms as an infinite dimensional coefficient space for the Hopf-cyclic cohomology of the Hopf algebra $\mathcal{H}_n$.
Let us recall from [27] that a vector space V is called a right-left stable-anti-Yetter-Drinfeld (SAYD) module over H if it is a right H-module, a left H-comodule, and

$$\nabla(v \cdot h) = S(h_{(3)}) v_{<-1>} h_{(1)} \otimes v_{<0>} \cdot h_{(2)}, \qquad v_{<0>} \cdot v_{<-1>} = v, \quad (2.4)$$

for any $v \in V$ and any $h \in H$.
Adopting the notation of [17], we let Ω_n^q to denote the space of formal exterior differential q-forms on $\mathbb{R}^n$. In particular, Ω_n^0 is the space of formal power series in $x^1, \ldots, x^n$, and

$$\Omega_n^1 := \{ f_i dx^i \mid f_i \text{ is a formal power series of } x^1, \ldots, x^n \}$$

is the space of formal differential 1-forms. The space Ω_n^1 is an infinite dimensional vector space, and it has a natural W_n-module structure, [17, Subsect. 2.2.4]. We shall,

in particular, consider the space $\Omega_n^{\leq 1} := \Omega_n^0 \oplus \Omega_n^1$, which is naturally a $U(\mathfrak{n})$-module. Transposing the action $U(\mathfrak{n}) \otimes \Omega_n^{\leq 1} \to \Omega_n^{\leq 1}$, we obtain

$$\Omega_n^{\leq 1 *} \longrightarrow \left(U(\mathfrak{n}) \otimes \Omega_n^{\leq 1}\right)^* .$$

which factors through the embedding

$$U(\mathfrak{n})^* \otimes \Omega_n^{\leq 1 *} \longrightarrow \left(U(\mathfrak{n}) \otimes \Omega_n^{\leq 1}\right)^* ,$$

see for instance [1, Sect. 5.3].
It then follows from $\Omega_n^{\lambda *} = \Omega_n^{1-\lambda}$, see [47], and the nondegenerate pairing between $U(\mathfrak{n})$ and $\mathcal{F}(N)$ that we have a (left, and then using the antipode) right coaction

$$\blacktriangledown : \Omega_n^{\leq 1} \longrightarrow \Omega_n^{\leq 1} \otimes \mathcal{F}(N), \qquad \omega \mapsto \omega_{<0>} \otimes \omega_{<1>}, \tag{2.5}$$

so that, given any $v \in U(\mathfrak{n})$, $\omega_{<0>}\, \omega_{<1>}(v) = v \triangleright \omega$, see also [1, (5.42)].

Remark 2.4 We remark that in the expense of passing to the topological vector spaces (in the sense of [2,3]) and their tensor product (for which we refer the reader to [54,58]) we may always dualize the above left action to a right coaction.

Following [50,53], we shall observe that $\Omega_n^{\leq 1}$ is an induced SAYD module over the Hopf algebra $\mathcal{F}(N) \blacktriangleright\!\!\blacktriangleleft U(\mathfrak{s})$. We therefore recall its definition.

Definition 2.5 Let $\mathfrak{g}$ be a Lie algebra, and $\mathcal{F}$ a $\mathfrak{g}$-Hopf algebra. Let also M be a (left) $\mathfrak{g}$-module, and a right $\mathcal{F}$-comodule via $\blacktriangledown : M \to M \otimes \mathcal{F}$. We then call M an induced $(\mathfrak{g}, \mathcal{F})$-module if

$$\blacktriangledown(X \cdot m) = X \bullet \blacktriangledown(m) \tag{2.6}$$

for any $X \in \mathfrak{g}$, any $m \in M$, and any $f \in \mathcal{F}$.

Lemma 2.6 *The space $\Omega_n^{\leq 1}$ is an induced $(\mathfrak{s}, \mathcal{F}(N))$-module.*

Proof We first recall that $\mathcal{F}(N)$ being a $\mathfrak{s}$-Hopf algebra was observed already in Proposition 2.3. We are thus left to show (2.6). To this end we observe that

$$\begin{aligned}
&\langle X \bullet (\omega_{<0>} \otimes \omega_{<1>}),\ v\rangle \\
&\quad = \langle X^{[0]} \cdot \omega_{<0>} \otimes X^{[1]} \omega_{<1>},\ v\rangle + \langle \omega_{<0>} \otimes X \triangleright \omega_{<1>},\ v\rangle \\
&\quad = (v_{(1)} \triangleright X) \cdot (v_{(2)} \triangleright \omega) + (X \triangleleft v) \cdot \omega = v \cdot (X \cdot \omega) = \langle \blacktriangledown(X \cdot \omega),\ v\rangle
\end{aligned}$$

for any $v \in U(\mathfrak{n})$, where the right coaction is the one given by (2.5), and the third equality follows from [49, (3.35)]. The claim thus follows from the non-degeneracy of the pairing between $U(\mathfrak{n})$ and $\mathcal{F}(N)$. □

As a result of [50, Prop. 3.4], $\Omega_n^{\leq 1}$ is a left / right YD-module over the bicrossproduct Hopf algebra $\mathcal{F}(N) \blacktriangleright\!\!\blacktriangleleft U(\mathfrak{s})$ via the action

$$\mathcal{F}(N) \blacktriangleright\!\!\blacktriangleleft U(\mathfrak{s}) \otimes \Omega_n^{\leq 1} \longrightarrow \Omega_n^{\leq 1}, \qquad (f \blacktriangleright\!\!\blacktriangleleft u) \cdot \omega := \varepsilon(f) u \cdot \omega,$$

and the coaction

$$\Omega_n^{\leq 1} \longrightarrow \Omega_n^{\leq 1} \otimes \mathcal{F}(N) \bowtie U(\mathfrak{s}), \qquad \blacktriangledown(\omega) := \omega_{<0>} \otimes (\omega_{<1>} \bowtie 1),$$

for any $\omega \in \Omega_n^{\leq 1}$, any $u \in U(\mathfrak{s})$, and any $f \in \mathcal{F}(N)$. Finally, since $(\delta, 1)$ is a MPI on the Hopf algebra $\mathcal{F}(N) \bowtie U(\mathfrak{s})$, see for instance [8,44] or [50, Thm. 3.2], we conclude that $\Omega_{n\delta}^{\leq 1} := {}^1\mathbb{C}_\delta \otimes \Omega_n^{\leq 1}$ is a right / left SAYD module over $\mathcal{F}(N) \bowtie U(\mathfrak{s})$ via the action

$$\Omega_{n\delta}^{\leq 1} \otimes \mathcal{F}(N) \bowtie U(\mathfrak{s}) \longrightarrow \Omega_{n\delta}^{\leq 1}, \qquad \omega \cdot (f \bowtie u) := \varepsilon(f)\delta(u_{(1)})S(u_{(2)}) \cdot \omega,$$

and the coaction

$$\Omega_{n\delta}^{\leq 1} \longrightarrow \mathcal{F}(N) \bowtie U(\mathfrak{s}) \otimes \Omega_{n\delta}^{\leq 1}, \qquad \blacktriangledown(\omega) := (S(\omega_{<1>}) \bowtie 1) \otimes \omega_{<0>}.$$

3 Lie algebra cohomology $H^*(W_n, \Omega_n^{\leq 1})$

In this section we recall the Lie-algebra cohomology with coefficients. In particular, we shall discuss the cohomology of the infinite dimensional Lie algebra of formal vector fields, with coefficients in the space of formal differential 1-forms, [17,22].

3.1 Lie algebra cohomology with coefficients

Let $\mathfrak{g}$ be a Lie algebra, and M a $\mathfrak{g}$-module. Then the graded space

$$C^*(\mathfrak{g}, M) = \bigoplus_{k\geq 0} C^k(\mathfrak{g}, M), \qquad C^k(\mathfrak{g}, M) := \mathrm{Hom}(\wedge^k \mathfrak{g}, M)$$

is a differential graded space via

$$\begin{aligned} &d_{\mathrm{CE}} : C^k(\mathfrak{g}, M) \longrightarrow C^{k+1}(\mathfrak{g}, M), \\ &d_{\mathrm{CE}}c(X_1, \ldots, X_{k+1}) \\ &\quad := \sum_{1\leq r<s\leq k+1} (-1)^{r+s-1} c([X_r, X_s], X_1, \ldots, \widehat{X_r}, \ldots, \widehat{X_s}, \ldots, X_{k+1}) \\ &\quad + \sum_{t=1}^{k+1} (-1)^t X_t \cdot c(X_1, \ldots, \widehat{X_t}, \ldots, X_{k+1}), \end{aligned}$$

or alternatively via

$$\begin{aligned} d_{\mathrm{CE}}(m) &= m \cdot X_i \otimes \theta^i, \\ d_{\mathrm{CE}}(m \otimes \mu) &= m \cdot X_i \otimes \theta^i \wedge \mu + m \otimes d_{\mathrm{DR}}(\mu), \end{aligned}$$

where the basis $\{\theta^i \mid 1 \leq i \leq n\}$ of $\mathfrak{g}^*$ is dual to $\{X_i \mid 1 \leq i \leq n\}$ of $\mathfrak{g}$, and $d_{\mathrm{DR}} : \wedge^p \mathfrak{g}^* \to \wedge^{p+1} \mathfrak{g}^*$ is the deRham coboundary (which is a derivation of order 1) given by

$$d_{\mathrm{DR}}(\theta^k) = \frac{1}{2} C_{ij}^k \theta^i \wedge \theta^j.$$

The homology of the differential graded space $(C^*(\mathfrak{g}, M), d_{\mathrm{CE}})$ is called the Lie algebra cohomology of $\mathfrak{g}$, with coefficients in M, and is denoted by $H^*(\mathfrak{g}, M)$.
We shall recall from [30] the multiplicative structure on the Lie algebra cohomology. Let M, M', and P be $\mathfrak{g}$-modules. Then M and M' are said to be paired to P if there exists a bilinear mapping $M \times M' \to P$, $(m, m') \mapsto m \cup m'$, such that

$$X \cdot (m \cup m') := X \cdot m \cup m' + m \cup X \cdot m',$$

for any $m \in M$, any $m' \in M'$, and any $X \in \mathfrak{g}$. Let also $S = \{s_1, \ldots, s_p\}$ be an ordered subset of integers in $\{1, 2, \ldots, p+q\}$, and $T = \{t_1, t_2, \ldots, t_q\}$ be its ordered complement. For each $1 \leq j \leq q$, let $S(j)$ denote the number of indices i for which $s_i > t_j$, and let $\nu(S) := \sum_{j=1}^q S(j)$. Then,

$$(c \cup c')(X_1, \ldots, X_{p+q}) := \sum_S (-1)^{\nu(S)} c(X_{s_1}, \ldots, X_{s_p}) \cup c'(X_{t_1}, \ldots, X_{t_q}) \quad (3.1)$$

defines an element $c \cup c' \in C^{p+q}(\mathfrak{g}, P)$, called the cup product of $c \in C^p(\mathfrak{g}, M)$ and $c' \in C^q(\mathfrak{g}, M')$, with the property that

$$d_{\mathrm{CE}}(c \cup c') = d_{\mathrm{CE}}(c) \cup c' + (-1)^p c \cup d_{\mathrm{CE}}(c').$$

Alternatively, if $c = m \otimes \mu \in C^p(\mathfrak{g}, M)$ and $c' = m' \otimes \mu' \in C^q(\mathfrak{g}, M')$, the cup product is given by

$$c \cup c' = m \cup m' \otimes \mu \wedge \mu' \in C^{p+q}(\mathfrak{g}, P), \quad (3.2)$$

see for instance [5].
In particular, the spaces Ω_n^0 and Ω_n^1 of formal differential forms are paired into $\Omega_n^{\leq 1}$, and thus the cohomology $H^*(W_n, \Omega_n^{\leq 1})$ possess a multiplicative structure, and a basis of $H^*(W_n, \Omega_n^{\leq 1})$ is given by $\lambda_k \in H^{2k-1}(W_n, \Omega_n^0)$, $1 \leq k \leq n$, and $\Lambda \in H^1(W_n, \Omega_n^1)$, subject to the relations

$$\lambda_i \cup \lambda_j = -\lambda_j \cup \lambda_i, \qquad \lambda_k \cup \Lambda = \Lambda \cup \lambda_k. \quad (3.3)$$

In particular, for $n = 1$, the generators may be represented by

$$\lambda(\xi) = \mathrm{div}(\xi), \quad (3.4)$$

and

$$\Lambda(\xi) = d\mathrm{div}(\xi), \quad (3.5)$$

see [17, Thm. 2.2.7].

3.2 Lie algebra cohomology $H^*(\mathfrak{s} \bowtie \mathfrak{n}, \Omega_n^{\leq 1})$

In this subsection we recall the bicomplex associated to the matched pair decomposition $W_n = \mathfrak{s} \bowtie \mathfrak{n}$, computing the Lie algebra cohomology of the Lie algebra W_n.
Along the lines of [50, Sect. 4.3], we consider the bicomplex

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow d_{\mathrm{CE}} & & \uparrow d_{\mathrm{CE}} & & \uparrow d_{\mathrm{CE}} & & \\
\Omega_n^{\leq 1} \otimes \wedge^2 \mathfrak{s}^* & \xrightarrow{\overrightarrow{d}_{\mathrm{CE}}} & \Omega_n^{\leq 1} \otimes \wedge^2 \mathfrak{s}^* \otimes \mathfrak{n}^* & \xrightarrow{\overrightarrow{d}_{\mathrm{CE}}} & \Omega_n^{\leq 1} \otimes \wedge^2 \mathfrak{s}^* \otimes \wedge^2 \mathfrak{n}^* & \xrightarrow{\overrightarrow{d}_{\mathrm{CE}}} & \cdots \\
\uparrow d_{\mathrm{CE}} & & \uparrow d_{\mathrm{CE}} & & \uparrow d_{\mathrm{CE}} & & \\
\Omega_n^{\leq 1} \otimes \mathfrak{s}^* & \xrightarrow{\overrightarrow{d}_{\mathrm{CE}}} & \Omega_n^{\leq 1} \otimes \mathfrak{s}^* \otimes \mathfrak{n}^* & \xrightarrow{\overrightarrow{d}_{\mathrm{CE}}} & \Omega_n^{\leq 1} \otimes \mathfrak{s}^* \otimes \wedge^2 \mathfrak{n}^* & \xrightarrow{\overrightarrow{d}_{\mathrm{CE}}} & \cdots \\
\uparrow d_{\mathrm{CE}} & & \uparrow d_{\mathrm{CE}} & & \uparrow d_{\mathrm{CE}} & & \\
\Omega_n^{\leq 1} & \xrightarrow{\overrightarrow{d}_{\mathrm{CE}}} & \Omega_n^{\leq 1} \otimes \mathfrak{n}^* & \xrightarrow{\overrightarrow{d}_{\mathrm{CE}}} & \Omega_n^{\leq 1} \otimes \wedge^2 \mathfrak{n}^* & \xrightarrow{\overrightarrow{d}_{\mathrm{CE}}} & \cdots
\end{array} \tag{3.6}$$

The cohomology $H^*(W_n, \Omega_n^{\leq 1})$ can be computed by the total complex of the bicomplex (3.6). This is achieved explicitly by

$$\begin{aligned}
&\natural : C^n(\mathfrak{s} \bowtie \mathfrak{n}, \Omega_n^{\leq 1}) \longrightarrow \mathrm{Tot}^n(\Omega_n^{\leq 1}, \mathfrak{s}^*, \mathfrak{n}^*) \\
&\natural(\Phi)(X_1, \ldots, X_p \mid \xi_1, \ldots, \xi_q) = \Phi(X_1 \oplus 0, \ldots, X_p \oplus 0, 0 \oplus \xi_1, \ldots, 0 \oplus \xi_q),
\end{aligned} \tag{3.7}$$

whose inverse is given by

$$\begin{aligned}
&\natural^{-1}(\omega \otimes \mu \otimes \nu)(X_1 \oplus \xi_1, \ldots, X_{p+q} \oplus \xi_{p+q}) \\
&\quad = \sum_{\sigma \in Sh(p,q)} (-1)^\sigma \omega\mu(X_{\sigma(1)}, \ldots, X_{\sigma(p)})\nu(\xi_{\sigma(p+1)}, \ldots, \xi_{\sigma(p+q)}),
\end{aligned}$$

where $Sh(p, q)$ denotes the set of (p, q)-shuffles. It follows from [45, Lemma 2.7] that (3.7) is an isomorphism of complexes.
Finally, let us use (3.7), and its inverse, to carry the cup product construction (3.1) on $C^*(\mathfrak{s} \bowtie \mathfrak{n}, \Omega_n^{\leq 1})$ to $\mathrm{Tot}^*(\Omega_n^{\leq 1}, \mathfrak{s}^*, \mathfrak{n}^*)$. Given any $a \otimes \mu \otimes \nu \in C^{p,q}(\Omega_n^{\leq 1}, \mathfrak{s}^*, \mathfrak{n}^*)$ in $\mathrm{Tot}^{p+q}(\Omega_n^{\leq 1}, \mathfrak{s}^*, \mathfrak{n}^*)$, and any $\omega \otimes \mu' \otimes \nu' \in C^{p',q'}(\Omega_n^{\leq 1}, \mathfrak{s}^*, \mathfrak{n}^*)$ in $\mathrm{Tot}^{p'+q'}(\Omega_n^{\leq 1}, \mathfrak{s}^*, \mathfrak{n}^*)$, we set

$$(a \otimes \mu \otimes \nu) \cup (\omega \otimes \mu' \otimes \nu') := \natural(\natural^{-1}(a \otimes \mu \otimes \nu) \cup \natural^{-1}(\omega \otimes \mu' \otimes \nu')).$$

Accordingly,

$$\begin{aligned}(a\otimes\mu\otimes\nu)\cup(\omega\otimes\mu'\otimes\nu') &:= \natural(\natural^{-1}(a\otimes\mu\otimes\nu)\cup\natural^{-1}(\omega\otimes\mu'\otimes\nu'))\\ &= \natural((a\otimes\mu\wedge\nu)\cup(\omega\otimes\mu'\wedge\nu')) = \natural(a\omega\otimes\mu\wedge\nu\wedge\mu'\wedge\nu')\\ &= (-1)^{qp'}\natural(a\omega\otimes\mu\wedge\mu'\wedge\nu\wedge\nu') = (-1)^{qp'}a\omega\otimes\mu\wedge\mu'\otimes\nu\wedge\nu'.\end{aligned} \tag{3.8}$$

4 Hopf-cyclic cohomology $HP(\mathcal{H}_n{}^{\mathrm{cop}}, \Omega^{\leq 1}_{n\delta})$

In this section we shall prove one of the main results of the paper, namely; a multiplicative structure on the bicomplex computing the Hopf-cyclic cohomology of $\mathcal{H}_n$.

