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The Picard group of motivic $\mathcal{A}_{\mathbb{C}}(1)$

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Abstract We show that the Picard group $\mathrm{Pic}(\mathcal{A}_{\mathbb{C}}(1))$ of the stable category of modules over $\mathbb{C}$-motivic $\mathcal{A}_{\mathbb{C}}(1)$ is isomorphic to $\mathbb{Z}^4$. By comparison, the Picard group of classical $\mathcal{A}(1)$ is $\mathbb{Z}^2 \oplus \mathbb{Z}/2$. One extra copy of $\mathbb{Z}$ arises from the motivic bigrading. The joker is a well-known exotic element of order 2 in the Picard group of classical $\mathcal{A}(1)$. The $\mathbb{C}$-motivic joker has infinite order.

Keywords Picard group · Stable module category · Motivic homotopy theory · Steenrod algebra

Mathematics Subject Classification 14F42 · 20G05 · 14C22

1 Introduction

1.1 The Picard group of classical $\mathcal{A}(1)$

Let $\mathcal{A}(1)$ be the subalgebra of the classical mod 2 Steenrod algebra generated by Sq^1 and Sq^2. The stable module category $\mathrm{Stab}(\mathcal{A}(1))$ is the category whose objects are

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the finitely generated graded left $\mathcal{A}(1)$-modules, and whose morphisms are the usual $\mathcal{A}(1)$-module maps, modulo maps that factor through projective $\mathcal{A}(1)$-modules.

The stable module category $\mathrm{Stab}(\mathcal{A}(1))$ is equipped with a tensor product over $\mathbb{F}_2$. The unit of this pairing is $\mathbb{F}_2$, and an object M of $\mathrm{Stab}(\mathcal{A}(1))$ is invertible if there exists another $\mathcal{A}(1)$-module N such that $M \otimes_{\mathbb{F}_2} N$ is stably isomorphic to $\mathbb{F}_2$. The Picard group $\mathrm{Pic}(\mathcal{A}(1))$ is the set of invertible stable isomorphism classes, with group operation given by tensor product over $\mathbb{F}_2$.

In homological degree greater than zero, Ext groups over $\mathcal{A}(1)$ are invariants of stable isomorphism classes of $\mathcal{A}(1)$-modules. Thus, $\mathrm{Stab}(\mathcal{A}(1))$ is the natural category on which these Ext groups over $\mathcal{A}(1)$ are defined. These Ext groups are of topological interest because of the Adams spectral sequence

$$E_2 = \mathrm{Ext}_{\mathcal{A}(1)}(H\mathbb{F}_2^*(X), \mathbb{F}_2) \longrightarrow ko_*(X)_2^\wedge,$$

converging to 2-completed *ko*-homology.

Adams and Priddy computed $\mathrm{Pic}(\mathcal{A}(1))$ while studying infinite loop space structures on the classifying space BSO [3, Section 3]. They found that the Picard group is isomorphic to $\mathbb{Z}^2 \oplus \mathbb{Z}/2$. One copy of $\mathbb{Z}$ comes from the grading; this corresponds to shifting the grading of an $A(1)$-module without changing its structure. The other copy of $\mathbb{Z}$ comes from the algebraic loop functor that is a formal part of the stable module category; see Definition 2.15 below for more details.

The copy of $\mathbb{Z}/2$ in $\mathrm{Pic}(\mathcal{A}(1))$ is the most interesting part of the calculation. It is exotic in the sense that it doesn't follow from the formal theory of stable module categories and Picard groups. The copy of $\mathbb{Z}/2$ is generated by the joker J shown in Fig. 1. It turns out that $J \otimes_{\mathbb{F}_2} J$ is stably isomorphic to $\mathbb{F}_2$, so J has order 2 in $\mathrm{Pic}(\mathcal{A}(1))$.

1.2 The $\mathbb{C}$-motivic setting

There has been much recent work on the computational side of motivic homotopy theory. In particular, the algebraic properties of the motivic Steenrod algebra have come under close scrutiny. As part of this program, it is natural to ask about the Picard

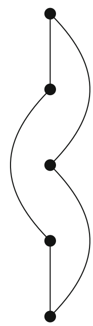


Fig. 1 The classical $\mathcal{A}(1)$-module J. Dots indicate copies of $\mathbb{F}_2$. The height of a dot reflects its grading. The central dot represents a class in degree zero. Straight lines indicate the Sq^1 action. Curved lines indicate the Sq^2 action.

group of the motivic version of $\mathcal{A}(1)$. The goal of this article is to carry out this computation for $\mathbb{C}$-motivic $\mathcal{A}_{\mathbb{C}}(1)$, which is the simplest motivic case.

The fundamental difficulty in the motivic situation is that the ground ring $\mathbb{M}_2$, *i.e.* the cohomology of a point, is not a field. Rather, it is a bigraded polynomial ring $\mathbb{F}_2[\tau]$, where $|\tau| = (0, 1)$, see for example [14]. Therefore, we must be careful to insert $\mathbb{M}_2$-freeness hypotheses at the appropriate places.

We will show that $\text{Pic}(\mathcal{A}_{\mathbb{C}}(1))$ is isomorphic to $\mathbb{Z}^4$. Two copies of $\mathbb{Z}$ arise from the motivic bigrading, and one copy of $\mathbb{Z}$ comes from the algebraic loop functor. This leaves one copy of $\mathbb{Z}$, which is generated by the motivic joker $J_{\mathbb{C}}$ (see Fig. 4). It turns out that the motivic joker has infinite order. The order of the motivic joker is the essential new aspect of the motivic calculation. Our main result is:

Theorem (5.6) *There is an isomorphism*

$$\mathbb{Z}^4 \longrightarrow \text{Pic}(\mathcal{A}_{\mathbb{C}}(1))$$

sending (a, b, c, d) *to the class of* $\Sigma^{a,b}\Omega^c J_{\mathbb{C}}^d$.

In Theorem 5.6, Σ is the bigraded shift functor, while Ω is the algebraic loop functor.

There are two main ideas in the proof. First, the Hopf algebra $\mathcal{A}_{\mathbb{C}}(1)/\tau$ is isomorphic to the group algebra of the dihedral group D_8 of order 8, so $\mathcal{A}_{\mathbb{C}}(1)/\tau$ is well-understood. In particular, the Picard group of $\mathcal{A}_{\mathbb{C}}(1)/\tau$ is known.

Second, consider the functor that takes an $\mathcal{A}_{\mathbb{C}}(1)$-module M to its quotient M/τ. In general, quotienting is not an exact functor. However, it turns out to be well-behaved for $\mathcal{A}_{\mathbb{C}}(1)$-modules that are $\mathbb{M}_2$-free. Using this well-behaved functor, we can pull back information about the Picard group of $\mathcal{A}_{\mathbb{C}}(1)/\tau$ to information about the Picard group of $\mathcal{A}_{\mathbb{C}}(1)$.

The difference between the $\mathbb{C}$-motivic and classical Picard groups is a familiar one. Frequently, motivic computations are larger than classical ones. However, they are also often more regular. This situation is clearly displayed in our work, where the motivic Picard group is free, while the classical Picard group has torsion.

There is an explicit way to compare the motivic and classical situations. Roughly speaking, setting $\tau = 1$ in the motivic setting recovers the classical setting.

Proposition (5.7) *Setting* $\tau = 1$ *induces the surjective group homomorphism*

$$\begin{aligned} \text{Pic}(\mathcal{A}_{\mathbb{C}}(1)) \cong \mathbb{Z}^4 &\longrightarrow \text{Pic}(\mathcal{A}(1)) \cong Z^2 \oplus \mathbb{Z}/2 \\ [\Sigma^{a,b}\Omega^c J_{\mathbb{C}}^d] &\longmapsto [\Sigma^a \Omega^c J^d]. \end{aligned}$$

We do not consider the Picard group of motivic $\mathcal{A}_k(1)$ over other base fields k. The $\mathbb{C}$-motivic phenomena described in this paper will occur over other base fields, but it is possible that additional complications arise.