4.1 Hopf-cyclic bicomplex

Let V be a right-left SAYD module over H. Then,

$$C(H,V) := \bigoplus_{q\geq 0} C^q(H,V), \quad C^q(H,V) := V\otimes H^{\otimes q} \tag{4.1}$$

is a cocyclic module via

$$\begin{aligned}\mathfrak{d}_0(v\otimes h^1\otimes\cdots\otimes h^q) &= v\otimes 1\otimes h^1\otimes\cdots\otimes h^q,\\ \mathfrak{d}_i(v\otimes h^1\otimes\cdots\otimes h^q) &= v\otimes h^1\otimes\cdots\otimes h^i{}_{(1)}\otimes h^i{}_{(2)}\otimes\cdots\otimes h^q,\\ \mathfrak{d}_{q+1}(v\otimes h^1\otimes\cdots\otimes h^q) &= v_{<0>}\otimes h^1\otimes\cdots\otimes h^q\otimes v_{<-1>},\\ \mathfrak{s}_j(v\otimes h^1\otimes\cdots\otimes h^q) &= v\otimes h^1\otimes\cdots\otimes\varepsilon(h^{j+1})\otimes\cdots\otimes h^q,\\ \mathfrak{t}(v\otimes h^1\otimes\cdots\otimes h^q) &= v_{<0>}h^1{}_{(1)}\otimes S(h^1{}_{(2)})\cdot(h^2\otimes\cdots\otimes h^q\otimes v_{<-1>}).\end{aligned}$$

Using these operators one defines the Hochschild coboundary

$$b : C^q_H(C,V)\to C^{q+1}_H(C,V), \qquad b := \sum_{i=0}^{q+1}(-1)^i\mathfrak{d}_i,$$

and the Connes boundary operator

$$B : C^{q+1}_H(C,V)\to C^q_H(C,V),\ B := \left(\sum_{i=0}^{q}(-1)^{qi}\mathfrak{t}_q^i\right)\mathfrak{s}_q\mathfrak{t}_{q+1}\left(\mathrm{Id}-(-1)^{q+1}\mathfrak{t}_{q+1}\right).$$

The cyclic cohomology of (4.1) is called the Hopf-cyclic cohomology of the Hopf algebra H, with coefficients in V, and it is denoted by $HC^*(H,V)$. Its periodized version, on the other hand, is denoted by $HP^*(H,V)$.
Using the bicrossproduct structure of $\mathcal{H}_n$, it is shown in [44] that the Hopf-cyclic cohomology $HC^*(\mathcal{H}_n{}^{\mathrm{cop}}, \Omega^{\leq 1}_{n\delta})$ can be calculated by the diagonal subcomplex of the

total of the bicomplex whose vertical coboundary $\uparrow\mathrm{Tot} := \uparrow b + \uparrow B$, and horizontal coboundary $\overrightarrow{\mathrm{Tot}} := \overrightarrow{b} + \overrightarrow{B}$ out of

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow b \downarrow \uparrow B & & \uparrow b \downarrow \uparrow B & & \uparrow b \downarrow \uparrow B & & \\
\Omega^{\leq 1}_{n\delta} \otimes \mathcal{U}^{\otimes 2} & \underset{\overrightarrow{B}}{\overset{\overrightarrow{b}}{\rightleftarrows}} & \Omega^{\leq 1}_{n\delta} \otimes \mathcal{U}^{\otimes 2} \otimes \mathcal{F} & \underset{\overrightarrow{B}}{\overset{\overrightarrow{b}}{\rightleftarrows}} & \Omega^{\leq 1}_{n\delta} \otimes \mathcal{U}^{\otimes 2} \otimes \mathcal{F}^{\otimes 2} & \underset{\overrightarrow{B}}{\overset{\overrightarrow{b}}{\rightleftarrows}} & \cdots \\
\uparrow b \downarrow \uparrow B & & \uparrow b \downarrow \uparrow B & & \uparrow b \downarrow \uparrow B & & \\
\Omega^{\leq 1}_{n\delta} \otimes \mathcal{U} & \underset{\overrightarrow{B}}{\overset{\overrightarrow{b}}{\rightleftarrows}} & \Omega^{\leq 1}_{n\delta} \otimes \mathcal{U} \otimes \mathcal{F} & \underset{\overrightarrow{B}}{\overset{\overrightarrow{b}}{\rightleftarrows}} & \Omega^{\leq 1}_{n\delta} \otimes \mathcal{U} \otimes \mathcal{F}^{\otimes 2} & \underset{\overrightarrow{B}}{\overset{\overrightarrow{b}}{\rightleftarrows}} & \cdots \\
\uparrow b \downarrow \uparrow B & & \uparrow b \downarrow \uparrow B & & \uparrow b \downarrow \uparrow B & & \\
\Omega^{\leq 1}_{n\delta} & \underset{\overrightarrow{B}}{\overset{\overrightarrow{b}}{\rightleftarrows}} & \Omega^{\leq 1}_{n\delta} \otimes \mathcal{F} & \underset{\overrightarrow{B}}{\overset{\overrightarrow{b}}{\rightleftarrows}} & \Omega^{\leq 1}_{n\delta} \otimes \mathcal{F}^{\otimes 2} & \underset{\overrightarrow{B}}{\overset{\overrightarrow{b}}{\rightleftarrows}} & \cdots
\end{array} \tag{4.2}$$

where $\mathcal{U} := U(\mathfrak{s})$, and $\mathcal{F} := \mathcal{F}(N)$. The identification is given by

$$\begin{aligned}
&\Psi_{\blacktriangleright\!\!<} : D^n(U(\mathfrak{s}), \mathcal{F}(N), \Omega^{\leq 1}_{n\delta}) \longrightarrow C^n({\mathcal{H}_n}^{\mathrm{cop}}, \Omega^{\leq 1}_{n\delta}),\\
&\Psi_{\blacktriangleright\!\!<}(\omega \otimes u^1 \otimes \ldots u^n \otimes f^1 \otimes \cdots \otimes f^n)\\
&\quad = \omega \otimes f^1 \blacktriangleright\!\!< {u^1}_{<0>} \otimes f^2 {u^1}_{<1>} \blacktriangleright\!\!< {u^2}_{<0>} \otimes \cdots \otimes f^n {u^1}_{<n-1>} \ldots {u^{n-1}}_{<1>} \blacktriangleright\!\!< u^n,
\end{aligned} \tag{4.3}$$

whose inverse is

$$\begin{aligned}
&\Psi^{-1}_{\blacktriangleright\!\!<} : C^n({\mathcal{H}_n}^{\mathrm{cop}}, \Omega^{\leq 1}_{n\delta}) \longrightarrow D^n(U(\mathfrak{s}), \mathcal{F}(N), \Omega^{\leq 1}_{n\delta}),\\
&\Psi^{-1}_{\blacktriangleright\!\!<}(\omega \otimes f^1 \blacktriangleright\!\!< u^1 \otimes \ldots \otimes f^n \blacktriangleright\!\!< u^n)\\
&\quad = \omega \otimes {u^1}_{<0>} \otimes \cdots \otimes {u^{n-1}}_{<0>} \otimes u^n \otimes f^1\\
&\qquad \otimes f^2 S({u^1}_{<n-1>}) \otimes f^3 S({u^1}_{<n-2>} {u^2}_{<n-2>}) \otimes \cdots \otimes f^n S({u^1}_{<1>} \ldots {u^{n-1}}_{<1>}).
\end{aligned} \tag{4.4}$$

It follows from [50, Prop. 4.4] that the application of

$$\begin{aligned}
&ant : \Omega^{\leq 1}_{n\delta} \otimes \wedge^p \mathfrak{s} \otimes \mathcal{F}(N)^{\otimes q} \longrightarrow \Omega^{\leq 1}_{n\delta} \otimes U(\mathfrak{s})^{\otimes p} \otimes \mathcal{F}(N)^{\otimes q}\\
&ant(\omega \otimes X_1 \wedge \ldots \wedge X_p \otimes f^1 \otimes \cdots \otimes f^q)\\
&\quad = \frac{1}{p!} \sum_{\sigma \in S_p} (-1)^\sigma \omega \otimes X_{\sigma(1)} \otimes \cdots \otimes X_{\sigma(p)} \otimes f^1 \otimes \cdots \otimes f^q
\end{aligned} \tag{4.5}$$

reduces the bicomplex (4.2) to

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\downarrow{\scriptstyle \partial_{\mathrm{CE}}} & & \downarrow{\scriptstyle \partial_{\mathrm{CE}}} & & \downarrow{\scriptstyle \partial_{\mathrm{CE}}} & & \\
\Omega^{\leq 1}_{n\delta} \otimes \wedge^2 \mathfrak{s} & \xrightarrow{b_N} & \Omega^{\leq 1}_{n\delta} \otimes \wedge^2 \mathfrak{s} \otimes \mathcal{F}(N) & \xrightarrow{b_N} & \Omega^{\leq 1}_{n\delta} \otimes \wedge^2 \mathfrak{s} \otimes \mathcal{F}(N)^{\otimes 2} & \xrightarrow{b_N} & \cdots \\
\downarrow{\scriptstyle \partial_{\mathrm{CE}}} & & \downarrow{\scriptstyle \partial_{\mathrm{CE}}} & & \downarrow{\scriptstyle \partial_{\mathrm{CE}}} & & \\
\Omega^{\leq 1}_{n\delta} \otimes \mathfrak{s} & \xrightarrow{b_N} & \Omega^{\leq 1}_{n\delta} \otimes \mathfrak{s} \otimes \mathcal{F}(N) & \xrightarrow{b_N} & \Omega^{\leq 1}_{n\delta} \otimes \mathfrak{s} \otimes \mathcal{F}(N)^{\otimes 2} & \xrightarrow{b_N} & \cdots \\
\downarrow{\scriptstyle \partial_{\mathrm{CE}}} & & \downarrow{\scriptstyle \partial_{\mathrm{CE}}} & & \downarrow{\scriptstyle \partial_{\mathrm{CE}}} & & \\
\Omega^{\leq 1}_{n\delta} & \xrightarrow{b_N} & \Omega^{\leq 1}_{n\delta} \otimes \mathcal{F}(N) & \xrightarrow{b_N} & \Omega^{\leq 1}_{n\delta} \otimes \mathcal{F}(N)^{\otimes 2} & \xrightarrow{b_N} & \cdots
\end{array} \tag{4.6}$$

where

$$\partial_{\mathrm{CE}} : \Omega^{\leq 1}_{n\delta} \otimes \wedge^p \mathfrak{s} \otimes \mathcal{F}(N)^{\otimes q} \longrightarrow \Omega^{\leq 1}_{n\delta} \otimes \wedge^{p-1} \mathfrak{s} \otimes \mathcal{F}(N)^{\otimes q}$$

is the Lie algebra homology boundary of the Lie algebra $\mathfrak{s}$, with coefficients in the $\mathfrak{s}$-module $\Omega^{\leq 1}_{n\delta} \otimes \mathcal{F}(N)^{\otimes q}$, and

$$\begin{aligned}
&b_N : \Omega^{\leq 1}_{n\delta} \otimes \wedge^p \mathfrak{s} \otimes \mathcal{F}(N)^{\otimes q} \longrightarrow \Omega^{\leq 1}_{n\delta} \otimes \wedge^p \mathfrak{s} \otimes \mathcal{F}(N)^{\otimes q+1}, \\
&b_N(\omega \otimes \alpha \otimes f^1 \otimes \cdots \otimes f^q) = \omega \otimes \alpha \otimes 1 \otimes f^1 \otimes \cdots \otimes f^q \\
&\quad + \sum_{i=1}^{q} (-1)^i \omega \otimes \alpha \otimes f^1 \otimes \cdots \otimes \Delta(f^i) \otimes \cdots \otimes f^q \\
&\quad + (-1)^{q+1} \omega_{<0>} \otimes \alpha_{<0>} \otimes f^1 \otimes \cdots \otimes f^q \otimes S(\alpha_{<1>}) S(\omega_{<1>}),
\end{aligned}$$

see [50, Prop. 4.4], or [44, Prop. 3.21]. We recall here that the right $\mathcal{F}(N)$-coaction on $\mathfrak{s}$ is given by (2.1), and it is extended to $\wedge^* \mathfrak{s}$ by multiplication.
Finally, by the Poincaré duality,

$$\begin{aligned}
&\mathfrak{D}_{\mathfrak{s}} : \Omega^{\leq 1}_{n\delta} \otimes \wedge^p \mathfrak{s}^* \otimes \mathcal{F}(N)^{\otimes q} \longrightarrow \Omega^{\leq 1}_{n\delta} \otimes \wedge^{n^2+n-p} \mathfrak{s} \otimes \mathcal{F}(N)^{\otimes q}, \\
&\mathfrak{D}_{\mathfrak{s}}(\omega \otimes \mu \otimes f^1 \otimes \cdots \otimes f^q) = \omega \otimes \iota_\mu(\varpi) \otimes f^1 \otimes \cdots \otimes f^q,
\end{aligned} \tag{4.7}$$

where $\varpi \in \wedge^{n^2+n}\mathfrak{s}$ is the covolume form and $\iota_\mu : \wedge^*\mathfrak{s} \to \wedge^{*-p}\mathfrak{s}$ is the contruction by $\mu \in \wedge^p\mathfrak{s}^*$, we identify the total complex of the bicomplex (4.6) with that of

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow{\scriptstyle d_{\mathrm{CE}}} & & \uparrow{\scriptstyle d_{\mathrm{CE}}} & & \uparrow{\scriptstyle d_{\mathrm{CE}}} & & \\
\Omega^{\leq 1}_{n\delta} \otimes \wedge^2\mathfrak{s}^* & \xrightarrow{b^*_N} & \Omega^{\leq 1}_{n\delta} \otimes \wedge^2\mathfrak{s}^* \otimes \mathcal{F}(N) & \xrightarrow{b^*_N} & \Omega^{\leq 1}_{n\delta} \otimes \wedge^2\mathfrak{s}^* \otimes \mathcal{F}(N)^{\otimes 2} & \xrightarrow{b^*_N} & \cdots \\
\uparrow{\scriptstyle d_{\mathrm{CE}}} & & \uparrow{\scriptstyle d_{\mathrm{CE}}} & & \uparrow{\scriptstyle d_{\mathrm{CE}}} & & \\
\Omega^{\leq 1}_{n\delta} \otimes \mathfrak{s}^* & \xrightarrow{b^*_N} & \Omega^{\leq 1}_{n\delta} \otimes \mathfrak{s}^* \otimes \mathcal{F}(N) & \xrightarrow{b^*_N} & \Omega^{\leq 1}_{n\delta} \otimes \mathfrak{s}^* \otimes \mathcal{F}(N)^{\otimes 2} & \xrightarrow{b^*_N} & \cdots \\
\uparrow{\scriptstyle d_{\mathrm{CE}}} & & \uparrow{\scriptstyle d_{\mathrm{CE}}} & & \uparrow{\scriptstyle d_{\mathrm{CE}}} & & \\
\Omega^{\leq 1}_{n\delta} & \xrightarrow{b^*_N} & \Omega^{\leq 1}_{n\delta} \otimes \mathcal{F}(N) & \xrightarrow{b^*_N} & \Omega^{\leq 1}_{n\delta} \otimes \mathcal{F}(N)^{\otimes 2} & \xrightarrow{b^*_N} & \cdots
\end{array} \tag{4.8}$$

where

$$\begin{aligned}
&b^*_N : \Omega^{\leq 1}_{n\delta} \otimes \wedge^p\mathfrak{s}^* \otimes \mathcal{F}(N)^{\otimes q} \longrightarrow \Omega^{\leq 1}_{n\delta} \otimes \wedge^p\mathfrak{s}^* \otimes \mathcal{F}(N)^{\otimes q+1} \\
&b^*_N(\omega \otimes \mu \otimes f^1 \otimes \cdots \otimes f^q) = \omega \otimes \mu \otimes 1 \otimes f^1 \otimes \cdots \otimes f^q \\
&\quad + \sum_{i=1}^{q} (-1)^i \omega \otimes \mu \otimes f^1 \otimes \cdots \otimes \Delta(f^i) \otimes \cdots \otimes f^q \\
&\quad + (-1)^{q+1} \omega_{<0>} \otimes \mu_{<0>} \otimes f^1 \otimes \cdots \otimes f^q \otimes S(\omega_{<1>})\mu_{<-1>},
\end{aligned} \tag{4.9}$$

see [50, Prop. 4.6], and

$$\begin{aligned}
&d_{\mathrm{CE}} : \Omega^{\leq 1}_{n\delta} \otimes \wedge^p\mathfrak{s}^* \otimes \mathcal{F}(N)^{\otimes q} \longrightarrow \Omega^{\leq 1}_{n\delta} \otimes \wedge^{p+1}\mathfrak{s}^* \otimes \mathcal{F}(N)^{\otimes q} \\
&d_{\mathrm{CE}}(\omega \otimes \mu \otimes \widetilde{f}) = \omega \otimes d_{\mathrm{DR}}(\mu) \otimes \widetilde{f} - X_i \cdot \omega \otimes \theta^i \wedge \mu \otimes \widetilde{f} \\
&\qquad\qquad - \omega \otimes \theta^i \wedge \mu \otimes X_i \bullet \widetilde{f}
\end{aligned} \tag{4.10}$$

is the Lie algebra cohomology coboundary of the Lie algebra $\mathfrak{s}$, with coefficients in the $\mathfrak{s}$-module $\Omega^{\leq 1}_n \otimes \mathcal{F}(N)^{\otimes q}$, see for instance [50, (4.1)]. The map $d_{\mathrm{DR}} : \wedge^p\mathfrak{s}^* \to \wedge^{p+1}\mathfrak{s}^*$ is the deRham differential of forms, and the left $\mathcal{F}(N)$-coaction on $\wedge^*\mathfrak{s}^*$ is obtained by transposing the right $\mathcal{F}(N)$-coaction

$$\begin{aligned}
&\mathfrak{s}^* \longrightarrow \mathcal{F}(N) \otimes \mathfrak{s}^*, \\
&\theta^i \mapsto 1 \otimes \theta^i, \qquad \theta^i_j \mapsto 1 \otimes \theta^i_j + \eta^i_{jk} \otimes \theta^k,
\end{aligned} \tag{4.11}$$

which can be extended to $\wedge^*\mathfrak{s}^*$ multiplicatively.