Our computation of the Picard group of motivic $\mathcal{A}_{\mathbb{C}}(1)$ is potentially useful for the following problem. From our perspective, the most essential property of the $\mathbb{C}$-motivic spectrum $ko_{\mathbb{C}}$ is that its cohomology is isomorphic to $\mathcal{A}_{\mathbb{C}}/\!/\mathcal{A}_{\mathbb{C}}(1)$ [7]. One might ask whether such a $\mathbb{C}$-motivic spectrum is unique. Suppose that X and Y are $\mathbb{C}$-motivic spectra whose cohomology modules are both isomorphic to $\mathcal{A}_{\mathbb{C}}//\mathcal{A}_{\mathbb{C}}(1)$. In order to construct an equivalence between X and Y, one could compute the maps between X and Y via the motivic Adams spectral sequence, whose E_2-page takes the form $\mathrm{Ext}_{\mathcal{A}_{\mathbb{C}}}(\mathcal{A}_{\mathbb{C}}//\mathcal{A}_{\mathbb{C}}(1), \mathcal{A}_{\mathbb{C}}//\mathcal{A}_{\mathbb{C}}(1))$. By a standard change of rings theorem, this E_2-page is equal to $\mathrm{Ext}_{\mathcal{A}_{\mathbb{C}}(1)}(\mathbb{M}_2, \mathcal{A}_{\mathbb{C}}//\mathcal{A}_{\mathbb{C}}(1))$. It is possible that this Adams spectral sequence is analyzable, because $\mathcal{A}//\mathcal{A}_{\mathbb{C}}(1)$ probably splits as an $\mathcal{A}_{\mathbb{C}}(1)$-module into summands that belong to the Picard group. We leave the details for future work.

2 Stable module theory of Hopf $\mathbb{M}_2$-algebras

2.1 Finite motivic Hopf algebras

Although the arguments and results of this paper can be understood in purely algebraic terms, the motivation for doing this computation comes from stable $\mathbb{C}$-motivic homotopy theory. This is the reason why the expression *motivic* appears often in our terminology. Note however, that no background on motivic stable homotopy theory is required to read this paper. For the reader who is more familiar with motivic stable homotopy theory, note that our notations are consistent with [8].

The cohomology of any 2-local $\mathbb{C}$-motivic spectrum is a module over the motivic cohomology of the 2-local motivic sphere spectrum $H^{*,*}(S^{0,0}; \mathbb{F}_2)$, which is our base ring. We write $\mathbb{M}_2$ for this ring; it is isomorphic to $\mathbb{F}_2[\tau]$ with τ in bidegree $(0, 1)$. Any such module is bigraded by indices (s, w), where s corresponds to the classical internal degree and w is the motivic weight.

Let $\mathcal{A}_{\mathbb{C}}$ be the $\mathbb{C}$-motivic Steenrod algebra at the prime 2. This bigraded Hopf algebra over $\mathbb{M}_2$ was first computed in [15], and its structure is thoroughly understood. In this paper, we will be interested in its small Hopf subalgebra $\mathcal{A}_{\mathbb{C}}(1)$, whose structure is recalled at the beginning of Sect. 4. As an algebra, it has a presentation

$$\mathcal{A}_{\mathbb{C}}(1) \cong \frac{\mathbb{M}_2[\mathrm{Sq}^1, \mathrm{Sq}^2]}{\mathrm{Sq}^1\,\mathrm{Sq}^1, \mathrm{Sq}^2\,\mathrm{Sq}^2 + \tau\,\mathrm{Sq}^1\,\mathrm{Sq}^2\,\mathrm{Sq}^1, \mathrm{Sq}^1\,\mathrm{Sq}^2\,\mathrm{Sq}^1\,\mathrm{Sq}^2 + \mathrm{Sq}^2\,\mathrm{Sq}^1\,\mathrm{Sq}^2\,\mathrm{Sq}^1},$$

where $|Sq^1| = (1, 0)$, and $|Sq^2| = (2, 1)$.

This Hopf algebra over $\mathbb{M}_2$ is the motivic analogue of the subalgebra $\mathcal{A}(1)$ of the classical modulo 2 Steenrod algebra. Indeed, the classical Hopf algebra $\mathcal{A}(1)$ has a presentation

$$\mathcal{A}(1) \cong \frac{\mathbb{F}_2[\mathrm{Sq}^1, \mathrm{Sq}^2]}{\mathrm{Sq}^1\,\mathrm{Sq}^1, \mathrm{Sq}^2\,\mathrm{Sq}^2 + \mathrm{Sq}^1\,\mathrm{Sq}^2\,\mathrm{Sq}^1},$$

so that setting $\tau = 1$ in $\mathcal{A}_{\mathbb{C}}(1)$ recovers $\mathcal{A}(1)$.

A fundamental difference between motivic $\mathcal{A}_{\mathbb{C}}(1)$ and classical $\mathcal{A}(1)$ is that the base ring $\mathbb{M}_2$ is not a field. This is far from harmless, and we must add freeness over $\mathbb{M}_2$ as a hypothesis for most objects under consideration. For this reason, all Hopf $\mathbb{M}_2$-algebras under consideration in this paper will be assumed to satisfy the following hypothesis:

Hypothesis 2.1 A Hopf algebra over $\mathbb{M}_2$ will always be a cocommutative bigraded Hopf algebra over $\mathbb{M}_2$ that is finitely generated and free as an $\mathbb{M}_2$-module.

Example 2.2 Recall the subalgebras $\mathcal{A}(n)$ and $\mathcal{E}(n)$ of the classical Steenrod algebra [2]. These subalgebras have $\mathbb{C}$-motivic analogues, and they satisfy Hypothesis 2.1.

Throughout the article, A will represent a Hopf $\mathbb{M}_2$-algebra (in particular, it satisfies Hypothesis 2.1), while $\mathcal{A}_{\mathbb{C}}$ represents the $\mathbb{C}$-motivic Steenrod algebra. Note that $\mathcal{A}_{\mathbb{C}}$ is not finitely generated as an $\mathbb{M}_2$-module. However, we are primarily interested in the subalgebra $\mathcal{A}_{\mathbb{C}}(1)$ of $\mathcal{A}_{\mathbb{C}}$ generated by Sq^1 and Sq^2, and $\mathcal{A}_{\mathbb{C}}(1)$ is a finitely generated $\mathbb{M}_2$-module.

Lemma 2.3 *A finitely generated $\mathbb{M}_2$-module M is free if and only if it has no τ-torsion elements.*

Proof The ring $\mathbb{M}_2$ is a graded principal ideal domain whose graded ideals are of the form (τ^k). Therefore, a finitely generated $\mathbb{M}_2$-module is a direct sum of a free module and cyclic modules of the form $\mathbb{M}_2/\tau^k$. □

Lemma 2.4 *Let A be a finite Hopf $\mathbb{M}_2$-algebra.*

(1) An A-module M is finitely generated if and only if it is finitely generated as an $\mathbb{M}_2$-module.
(2) If M is a finitely generated projective A-module, then it is free as an $\mathbb{M}_2$-module.

Proof The first assertion follows since A is finitely generated as an $\mathbb{M}_2$-module, and conversely $\mathbb{M}_2$ is finitely generated as an A-module (with the trivial A-module structure).

For the second assertion, suppose that M is a finitely generated projective A-module. Then it is a summand of a free A-module F, which is a free $\mathbb{M}_2$-module, since A is $\mathbb{M}_2$-free. It follows that M is $\mathbb{M}_2$-free by Lemma 2.3. □

2.2 The stable category

We now recall the basic framework of stable module categories, as applied to a finite Hopf $\mathbb{M}_2$-algebra A satisfying Hypothesis 2.1. The case of a finite Hopf $\mathbb{M}_2$-algebra A satisfying Hypothesis 2.1 is similar to the case when A is a finite dimensional graded connected Hopf algebra over a field, for which a good reference is [10, Section 14.1]. However, since the base ring $\mathbb{M}_2$ of a Hopf $\mathbb{M}_2$-algebra is not a field, one has to pay attention to the underlying theory of $\mathbb{M}_2$-modules, and observe that some results require an additional $\mathbb{M}_2$-freeness hypothesis. We explain below in Lemma 2.11 why this is not restrictive for our goal, which is to compute the Picard group of stable A-modules.

Definition 2.5 Let ${}_A\mathbf{Mod}$ be the category of bigraded finitely generated left A-modules, and let ${}_A\mathbf{Mod}^{\mathrm{f}}$ be the full subcategory of ${}_A\mathbf{Mod}$ consisting of left A-modules that are free over $\mathbb{M}_2$.

Definition 2.6 Let $\mathrm{Stab}(A)$ be the category whose objects are the same as in ${}_A\mathbf{Mod}^{\mathrm{f}}$, and whose morphisms are given by

$$\mathrm{Hom}_{\mathrm{Stab}(A)}(M, N) = \mathrm{Hom}_A(M, N) \big/ \sim,$$

where two morphisms f and g are equivalent if their difference factors through a projective A-module. If M and N are objects of ${}_A\mathbf{Mod}^{\mathrm{f}}$, then we write $M \simeq N$ if M and N are stably equivalent, i.e., if they are isomorphic in the stable category $\mathrm{Stab}(A)$.

Remark 2.7 Instead of restricting to ${}_A\mathbf{Mod}^{\mathrm{f}}$, one could consider the full stable category of A-modules, without any $\mathbb{M}_2$-freeness hypothesis. Since projective A-modules are $\mathbb{M}_2$-free by Lemma 2.4, a morphism between finitely generated $\mathbb{M}_2$-free A-modules factors through a projective in ${}_A\mathbf{Mod}$ if and only if it factors through a projective in ${}_A\mathbf{Mod}^{\mathrm{f}}$. Thus, the inclusion of the subcategory ${}_A\mathbf{Mod}^{\mathrm{f}}$ in ${}_A\mathbf{Mod}$ induces a fully faithful functor between stable categories.

Our main interest is the Picard group $\mathrm{Pic}(A)$ of the stable category of some Hopf $\mathbb{M}_2$-algebra A. We will see below in Lemma 2.11 that all representatives of every element in $\mathrm{Pic}(A)$ are actually free over $\mathbb{M}_2$ and thus captured by $\mathrm{Stab}(A)$. In other words, the assumptions about $\mathbb{M}_2$-freeness in Definitions 2.5 and 2.6 are no loss of generality.

In the same vein, it is essential that we use constructions that preserve $\mathbb{M}_2$-freeness. For example for any finitely generated A-module M (not necessarily $\mathbb{M}_2$-free), the algebraic loop ΩM (defined below in Definition 2.15) is free over $\mathbb{M}_2$ by Lemma 2.3, as it is the kernel of a map from a finitely generated free $\mathbb{M}_2$-module.

The stable category $\mathrm{Stab}(A)$ is naturally enriched over A-modules, since the equivalence relation on morphisms is A-linear. The category $\mathrm{Stab}(A)$ has additional structure that we describe next.

Proposition 2.8 *The category* $\mathrm{Stab}(A)$ *is a closed symmetric monoidal category, where the monoidal structure is given by*

$$M \otimes N := M \otimes_{\mathbb{M}_2} N,$$

with the A-module structure induced by the diagonal $A \longrightarrow A \otimes_{\mathbb{M}_2} A$, and the internal hom *is defined by* $\mathrm{hom}(M, N) = \mathrm{hom}_{\mathbb{M}_2}(M, N)$*, with the A-module structure given by conjugation at the source and target.*

Proof When working over a field, this is a standard result from the theory of stable modules; see [10, Proposition 15.2.19] for example. The proof goes as in the classical case, with the $\mathbb{M}_2$-freeness assumption used in a crucial way. First, the unit $\mathbb{M}_2$ is clearly $\mathbb{M}_2$-free. The tensor product of $\mathbb{M}_2$-free modules is $\mathbb{M}_2$-free, so the tensor product over $\mathbb{M}_2$ induces a tensor product in ${}_A\mathbf{Mod}^{\mathrm{f}}$. Then, since finitely generated projective A-modules are $\mathbb{M}_2$-free by Lemma 2.4, the classical argument using the shearing isomorphism implies that the tensor product passes to the stable category. This structure is symmetric monoidal since A is cocommutative. Finally, the $\mathbb{M}_2$-module consisting of $\mathbb{M}_2$-linear maps between $\mathbb{M}_2$-free A-modules is $\mathbb{M}_2$-free, so that hom is well defined in ${}_A\mathbf{Mod}^{\mathrm{f}}$. Moreover, since our modules are finitely generated, it is clear that

$$\mathrm{hom}_{{}_A\mathbf{Mod}^{\mathrm{f}}}(M, N) = \mathrm{hom}_{{}_A\mathbf{Mod}^{\mathrm{f}}}(\mathbb{M}_2, \mathrm{hom}_{{}_A\mathbf{Mod}^{\mathrm{f}}}(M, \mathbb{M}_2) \otimes N).$$

Since the tensor product descends to the stable category, the functor hom does as well. Finally, the fact that the functors $\otimes$ and hom define a symmetric closed monoidal structure is *mutatis mutandis* the same as the proof of this fact when working over a field. □

2.3 Picard groups

Definition 2.9 Let A be a finite Hopf $\mathbb{M}_2$-algebra satisfying Hypothesis 2.1. The Picard group $\mathrm{Pic}(A)$ is the group (of isomorphism classes) of invertible objects of $\mathrm{Stab}(A)$ under the monoidal structure, *i.e.,* the group of stably invertible modules with the tensor product as group law.

Note that $\mathrm{Pic}(A)$ is an abelian group because $\mathrm{Stab}(A)$ is symmetric monoidal.

Remark 2.10 In Definition 2.9, we are only considering finitely generated A-modules. This is no loss of generality because every invertible object must be finitely generated. This follows from [11, Proposition 2.1.3], for example.

Lemma 2.11 *Let M be a finitely generated A-module. Suppose that there exists an A-module N and two maps*

$$M \otimes N \xrightarrow{f} \mathbb{M}_2 \xrightarrow{g} M \otimes N,$$

such that $gf - id_{M\otimes N}$ factors through a projective A-module. Then M is $\mathbb{M}_2$-free.

The point of Lemma 2.11 is that there is no harm in considering only $\mathbb{M}_2$-free modules in the Picard group.