4.2 A multiplicative structure on $C^{*,*}(\Omega^{\leq 1}_{n\delta}, \mathfrak{s}^*, \mathcal{F}(N))$

In this subsection we introduce a multiplicative structure on the bicomplex

$$\begin{aligned} C^{*,*}(\Omega^{\leq 1}_{n\delta}, \mathfrak{s}^*, \mathcal{F}(N)) &= \bigoplus_{p,q\geq 0} C^{p,q}(\Omega^{\leq 1}_{n\delta}, \mathfrak{s}^*, \mathcal{F}(N)), \\ C^{p,q}(\Omega^{\leq 1}_{n\delta}, \mathfrak{s}^*, \mathcal{F}(N)) &:= \Omega^{\leq 1}_{n\delta} \otimes \wedge^p \mathfrak{s}^* \otimes \mathcal{F}(N)^{\otimes q} \end{aligned} \tag{4.12}$$

given by (4.8). To this end, for any $a\otimes\eta\otimes \widetilde{f} \in C^{p,q}(\Omega^0_{n\delta}, \mathfrak{s}^*, \mathcal{F}(N))$, and $\omega\otimes\mu'\otimes\widetilde{g} \in C^{p',q'}(\Omega^1_{n\delta}, \mathfrak{s}^*, \mathcal{F}(N))$ let

$$(a\otimes\mu\otimes\widetilde{f})\cup(\omega\otimes\mu'\otimes\widetilde{g}) := a_{<0>}\omega\otimes\mu_{<0>}\wedge\mu'\otimes\widetilde{f}\otimes S(a_{<1>})\mu_{<-1>}\cdot\widetilde{g}. \tag{4.13}$$

Proposition 4.1 *The horizontal coboundary (4.9) acts as a graded derivation, i.e. for any $a\otimes\mu\otimes\widetilde{f} \in C^{p,q}(\Omega^0_{n\delta}, \mathfrak{s}^*, \mathcal{F}(N))$, and $\omega\otimes\mu'\otimes\widetilde{g} \in C^{p',q'}(\Omega^1_{n\delta}, \mathfrak{s}^*, \mathcal{F}(N))$,*

$$\begin{aligned} &b^*_N\Big((a\otimes\mu\otimes\widetilde{f})\cup(\omega\otimes\mu'\otimes\widetilde{g})\Big) \\ &\quad= b^*_N\Big(a\otimes\mu\otimes\widetilde{f}\Big)\cup(\omega\otimes\mu'\otimes\widetilde{g}) + (-1)^q(a\otimes\mu\otimes\widetilde{f})\cup b^*_N\Big(\omega\otimes\mu'\otimes\widetilde{g}\Big). \end{aligned}$$

Proof Given $a\otimes\mu\otimes\widetilde{f} \in C^{p,q}(\Omega^0_{n\delta}, \mathfrak{s}^*, \mathcal{F}(N))$, let

$$\Delta_i(\widetilde{f}) := f^1\otimes\cdots\otimes\Delta(f^i)\otimes\cdots\otimes f^q.$$

We then note that

$$\begin{aligned} &b^*_N\Big((a\otimes\mu\otimes\widetilde{f})\cup(\omega\otimes\mu'\otimes\widetilde{g})\Big) \\ &\quad= a_{<0>}\omega\otimes\mu_{<0>}\wedge\mu'\otimes 1\otimes\widetilde{f}\otimes S(a_{<1>})\mu_{<-1>}\cdot\widetilde{g} \\ &\qquad+\sum_{i=1}^{q}(-1)^i a_{<0>}\omega\otimes\mu_{<0>}\wedge\mu'\otimes\Delta_i(\widetilde{f})\otimes S(a_{<1>})\mu_{<-1>}\cdot\widetilde{g} \\ &\qquad+\sum_{i=q+1}^{q+q'}(-1)^i a_{<0>}\omega\otimes\mu_{<0>}\wedge\mu'\otimes\widetilde{f}\otimes S(a_{<1>})\mu_{<-1>}\cdot\Delta_i(\widetilde{g}) \\ &\qquad+(-1)^{q+q'+1}(a_{<0>}\omega)_{<0>}\otimes\mu_{<0>}\wedge\mu'_{<0>}\otimes\widetilde{f} \\ &\qquad\otimes S(a_{<1>})\mu_{<-2>}\cdot\widetilde{g}\otimes S((a_{<0>}\omega)_{<1>})\mu_{<-1>}\mu'_{<-1>}, \end{aligned}$$

which can be rewritten as

$$
\begin{aligned}
& a_{<0>}\omega \otimes \mu_{<0>} \wedge \mu' \otimes 1 \otimes \widetilde{f} \otimes S(a_{<1>})\mu_{<-1>} \cdot \widetilde{g} \\
& \quad + \sum_{i=1}^{q} (-1)^i a_{<0>}\omega \otimes \mu_{<0>} \wedge \mu' \otimes \Delta_i(\widetilde{f}) \otimes S(a_{<1>})\mu_{<-1>} \cdot \widetilde{g} \\
& \quad + (-1)^{q+1} a_{<0><0>}\omega \otimes \mu_{<0><0>} \wedge \mu' \otimes \widetilde{f} \\
& \quad \otimes S(a_{<1>})\mu_{<-1>} \otimes S(a_{<0><1>})\mu_{<0><-1>} \cdot \widetilde{g}) \\
& \quad + (-1)^q \Big\{ a_{<0>}\omega \otimes \mu_{<0>} \wedge \mu' \otimes \widetilde{f} \otimes S(a_{<1>})\mu_{<-1>} \cdot (1 \otimes \widetilde{g}) \\
& \quad + \sum_{i=1}^{q'} (-1)^i a_{<0>}\omega \otimes \mu_{<0>} \wedge \mu' \otimes \widetilde{f} \otimes S(a_{<1>})\mu_{<-1>} \cdot \Delta_i(\widetilde{g}) \\
& \quad + (-1)^{q'+1} (a\omega)_{<0>} \otimes \mu_{<0>} \wedge \mu'_{<0>} \otimes \widetilde{f} \\
& \quad \otimes S(a_{<1>})\mu_{<-2>} \cdot \widetilde{g} \otimes S((a_{<0>}\omega)_{<1>})\mu_{<-1>}\mu'_{<-1>} \Big\},
\end{aligned}
$$

the first three lines of which being $b_N^*\left(a \otimes \mu \otimes \widetilde{f}\right) \cup (\omega \otimes \mu' \otimes \widetilde{g})$, and the last three being $(-1)^q (a \otimes \mu \otimes \widetilde{f}) \cup b_N^*\left(\omega \otimes \mu' \otimes \widetilde{g}\right)$. □

On the next move we deal with the vertical coboundary. To this end, we record a series of lemmas below. The first one is about the action (2.3).

Lemma 4.2 *For any $X \in \mathfrak{s}$, any $\widetilde{f} \in \mathcal{F}(N)^{\otimes r}$, and any $\widetilde{g} \in \mathcal{F}(N)^{\otimes s}$,*

$$X \bullet (\widetilde{f} \otimes \widetilde{g}) = X_{<0>} \bullet \widetilde{f} \otimes X_{<1>} \cdot \widetilde{g} + \widetilde{f} \otimes X \bullet \widetilde{g}.$$

Proof It follows at once from the definition of the action (2.3) that

$$
\begin{aligned}
X \bullet (f^1 \otimes \cdots \otimes f^q) &= (1 \blacktriangleright\!\!\blacktriangleleft X) \cdot (f^1 \otimes \cdots \otimes f^q) \\
&= X_{<0>} \triangleright f^1 \otimes X_{<1>} \cdot (f^2 \otimes \cdots \otimes f^q) \\
&\quad + f^1 \otimes X \bullet (f^2 \otimes \cdots \otimes f^q).
\end{aligned}
$$

The claim then follows immediately. □

The rest of the lemmas point out some auxiliary results by the commutativity of the horizontal and the vertical coboundary maps of the bicomplex (4.8).

Lemma 4.3 *For any $\mu \in \wedge^p \mathfrak{s}^*$,*

$$d_{\mathrm{DR}}(\mu)_{<-1>} \otimes d_{\mathrm{DR}}(\mu)_{<0>} = \mu_{<-1>} \otimes d_{\mathrm{DR}}(\mu_{<0>}) - X_i \triangleright \mu_{<-1>} \otimes \theta^i \wedge \mu_{<0>}.$$

Proof We use the commutativity of the horizontal and the vertical coboundary maps of (4.8). Fixing a trivial coefficient, we have on one hand

$$b_N^* d_{\mathrm{CE}}(\mu) = b_N^*(d_{\mathrm{DR}}(\mu)) = d_{\mathrm{DR}}(\mu) \otimes 1 - d_{\mathrm{DR}}(\mu)_{<0>} \otimes d_{\mathrm{DR}}(\mu)_{<-1>},$$

and on the other hand

$$\begin{aligned} d_{\mathrm{CE}} b_N^*(\mu) &= d_{\mathrm{CE}}(\mu \otimes 1 - \mu_{<0>} \otimes \mu_{<-1>}) \\ &= d_{\mathrm{DR}}(\mu) \otimes 1 - d_{\mathrm{DR}}(\mu_{<0>}) \otimes \mu_{<-1>} + \theta^i \wedge \mu_{<0>} \otimes X_i \triangleright \mu_{<-1>}. \end{aligned}$$

The result follows. □

Lemma 4.4 *For any $a \in \Omega_n^0$, and any $\mu \in \wedge^p \mathfrak{s}^*$,*

$$\begin{aligned} &(X_i \cdot a)_{<0>} \otimes (\theta^i \wedge \mu)_{<0>} \otimes S((X_i \cdot a)_{<1>})(\theta^i \wedge \mu)_{<-1>} \\ &\quad = X_i \cdot a_{<0>} \otimes \theta^i \wedge \mu_{<0>} \otimes S(a_{<1>})\mu_{<-1>} \\ &\qquad + a_{<0>} \otimes \theta^i \wedge \mu_{<0>} \otimes X_i \triangleright S(a_{<1>})\mu_{<-1>}. \end{aligned}$$

Proof On one hand we have

$$\begin{aligned} b_N^* d_{\mathrm{CE}}(a \otimes \mu) &= b_N^*\Big(a \otimes d_{\mathrm{DR}}(\mu) - X_i \cdot a \otimes \theta^i \wedge \mu\Big) \\ &= a \otimes d_{\mathrm{DR}}(\mu) \otimes 1 - a_{<0>} \otimes d_{\mathrm{DR}}(\mu_{<0>}) \otimes S(a_{<1>})\mu_{<-1>} \\ &\quad + a_{<0>} \otimes \theta^i \wedge \mu_{<0>} \otimes S(a_{<1>})(X_i \triangleright \mu_{<-1>}) - X_i \cdot a \otimes \theta^i \wedge \mu \\ &\quad + (X_i \cdot a)_{<0>} \otimes (\theta^i \wedge \mu)_{<0>} \otimes S((X_i \cdot a)_{<1>})(\theta^i \wedge \mu)_{<-1>}, \end{aligned}$$

where we used Lemma 4.3 on the second equality, and on the other hand,

$$\begin{aligned} d_{\mathrm{CE}} b_N^*(a \otimes \mu) &= d_{\mathrm{CE}}\Big(a \otimes \mu \otimes 1 - a_{<0>} \otimes \mu_{<0>} \otimes S(a_{<1>})\mu_{<-1>}\Big) \\ &= a \otimes d_{\mathrm{DR}}(\mu) \otimes 1 - X_i \cdot a \otimes \theta^i \wedge \mu \otimes 1 \\ &\quad - a_{<0>} \otimes d_{\mathrm{DR}}(\mu_{<0>}) \otimes S(a_{<1>})\mu_{<-1>} \\ &\quad + X_i \cdot a_{<0>} \otimes \theta^i \wedge \mu_{<0>} \otimes S(a_{<1>})\mu_{<-1>} \\ &\quad + a_{<0>} \otimes \theta^i \wedge \mu_{<0>} \otimes X_i \triangleright (S(a_{<1>})\mu_{<-1>}). \end{aligned}$$

In view of the commutativity of the horizontal and the vertical coboundaries of the bicomplex (4.8), a comparison of the two equations yields the claim. □

Lemma 4.5 *For any $\widetilde{f} \in \mathcal{F}(N)^{\otimes q}$,*

$$\theta^i{}_{<0>} \otimes X_i \bullet \widetilde{f} \otimes \theta^i{}_{<-1>} = \theta^i \otimes X_i \bullet \widetilde{f} \otimes 1.$$

Proof We have

$$b_N^* d_{\mathrm{CE}}(\widetilde{f}) = b_N^*\Big(-\theta^i \otimes X_i \bullet \widetilde{f}\Big) = -\theta^i \otimes 1 \otimes X_i \bullet \widetilde{f} + \theta^i{}_{<0>} \otimes X_i \bullet \widetilde{f} \otimes \theta^i{}_{<-1>},$$

and

$$d_{\mathrm{CE}} b_N^*(\widetilde{f}) = d_{\mathrm{CE}}\Big(1 \otimes \widetilde{f} - \widetilde{f} \otimes 1\Big) = -\theta^i \otimes X_i \bullet (1 \otimes \widetilde{f}) + \theta^i \otimes X_i \bullet (\widetilde{f} \otimes 1)$$
$$= -\theta^i \otimes 1 \otimes X_i \bullet \widetilde{f} + \theta^i \otimes X_i \bullet \widetilde{f} \otimes 1,$$

where we used Lemma 4.2 on the last equation. The result then follows once again by the commutativity of the horizontal and the vertical coboundaries of (4.8). □

Proposition 4.6 *The vertical coboundary* (4.10) *acts as a graded differential, i.e. for any* $a \otimes \mu \otimes \widetilde{f} \in C^{p,q}(\Omega^0_{n\delta}, \mathfrak{s}^*, \mathcal{F}(N))$, *and* $\omega \otimes \mu' \otimes \widetilde{g} \in C^{p',q'}(\Omega^1_{n\delta}, \mathfrak{s}^*, \mathcal{F}(N))$,

$$d_{\mathrm{CE}}\Big((a \otimes \mu \otimes \widetilde{f}) \cup (\omega \otimes \mu' \otimes \widetilde{g})\Big)$$
$$= d_{\mathrm{CE}}\Big(a \otimes \mu \otimes \widetilde{f}\Big) \cup (\omega \otimes \mu' \otimes \widetilde{g}) + (-1)^p (a \otimes \mu \otimes \widetilde{f}) \cup d_{\mathrm{CE}}\Big(\omega \otimes \mu' \otimes \widetilde{g}\Big).$$

Proof We first observe that

$$\begin{aligned} & d_{\mathrm{CE}}\Big((a \otimes \mu \otimes \widetilde{f}) \cup (\omega \otimes \mu' \otimes \widetilde{g})\Big) \\ & = a_{<0>}\omega \otimes d_{\mathrm{DR}}(\mu_{<0>} \wedge \mu') \otimes \widetilde{f} \otimes S(a_{<1>})\mu_{<-1>} \cdot \widetilde{g} + \\ & \quad - X_i \cdot (a_{<0>}\omega) \otimes \theta^i \wedge \mu_{<0>} \wedge \mu' \otimes \widetilde{f} \otimes S(a_{<1>})\mu_{<-1>} \cdot \widetilde{g} + \\ & \quad - a_{<0>}\omega \otimes \theta^i \wedge \mu_{<0>} \wedge \mu' \otimes X_i \bullet (\widetilde{f} \otimes S(a_{<1>})\mu_{<-1>} \cdot \widetilde{g}). \end{aligned}$$