Proof Let $h : M \otimes N \longrightarrow P$ be the first map appearing in the supposed factorization of $gf - id_{M\otimes N}$. Then the morphism $M \otimes N \xrightarrow{h\oplus f} P \oplus \mathbb{M}_2$ is injective. By Lemma 2.4, the target is $\mathbb{M}_2$-free. This implies that $M \otimes N$ is $\mathbb{M}_2$-free as well. Finally, $M \otimes N$ being $\mathbb{M}_2$-free implies that both M and N are $\mathbb{M}_2$-free as well by Lemma 2.3. □

Definition 2.12 Let

$$D : {}_A\mathbf{Mod}^{\mathrm{op}} \longrightarrow {}_A\mathbf{Mod} : M \longmapsto DM = \mathrm{Hom}_{\mathbb{M}_2}(M, \mathbb{M}_2),$$

be the $\mathbb{M}_2$-linear dual functor. The A-module structure on DM is defined by $af(m) = f(c(a)m)$, for $a \in A$, $m \in M$ and $f \in DM$, where c denotes the conjugation in A.

Lemma 2.13 *The $\mathbb{M}_2$-linear dual functor D induces a functor*

$$D : \mathrm{Stab}(A)^{\mathrm{op}} \longrightarrow \mathrm{Stab}(A).$$

Proof The dual functor D preserves $\mathbb{M}_2$-freeness because D is defined as Hom over $\mathbb{M}_2$.

It suffices to check that if P is A-projective, then DP is A-projective. Since the dual respects direct sums, it is enough to show that DA is projective. This follows as in [10, Theorem 12.2.9] by considering a retraction

$$DA \longrightarrow DA \otimes A \longrightarrow DA,$$

and observing that the "shearing map" [10, Proposition 12.1.4] makes $DA \otimes A$ into a free A-module. □

Lemma 2.14 shows that the dual functor D corresponds to inversion in the Picard group.

Lemma 2.14 *Let M be an A-module. The evaluation morphism $DM \otimes M \xrightarrow{ev} \mathbb{M}_2$ is a stable equivalence if and only if M is invertible. In particular, the inverse of any element* $[M]$ *in* $\mathrm{Pic}(A)$ *is its dual* $[DM]$.

Proof This fact is standard in stable module theory; see [6, Proposition A.2.8]. □

We next describe the algebraic loop functor that is part of the structure of a stable module category.

Definition 2.15 Let Ω be the endo-functor of $\mathrm{Stab}(A)$ given by

$$\Omega M = \ker(\epsilon) \otimes M,$$

where $\epsilon : A \longrightarrow \mathbb{M}_2$ is the augmentation of A. For $k \geq 0$, define $\Omega^k M$ inductively to be $\Omega(\Omega^{k-1}M)$. For $k < 0$, define $\Omega^k M$ to be $D(\Omega^{-k}DM)$.

Note that ΩM is $\mathbb{M}_2$-free because it is a tensor product of $\mathbb{M}_2$-free A-modules.

As we will see, up to a stable equivalence ΩM can be constructed using any projective cover of M. First, recall Schanuel's lemma (see for instance [9, Lemma 5.1]):

Lemma 2.16 (Schanuel's lemma) *Let R be a ring and M be an R-module. Let $K \to P_1 \to M$ and $L \to P_2 \to M$ be two short exact sequences with P_1 and P_2 two projective R-modules. Then $K \oplus P_2 \cong L \oplus P_1$.*

Let $f : P \longrightarrow M$ be any projective cover of M. By Schanuel's lemma, ΩM is stably equivalent to $\ker(f)$. The next result justifies the notation Ω^{-1} by showing that Ω^{-1} is a stable inverse of Ω. This is the analogue of [4, Proposition 2.10] in our setting.

Lemma 2.17 *The functor Ω is stably invertible, and its inverse is Ω^{-1}.*

Proof Let $P \longrightarrow M$ be a projective cover, so there is a short exact sequence

$$\Omega M \longrightarrow P \longrightarrow M.$$

Apply D to obtain the short exact sequence

$$DM \longrightarrow DP \longrightarrow D\Omega M.$$

Note that DP is free, since A is self-dual and $A \otimes M$ is free over A by the shearing map. Therefore, $DP \longrightarrow D\Omega M$ is a projective cover, and $DM \simeq \Omega D\Omega M$. This gives a stable equivalence $M \simeq \Omega^{-1}\Omega M$. □

Lemma 2.18 *Let M be stably invertible with inverse M^{-1}. Then ΩM is invertible. Moreover, the inverse of ΩM is $\Omega^{-1}M^{-1}$.*

Proof This is a standard part of the theory of stable modules when working over a field. Here the result follows from the chain of stable isomorphisms $\Omega M \otimes \Omega^{-1} M^{-1} \cong \Omega\mathbb{M}_2 \otimes \Omega^{-1}\mathbb{M}_2 \otimes M \otimes M^{-1}$. The latter is $\Omega\Omega^{-1}\mathbb{M}_2 \otimes M \otimes M^{-1}$, and both $\Omega\Omega^{-1}\mathbb{M}_2$ and $M \otimes M^{-1}$ are stably equivalent to $\mathbb{M}_2$ (the former by Lemma 2.17, the latter by assumption). □

Lemma 2.18 implies that there is a group homomorphism

$$\eta : \mathbb{Z}^3 \longrightarrow \mathrm{Pic}(A),$$

sending (m, n, s) to the stable class of $\Sigma^{m,n}\Omega^s\mathbb{M}_2$. Here $\Sigma^{m,n}$ is the suspension functor that shifts bidegrees by (m, n). This homomorphism constructs many elements in the Picard group of A. Such elements exist for essentially formal reasons and do not really reflect the structure of the underlying algebra A. In a sense, the image of η consists of uninteresting invertible elements.

3 τ quotients

Let A be a Hopf $\mathbb{M}_2$-algebra satisfying Hypothesis 2.1. Then $A/\tau = \mathbb{F}_2 \otimes_{\mathbb{M}_2} A$ is a bigraded Hopf $\mathbb{F}_2$-algebra. Since A/τ is defined over a field $\mathbb{F}_2$, it is generally easier to understand than A itself. We shall use a change of basis functor that relates our finite Hopf $\mathbb{M}_2$-algebra A to the bigraded Hopf $\mathbb{F}_2$-algebra A/τ. Since A/τ is a Hopf algebra over a field, one can consider the category ${}_{A/\tau}\mathbf{Mod}^{\mathrm{f}}$ of finitely generated A/τ-modules, and the usual stable category of bigraded A/τ-modules associated to it. We will denote it by $\mathrm{Stab}(A/\tau)$, and $\mathrm{Pic}(A/\tau)$ is the group of invertible objects in $\mathrm{Stab}(A/\tau)$. Note that $\mathcal{A}/\tau$ and $\mathrm{Pic}(A/\tau)$ are still bigraded.

Proposition 3.1 *Tensoring with the $\mathbb{M}_2$-module $\mathbb{F}_2$ induces a strongly monoidal functor*

$${}_A\mathbf{Mod}^{\mathrm{f}} \xrightarrow{(-)/\tau} {}_{A/\tau}\mathbf{Mod}^{\mathrm{f}},$$

between categories of bigraded modules that preserves exact sequences. This functor passes to the stable category of bigraded modules and thus induces a strongly monoidal functor

$$\mathrm{Stab}(A) \xrightarrow{(-)/\tau} \mathrm{Stab}(A/\tau).$$

Proof The unit $\mathbb{M}_2$ of the monoidal structure of ${}_A\mathbf{Mod}^{\mathrm{f}}$ is sent to the unit $\mathbb{F}_2$. The functor is strongly monoidal since

$$M\big/\tau \otimes N\big/\tau \cong M \otimes N\big/\tau,$$

for all bigraded $\mathbb{M}_2$-modules M and N.

Consider a short exact sequence in ${}_A\mathbf{Mod}^{\mathrm{f}}$. The sequence is split exact on the underlying free $\mathbb{M}_2$-modules. It is still split exact as a sequence of $\mathbb{F}_2$-modules after tensoring with $\mathbb{F}_2$. This shows that $(-)/\tau$ is exact.