Using the fact that the deRham coboundary is a graded differential we arrive at

$$\begin{aligned} & d_{\mathrm{CE}}\Big((a \otimes \mu \otimes \widetilde{f}) \cup (\omega \otimes \mu' \otimes \widetilde{g})\Big) \\ & = a_{<0>}\omega \otimes d_{\mathrm{DR}}(\mu_{<0>}) \wedge \mu' \otimes \widetilde{f} \otimes S(a_{<1>})\mu_{<-1>} \cdot \widetilde{g} + \\ & \quad (-1)^p a_{<0>}\omega \otimes \mu_{<0>} \wedge d_{\mathrm{DR}}(\mu') \otimes \widetilde{f} \otimes S(a_{<1>})\mu_{<-1>} \cdot \widetilde{g} + \\ & \quad - X_i \cdot (a_{<0>}\omega) \otimes \theta^i \wedge \mu_{<0>} \wedge \mu' \otimes \widetilde{f} \otimes S(a_{<1>})\mu_{<-1>} \cdot \widetilde{g} + \\ & \quad - a_{<0>}\omega \otimes \theta^i \wedge \mu_{<0>} \wedge \mu' \otimes X_i \bullet (\widetilde{f} \otimes S(a_{<1>})\mu_{<-1>} \cdot \widetilde{g}). \end{aligned}$$

Next, we recall that the Lie algebra $\mathfrak{s} \subseteq W_n$ atcs on Ω^0_n by derivations. Thus,

$$\begin{aligned} & d_{\mathrm{CE}}\Big((a \otimes \mu \otimes \widetilde{f}) \cup (\omega \otimes \mu' \otimes \widetilde{g})\Big) \\ & = a_{<0>}\omega \otimes d_{\mathrm{DR}}(\mu_{<0>}) \wedge \mu' \otimes \widetilde{f} \otimes S(a_{<1>})\mu_{<-1>} \cdot \widetilde{g} + \\ & \quad (-1)^p a_{<0>}\omega \otimes \mu_{<0>} \wedge d_{\mathrm{DR}}(\mu') \otimes \widetilde{f} \otimes S(a_{<1>})\mu_{<-1>} \cdot \widetilde{g} + \\ & \quad - (X_i \cdot a_{<0>})\omega \otimes \theta^i \wedge \mu_{<0>} \wedge \mu' \otimes \widetilde{f} \otimes S(a_{<1>})\mu_{<-1>} \cdot \widetilde{g} + \\ & \quad - a_{<0>}(X_i \cdot \omega) \otimes \theta^i \wedge \mu_{<0>} \wedge \mu' \otimes \widetilde{f} \otimes S(a_{<1>})\mu_{<-1>} \cdot \widetilde{g} + \\ & \quad - a_{<0>}\omega \otimes \theta^i \wedge \mu_{<0>} \wedge \mu' \otimes X_i \bullet (\widetilde{f} \otimes S(a_{<1>})\mu_{<-1>} \cdot \widetilde{g}). \end{aligned}$$

Then using Lemma 4.2 we get

$$
\begin{aligned}
&d_{\mathrm{CE}}\Big((a\otimes\mu\otimes\widetilde{f})\cup(\omega\otimes\mu'\otimes\widetilde{g})\Big)\\
&\quad= a_{<0>}\omega\otimes d_{\mathrm{DR}}(\mu_{<0>})\wedge\mu'\otimes\widetilde{f}\otimes S(a_{<1>})\mu_{<-1>}\cdot\widetilde{g}+\\
&\quad\ (-1)^p a_{<0>}\omega\otimes\mu_{<0>}\wedge d_{\mathrm{DR}}(\mu')\otimes\widetilde{f}\otimes S(a_{<1>})\mu_{<-1>}\cdot\widetilde{g}+\\
&\quad\ -(X_i\cdot a_{<0>})\omega\otimes\theta^i\wedge\mu_{<0>}\wedge\mu'\otimes\widetilde{f}\otimes S(a_{<1>})\mu_{<-1>}\cdot\widetilde{g}+\\
&\quad\ -a_{<0>}(X_i\cdot\omega)\otimes\theta^i\wedge\mu_{<0>}\wedge\mu'\otimes\widetilde{f}\otimes S(a_{<1>})\mu_{<-1>}\cdot\widetilde{g}+\\
&\quad\ -a_{<0>}\omega\otimes\theta^i\wedge\mu_{<0>}\wedge\mu'\otimes X_{i_{<0>}}\bullet\widetilde{f}\otimes X_{i_{<1>}}S(a_{<1>})\mu_{<-1>}\cdot\widetilde{g}+\\
&\quad\ -a_{<0>}\omega\otimes\theta^i\wedge\mu_{<0>}\wedge\mu'\otimes\widetilde{f}\otimes X_i\bullet(S(a_{<1>})\mu_{<-1>}\cdot\widetilde{g}),
\end{aligned}
$$

where

$$
\begin{aligned}
&a_{<0>}\omega\otimes\theta^i\wedge\mu_{<0>}\wedge\mu'\otimes\widetilde{f}\otimes X_i\bullet(S(a_{<1>})\mu_{<-1>}\cdot\widetilde{g})\\
&\quad= a_{<0>}\omega\otimes\theta^i\wedge\mu_{<0>}\wedge\mu'\otimes\widetilde{f}\otimes(1\bowtie X_i)(S(a_{<1>})\mu_{<-1>}\bowtie 1)\cdot\widetilde{g}+\\
&\quad= a_{<0>}\omega\otimes\theta^i\wedge\mu_{<0>}\wedge\mu'\otimes\widetilde{f}\otimes(X_i\triangleright S(a_{<1>})\mu_{<-1>})\cdot\widetilde{g}+\\
&\quad\ + a_{<0>}\omega\otimes\theta^i\wedge\mu_{<0>}\wedge\mu'\otimes\widetilde{f}\otimes S(a_{<1>})\mu_{<-1>}\cdot(X_i\bullet\widetilde{g}).
\end{aligned}
$$

As a result,

$$
\begin{aligned}
&d_{\mathrm{CE}}\Big((a\otimes\mu\otimes\widetilde{f})\cup(\omega\otimes\mu'\otimes\widetilde{g})\Big)\\
&\quad= a_{<0>}\omega\otimes d_{\mathrm{DR}}(\mu_{<0>})\wedge\mu'\otimes\widetilde{f}\otimes S(a_{<1>})\mu_{<-1>}\cdot\widetilde{g}+\\
&\quad\ (-1)^p a_{<0>}\omega\otimes\mu_{<0>}\wedge d_{\mathrm{DR}}(\mu')\otimes\widetilde{f}\otimes S(a_{<1>})\mu_{<-1>}\cdot\widetilde{g}+\\
&\quad\ -(X_i\cdot a_{<0>})\omega\otimes\theta^i\wedge\mu_{<0>}\wedge\mu'\otimes\widetilde{f}\otimes S(a_{<1>})\mu_{<-1>}\cdot\widetilde{g}+\\
&\quad\ -a_{<0>}(X_i\cdot\omega)\otimes\theta^i\wedge\mu_{<0>}\wedge\mu'\otimes\widetilde{f}\otimes S(a_{<1>})\mu_{<-1>}\cdot\widetilde{g}+\\
&\quad\ -a_{<0>}\omega\otimes\theta^i\wedge\mu_{<0>}\wedge\mu'\otimes X_{i_{<0>}}\bullet\widetilde{f}\otimes X_{i_{<1>}}S(a_{<1>})\mu_{<-1>}\cdot\widetilde{g}+\\
&\quad\ -a_{<0>}\omega\otimes\theta^i\wedge\mu_{<0>}\wedge\mu'\otimes\widetilde{f}\otimes(X_i\triangleright S(a_{<1>})\mu_{<-1>})\cdot\widetilde{g}+\\
&\quad\ -a_{<0>}\omega\otimes\theta^i\wedge\mu_{<0>}\wedge\mu'\otimes\widetilde{f}\otimes S(a_{<1>})\mu_{<-1>}\cdot(X_i\bullet\widetilde{g}).
\end{aligned}
\tag{4.14}
$$

On the other hand,

$$
\begin{aligned}
&d_{\mathrm{CE}}\Big(a\otimes\mu\otimes\widetilde{f}\Big)\cup(\omega\otimes\mu'\otimes\widetilde{g})\\
&\quad= (a\otimes d_{\mathrm{DR}}(\mu)\otimes\widetilde{f}-X_i\triangleright a\otimes\theta^i\wedge\mu\otimes\widetilde{f}\\
&\quad\ -a\otimes\theta^i\wedge\mu\otimes X_i\bullet\widetilde{f})\cup(\omega\otimes\mu'\otimes\widetilde{g})\\
&\quad= a_{<0>}\omega\otimes d_{\mathrm{DR}}(\mu)_{<0>}\wedge\mu'\otimes\widetilde{f}\otimes S(a_{<1>})d_{\mathrm{DR}}(\mu)_{<-1>}\cdot\widetilde{g}+\\
&\quad\ -(X_i\cdot a)_{<0>}\omega\otimes(\theta^i\wedge\mu)_{<0>}\wedge\mu'\otimes\widetilde{f}\otimes S((X_i\triangleright a)_{<1>})(\theta^i\wedge\mu)_{<-1>}\cdot\widetilde{g}+\\
&\quad\ -a_{<0>}\omega\otimes(\theta^i\wedge\mu)_{<0>}\otimes X_i\bullet\widetilde{f}\otimes S(a_{<1>})(\theta^i\wedge\mu)_{<-1>}\cdot\widetilde{g},
\end{aligned}
$$

from which we arrive, in view of Lemma 4.3, at

$$\begin{aligned}
&d_{\mathrm{CE}}\Big(a\otimes\mu\otimes\widetilde{f}\Big)\cup(\omega\otimes\mu'\otimes\widetilde{g})\\
&\quad= a_{<0>}\omega\otimes d_{\mathrm{DR}}(\mu_{<0>})\wedge\mu'\otimes\widetilde{f}\otimes S(a_{<1>})\mu_{<-1>}\cdot\widetilde{g}+\\
&\qquad- a_{<0>}\omega\otimes\theta^i\wedge\mu_{<0>}\wedge\mu'\otimes\widetilde{f}\otimes S(a_{<1>})(X_i\triangleright\mu_{<-1>})\cdot\widetilde{g}+\\
&\qquad- (X_i\cdot a)_{<0>}\omega\otimes(\theta^i\wedge\mu)_{<0>}\wedge\mu'\otimes\widetilde{f}\otimes S((X_i\triangleright a)_{<1>})(\theta^i\wedge\mu)_{<-1>}\cdot\widetilde{g}+\\
&\qquad- a_{<0>}\omega\otimes(\theta^i\wedge\mu)_{<0>}\otimes X_i\bullet\widetilde{f}\otimes S(a_{<1>})(\theta^i\wedge\mu)_{<-1>}\cdot\widetilde{g}.
\end{aligned}$$

Invoking next Lemmas 4.4 and 4.5,

$$\begin{aligned}
&d_{\mathrm{CE}}\Big(a\otimes\mu\otimes\widetilde{f}\Big)\cup(\omega\otimes\mu'\otimes\widetilde{g})\\
&\quad= a_{<0>}\omega\otimes d_{\mathrm{DR}}(\mu_{<0>})\wedge\mu'\otimes\widetilde{f}\otimes S(a_{<1>})\mu_{<-1>}\cdot\widetilde{g}+\\
&\qquad- a_{<0>}\omega\otimes\theta^i\wedge\mu_{<0>}\wedge\mu'\otimes\widetilde{f}\otimes S(a_{<1>})(X_i\triangleright\mu_{<-1>})\cdot\widetilde{g}+\\
&\qquad- (X_i\cdot a_{<0>})\omega\otimes\theta^i\wedge\mu_{<0>}\wedge\mu'\otimes\widetilde{f}\otimes S(a_{<1>})\mu_{<-1>}\cdot\widetilde{g}+\\
&\qquad- a_{<0>}\omega\otimes(\theta^i\wedge\mu)_{<0>}\wedge\mu'\otimes\widetilde{f}\otimes(X_i\triangleright S(a_{<1>}))\mu_{<-1>}\cdot\widetilde{g}+\\
&\qquad- a_{<0>}\omega\otimes\theta^i\wedge\mu_{<0>}\otimes X_i\bullet\widetilde{f}\otimes S(a_{<1>})\mu_{<-1>}\cdot\widetilde{g}.
\end{aligned}\tag{4.15}$$

Finally we see that

$$\begin{aligned}
&(a\otimes\mu\otimes\widetilde{f})\cup d_{\mathrm{CE}}\Big(\omega\otimes\mu'\otimes\widetilde{g}\Big)\\
&\quad=(a\otimes\mu\otimes\widetilde{f})\cup(\omega\otimes d_{\mathrm{DR}}(\mu')\otimes\widetilde{g}-X_i\cdot\omega\otimes\theta^i\wedge\mu'\otimes\widetilde{g}\\
&\qquad-\omega\otimes\theta^i\wedge\mu'\otimes X_i\bullet\widetilde{g})\\
&\quad=a_{<0>}\omega\otimes\mu_{<0>}\wedge d_{\mathrm{DR}}(\mu')\otimes S(a_{<1>})\mu_{<-1>}\cdot\widetilde{g}+\\
&\qquad- a_{<0>}(X_i\cdot\omega)\otimes\mu_{<0>}\wedge\theta^i\wedge\mu'\otimes S(a_{<1>})\mu_{<-1>}\cdot\widetilde{g}+\\
&\qquad- a_{<0>}\omega\otimes\mu_{<0>}\wedge\theta^i\wedge\mu'\otimes S(a_{<1>})\mu_{<-1>}\cdot(X_i\bullet\widetilde{g}).
\end{aligned}\tag{4.16}$$

The claim now follows immediately from the comparison of (4.14), (4.15) and (4.16). □

We are ready to express the main result of the section.

Theorem 4.7 *The coboundary*

$$d_{\mathrm{CE}}+(-1)^p b_N^*: C^{p,q}(\Omega^{\leq 1}_{n\delta},\mathfrak{s}^*,\mathcal{F}(N))\to C^{p+1,q}(\Omega^{\leq 1}_{n\delta},\mathfrak{s}^*,\mathcal{F}(N))\oplus C^{p,q+1}(\Omega^{\leq 1}_{n\delta},\mathfrak{s}^*,\mathcal{F}(N))$$

of the total complex of the bicomplex (4.8) acts as a graded differential with respect to the product structure given by

$$(a\otimes\mu\otimes\widetilde{f})*(\omega\otimes\mu'\otimes\widetilde{g})=(-1)^{qp'}(a\otimes\mu\otimes\widetilde{f})\cup(\omega\otimes\mu'\otimes\widetilde{g})$$

for any $a \otimes \mu \otimes \widetilde{f} \in \Omega^0_{n\delta} \otimes \wedge^p \mathfrak{s}^* \otimes \mathcal{F}(N)^{\otimes q}$, *and any* $\omega \otimes \mu' \otimes \widetilde{g} \in \Omega^1_{n\delta} \otimes \wedge^{p'} \mathfrak{s}^* \otimes \mathcal{F}(N)^{\otimes q'}$.

Proof We have

$$\begin{aligned}
&(d_{\mathrm{CE}} + (-1)^{p+p'} b_N^*)\Big((a \otimes \mu \otimes \widetilde{f}) * (\omega \otimes \mu' \otimes \widetilde{g})\Big) \\
&\quad = (-1)^{qp'} d_{\mathrm{CE}}\Big((a \otimes \mu \otimes \widetilde{f}) \cup (\omega \otimes \mu' \otimes \widetilde{g})\Big) \\
&\qquad + (-1)^{qp'+p+p'} b_N^*\Big((a \otimes \mu \otimes \widetilde{f}) \cup (\omega \otimes \mu' \otimes \widetilde{g})\Big).
\end{aligned}$$

In view of Propositions 4.1, and 4.6,

$$\begin{aligned}
&(d_{\mathrm{CE}} + (-1)^{p+p'} b_N^*)\Big((a \otimes \mu \otimes \widetilde{f}) * (\omega \otimes \mu' \otimes \widetilde{g})\Big) \\
&\quad = (-1)^{qp'} d_{\mathrm{CE}}\Big(a \otimes \mu \otimes \widetilde{f}\Big) \cup (\omega \otimes \mu' \otimes \widetilde{g}) \\
&\qquad + (-1)^{qp'+p+p'} (a \otimes \mu \otimes \widetilde{f}) \cup d_{\mathrm{CE}}\Big(\omega \otimes \mu' \otimes \widetilde{g}\Big) \\
&\qquad + (-1)^{qp'+p} b_N^*\Big(a \otimes \mu \otimes \widetilde{f}\Big) \cup (\omega \otimes \mu' \otimes \widetilde{g}) \\
&\qquad + (-1)^{qp'+p+p'+q} (a \otimes \mu \otimes \widetilde{f}) \cup b_N^*\Big(\omega \otimes \mu' \otimes \widetilde{g}\Big) \\
&\quad = (d_{\mathrm{CE}} + (-1)^{p} b_N^*)\Big(a \otimes \mu \otimes \widetilde{f}\Big) * (\omega \otimes \mu' \otimes \widetilde{g}) \\
&\qquad + (-1)^{p+q} (a \otimes \mu \otimes \widetilde{f}) * (d_{\mathrm{CE}} + (-1)^{p'} b_N^*)\Big(\omega \otimes \mu' \otimes \widetilde{g}\Big).
\end{aligned}$$

□

5 The transfer of classes

5.1 Multiplicativity of the characteristic homomorphism

We show that the chacracteristic homomorphism of [50, Thm. 4.10], identifying the Hopf-cyclic cohomology of $\mathcal{H}_n$ with the Lie algebra cohomology of W_n, with non-trivial coefficients, respects the multiplicative structures on its domain and the range.