The functor sends free A-modules to free A/τ-modules. By additivity, we conclude that it sends projective A-modules to projective A/τ-modules and thus descends to the stable categories. □

We now come to the first major result that will allow us to understand the stable module category of a Hopf $\mathbb{M}_2$-algebra A. Lemma 3.2 identifies projective A-modules in terms of their quotients by τ.

Lemma 3.2 *Let A be a Hopf $\mathbb{M}_2$-algebra, and let M be a finitely generated A-module that is $\mathbb{M}_2$-free. The following conditions are equivalent:*

(1) M is projective as an A-module.
(2) M/τ is projective as an A/τ-module.
(3) M/τ is free as an A/τ-module.

Proof Recall that A/τ is a Frobenius algebra since it is a finite dimensional Hopf algebra over the field $\mathbb{F}_2$ [10, Theorem 12.2.9]. In particular, projective A/τ-modules and free A/τ-modules are the same. This shows that conditions (2) and (3) are equivalent.

Now suppose that M is a projective A-module. Then M/τ is a projective A/τ-module by Proposition 3.1. This shows that condition (1) implies condition (2).

To show that condition (3) implies condition (1), suppose that M/τ is a free A/τ-module and fix an isomorphism $f : \oplus A/\tau \longrightarrow M/\tau$. The map $\oplus A \longrightarrow \oplus A/\tau$ is a surjective map, with projective source. Therefore the composite $\oplus A \longrightarrow \oplus A/\tau \longrightarrow M/\tau$ is also surjective with projective source. In particular, it lifts through the surjective A-module map $M \longrightarrow M/\tau$, giving a surjective morphism $\tilde{f} : \oplus A \longrightarrow M$. Finally, $\tilde{f}$ reduces to the isomorphism f modulo τ, so $\ker(\tilde{f})$ is zero. Thus, M is projective. □

Lemma 3.3 *Let M and N be finitely generated A-modules that are also $\mathbb{M}_2$-free, and let $f : M \longrightarrow N$ be a map such that $\overline{f} : M/\tau \longrightarrow N/\tau$ is injective. Then f is also injective, and the cokernel of f is $\mathbb{M}_2$-free.*

Proof Let $y \in M$ such that $f(y) = 0$. One can write $y = \tau^k x$ for k maximal, since M is a finitely generated $\mathbb{M}_2$-free module. Then

$$0 = f(y) = f(\tau^k x) = \tau^k \cdot f(x),$$

and so $f(x) = 0$ since N is $\mathbb{M}_2$-free. It follows that the corresponding element $\overline{x} \in M/\tau$ also satisfies $\overline{f}(\overline{x}) = 0$. By injectivity of $\overline{f}$, we have $\overline{x} = 0$, and thus $x = 0$ by $\mathbb{M}_2$-freeness of M. This shows that $y = 0$ and thus that f is injective.

Now consider the cokernel N/M of f. Since N/M is finitely generated, it suffices by Lemma 2.3 to consider the annihilator of τ in N/M. We will show that this annihilator is zero.

Let x be an element of N, and let $\overline{x}$ be the element of N/M that it represents. Suppose that $\tau\overline{x}$ is zero. Then τx belongs to M. Since f/τ is injective and $(f/\tau)(\overline{\tau x})$ is zero, we conclude as in the first paragraph that τx equals τy for some y in M. Since N is $\mathbb{M}_2$-free, it follows that x equals y. In particular, x belongs to M. In other words, $\overline{x}$ is zero. □

The strong monoidal exact functor

$$-/\tau : \mathrm{Stab}(A) \longrightarrow \mathrm{Stab}(A/\tau),$$

of Proposition 3.1 induces a group homomorphism

$$R_\tau : \mathrm{Pic}(A) \longrightarrow \mathrm{Pic}(A/\tau).$$

Proposition 3.4 *The map* $R_\tau : \mathrm{Pic}(A) \longrightarrow \mathrm{Pic}(A/\tau)$ *is injective.*

Proof Let M be a finitely generated A-module such that M is $\mathbb{M}_2$-free, and suppose that $[M]$ in $\mathrm{Pic}(A)$ belongs to the kernel of R_τ. Equivalently, M/τ is stably equivalent to the A/τ-module $\mathbb{F}_2$ concentrated in bidegree (0, 0). Since A/τ is a finite dimensional Frobenius algebra over $\mathbb{F}_2$, we can use [10, Proposition 14.11] to see that M/τ is isomorphic to $\mathbb{F}_2 \oplus F/\tau$, where F is a free A-module. Let j be the injection $F/\tau \longrightarrow M/\tau$.

There is a commutative diagram

$$\begin{array}{ccc} M & \twoheadrightarrow & M/\tau \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle j} \\ F & \twoheadrightarrow & F/\tau, \end{array}$$

in which the dashed arrow exists because F is A-projective and $M \longrightarrow M/\tau$ is a surjection. By Lemma 3.3, i is injective because j is injective.

We now compute the cokernel C of i. Lemma 3.3 implies that C is $\mathbb{M}_2$-free. Then Proposition 3.1 says that C/τ is isomorphic to the cokernel of j, which is $\mathbb{F}_2$ by inspection. We conclude that C is isomorphic to the A-module $\mathbb{M}_2$ generated in bidegree (0, 0).

Thus, there is a short exact sequence

$$F \hookrightarrow M \twoheadrightarrow \mathbb{M}_2,$$

so $M \twoheadrightarrow \mathbb{M}_2$ is a stable equivalence and $[M]$ is trivial in $\mathrm{Pic}(A)$. □

Remark 3.5 The proof of Proposition 3.4 implicitly contains the appropriate generalization of the notion of a reduced module in the motivic setting. When working over a field, a module M over a Hopf algebra is said to be reduced if it contains no free summands. In the motivic setting, the correct notion seems to be: an $\mathbb{M}_2$-free A-module M is free if it contains no free submodule F such that the cokernel of the inclusion $F \subset M$ is still $\mathbb{M}_2$-free.

4 The Hopf $\mathbb{M}_2$-algebra $\mathcal{A}_{\mathbb{C}}(1)$

In this section, we introduce the specific finite Hopf $\mathbb{M}_2$-algebra $\mathcal{A}_{\mathbb{C}}(1)$ whose Picard group we will compute.

Definition 4.1 The finite Hopf $\mathbb{M}_2$-algebra $\mathcal{A}_{\mathbb{C}}(1)$ is the $\mathbb{M}_2$-subalgebra of the motivic Steenrod algebra generated by Sq^1 and Sq^2.

Lemma 4.2 *The Hopf* $\mathbb{M}_2$*-algebra* $\mathcal{A}_{\mathbb{C}}(1)$ *is isomorphic to*

$$\frac{\mathbb{M}_2[\mathrm{Sq}^1, \mathrm{Sq}^2]}{\mathrm{Sq}^1 \mathrm{Sq}^1, \mathrm{Sq}^2 \mathrm{Sq}^2 + \tau \mathrm{Sq}^1 \mathrm{Sq}^2 \mathrm{Sq}^1, \mathrm{Sq}^1 \mathrm{Sq}^2 \mathrm{Sq}^1 \mathrm{Sq}^2 + \mathrm{Sq}^2 \mathrm{Sq}^1 \mathrm{Sq}^2 \mathrm{Sq}^1}.$$

The element Sq^1 *is primitive, and* $\Delta(\mathrm{Sq}^2) = \mathrm{Sq}^2 \otimes 1 + \tau \mathrm{Sq}^1 \otimes \mathrm{Sq}^1 + 1 \otimes \mathrm{Sq}^2$.

Proof This follows immediately from Voevodsky's description of the motivic Steenrod algebra [15]. □

See Fig. 2 for a picture of $\mathcal{A}_{\mathbb{C}}(1)$.

Remark 4.3 Classical $\mathcal{A}(1)$ is obtained from motivic $\mathcal{A}_{\mathbb{C}}(1)$ by setting $\tau = 1$ and by supressing the weight grading. Setting $\tau = 1$ gives a monoidal functor $(-)/(\tau - 1) : {}_{\mathbb{M}_2}\mathbf{Mod} \longrightarrow {}_{\mathbb{F}_2}\mathbf{Mod}$, where ${}_{\mathbb{M}_2}\mathbf{Mod}$ is the category of bigraded $\mathbb{M}_2$-modules, and ${}_{\mathbb{F}_2}\mathbf{Mod}$ is the category of graded $\mathbb{F}_2$-vector spaces. Indeed $|\tau| = (0, -1)$ and $|1| = (0, 0)$, so that the relation $(\tau - 1)$ is homogeneous in the internal degree (but not in the motivic weight).

The functor $(-)/(\tau - 1)$ sends $\mathcal{A}_{\mathbb{C}}(1)$ to $\mathcal{A}(1)$, as a Hopf $\mathbb{F}_2$-algebra. Consequently, we get a monoidal functor

$$(-)/(\tau - 1) : {}_{\mathcal{A}_{\mathbb{C}}(1)}\mathbf{Mod} \longrightarrow {}_{\mathcal{A}(1)}\mathbf{Mod}.$$

Since the image of a free $\mathcal{A}_{\mathbb{C}}(1)$-module is a free $\mathcal{A}(1)$-module, the functor $(-)/(\tau - 1)$ induces a monoidal functor

$$(-)/(\tau - 1) : \mathrm{Stab}(\mathcal{A}_{\mathbb{C}}(1)) \longrightarrow \mathrm{Stab}(\mathcal{A}(1)),$$

on stable module categories. In particular, we get a group homomorphism

$$\mathrm{Pic}(\mathcal{A}_{\mathbb{C}}(1)) \longrightarrow \mathrm{Pic}(\mathcal{A}(1)).$$

We will explicitly describe this group homomorphism in Proposition 5.7.

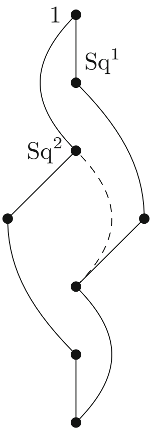


Fig. 2 The Hopf $\mathbb{M}_2$-algebra $\mathcal{A}_{\mathbb{C}}(1)$. Dots indicate copies of $\mathbb{M}_2$. The height of a dot reflects its internal degree. The class 1 is in bidegree (0, 0). Straight lines indicate the Sq^1 action. Curved lines indicate the Sq^2 action. The dashed curved line indicates that Sq^2 of an $\mathbb{M}_2$-generator equals τ times an $\mathbb{M}_2$-generator, i.e., that $\mathrm{Sq}^2\,\mathrm{Sq}^2 = \tau\,\mathrm{Sq}^1\,\mathrm{Sq}^2\,\mathrm{Sq}^1$.

When writing $\mathcal{A}_{\mathbb{C}}(1)$-modules we use the following conventions. A straight line represents the action of Sq^1, a curved line represents the action of Sq^2, and a dashed line represents that a squaring operation hits τ times a generator. For example, the dotted line in Fig. 2 shows the relation $\mathrm{Sq}^2\,\mathrm{Sq}^2 = \tau\,\mathrm{Sq}^1\,\mathrm{Sq}^2\,\mathrm{Sq}^1$.

Lemma 4.4 *As ungraded Hopf algebras, $\mathcal{A}_{\mathbb{C}}(1)/\tau$ is isomorphic to the group algebra $\mathbb{F}_2[D_8]$ of the dihedral group D_8 of order* 8.

Proof Lemma 4.2 implies that $\mathcal{A}_{\mathbb{C}}(1)/\tau$ is isomorphic to

$$\frac{\mathbb{F}_2[\mathrm{Sq}^1, \mathrm{Sq}^2]}{\mathrm{Sq}^1\,\mathrm{Sq}^1, \mathrm{Sq}^2\,\mathrm{Sq}^2, \mathrm{Sq}^1\,\mathrm{Sq}^2\,\mathrm{Sq}^1\,\mathrm{Sq}^2 + \mathrm{Sq}^2\,\mathrm{Sq}^1\,\mathrm{Sq}^2\,\mathrm{Sq}^1}.$$

For our purposes, a convenient presentation of D_8 consists of two generators x and y with the relations x^2, y^2, and $(xy)^4$. The isomorphism from $\mathcal{A}_{\mathbb{C}}(1)/\tau$ to $\mathbb{F}_2[D_8]$ takes Sq^1 to $1 + x$ and Sq^2 to $1 + y$. □

Recall that a Hopf subalgebra B of a Hopf $\mathbb{F}_2$-algebra A is elementary if it is isomorphic to an exterior algebra. Note that $Q_0 = \mathrm{Sq}^1$ and $Q_1 = \mathrm{Sq}^2\,\mathrm{Sq}^1 + \mathrm{Sq}^1\,\mathrm{Sq}^2$ are elements of $\mathcal{A}_{\mathbb{C}}(1)$ whose squares are zero. These elements are the motivic analogues of the first two Milnor primitives in the classical Steenrod algebra modulo 2.

Lemma 4.5 *The maximal elementary sub-Hopf algebras of $\mathcal{A}_{\mathbb{C}}(1)/\tau$ are the exterior algebras $E(Q_0, Q_1)$ and $E(\mathrm{Sq}^2, Q_1)$.*

Proof Lemma 4.4 says that A/τ is isomorphic to the group algebra $\mathbb{F}_2[D_8]$ of the dihedral group of order 8. The elementary sub-Hopf algebras of $\mathbb{F}_2[D_8]$ correspond to the elementary abelian 2-subgroups of D_8. The group D_8 has two maximal elementary abelian subgroups. Tracing back through the isomorphism of Lemma 4.4, one can identify the two maximal elementary sub-Hopf algebras of $\mathcal{A}_{\mathbb{C}}(1)/\tau$. □

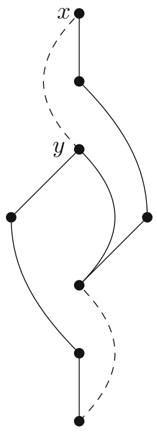


Fig. 3 The $\mathcal{A}_{\mathbb{C}}(1)$-module $\widetilde{\mathcal{A}}_{\mathbb{C}}(1)$

4.1 Margolis homology

We now turn to an algebraic invariant detecting projectivity of $\mathcal{A}_{\mathbb{C}}(1)$-modules, analogous to Margolis's techniques using P_t^s-homology [10].

Definition 4.6 Let x be an element of A such that x^2 is zero. For any A-module M, define the Margolis homology $H(M; x)$ to be the annihilator of x modulo the submodule xM.

Classically, an $\mathcal{A}(1)$-module M is projective if and only if $H(M; Q_0)$ and $H(M; Q_1)$ are both zero [1, Theorem 3.1], which is a direct consequence of a more general result [12, Theorem 1.2–1.4]. Our goal is to generalize this result to the motivic situation. Unfortunately, the motivic situation is more complicated. If M is an $\mathcal{A}_{\mathbb{C}}(1)$-module and $H(M; Q_0)$ and $H(M; Q_1)$ both vanish, then M is not necessarily projective.