Theorem 5.1 *For the Lie algebra* $W_n = \mathfrak{s} \bowtie \mathfrak{n}$, *the Hopf algebra* $\mathcal{F}(N) \blacktriangleright\!\!\blacktriangleleft U(\mathfrak{s})$, *and the induced* $\mathcal{F}(N) \blacktriangleright\!\!\blacktriangleleft U(\mathfrak{s})$*-module* $\Omega^{\leq 1}_n$,

$$HP^*(\mathcal{F}(N) \blacktriangleright\!\!\blacktriangleleft U(\mathfrak{s}), \Omega^{\leq 1}_{n\delta}) \cong \bigoplus_{m=* \bmod 2} H^m(W_n, \Omega^{\leq 1}_n).$$

Proof In view of [50, Thm. 4.10], we need to show that the van Est type map

$$\begin{aligned}&\mathcal{V}:\Omega_n^{\leq 1}\otimes\wedge^p\mathfrak{s}^*\otimes\mathcal{F}(N)^{\otimes q}\longrightarrow\Omega_n^{\leq 1}\otimes\wedge^p\mathfrak{s}^*\otimes\wedge^q\mathfrak{n}^*\\&\mathcal{V}(\omega\otimes\mu\otimes f^1\otimes\cdots\otimes f^q)(X_1,\ldots,X_p\mid\xi_1,\ldots,\xi_q)\\&\quad=\mu(X_1,\ldots,X_p)\sum_{\sigma\in S_q}(-1)^\sigma\langle\xi_{\sigma(1)},\ f^1\rangle\ldots\langle\xi_{\sigma(q)},\ f^q\rangle\omega\end{aligned}\tag{5.1}$$

from the total complex of the bicomplex (4.8) to that of (3.6) is a quasi-isomorphism. This, in turn, follows at once from [50, Lemma 4.1] given the non-degenerate pairing [49, (3.50)], see also [8], between $\mathcal{F}(N)$ and $U(\mathfrak{n})$. □

The following is our main result.

Theorem 5.2 *The quasi-isomorphism* (5.1) *is multiplicative, i.e.*

$$\mathcal{V}\Big((a\otimes\mu\otimes\widetilde{f})*(\omega\otimes\mu'\otimes\widetilde{g})\Big)=\mathcal{V}(a\otimes\mu\otimes\widetilde{f})\cup\mathcal{V}(\omega\otimes\mu'\otimes\widetilde{g}).$$

Proof For $a\otimes\mu\otimes\widetilde{f}\in C^{p,q}(\Omega^{\leq 1}_{n\delta},\mathfrak{s}^*,\mathcal{F}(N))$, and $\omega\otimes\mu'\otimes\widetilde{g}\in C^{p',q'}(\Omega^{\leq 1}_{n\delta},\mathfrak{s}^*,\mathcal{F}(N))$, we observe that

$$\begin{aligned}&\mathcal{V}\Big((a\otimes\mu\otimes\widetilde{f})*(\omega\otimes\mu'\otimes\widetilde{g})\Big)(X_1,\ldots,X_{p+p'}\mid\xi_1,\ldots,\xi_{q+q'})\\&\quad=(-1)^{qp'}\mathcal{V}\Big(a_{<0>}\omega\otimes\mu_{<0>}\wedge\mu'\otimes\widetilde{f}\otimes S(a_{<1>})\mu_{<1>}\cdot\widetilde{g}\Big)\\&\qquad\times(X_1,\ldots,X_{p+p'}\mid\xi_1,\ldots,\xi_{q+q'})\\&\quad=(-1)^{qp'}\Big\langle\mu_{<0>}\wedge\mu',X_1,\ldots,X_{p+p'}\Big\rangle\\&\qquad\times\sum_{\sigma\in S_{q+q'}}(-1)^\sigma\Big\langle\widetilde{f}\otimes S(a_{<1>})\mu_{<-1>}\cdot\widetilde{g},\xi_{\sigma(1)},\ldots,\xi_{\sigma(q+q')}\Big\rangle a_{<0>}\omega\\&\quad=(-1)^{qp'}\Big\langle\mu\wedge\mu',X_1,\ldots,X_{p+p'}\Big\rangle\sum_{\sigma\in S_{q+q'}}(-1)^\sigma\Big\langle\widetilde{f}\otimes\widetilde{g},\xi_{\sigma(1)},\ldots,\xi_{\sigma(q+q')}\Big\rangle a\omega,\end{aligned}$$

where on the last equality we used the fact that the Lie algebra elements are primitive, and that the (non-identity) elements of $\mathcal{F}(N)$ are zero under the counit (when evaluated on the identity). Employing the anti-symmetrization map $\widetilde{ant}:C^n(\mathcal{F}(N),V)\to C^n(\mathfrak{n},V)$, see for instance [50, Subsect. 4.1], and setting $\widetilde{ant(f)}:=ant(f^1)\wedge\cdots\wedge ant(f^q)$ corresponding to $\widetilde{f}:=f^1\otimes\cdots\otimes f^q$, we may rewrite the cup product as

$$\mathcal{V}\Big((a\otimes\mu\otimes\widetilde{f})*(\omega\otimes\mu'\otimes\widetilde{g})\Big)=(-1)^{qp'}a\omega\otimes\mu\wedge\mu'\otimes\widetilde{ant(f)}\wedge\widetilde{ant(g)}.$$

The claim now follows from (3.8). □

A few words on the above results are in order. We recall that the multiplicative generators of the cohomology on the range are already known, see [17, Thm. 2.2.7],

and the Sect. 3.1 above. In addition, it is shown in Theorem 5.1 that the van Est type map (5.1) is an isomorphism on the level of the cohomologies. Hence, by Theorem 5.2 the (multiplicative) generators of the Gelfand–Fuks cohomology (which are the characteristic classes of foliations) can be pulled back to the Hopf-cyclic cohomology of $\mathcal{H}_n$. More explicitly, $\mathcal{V}^{-1}(\lambda_k) \in C^{2k-1}(\mathcal{F}(N) \blacktriangleright\!\!< U(\mathfrak{s}), \Omega^{\leq 1}_{n\delta})$ and $\mathcal{V}^{-1}(\Lambda) \in C^1(\mathcal{F}(N) \blacktriangleright\!\!< U(\mathfrak{s}), \Omega^{\leq 1}_{n\delta})$, subject to the relations (3.3), form a basis for the Hopf-cyclic cohomology of $\mathcal{H}_n$, with coefficients in $\Omega^{\leq 1}_{n\delta}$. In the next subsection we shall illustrate this pull-back procedure for $n = 1$, and demonstrate the inverse images under (5.1) of the classes (3.4) and (3.5).

5.2 The Hopf-cyclic classes

We illustrate the transfer of classes in the case of $n = 1$. For the ease of the presentation we are going to work with the representatives in the completion of the Lie algebra W_1 with respect to the natural topology (the strict inductive limit topology of [2,3]), and of the Hopf algebra $\mathcal{H}_1$, and of the projected tensor product $\widehat{\otimes}_\pi$ (to which we shall keep referring as $\otimes$). For convenience, we refer the reader to [53] for the Hopf-cyclic cohomology for the topological Hopf-algebras.

Let us first note that we shall adopt the basis $\{e_i \mid i \geq -1\}$ of the Lie algebra W_1, [17, Subsect. 1.1.2], and the basis $\{f^i \mid i \geq 0\}$ of the W_1-module Ω^1_1, [1, Sect. 5.3], where the (left) W_1-action is given by

$$e_i \cdot f^j = (i + j + 1) f^{i+j}.$$

Below we shall also use the right action $f^j \cdot e_i := -e_i \cdot f^j$.

As it is noted, the cohomology $H^*(W_1, \Omega^{\leq 1}_1)$ is generated by the classes (3.4) and (3.5). More explicitly, if

$$\begin{aligned} \xi &= c_{-1}e_{-1} + c_0e_0 + c_1e_1 + \cdots \\ &= c_{-1}\frac{\partial}{\partial x} + c_0 x\frac{\partial}{\partial x} + c_1 x^2\frac{\partial}{\partial x} + \cdots \end{aligned}$$

one has

$$\lambda(\xi) = c_0 + 2c_1x + 3c_2x^2 + \cdots$$

that is, setting $\{\theta^i \mid i \geq -1\}$ such that $\langle \theta^i,\ e_j \rangle = \delta^i_j$,

$$\lambda = \mathbf{1} \otimes \theta^0 + \sum_{i\geq 1} (i + 1)x^i \otimes \theta^i \in C^{0,1}(\Omega^{\leq 1}_1, \mathfrak{s}^*, \mathfrak{n}^*) \oplus C^{1,0}(\Omega^{\leq 1}_1, \mathfrak{s}^*, \mathfrak{n}^*), \quad (5.2)$$

and similarly

$$\Lambda = \sum_{i\geq 1} (i + 1)if^{i-1} \otimes \theta^i \in C^{1,0}(\Omega^{\leq 1}_1, \mathfrak{s}^*, \mathfrak{n}^*) \quad (5.3)$$

on the bicomplex (3.6). We note also that

$$\begin{aligned}\overrightarrow{d}_{\mathrm{CE}}(\Lambda)(e_p, e_q) &= \mu([e_p, e_q]) - e_p \cdot \Lambda(e_q) + e_q \cdot \Lambda(e_p)\\ &= (q-p)\Lambda(e_{p+q}) - e_p \cdot \Lambda(e_q) + e_q \cdot \Lambda(e_p)\\ &= (q-p)(p+q+1)(p+q) f^{p+q-1} - e_p \cdot (q+1) q f^{q-1} + e_q \cdot (p+1) p f^{p-1}\\ &= (q-p)(p+q+1)(p+q) f^{p+q-1} - (q+1) q (p+q) f^{p+q}\\ &\quad + (p+1) p (p+q) f^{p+q} = 0,\end{aligned} \tag{5.4}$$

as well as,

$$\begin{aligned}\uparrow d_{\mathrm{CE}}(\Lambda) &= \mu \cdot e_{-1} \otimes \theta^{-1} + \Lambda \cdot e_0 \otimes \theta^0\\ &= \sum_{i\geq 1} (i+1) i f^{i-1} \cdot e_{-1} \otimes \theta^{-1} \otimes \theta^i + \sum_{i\geq 1} (i+1) i f^{i-1} \otimes \theta^{-1} \otimes \theta^i \cdot e_{-1}\\ &\quad + \sum_{i\geq 1} (i+1) i f^{i-1} \cdot e_0 \otimes \theta^0 \otimes \theta^i + \sum_{i\geq 1} (i+1) i f^{i-1} \otimes \theta^0 \otimes \theta^i \cdot e_0\\ &= -\sum_{i\geq 2} (i+1) i (i-1) f^{i-2} \otimes \theta^{-1} \otimes \theta^i + \sum_{i\geq 1} (i+2)(i+1) i f^{i-1} \otimes \theta^{-1} \otimes \theta^{i+1} +\\ &\quad - \sum_{i\geq 1} (i+1) i^2 f^{i-1} \otimes \theta^0 \otimes \theta^i + \sum_{i\geq 1} (i+1) i^2 f^{i-1} \otimes \theta^0 \otimes \theta^i = 0.\end{aligned} \tag{5.5}$$

As for $\lambda \in C^{0,1}(\Omega_1^{\leq 1}, \mathfrak{s}^*, \mathfrak{n}^*) \oplus C^{1,0}(\Omega_1^{\leq 1}, \mathfrak{s}^*, \mathfrak{n}^*)$, we observe that

$$\uparrow d_{\mathrm{CE}}(\mathbf{1} \otimes \theta^0) = \mathbf{1} \cdot e_{-1} \otimes \theta^{-1} \wedge \theta^0 + \mathbf{1} \cdot e_0 \otimes \theta^0 \wedge \theta^0 + \mathbf{1} \otimes d_{\mathrm{DR}}(\theta^0) = 0. \tag{5.6}$$

On the other hand,

$$\overrightarrow{d}_{\mathrm{CE}}(\mathbf{1} \otimes \theta^0)(e_p) = e_p \cdot (\mathbf{1} \otimes \theta^0) = -\mathbf{1} \otimes \theta^0 \cdot e_p = \begin{cases} 2(\mathbf{1} \otimes \theta^{-1}), & \text{if } p = 1,\\ 0, & \text{if } p > 1,\end{cases}$$

that is,

$$\overrightarrow{d}_{\mathrm{CE}}(\mathbf{1} \otimes \theta^0) = 2(\mathbf{1} \otimes \theta^{-1} \otimes \theta^1),$$

and

$$
\begin{aligned}
&\uparrow d_{\mathrm{CE}}\Big(\sum_{i\geq 1}(i+1)x^i\otimes\theta^i\Big)\\
&=\sum_{i\geq 1}(i+1)x^i\cdot e_{-1}\otimes\theta^{-1}\otimes\theta^i+\sum_{i\geq 1}(i+1)x^i\cdot e_0\otimes\theta^0\otimes\theta^i\\
&\quad+\sum_{i\geq 1}(i+1)x^i\otimes\theta^{-1}\otimes\theta^i\cdot e_{-1}+\sum_{i\geq 1}(i+1)x^i\otimes\theta^0\otimes\theta^i\cdot e_0\\
&=-\sum_{i\geq 1}(i+1)ix^{i-1}\otimes\theta^{-1}\otimes\theta^i-\sum_{i\geq 1}(i+1)ix^i\otimes\theta^0\otimes\theta^i\\
&\quad+\sum_{i\geq 1}(i+2)(i+1)x^i\otimes\theta^{-1}\otimes\theta^{i+1}+\sum_{i\geq 1}(i+1)ix^i\otimes\theta^0\otimes\theta^i\\
&=-2(\mathbf{1}\otimes\theta^{-1}\otimes\theta^1)-\sum_{i\geq 2}(i+1)ix^{i-1}\otimes\theta^{-1}\otimes\theta^i-\sum_{i\geq 1}(i+1)ix^i\otimes\theta^0\otimes\theta^i\\
&\quad+\sum_{i\geq 1}(i+2)(i+1)x^i\otimes\theta^{-1}\otimes\theta^{i+1}+\sum_{i\geq 1}(i+1)ix^i\otimes\theta^0\otimes\theta^i\\
&=-2(\mathbf{1}\otimes\theta^{-1}\otimes\theta^1).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&\vec{d}_{\mathrm{CE}}\Big(\sum_{i\geq 1}(i+1)x^i\otimes\theta^i\Big)(e_p,e_q)\\
&=\sum_{i\geq 1}(i+1)x^i\theta^i([e_p,e_q])-e_p\cdot\sum_{i\geq 1}(i+1)x^i\theta^i(e_q)+e_q\cdot\sum_{i\geq 1}(i+1)x^i\theta^i(e_p)\\
&=(q-p)(p+q+1)x^{p+q}-(q+1)(p+q+1)x^{p+q}\\
&\quad+(p+1)(p+q+1)x^{p+q}=0.
\end{aligned}
$$

Referring the reader to [41] for details on spectral sequences, we now investigate the generators of the cohomology $H^*(W_1,\Omega_1^{\leq 1})$ in the 1st page of the spectral sequence associated to the natural filtration of the bicomplex (3.6).

Proposition 5.3 *On the E_1-term of the spectral sequence corresponding to the natural filtration of the bicomplex (3.6), we have*

$$
[\mathbf{1}\otimes\theta^0]_1\in E_1^{0,1},\qquad\qquad \left[\sum_{i\geq 1}(i+1)if^{i-1}\otimes\theta^i\right]_1\in E_1^{1,0}.
$$

Proof The 0-page (E_0, d_0) of the spectral sequence consists of the vertical cohomology classes. As a result, we conclude from (5.6) and (5.5) that

$$[\mathbf{1}\otimes\theta^0]_0 \in E_0^{0,1}, \quad \left[\sum_{i\geq 1}(i+1)if^{i-1}\otimes\theta^i\right]_0 \in E_0^{1,0}.$$

On the E_1-level we have $d_1 : E_1^{p,q} \to E_1^{p+1,q}$, that is, horizontal coboundary map acting on the vertical cohomology classes. We then note that

$$d_1[\mathbf{1}\otimes\theta^0]_1 = \left[\overrightarrow{d}_{\mathrm{CE}}(\mathbf{1}\otimes\theta^0)\right]_1 = \left[-\uparrow d_{\mathrm{CE}}\left(\sum_{i\geq 1}(i+1)x^i\otimes\theta^i\right)\right]_1 = [0]_1,$$

hence

$$[\mathbf{1}\otimes\theta^0]_1 \in E_1^{0,1}.$$

On the other hand, (5.4) implies that

$$\left[\sum_{i\geq 1}(i+1)if^{i-1}\otimes\theta^i\right]_1 \in E_1^{1,0}.$$

□

We now pull these two classes back to the Hopf-cyclic bicomplex (4.2). Once again we recall from [1, Subsect. 4.2] the affine coordinates $\{\mathbf{x}_i \mid i \geq 1\}$ of N, which are given by

$$\mathbf{x}_i(e_J) = \begin{cases} 1, & \text{if } J = (i), \\ 0, & \text{otherwise}, \end{cases} \tag{5.7}$$

where $e_J := e_{j_1}\ldots e_{j_n}$ for $J = (j_1, \ldots, j_n)$.