Example 4.7 Let $\widetilde{\mathcal{A}}_{\mathbb{C}}(1)$ be the $\mathcal{A}_{\mathbb{C}}(1)$-module on two generators x and y of degrees (0, 0) and (2, 0) respectively, subject to the relations $\mathrm{Sq}^2 x = \tau y$ and $\mathrm{Sq}^1 \mathrm{Sq}^2 \mathrm{Sq}^1 x = \mathrm{Sq}^2 y$. Figure 3 represents $\widetilde{\mathcal{A}}(1)$ as an $\mathcal{A}_{\mathbb{C}}(1)$-module.

The Margolis homology groups $H(\widetilde{\mathcal{A}}(1); Q_0)$ and $H(\widetilde{\mathcal{A}}(1); Q_1)$ both vanish. However, $\widetilde{\mathcal{A}}(1)$ is not a projective $\mathcal{A}_{\mathbb{C}}(1)$-module.

It turns out that we need two additional criteria for projectivity beyond Q_0-homology and Q_1-homology. The presentation of $\mathcal{A}_{\mathbb{C}}(1)$ provided by Lemma 4.2 gives the relation $(Sq^2)^2 = \tau Sq^1 Sq^2 Sq^1$. In particular, $(Sq^2)^2 = 0$ modulo τ.

Proposition 4.8 *Let M be a finitely generated $\mathcal{A}_{\mathbb{C}}(1)$-module. Then M is projective if and only if:*

(1) M is free over $\mathbb{M}_2$; and
(2) $H(M/\tau; Q_0) = 0$; and
(3) $H(M/\tau; Q_1) = 0$; and
(4) $H(M/\tau; \mathrm{Sq}^2) = 0$.

Proof First suppose that M is projective. By inspection, conditions (2) through (4) are satisfied when M is $\mathcal{A}_{\mathbb{C}}(1)$. Therefore, these conditions are satisfied when M is free. Using that a projective module is a summand of a free module, conditions (2) through (4) are also satisfied for any projective M. Finally, Lemma 2.4 shows that condition (1) is satisfied.

Now suppose that conditions (1) through (4) are satisfied. By Lemma 3.2, it suffices to show that M/τ is $\mathcal{A}_{\mathbb{C}}(1)/\tau$-projective. By [12, Theorem 1.2–1.4], an $\mathcal{A}_{\mathbb{C}}(1)/\tau$-module is projective if and only if its restrictions to the quasi-elementary sub-Hopf algebras of $\mathcal{A}_{\mathbb{C}}(1)/\tau$ are. See [12, Definition 1.1] for the definition of quasi-elementary sub-Hopf algebras.

For group algebras, quasi-elementary sub-Hopf algebras coincide with elementary sub-Hopf algebras [13] (as observed in [12]). Since $\mathcal{A}_{\mathbb{C}}(1)/\tau$ is isomorphic to the group algebra $\mathbb{F}_2[D_8]$ by Lemmas 4.4, 4.5 shows that the quasi-elementary sub-Hopf algebras of $\mathcal{A}_{\mathbb{C}}(1)/\tau$ are the exterior algebra $E(Q_0, Q_1)$ and the exterior algebra $E(\mathrm{Sq}^2, Q_1)$. Conditions (2) and (3) imply that M/τ is $E(Q_0, Q_1)$-projective, and conditions (3) and (4) imply that M/τ is $E(\mathrm{Sq}^2, Q_1)$-projective. □

Remark 4.9 The exterior algebra $E(Q_0, Q_1)$ is the unique maximal quasi-elementary sub-Hopf algebra of the classial Hopf algebra $\mathcal{A}(1)$. This explains why condition (4) of Proposition 4.8 is absent from the classification of projective $\mathcal{A}(1)$-modules.

Corollary 4.10 *Let M and N be finitely generated $\mathcal{A}_{\mathbb{C}}(1)$-modules that are $\mathbb{M}_2$-free, and let $f : M \longrightarrow N$ be an $\mathcal{A}_{\mathbb{C}}(1)$-module map. Then f is a stable equivalence if and only if $f/\tau : M/\tau \longrightarrow N/\tau$ induces an isomorphism in Margolis homologies with respect to Q_0, Q_1, and* Sq^2.

Proof We may choose a free $\mathcal{A}_{\mathbb{C}}(1)$-module F and a surjective map $g : M \oplus F \longrightarrow N$ that restricts to f on M. Then f is a stable equivalence if and only if g is a stable equivalence, and f/τ induces isomorphisms in Margolis homologies if and only if g/τ induces isomorphisms in Margolis homologies. In other words, we may assume that f is surjective. (From a model categorical perspective, we have replaced f by an equivalent fibration.)

Let K be the kernel of f. The $\mathcal{A}_{\mathbb{C}}(1)$-module K is finitely generated and $\mathbb{M}_2$-free because it is a subobject of the finitely generated $\mathbb{M}_2$-free module M. The short exact sequence

$$0 \longrightarrow K \longrightarrow M \xrightarrow{f} N \longrightarrow 0,$$

induces a short exact sequence

$$0 \longrightarrow K/\tau \longrightarrow M/\tau \xrightarrow{f/\tau} N/\tau \longrightarrow 0,$$

by Proposition 3.1. This last short exact sequence induces long exact sequences in Margolis homologies with respect to Q_0, Q_1 and Sq^2. The long exact sequence shows that f/τ is an isomorphism in Margolis homologies if and only if K/τ has vanishing Margolis homologies. Now, Proposition 4.8 implies that K/τ has vanishing Margolis homologies if and only if K is projective. Finally, K is projective if and only if f is a stable equivalence. □

We establish a Künneth theorem for Margolis homology.

Proposition 4.11 *Let M and N be $\mathcal{A}_{\mathbb{C}}(1)$-modules that are free over $\mathbb{M}_2$. Then*

$$H(M/\tau \otimes N/\tau; x) \cong H(M/\tau; x) \otimes H(N/\tau; x),$$

when x is Q_0, Q_1, or Sq^2.

Proof Lemma 4.2 gives the coproduct formula

$$\Delta(\mathrm{Sq}^2) = \mathrm{Sq}^2 \otimes 1 + \tau\, \mathrm{Sq}^1 \otimes \mathrm{Sq}^1 + 1 \otimes \mathrm{Sq}^2 .$$

Therefore, Sq^2 is primitive modulo τ. In particular, it acts as a derivation on $M/\tau \otimes N/\tau$. The isomorphism in Sq^2-homology follows from the classical Künneth formula for chain complexes over $\mathbb{F}_2$.

The arguments for Q_0 and Q_1 are the same, except slightly easier because these elements are primitive even before quotienting by τ. □

Proposition 4.12 *Let M be a finitely generated $\mathcal{A}_{\mathbb{C}}(1)$-module that is $\mathbb{M}_2$-free. Then M is invertible if and only if M/τ has one-dimensional Margolis homologies with respect to Q_0, Q_1, and Sq^2.*

Proof First suppose that M is invertible. In other words, there exists an $\mathcal{A}_{\mathbb{C}}(1)$-module N and a stable equivalence

$$M \otimes N \xrightarrow{\simeq} \mathbb{M}_2.$$

Proposition 3.1 implies that there is a stable equivalence

$$(M \otimes N)/\tau \xrightarrow{\simeq} \mathbb{F}_2,$$

of $\mathcal{A}_{\mathbb{C}}(1)/\tau$-modules. Corollary 4.10 shows that

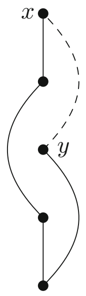
$$H((M \otimes N)/\tau; x) \longrightarrow H(\mathbb{F}_2; x),$$

is an isomorphism when x is Q_0, Q_1, or Sq^2. Now use Proposition 4.11 to deduce that $H(M/\tau; x) \otimes H(N/\tau; x)$ is isomorphic to $\mathbb{F}_2$. It follows that $H(M/\tau; x)$ is one-dimensional.