Proposition 5.4 *On the E_1-term of the spectral sequence corresponding to the natural filtration of the bicomplex (4.8), we have*

$$[\mathbf{1}\otimes\theta^0]_1 \in E_1^{0,1}, \left[\sum_{i\geq 1}(i+1)if^{i-1}\otimes\mathbf{x}_i\right]_1 \in E_1^{1,0}.$$

Proof We have by [50, Thm. 4.10] that the characteristic homomorphism (5.1) is an isomorphism on the E_1-level of the spectral sequences associated to the natural filtrations of the bicomplexes (3.6) and (4.8). It already follows from (5.5) that

$$\left[\sum_{i\geq 1}(i+1)if^{i-1}\otimes\mathbf{x}_i\right]_0 \in E_0^{1,0},$$

and since (5.1) is an isomorphism of the E_1-terms,

$$b_N^*\left(\sum_{i\geq 1}(i+1)if^{i-1}\otimes \mathbf{x}_i\right)\in E_1^{2,0}$$

is a vertical coboundary. But then, since it resides in the 0th row, we conclude that

$$b_N^*\left(\sum_{i\geq 1}(i+1)if^{i-1}\otimes \mathbf{x}_i\right)=0.$$

Furthermore, in view of [59, Mapping Lemma 5.2.4], (5.1) is also a map of E_r-terms as well, for any $r\geq 1$. Thus, from

$$b_N^*(\mathbf{1}\otimes\theta^0)=-2(\mathbf{1}\otimes\theta^{-1}\otimes\mathbf{x}_1)=d_{\mathrm{CE}}\left(\sum_{i\geq 1}(i+1)x^i\otimes\mathbf{x}_i\right),$$

we conclude that

$$b_N^*\left(\sum_{i\geq 1}(i+1)x^i\otimes\mathbf{x}_i\right)=0.$$

□

Corollary 5.5 *The total cohomology of the bicomplex* (4.8) *is generated by the classes*

$$\lambda':=\mathbf{1}\otimes\theta^0\oplus\sum_{i\geq 1}(i+1)x^i\otimes\mathbf{x}_i\in C^{0,1}(\Omega^{\leq 1}_{1\delta},\mathfrak{s}^*,\mathcal{F}(N))\oplus C^{1,0}(\Omega^{\leq 1}_{1\delta},\mathfrak{s}^*,\mathcal{F}(N)) \tag{5.8}$$

and

$$\Lambda':=\sum_{i\geq 1}(i+1)f^i\otimes\mathbf{x}_i\in C^{1,0}(\Omega^{\leq 1}_{1\delta},\mathfrak{s}^*,\mathcal{F}(N)). \tag{5.9}$$

Applying the Poincaré duality (4.7), we push the above classes to

$$\mathbf{1}\otimes e_{-1}\oplus\sum_{i\geq 1}(i+1)x^i\otimes e_{-1}\wedge e_0\otimes\mathbf{x}_i\in C^{0,1}(\Omega^{\leq 1}_{1\delta},\mathfrak{s},\mathcal{F}(N))$$
$$\oplus C^{1,2}(\Omega^{\leq 1}_{1\delta},\mathfrak{s},\mathcal{F}(N)),$$

and

$$\sum_{i\geq 1}(i+1)f^i\otimes e_{-1}\wedge e_0\otimes\mathbf{x}_i\in C^{1,2}(\Omega^{\leq 1}_{1\delta},\mathfrak{s},\mathcal{F}(N)).$$

We then observe that

$$\partial_{\mathrm{CE}}(\mathbf{1}\otimes e_{-1}\wedge e_0)=\mathbf{1}\otimes[e_{-1},\ e_0]=\mathbf{1}\otimes e_{-1}$$

and that

$$b_N(\mathbf{1}\otimes e_{-1}\wedge e_0)=0.$$

Hence the former class is cohomologous to

$$\sum_{i\geq 1}(i+1)x^i\otimes e_{-1}\wedge e_0\ \otimes\ \mathbf{x}_i\in C^{1,2}(\Omega^{\leq 1}_{1\delta},\mathfrak{s},\mathcal{F}(N)).$$

On the next move, we apply the anti-symmetrization map (4.5) to get

$$\frac{1}{2}\sum_{i\geq 1}(i+1)x^i\otimes(e_{-1}\otimes e_0-e_0\otimes e_{-1})\otimes\mathbf{x}_i\in C^{1,2}(\Omega^{\leq 1}_{1\delta},U(\mathfrak{s}),\mathcal{F}(N))$$

and

$$\frac{1}{2}\sum_{i\geq 1}(i+1)f^i\otimes(e_{-1}\otimes e_0-e_0\otimes e_{-1})\otimes\mathbf{x}_i\in C^{1,2}(\Omega^{\leq 1}_{1\delta},U(\mathfrak{s}),\mathcal{F}(N)).$$

We next carry the classes from the total complex to the diagonal subcomplex via the Alexander-Whitney map, see for instance [34]. This way we obtain

$$\begin{aligned}&\frac{1}{2}\sum_{i\geq 1}(i+1)x^i\otimes(e_{-1}\otimes e_0\otimes 1-e_0\otimes e_{-1}\otimes 1)\\&\quad\otimes 1\otimes 1\otimes\mathbf{x}_i\in\mathrm{Diag}^3(U(\mathfrak{s}),\mathcal{F}(N),\Omega^{\leq 1}_{1\delta}),\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{2}\sum_{i\geq 1}(i+1)f^i\otimes(e_{-1}\otimes e_0\otimes 1-e_0\otimes e_{-1}\otimes 1)\\&\quad\otimes 1\otimes 1\otimes\mathbf{x}_i\in\mathrm{Diag}^3(U(\mathfrak{s}),\mathcal{F}(N),\Omega^{\leq 1}_{1\delta}).\end{aligned}$$

Finally, we apply (4.3) to get the Hopf-cyclic representatives of the classes (3.4) and (3.5). As a result, we conclude the following.

Corollary 5.6 *The classes* (3.4) *and* (3.5) *are represented, in the Hopf-cyclic cohomology of the Hopf algebra* $\mathcal{H}_1$ *with coefficients in* $\Omega^{\leq 1}_{1\delta}$*, by the 3-cocycles given*

by

$$\lambda_{Hopf} := \frac{1}{2}\sum_{i\geq 0}(i+1)x^i \otimes e_{-1}\otimes e_0 \otimes \mathbf{x}_i \\ -\sum_{i\geq 0}(i+1)x^i \otimes e_0 \otimes \mathbf{x}_1 e_0 \otimes \mathbf{x}_i - \frac{1}{2}\sum_{i\geq 0}(i+1)x^i \otimes e_0 \otimes e_{-1} \otimes \mathbf{x}_i,$$

and

$$\Lambda_{Hopf} := \frac{1}{2}\sum_{i\geq 0}(i+1)f^i \otimes e_{-1}\otimes e_0 \otimes \mathbf{x}_i \\ -\sum_{i\geq 0}(i+1)f^i \otimes e_0 \otimes \mathbf{x}_1 e_0 \otimes \mathbf{x}_i - \frac{1}{2}\sum_{i\geq 0}(i+1)f^i \otimes e_0 \otimes e_{-1} \otimes \mathbf{x}_i.$$

Proof The claim follows from

$$\begin{aligned}\lambda_{Hopf} &= \Psi_{\blacktriangleright\!\!<}\left(\frac{1}{2}\sum_{i\geq 1}(i+1)x^i \otimes (e_{-1}\otimes e_0 \otimes 1 - e_0 \otimes e_{-1}\otimes 1)\otimes 1 \otimes 1 \otimes \mathbf{x}_i\right)\\ &= \frac{1}{2}\sum_{i\geq 1}(i+1)x^i \otimes (1 \blacktriangleright\!\!< e_{-1_{<0>}}) \otimes (e_{-1_{<1>}} \blacktriangleright\!\!< e_{0_{<0>}}) \otimes (\mathbf{x}_i e_{-1_{<2>}} e_{0_{<1>}} \blacktriangleright\!\!< 1)\\ &\quad - \frac{1}{2}\sum_{i\geq 1}(i+1)x^i \otimes (1 \blacktriangleright\!\!< e_{0_{<0>}}) \otimes (e_{0_{<1>}} \blacktriangleright\!\!< e_{-1_{<0>}}) \otimes (\mathbf{x}_i e_{0_{<2>}} e_{-1_{<1>}} \blacktriangleright\!\!< 1),\end{aligned}$$

and the similar arguments for $\Lambda_{Hopf} \in C^3(\mathcal{H}_1, \Omega^{\leq 1}_{1\delta})$. □

5.3 Connection with the group cohomology

We shall now construct more compact representatives of the cocycles (5.8) and (5.9) in the group cohomology of the group N. Let us consider the bigraded space

$$C^{*,*}_{\text{pol}}(N, \mathfrak{s}, \Omega^{\leq 1}_1) := \bigoplus_{p,q\geq 0} C^{p,q}_{\text{pol}}(N, \mathfrak{s}, \Omega^{\leq 1}_1),$$
$$C^{p,q}_{\text{pol}}(N, \mathfrak{s}, \Omega^{\leq 1}_1) := C^q_{\text{pol}}(N, \Omega^{\leq 1}_1 \otimes \wedge^p \mathfrak{s}^*) \tag{5.10}$$

where $C^q_{\text{pol}}(N, \Omega^{\leq 1}_{1\delta} \otimes \wedge^p \mathfrak{s}^*)$ refers to the space of polynomial q-cochains of the group cohomology of N, with coefficients in the N-module $\Omega^{\leq 1}_{1\delta} \otimes \wedge^p \mathfrak{s}^*$, see for instance [29, Sect. 2]. Namely, the set of (homogeneous) polynomial cochains

$$\phi : \underbrace{N \times \cdots \times N}_{(q+1)-many} \longrightarrow \Omega^{\leq 1}_1 \otimes \wedge^p \mathfrak{s}^*,$$

satisfying

$$\phi(\psi\psi_0, \ldots, \psi\psi_q) = \psi \cdot \phi(\psi_0, \ldots, \psi_q),$$

together with the coboundary

$$\begin{aligned} &b_N : C^q_{\mathrm{pol}}(N, \Omega^{\leq 1}_1 \otimes \wedge^p \mathfrak{s}^*) \longrightarrow C^{q+1}_{\mathrm{pol}}(N, \Omega^{\leq 1}_1 \otimes \wedge^p \mathfrak{s}^*), \\ &b_N(\phi)(\psi_0, \ldots, \psi_{q+1}) = \sum_{i=0}^{q+1} (-1)^i \, \phi(\psi_0, \ldots, \widehat{\psi_i}, \ldots, \psi_{q+1}). \end{aligned} \tag{5.11}$$

The action of the group N on Ω^1_1 is given explicitly by

$$f(x)dx \cdot \psi := f(\psi(x))\psi'(x)dx,$$

see [47, Sect. 1]. In addition, we introduce the coboundary

$$\begin{aligned} &b_{\mathfrak{s}} : C^q_{\mathrm{pol}}(N, \Omega^{\leq 1}_1 \otimes \wedge^p \mathfrak{s}^*) \longrightarrow C^q_{\mathrm{pol}}(N, \Omega^{\leq 1}_1 \otimes \wedge^{p+1} \mathfrak{s}^*), \\ &b_{\mathfrak{s}}(\phi)(\psi_0, \ldots, \psi_q) := d^{\Omega}_{\mathrm{CE}}(\phi(\psi_0, \ldots, \psi_q)) - \sum_{j=-1}^{0} \theta^j \wedge (e_j \triangleright \phi)(\psi_0, \ldots, \psi_q), \end{aligned} \tag{5.12}$$

where $d^{\Omega}_{\mathrm{CE}} : \Omega^{\leq 1}_1 \otimes \wedge^p \mathfrak{s}^* \to \Omega^{\leq 1}_1 \otimes \wedge^{p+1} \mathfrak{s}^*$ is the Lie algebra cohomology coboundary (with coefficients in $\Omega^{\leq 1}_1$), and for any $X \in \mathfrak{s}$ and $\phi \in C^q(N, \Omega^{\leq 1}_1 \otimes \wedge^p \mathfrak{s}^*)$,

$$(X \triangleright \phi)(\psi_1, \ldots, \psi_q) := \sum_{j=0}^{q} \frac{d}{dt}\bigg|_{t=0} \phi(\psi_0, \ldots, \psi_j \triangleleft \exp(tX), \ldots, \psi_q).$$

We thus have the following.

Proposition 5.7 *The coboundaries* (5.11) *and* b_s *commute, that is,*

$$b_N \circ b_{\mathfrak{s}} = b_{\mathfrak{s}} \circ b_N.$$

Proof On one hand we have

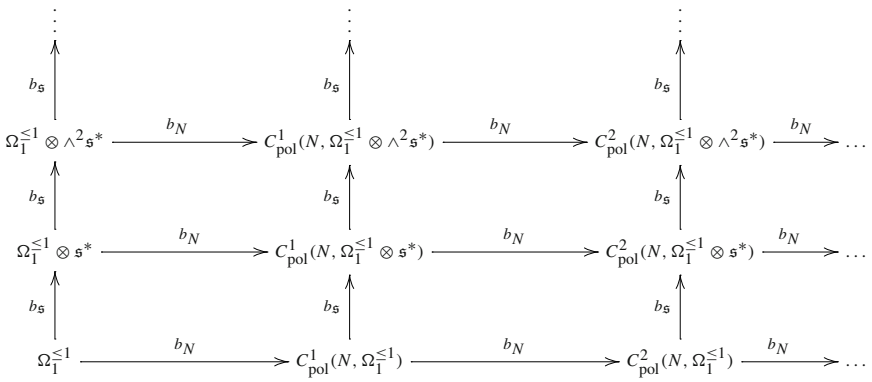
$$\begin{aligned}(b_N \circ b_{\mathfrak{s}})(\phi)(\psi_0,\dots,\psi_{q+1}) &= b_N(b_{\mathfrak{s}}(\phi))(\psi_0,\dots,\psi_{q+1})\\ &= \sum_{i=0}^{q+1}(-1)^i\, b_{\mathfrak{s}}(\phi)(\psi_0,\dots,\widehat{\psi_i},\dots,\psi_{q+1})\\ &= \sum_{i=0}^{q+1}(-1)^i\ \Big(d^{\Omega}_{\mathrm{CE}}(\phi(\psi_0,\dots,\widehat{\psi_i},\dots,\psi_{q+1}))\\ &\quad -\theta^j\wedge(e_j\triangleright\phi)(\psi_0,\dots,\widehat{\psi_i},\dots,\psi_{q+1})\Big),\end{aligned}$$

and on the other hand,

$$\begin{aligned}(b_{\mathfrak{s}}\circ b_N)(\phi)(\psi_0,\dots,\psi_{q+1}) &= b_{\mathfrak{s}}(b_N(\phi))(\psi_0,\dots,\psi_{q+1})\\ &= d^{\Omega}_{\mathrm{CE}}(b_N(\phi)(\psi_0,\dots,\psi_{q+1}))-\theta^j\wedge(e_j\triangleright b_N(\phi))(\psi_0,\cdots,\psi_q)\\ &= d^{\Omega}_{\mathrm{CE}}\left(\sum_{i=0}^{q+1}(-1)^i\,\phi(\psi_0,\dots,\widehat{\psi_i},\dots,\psi_{q+1})\right)\\ &\quad -\sum_{i=0}^{q+1}(-1)^i\,\theta^j\wedge(e_j\triangleright\phi)(\psi_0,\dots,\widehat{\psi_i},\dots,\psi_{q+1}).\end{aligned}$$

□

As a result, we arrive at the bicomplex



The bicomplex (5.10) is evidently a sub-bicomplex of

$$\begin{aligned}C^{*,*}_{\mathrm{cont}}(N,\mathfrak{s},\Omega^{\leq 1}_{\mathrm{l}}) &:= \bigoplus_{p,q\geq 0} C^{p,q}_{\mathrm{cont}}(N,\mathfrak{s},\Omega^{\leq 1}_{\mathrm{l}}),\\ C^{p,q}_{\mathrm{cont}}(N,\mathfrak{s},\Omega^{\leq 1}_{\mathrm{l}}) &:= C^q_{\mathrm{cont}}(N,\Omega^{\leq 1}_{\mathrm{l}}\otimes\wedge^p\mathfrak{s}^*)\end{aligned}\tag{5.13}$$

of continuous group cochains. Furthermore, $C^q_{\text{cont}}(N, \Omega^1_{1\delta} \otimes \wedge^p \mathfrak{s}^*)$ may be considered as the set of continuous (inhomogeneous cochains)

$$\overline{\phi} : \underbrace{N \times \cdots \times N}_{q-many} \longrightarrow \Omega^{\leq 1}_1 \otimes \wedge^p \mathfrak{s}^*,$$

via the identification $\overline{\phi}(\psi_1, \ldots, \psi_q) = \phi(\psi_1 \ldots \psi_q, \psi_2 \ldots \psi_q, \ldots, \psi_q, e)$. This way, the horizontal coboundary transforms into

$$\begin{aligned}
&\overline{b}_N : C^q(N, \Omega^{\leq 1}_1 \otimes \wedge^p \mathfrak{s}^*) \longrightarrow C^{q+1}(N, \Omega^{\leq 1}_1 \otimes \wedge^p \mathfrak{s}^*),\\
&\overline{b}_N(\overline{\phi})(\psi_1, \ldots, \psi_{q+1}) = \overline{\phi}(\psi_2, \ldots, \psi_{q+1})\\
&\quad + \sum_{i=1}^{q} (-1)^i \, \overline{\phi}(\psi_1, \ldots, \psi_i \psi_{i+1}, \ldots, \psi_{q+1})\\
&\quad + (-1)^{q+1} \overline{\phi}(\psi_1, \ldots, \psi_q) \cdot \psi_{q+1}.
\end{aligned}$$

Proposition 5.8 *We have the van Est-type isomorphism*

$$H^*_{\text{cont}}(\text{Diff}(\mathbb{R}), \Omega^{\leq 1}_1) \cong H^*(W_1, \Omega^{\leq 1}_1).$$

Proof In view of the van Est isomorphism [13, Prop. 1.5] on the vertical level, we note from [29, Lemma 1] that the E^1-term of the spectral sequence, associated to the natural filtration of the bicomplex (5.13), is identified with the E^1-term of the Cartan-Leray spectral sequence which computes, by [29, Prop. 5], the group cohomology $H^*_{\text{cont}}(\text{Diff}(\mathbb{R}), \Omega^{\leq 1}_1)$, regarding the decomposition $\text{Diff}(\mathbb{R}) = S \cdot N$.
The claim now follows from the identification of the bicomplex (5.13) with the Lie algebra cohomology bicomplex (3.6), which requires the commutativity of the inverse limit and the cohomology as follows. The (profinite) group N is given as an inverse limit, see [45, Eqn. (1.52)]:

$$N = \varprojlim_{k \to \infty} N_k.$$

We have projections $\pi_{ij} : N_i \to N_j$, from the group N_i of invertible i-jets at $0 \in \mathbb{R}$ to the group N_j of invertible j-jets at $0 \in \mathbb{R}$, for any $i \geq j$, see for instance [33, Sect. IV.13]. Therefore, the inverse system (N_i, π_{ij}) satisfies the Mittag-Leffler condition, [14], see also [12, Thm. 1]. Hence

$$H^*_{\text{cont}}(N, \Omega^{\leq 1}_1 \otimes \mathfrak{s}^*) = H^*_{\text{cont}}\left(\varprojlim_{k \to \infty} N_k, \Omega^{\leq 1}_1 \otimes \mathfrak{s}^*\right) = \varprojlim_{k \to \infty} H^*_{\text{cont}}\left(N_k, \Omega^{\leq 1}_1 \otimes \mathfrak{s}^*\right).$$

On the other hand, $\mathfrak{n}_k := \langle \{e_j \mid j \geq k\} \rangle$ being the Lie algebra of the group N_k, [33, Sect. IV.13], on the infinitesimal level we have the inverse system $(\mathfrak{n}_i, \pi_{ij})$ of Lie algebras with the projections $\pi_{ij} : \mathfrak{n}_i \to \mathfrak{n}_j$ for any $i \geq j$. As such,

$$\varinjlim_{k\to\infty} H^*_{\text{cont}}\left(N_k, \Omega^{\leq 1}_{\mathrm{I}} \otimes \mathfrak{s}^*\right) \cong \varinjlim_{k\to\infty} H^*_{\text{cont}}\left(\mathfrak{n}_k, \Omega^{\leq 1}_{\mathrm{I}} \otimes \mathfrak{s}^*\right)$$
$$= H^*_{\text{cont}}\left(\varinjlim_{k\to\infty} \mathfrak{n}_k, \Omega^{\leq 1}_{\mathrm{I}} \otimes \mathfrak{s}^*\right) = H^*(\mathfrak{n}, \Omega^{\leq 1}_{\mathrm{I}} \otimes \mathfrak{s}^*),$$

where we note for the first (van Est) isomorphism that the maximal compact subgroup of N_k, for any $k \geq 1$, is $SO(1)$. We refer the reader to [11, Thm. 2.4 & Thm. 2.5] for further identifications of these cohomologies. □

On the next step, let us consider the (coinvariant) bigraded space

$$C^{*,*}_{\text{coinv}}(\Omega^{\leq 1}_{n\delta}, \mathfrak{s}^*, \mathcal{F}(N)) = \bigoplus_{p,q\geq 0} C^{p,q}_{\text{coinv}}(\Omega^{\leq 1}_{n\delta}, \mathfrak{s}^*, \mathcal{F}(N)), \tag{5.14}$$

where

$$C^{p,q}_{\text{coinv}}(\Omega^{\leq 1}_{n\delta}, \mathfrak{s}^*, \mathcal{F}(N)) := \left(\Omega^{\leq 1}_{n\delta} \otimes \wedge^p \mathfrak{s}^* \otimes \mathcal{F}(N)^{\otimes q+1}\right)^{\mathcal{F}(N)}$$
$$= \left\{v \otimes \mu \otimes \widetilde{f} \mid v_{<0>} \otimes \mu_{<0>} \otimes \widetilde{f} \otimes S(v_{<1>}\mu_{<1>}) = v \otimes \mu \otimes \widetilde{f}_{<0>} \otimes \widetilde{f}_{<1>}\right\},$$

and

$$\widetilde{f}_{<0>} \otimes \widetilde{f}_{<1>} := f^0{}_{(1)} \otimes \cdots \otimes f^q{}_{(1)} \otimes f^0{}_{(2)} \ldots f^q{}_{(2)}.$$

As in [45, Prop. 1.15], (5.14) can be identified with the bicomplex (4.12).

Proposition 5.9 *The mapping*

$$\mathcal{I} : C^{p,q}(\Omega^{\leq 1}_{n\delta}, \mathfrak{s}^*, \mathcal{F}(N)) \longrightarrow C^{p,q}_{\text{coinv}}(\Omega^{\leq 1}_{n\delta}, \mathfrak{s}^*, \mathcal{F}(N))$$

given by

$$\mathcal{I}(v \otimes \mu \otimes f^1 \otimes \cdots \otimes f^q)$$
$$:= v_{<0>} \otimes \mu_{<0>} \otimes f^1{}_{(1)} \otimes S(f^1{}_{(2)}) f^2{}_{(1)} \otimes \cdots \otimes S(f^{q-1}{}_{(2)}) f^q{}_{(1)}$$
$$\otimes S(v_{<1>}\mu_{<1>} f^q{}_{(2)})$$

is an isomorphism.

Proof We first show that the image is indeed in the coinvariant bicomplex. To this end we note that

$$
\begin{aligned}
&\left(v_{<0>}\otimes\mu_{<0>}\right)_{<0>}\otimes f^1{}_{(1)}\otimes S(f^1{}_{(2)})f^2{}_{(1)}\otimes\cdots\\
&\quad\otimes S(f^{q-1}{}_{(2)})f^q{}_{(1)}\otimes S(v_{<1>}\mu_{<1>}f^q{}_{(2)})\otimes S(\left(v_{<0>}\otimes\mu_{<0>}\right)_{<1>})\\
&=v_{<0>}\otimes\mu_{<0>}\otimes f^1{}_{(1)}\otimes S(f^1{}_{(2)})f^2{}_{(1)}\otimes\cdots\\
&\quad\otimes S(f^{q-1}{}_{(2)})f^q{}_{(1)}\otimes S(v_{<2>}\mu_{<2>}f^q{}_{(2)})\otimes S(v_{<1>}\mu_{<1>})\\
&=v_{<0>}\otimes\mu_{<0>}\otimes f^1{}_{(1)}\otimes S(f^1{}_{(4)})f^2{}_{(1)}\otimes\cdots\otimes S(f^{q-1}{}_{(4)})f^q{}_{(1)}\otimes S(v_{<2>}\mu_{<2>}f^q{}_{(4)})\\
&\quad\otimes f^1{}_{(2)}S(f^1{}_{(3)})f^2{}_{(2)}\ldots S(f^{q-1}{}_{(3)})f^q{}_{(2)}S(v_{<1>}\mu_{<1>}f^q{}_{(3)})\\
&=v_{<0>}\otimes\mu_{<0>}\otimes\left(f^1{}_{(1)}\right)_{(1)}\otimes\left(S(f^1{}_{(2)})f^2{}_{(1)}\right)_{(1)}\otimes\cdots\\
&\quad\otimes\left(S(f^{q-1}{}_{(2)})f^q{}_{(1)}\right)_{(1)}\otimes S(v_{<1>}\mu_{<1>}f^q{}_{(2)})_{(1)}\\
&\quad\otimes\left(f^1{}_{(1)}\right)_{(2)}\left(S(f^1{}_{(2)})f^2{}_{(1)}\right)_{(2)}\ldots\left(S(f^{q-1}{}_{(2)})f^q{}_{(1)}\right)_{(2)}S(v_{<1>}\mu_{<1>}f^q{}_{(2)})_{(2)}.
\end{aligned}
$$

Next, we observe the invertibility by introducing

$$
\begin{aligned}
\mathcal{I}^{-1}(v\otimes\mu\otimes f^0\otimes\cdots\otimes f^q) &:= v\otimes\mu\otimes f^0{}_{(1)}\otimes f^0{}_{(2)}f^1{}_{(1)}\\
&\quad\otimes\cdots\otimes f^0{}_{(q)}\ldots f^{q-2}{}_{(2)}f^{q-1}\varepsilon(f^q).
\end{aligned}
$$

Indeed,

$$
\begin{aligned}
&\mathcal{I}(\mathcal{I}^{-1}(v\otimes\mu\otimes f^0\otimes\cdots\otimes f^q))\\
&\quad=\mathcal{I}(v\otimes\mu\otimes f^0{}_{(1)}\otimes f^0{}_{(2)}f^1{}_{(1)}\otimes\cdots\otimes f^0{}_{(q)}\ldots f^{q-2}{}_{(2)}f^{q-1}\varepsilon(f^q))\\
&\quad=v_{<0>}\otimes\mu_{<0>}\otimes f^0{}_{(1)(1)}\otimes S(f^0{}_{(1)(2)})f^0{}_{(2)(1)}f^1{}_{(1)(1)}\otimes\cdots\\
&\qquad\otimes S(f^0{}_{(q-1)(2)}\ldots f^{q-2}{}_{(1)(2)})f^0{}_{(q)(1)}\ldots f^{q-2}{}_{(2)(1)}f^{q-1}{}_{(1)}\\
&\qquad\otimes S(v_{<1>}\mu_{<1>}f^0{}_{(q)(2)}\ldots f^{q-2}{}_{(2)(2)}f^{q-1}{}_{(2)}))\varepsilon(f^q)\\
&\quad=v_{<0>}\otimes\mu_{<0>}\otimes f^0{}_{(1)}\otimes\cdots\otimes f^{q-1}{}_{(1)}\\
&\qquad\otimes\varepsilon(f^q)S(f^0{}_{(2)}\ldots f^{q-2}{}_{(2)}f^{q-1}{}_{(2)})S(v_{<1>}\mu_{<1>})\\
&\quad=v\otimes\mu\otimes f^0{}_{(1)}\otimes\cdots\otimes f^{q-1}{}_{(1)}\otimes\varepsilon(f^q{}_{(1)})\\
&\qquad S(f^0{}_{(2)}\ldots f^{q-2}{}_{(2)}f^{q-1}{}_{(2)})f^0{}_{(3)}\ldots f^{q-1}{}_{(3)}f^q{}_{(2)}\\
&\quad=v\otimes\mu\otimes f^0\otimes\cdots\otimes f^q,
\end{aligned}
$$

where we used the coinvariance condition in the fourth equality. Similarly we may observe

$$
\mathcal{I}^{-1}(\mathcal{I}(v\otimes\mu\otimes f^1\otimes\cdots\otimes f^q))=v\otimes\mu\otimes f^1\otimes\cdots\otimes f^q.
$$

□

As a result, by the transfer of structure, we obtain

$$d_{\mathrm{CE}}^{\mathrm{coinv}} := \mathcal{I} \circ d_{\mathrm{CE}} \circ \mathcal{I}^{-1} : C^{p,q}_{\mathrm{coinv}}(\Omega^{\leq 1}_{n\delta}, \mathfrak{s}^*, \mathcal{F}(N)) \longrightarrow C^{p+1,q}_{\mathrm{coinv}}(\Omega^{\leq 1}_{n\delta}, \mathfrak{s}^*, \mathcal{F}(N))$$
$$b_N^{*\,\mathrm{coinv}} := \mathcal{I} \circ b_N^* \circ \mathcal{I}^{-1} : C^{p,q}_{\mathrm{coinv}}(\Omega^{\leq 1}_{n\delta}, \mathfrak{s}^*, \mathcal{F}(N)) \longrightarrow C^{p,q+1}_{\mathrm{coinv}}(\Omega^{\leq 1}_{n\delta}, \mathfrak{s}^*, \mathcal{F}(N))$$

Now we identify, just as [45, Prop. 1.16] the coinvariant bicomplex (5.14) with the bicomplex (5.10).

Proposition 5.10 *The map*

$$\mathcal{J} : C^{*,*}_{\mathrm{coinv}}(\Omega^{\leq 1}_{n\delta}, \mathfrak{s}^*, \mathcal{F}(N)) \to C^*_{\mathrm{pol}}(N, \Omega^{\leq 1}_1 \otimes \wedge^* \mathfrak{s}^*),$$

given by

$$\mathcal{J}(v \otimes \mu \otimes f^0 \otimes \cdots \otimes f^q)(\psi_0, \ldots, \psi_q) = v \otimes \mu\, f^0(\psi_0) \ldots f^q(\psi_q),$$

is an isomorphism of bicomplexes.

Proof We begin with the well-definedness. Indeed,

$$\begin{aligned}
&\mathcal{J}(v \otimes \mu \otimes f^0 \otimes \cdots \otimes f^q)(\psi_0\psi, \ldots, \psi_q\psi) \\
&\quad = \mathcal{J}(v \otimes \mu \otimes f^0{}_{(1)} \otimes \cdots \otimes f^q{}_{(1)} \otimes f^0{}_{(2)} \ldots f^q{}_{(2)})(\psi_0, \ldots, \psi) \\
&\quad = \mathcal{J}(v_{<0>} \otimes \mu_{<0>} \otimes f^0 \otimes \cdots \otimes f^q \otimes S(v_{<1>}\mu_{<1>}))(\psi_0, \ldots, \psi_q, \psi) \\
&\quad = v_{<0>} \otimes \mu_{<0>}\, f^0(\psi_0) \ldots f^q(\psi_q) S(v_{<1>}\mu_{<1>})(\psi) \\
&\quad = v_{<0>} \otimes \mu_{<0>}\, f^0(\psi_0) \ldots f^q(\psi_q)(v_{<1>}\mu_{<1>})(\psi^{-1}) \\
&\quad = v_{<0>} \otimes \mu_{<0>}\, f^0(\psi_0) \ldots f^q(\psi_q) v_{<1>}(\psi^{-1})\mu_{<1>}(\psi^{-1}) \\
&\quad = \psi^{-1} \cdot (v \otimes \mu)\, f^0(\psi_0) \ldots f^q(\psi_q) = \Big((v \otimes \mu)\, f^0(\psi_0) \ldots f^q(\psi_q)\Big) \cdot \psi \\
&\quad = \mathcal{J}(v \otimes \mu \otimes f^0 \otimes \cdots \otimes f^q)(\psi_0, \ldots, \psi_q) \cdot \psi.
\end{aligned}$$

Let us now check the compatibility with the coboundaries. To begin with, we have

$$\begin{aligned}
&b_N(\mathcal{J}(v \otimes \mu \otimes f^0 \otimes \cdots \otimes f^q))(\psi_0, \ldots, \psi_{q+1}) \\
&\quad = \sum_{i=0}^{q+1} (-1)^i\, \mathcal{J}(v \otimes \mu \otimes f^0 \otimes \cdots \otimes f^q)(\psi_0, \ldots, \widehat{\psi_i}, \ldots, \psi_{q+1}) \\
&\quad = \sum_{i=0}^{q+1} (-1)^i\, v \otimes \mu\, f^0(\psi_0) \ldots f^i(\psi_{i+1}) \ldots f^q(\psi_{q+1}).
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
b_N^{*\mathrm{coinv}}(v\otimes\mu\otimes f^0\otimes\cdots\otimes f^q) &= \mathcal{I}\circ b_N^*\circ\mathcal{I}^{-1}(v\otimes\mu\otimes f^0\otimes\cdots\otimes f^q)\\
&= \mathcal{I}\circ b_N^*\Big(v\otimes\mu\otimes f^0{}_{(1)}\otimes f^0{}_{(2)}f^1{}_{(1)}\otimes\cdots\otimes f^0{}_{(q)}\ldots f^{q-2}{}_{(2)}f^{q-1}\varepsilon(f^q)\Big)\\
&= \mathcal{I}\Big(v\otimes\mu\otimes 1\otimes f^0{}_{(1)}\otimes f^0{}_{(2)}f^1{}_{(1)}\otimes\cdots\otimes f^0{}_{(q)}\ldots f^{q-2}{}_{(2)}f^{q-1}\varepsilon(f^q)\\
&\quad+\sum_{i=0}^{q}(-1)^i\, v\otimes\mu\otimes f^0{}_{(1)}\otimes f^0{}_{(2)}f^1{}_{(1)}\otimes\cdots\otimes\Delta(f^0{}_{(i)}f^1{}_{(i-1)}\ldots f^{i-1}{}_{(1)})\otimes\cdots\\
&\quad\otimes f^0{}_{(q)}\ldots f^{q-2}{}_{(2)}f^{q-1}\varepsilon(f^q)\\
&\quad+(-1)^{q+1}\, v_{<0>}\otimes\mu_{<0>}\otimes f^0{}_{(1)}\otimes f^0{}_{(2)}f^1{}_{(1)}\otimes\cdots\\
&\quad\otimes f^0{}_{(q)}\ldots f^{q-2}{}_{(2)}f^{q-1}\varepsilon(f^q)\otimes S(v_{<1>})\mu_{<-1>}\Big).
\end{aligned}
$$

Now we note, in view of the coinvariance property, that

$$
\begin{aligned}
&\mathcal{I}\Big(v\otimes\mu\otimes 1\otimes f^0{}_{(1)}\otimes f^0{}_{(2)}f^1{}_{(1)}\otimes\cdots\otimes f^0{}_{(q)}\ldots f^{q-2}{}_{(2)}f^{q-1}\varepsilon(f^q)\Big)\\
&\quad= v_{<0>}\otimes\mu_{<0>}\otimes 1\otimes S(1)f^0{}_{(1)(1)}\otimes S(f^0{}_{(1)(2)})f^0{}_{(2)(1)}f^1{}_{(1)(1)}\otimes\cdots\\
&\qquad\otimes S(f^0{}_{(q-1)(2)}\ldots f^{q-2}{}_{(1)(2)})f^0{}_{(q)(1)}\ldots f^{q-2}{}_{(2)(1)}f^{q-1}{}_{(1)}\varepsilon(f^q)\\
&\qquad\otimes S(v_{<1>}\mu_{<1>}f^0{}_{(q)(2)}\ldots f^{q-2}{}_{(1)(2)}f^{q-1}{}_{(2)})\\
&\quad= v\otimes\mu\otimes 1\otimes f^0\otimes\cdots\otimes f^q,
\end{aligned}
$$

that

$$
\begin{aligned}
&\mathcal{I}\Big(v\otimes\mu\otimes f^0{}_{(1)}\otimes f^0{}_{(2)}f^1{}_{(1)}\otimes\cdots\otimes\Delta(f^0{}_{(i)}f^1{}_{(i-1)}\ldots f^{i-1}{}_{(1)})\otimes\cdots\\
&\qquad\otimes f^0{}_{(q)}\ldots f^{q-2}{}_{(2)}f^{q-1}\varepsilon(f^q)\Big)\\
&\quad= v_{<0>}\otimes\mu_{<0>}\otimes f^0{}_{(1)(1)}\otimes S(f^0{}_{(1)(2)})f^0{}_{(2)(1)}f^1{}_{(1)(1)}\otimes\cdots\\
&\qquad\otimes S(f^0{}_{(i)(1)(2)}f^1{}_{(i-1)(1)(2)}\ldots f^{i-1}{}_{(1)(1)(2)})f^0{}_{(i)(2)(1)}f^1{}_{(i-1)(2)(1)}\ldots f^{i-1}{}_{(1)(2)(1)}\otimes\cdots\\
&\qquad\otimes S(v_{<1>}\mu_{<1>}f^0{}_{(q)(2)}\ldots f^{q-2}{}_{(2)(2)}f^{q-1}{}_{(2)})\\
&\quad= v\otimes\mu\otimes f^0\otimes\cdots\otimes f^{i-1}\otimes 1\otimes f^i\otimes\cdots\otimes f^q,
\end{aligned}
$$

and that

$$
\begin{aligned}
&\mathcal{I}\Big(v_{<0>}\otimes\mu_{<0>}\otimes f^0{}_{(1)}\otimes f^0{}_{(2)}f^1{}_{(1)}\otimes\cdots\\
&\qquad\otimes f^0{}_{(q)}\ldots f^{q-2}{}_{(2)}f^{q-1}\varepsilon(f^q)\otimes S(v_{<1>})\mu_{<-1>}\Big)\\
&\quad= v_{<0><0>}\otimes\mu_{<0><0>}\otimes f^0{}_{(1)(1)}\otimes S(f^0{}_{(1)(2)})f^0{}_{(2)(1)}f^1{}_{(1)(1)}\otimes\cdots\\
&\qquad\otimes S(f^0{}_{(q)(2)}\ldots f^{q-2}{}_{(2)(2)}f^{q-1}{}_{(2)}\varepsilon(f^q))S(v_{<1>}\mu_{<1>})_{(1)}\\
&\qquad\otimes S(v_{<0><1>}\mu_{<0><1>}S(v_{<1>}\mu_{<1>})_{(2)})\\
&\quad= v\otimes\mu\otimes f^0\otimes\cdots\otimes f^q\otimes 1.
\end{aligned}
$$

As a result,

$$\begin{aligned}\mathcal{J}(b_N^{*\mathrm{coinv}}(v\otimes\mu\otimes f^0\otimes\cdots\otimes f^q))(\psi_0,\ldots,\psi_{q+1})\\ =\sum_{i=0}^{q+1}(-1)^i\, v\otimes\mu\, f^0(\psi_0)\ldots f^i(\psi_{i+1})\ldots f^q(\psi_{q+1}).\end{aligned}$$

We thus observe that $\mathcal{J}\circ b_N^{*\mathrm{coinv}} = b_N\circ\mathcal{J}$. Let us proceed to $b_{\mathfrak{s}}\circ\mathcal{J}=\mathcal{J}\circ d_{\mathrm{CE}}^{\mathrm{coinv}}$, that is, $b_{\mathfrak{s}}\circ\mathcal{J}=\mathcal{J}\circ\mathcal{I}\circ d_{\mathrm{CE}}\circ\mathcal{I}^{-1}$, the compatibility with the vertical coboundary maps. To this end, we shall verify

$$\mathcal{I}^{-1}\circ\mathcal{J}^{-1}\circ b_{\mathfrak{s}}\circ\mathcal{J}=d_{\mathrm{CE}}\circ\mathcal{I}^{-1}.$$

On the one hand we have

$$\begin{aligned}&d_{\mathrm{CE}}\circ\mathcal{I}^{-1}(v\otimes\mu\otimes f^0\otimes\cdots\otimes f^q)\\ &\quad=d_{\mathrm{CE}}\left(v\otimes\mu\otimes f^0{}_{(1)}\otimes f^0{}_{(2)}f^1{}_{(1)}\otimes\cdots\otimes f^0{}_{(q)}\ldots f^{q-2}{}_{(2)}f^{q-1}\varepsilon(f^q)\right)\\ &\quad=d_{\mathrm{CE}}^{\Omega}(v\otimes\mu)\otimes f^0{}_{(1)}\otimes f^0{}_{(2)}f^1{}_{(1)}\otimes\cdots\otimes f^0{}_{(q)}\ldots f^{q-2}{}_{(2)}f^{q-1}\varepsilon(f^q)\\ &\qquad-v\otimes\theta^j\wedge\mu\otimes e_j\bullet\left(f^0{}_{(1)}\otimes f^0{}_{(2)}f^1{}_{(1)}\otimes\cdots\otimes f^0{}_{(q)}\ldots f^{q-2}{}_{(2)}f^{q-1}\varepsilon(f^q)\right),\end{aligned}$$

and on the other hand

$$\begin{aligned}&\mathcal{I}^{-1}\circ\mathcal{J}^{-1}\circ b_{\mathfrak{s}}\circ\mathcal{J}(v\otimes\mu\otimes f^0\otimes\cdots\otimes f^q)\\ &\quad=\mathcal{I}^{-1}\left(d_{\mathrm{CE}}^{\Omega}(v\otimes\mu)\otimes f^0\otimes\cdots\otimes f^q\right)\\ &\qquad-\mathcal{I}^{-1}\left(v\otimes\theta^j\wedge\mu\otimes e_j\triangleright(f^0\otimes\cdots\otimes f^q)\right)\\ &\quad=d_{\mathrm{CE}}^{\Omega}(v\otimes\mu)\otimes f^0{}_{(1)}\otimes f^0{}_{(2)}f^1{}_{(1)}\otimes\cdots\otimes f^0{}_{(q)}\ldots f^{q-2}{}_{(2)}f^{q-1}\varepsilon(f^q)\\ &\qquad-v\otimes\theta^j\wedge\mu\otimes e_j\bullet\left(f^0{}_{(1)}\otimes f^0{}_{(2)}f^1{}_{(1)}\otimes\cdots\otimes f^0{}_{(q)}\ldots f^{q-2}{}_{(2)}f^{q-1}\varepsilon(f^q)\right),\end{aligned}$$

where the last equality is a direct consequence of [45, Eq. (1.50)]. □

As a result, we now have the classes

$$\mathcal{J}\circ\mathcal{I}\left([\mathbf{1}\otimes\theta^0]_1\right)\in E_1^{0,1},\qquad \mathcal{J}\circ\mathcal{I}\left(\left[\sum_{i\geq1}(i+1)if^{i-1}\otimes\mathbf{x}_i\right]_1\right)\in E_1^{1,0}$$

in the bicomplex (5.10). We note further that, since they are not coboundaries of continuous 0-cochains, they survive in the continuous bicomplex (5.13) which is the E_1-term of the Cartan–Leray spectral sequence computing the group cohomology $H^*_{\mathrm{cont}}(\mathrm{Diff}(\mathbb{R}),\Omega_1^{\leq1})$.

Proposition 5.11 *The cochain*

$$d\ell \in C^1_{\text{cont}}(N, \Omega^{\leq 1}_1 \otimes \wedge^0 \mathfrak{s}^*),$$

given by

$$d\ell : \psi \mapsto d\log(\psi'(x))dx = \frac{\psi''(x)}{\psi'(x)}\, dx,$$

is a cocycle in the bicomplex (5.13).

Proof Let us first observe that

$$\begin{aligned} b_N(d\ell)(\psi_1, \psi_2) &= (d\ell)(\psi_2) - (d\ell)(\psi_1\psi_2) + (d\ell)(\psi_1)\cdot\psi_2 \\ &= \frac{\psi_2''(x)}{\psi_2'(x)}\, dx - \frac{(\psi_1\psi_2)''(x)}{(\psi_1\psi_2)'(x)}\, dx + \frac{\psi_1''(x)}{\psi_1'(x)}\, dx \cdot \psi_2 \\ &= \frac{\psi_2''(x)}{\psi_2'(x)}\, dx - \left[\frac{\psi_1''(\psi_2(x))\psi_2'(x)^2 + \psi_1'(\psi_2(x))\psi_2''(x)}{\psi_1'(\psi_2(x))\psi_2'(x)}\, dx\right] \\ &\quad + \frac{\psi_1''(\psi_2(x))}{\psi_1'(\psi_2(x))}\psi_2'(x)\, dx = 0. \end{aligned}$$

Next, we see that

$$\begin{aligned} b_{\mathfrak{s}}(d\ell)(\psi) &= d^{\Omega}_{\text{CE}}(d\ell(\psi)) - \theta^{-1} \wedge (e_{-1} \triangleright d\ell)(\psi) - \theta^0 \wedge (e_0 \triangleright d\ell)(\psi) \\ &= d\ell(\psi)\cdot e_{-1} \otimes \theta^{-1} + d\ell(\psi)\cdot e_0 \otimes \theta^0 \\ &\quad - d\ell(\psi \triangleleft e_{-1}) \otimes \theta^{-1} - d\ell(\psi \triangleleft e_0) \otimes \theta^0 \\ &= (d\ell(\psi)\cdot e_{-1} - d\ell(\psi \triangleleft e_{-1})) \otimes \theta^{-1} \\ &\quad + (d\ell(\psi)\cdot e_0 - d\ell(\psi \triangleleft e_0)) \otimes \theta^0. \end{aligned}$$

We thus have to recall that

$$\exp(te_{-1}) : \mathbb{R} \longrightarrow \mathbb{R}, \qquad \exp(te_{-1})(x) = x + t,$$

and

$$\exp(te_0) : \mathbb{R} \longrightarrow \mathbb{R}, \qquad \exp(te_0)(x) = tx,$$

and on the other hand, for any $\phi \in \text{Diff}(\mathbb{R})$,

$$\phi = \varphi\psi, \qquad \varphi(x) = \phi'(0)x + \phi(0), \qquad \psi = \varphi^{-1}\phi.$$

Then since the mutual actions satisfy $\psi\varphi = (\psi \triangleright \varphi)(\psi \triangleleft \varphi)$, we have

$$(\psi \triangleleft \varphi)(x) = \frac{(\psi\varphi)(x)}{(\psi\varphi)'(0)} - \frac{(\psi\varphi)(0)}{(\psi\varphi)'(0)}.$$

In particular,

$$(\psi \triangleleft \exp(te_{-1}))(x) = \frac{\psi(x+t)}{\psi'(t)} - \frac{\psi(t)}{\psi'(t)},$$

and keeping $\psi(0) = 0$ and $\psi'(0) = 1$ in mind,

$$(\psi \triangleleft \exp(te_0))(x) = \frac{\psi(tx)}{t}.$$

Hence, we have

$$d\ell(\psi \triangleleft \exp(te_{-1}))(x) = \frac{\psi''(x+t)}{\psi'(x+t)},$$

and

$$d\ell(\psi \triangleleft \exp(te_0))(x) = \frac{t\psi''(tx)}{\psi'(tx)}.$$

As a result,

$$\begin{aligned} b_{\mathfrak{s}}(d\ell)(\psi) = &\left.\frac{d}{dt}\right|_{t=0} \left[\frac{\psi''(x+t)}{\psi'(x+t)}(x+t)' - \frac{\psi''(x+t)}{\psi'(x+t)}\right] \otimes \theta^{-1} \\ &+ \left.\frac{d}{dt}\right|_{t=0} \left[\frac{\psi''(tx)}{\psi'(tx)}(tx)' - \frac{t\psi''(tx)}{\psi'(tx)}\right] \otimes \theta^0 = 0. \end{aligned}$$

□

Proposition 5.12 *The cochain*

$$\mathbf{1} \otimes \theta^0 + \ell \in C^0_{\mathrm{cont}}(N, \Omega^{\leq 1}_{\mathrm{I}} \otimes \wedge^1 \mathfrak{s}^*) \oplus C^1_{\mathrm{cont}}(N, \Omega^{\leq 1}_{\mathrm{I}} \otimes \wedge^0 \mathfrak{s}^*),$$

where

$$\ell : \psi \mapsto \log(\psi'(x)),$$

is a cocycle in the bicomplex (5.13).

Proof To begin with, we already have

$$b_{\mathfrak{s}}(\mathbf{1} \otimes \theta^0) = \mathbf{1} \cdot e_j \otimes \theta^j \wedge \theta^0 + \mathbf{1} \otimes d_{\mathrm{DR}}(\theta^0) = 0.$$

On the other hand, for the horizontal coboundary we observe from (4.11), and [1, Eqn. (3.33)] that

$$\begin{aligned} b_N(\mathbf{1}\otimes\theta^0)(\psi) &= (\mathbf{1}\otimes\theta^0)\cdot\psi = \mathbf{1}\otimes\theta^0{}_{<-1>}(\psi)\,\theta^0{}_{<0>} \\ &= \mathbf{1}\otimes\delta_1(\psi)\theta^{-1} = -\mathbf{1}\otimes\psi''(0)\theta^{-1}. \end{aligned}$$

We proceed to $\ell \in C^1_{\text{cont}}(N, \Omega^{\leq 1}_1 \otimes \wedge^0\mathfrak{s}^*)$. On one hand we have

$$\begin{aligned} b_{\mathfrak{s}}(\ell)(\psi) &= d^{\Omega}_{\text{CE}}(\ell(\psi)) - \theta^{-1}\wedge(e_{-1}\triangleright\ell)(\psi) - \theta^0\wedge(e_0\triangleright\ell)(\psi) \\ &= (\ell(\psi)\cdot e_{-1} - \ell(\psi\triangleleft e_{-1}))\otimes\theta^{-1} + (\ell(\psi)\cdot e_0 - \ell(\psi\triangleleft e_0))\otimes\theta^0 \\ &= \frac{d}{dt}\Big|_{t=0}\left[\log(\psi'(x+t))(x+t)' - \log\left(\frac{\psi'(x+t)}{\psi'(t)}\right)\right]\otimes\theta^{-1} \\ &\quad + \frac{d}{dt}\Big|_{t=0}\left[\log(\psi'(tx))(tx)' - \log(\psi'(tx))\right]\otimes\theta^0 \\ &= \psi''(0)\theta^{-1}, \end{aligned}$$

and on the other hand,

$$\begin{aligned} b_N(\ell)(\psi_1,\psi_2) &= \ell(\psi_2) - \ell(\psi_1\psi_2) + \ell(\psi_1)\cdot\psi_2 \\ &= \log(\psi_2'(x)) - \log(\psi_1'(\psi_2(x))\psi_2'(x)) + \log(\psi_1'(\psi_2(x))) = 0. \end{aligned}$$

□

Since there are the only two classes in $H^1(\text{Diff}(\mathbb{R}), \Omega^{\leq 1}_1)$, by Proposition 5.8, we conclude

$$[\mathcal{J}\circ\mathcal{I}(\lambda')]_1 = [\mathbf{1}\otimes\theta^0]_1 + [\ell]_1, \qquad [\mathcal{J}\circ\mathcal{I}(\Lambda')] = [d\ell]_1.$$

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