Now assume that M/τ has one-dimensional Margolis homologies. Note that

$$H(D(M/\tau); x) \cong \mathrm{Hom}_{\mathbb{F}_2}(H(M/\tau; x); \mathbb{F}_2),$$

Fig. 4 The $\mathcal{A}_{\mathbb{C}}(1)$-module $J_{\mathbb{C}}$



when x is Q_0, Q_1, or Sq^2. Therefore, $D(M/\tau)$ also has one-dimensional Margolis homologies. By Proposition 4.11, $M/\tau \otimes D(M/\tau)$ also has one-dimensional Margolis homologies. Hence the evaluation map

$$M/\tau \otimes D(M/\tau) \longrightarrow \mathbb{F}_2,$$

induces an isomorphism in Margolis homologies because both sides are one-dimensional. Note that $M/\tau \otimes D(M/\tau)$ is isomorphic to $(M \otimes DM)/\tau$ by Proposition 3.1. Finally, Corollary 4.10 shows that the evaluation map

$$M \otimes DM \longrightarrow \mathbb{M}_2,$$

is a stable equivalence. This shows that M is invertible with inverse DM. □

5 The Picard group of $\mathcal{A}_{\mathbb{C}}(1)$

Definition 5.1 Let $J_{\mathbb{C}}$ be the $\mathcal{A}_{\mathbb{C}}(1)$-module on two generators x and y of degrees $(0, 0)$ and $(2, 0)$ respectively, subject to the relations $\mathrm{Sq}^2 x = \tau y$, $\mathrm{Sq}^1 \mathrm{Sq}^2 \mathrm{Sq}^1 x = \mathrm{Sq}^2 y$, and $\mathrm{Sq}^1 y = 0$.

Figure 4 represents $J_{\mathbb{C}}$ as an $\mathcal{A}_{\mathbb{C}}(1)$-module.

Lemma 5.2 *The $\mathcal{A}_{\mathbb{C}}(1)$-module $J_{\mathbb{C}}$ is invertible, and the order of* $[J_{\mathbb{C}}]$ *in* $\mathrm{Pic}(\mathcal{A}_{\mathbb{C}}(1))$ *is infinite.*

Proof Proposition 4.12 implies that $J_{\mathbb{C}}$ is invertible. The Q_0-homology and Q_1-homology of $J_{\mathbb{C}}/\tau$ are generated by x, while the Sq^2-homology of $J_{\mathbb{C}}/\tau$ is generated by y.

The degrees of x and y are different. Therefore, the Sq^2-homology and the Q_0-homology of any tensor power $J_{\mathbb{C}}^{\otimes n}$ of $J_{\mathbb{C}}$ are in different degrees. On the other hand, the Sq^2-homology and the Q_0-homology of $\mathbb{M}_2$ are in the same degree. This shows that $J_{\mathbb{C}}^{\otimes n}$ is not stably equivalent to $\mathbb{M}_2$. □

Remark 5.3 The classical joker is self-dual as an $\mathcal{A}(1)$-module. Therefore, it represents an element of order two in $\mathrm{Pic}(\mathcal{A}(1))$. On the other hand, Fig. 4 shows that the motivic joker is not self-dual. Indeed, $DJ_{\mathbb{C}}$ is, up to a suspension, the $\mathcal{A}_{\mathbb{C}}(1)$-module on one generator x of degree $(0, 0)$ subject to the relation $\mathrm{Sq}^1\,\mathrm{Sq}^2\,x = 0$.

Lemma 5.4 *There is an isomorphism*

$$\phi : \mathrm{Pic}(\mathcal{A}_{\mathbb{C}}(1)/\tau) \xrightarrow{\cong} \mathbb{Z}^2 \oplus \mathrm{Pic}(\mathbb{F}_2[D_8]),$$

sending a class $[M]$ *to* $(a, b, [\overline{M}])$*, where* (a, b) *is the bidegree of the unique non-zero Margolis* Q_0*-homology class of* M*, and* $\overline{M}$ *is the ungraded* $\mathbb{F}_2[D_8]$*-module underlying* M*.*

Proof By Lemma 4.4, the Hopf algebras $\mathcal{A}_{\mathbb{C}}(1)/\tau$ and $\mathbb{F}_2[D_8]$ are isomorphic as ungraded Hopf algebras over $\mathbb{F}_2$.

Recall from [5, Theorem 5.4] that the ungraded Picard group of $\mathbb{F}_2[D_8]$ is isomorphic to $\mathbb{Z}^2$, generated by $\Omega\mathbb{F}_2$ and $J_{\mathbb{C}}/\tau$ (this $\mathbb{F}_2[D_8]$-module is called ΩL in *loc cit*).

Let $[M]$ be an element of the kernel of ϕ. Then, as an ungraded $\mathbb{F}_2[D_8]$-module, $\overline{M} \simeq \mathbb{F}_2$. Equivalently, $M \simeq \Sigma^{a,b}\mathbb{F}_2$ as graded modules. But $\deg(H(M, Q_0)) = (0, 0)$ since $[M] \in \ker(\phi)$. We conclude that $M \simeq \mathbb{F}_2$, where $\mathbb{F}_2$ is concentrated in bidegree $(0, 0)$. This shows that ϕ is injective.

Let $(a, b, x) \in \mathbb{Z}^2 \oplus \mathrm{Pic}(\mathbb{F}_2[D_8])$. We can always choose a representative of x of the form $\Omega^c(J_{\mathbb{C}}/\tau)^d$ by [5, Theorem 5.4] . Then $\phi(\Sigma^{a,b}\Omega^c(J_{\mathbb{C}}/\tau)^d) = (a, b, x)$. This shows that ϕ is surjective. □

Remark 5.5 Another version of Lemma 5.4 considers the subgroup of $\mathrm{Pic}(\mathcal{A}_{\mathbb{C}}(1)/\tau)$ consisting of modules whose Q_0-homology is concentrated in degree $(0, 0)$. This subgroup is isomorphic to $\mathrm{Pic}(\mathbb{F}_2[D_8])$.

Theorem 5.6 *There is an isomorphism*

$$\mathbb{Z}^4 \longrightarrow \mathrm{Pic}(\mathcal{A}_{\mathbb{C}}(1)),$$

sending (a, b, c, d) *to the class of* $\Sigma^{a,b}\Omega^c J_{\mathbb{C}}^d$*.*

Proof Recall the homomorphism

$$R_\tau : \mathrm{Pic}(\mathcal{A}_{\mathbb{C}}(1)) \longrightarrow \mathrm{Pic}(A/\tau),$$

from Proposition 3.4. Consider the composition

$$\mathbb{Z}^4 \longrightarrow \mathrm{Pic}(\mathcal{A}_{\mathbb{C}}(1)) \xrightarrow{R_\tau} \mathrm{Pic}(\mathcal{A}_{\mathbb{C}}(1)/\tau) \xrightarrow{\cong} \mathbb{Z}^4,$$

where the last isomorphism is the map ϕ of Lemma 5.4.

By direct computation, the first map sends $(a, b, c, d) \in \mathbb{Z}^4$ to $[\Sigma^{a,b}\Omega^c J_{\mathbb{C}}^d]$, which is sent to $[\Sigma^{a,b}\Omega^c(J_{\mathbb{C}}/\tau)^d] \in \mathrm{Pic}(\mathcal{A}_{\mathbb{C}}(1)/\tau)$. Finally, $\phi([\Sigma^{a,b}\Omega^c(J_{\mathbb{C}}/\tau)^d]) = (a, b, c, d)$ by construction of ϕ. Thus the composition is an isomorphism. This shows that R_τ is surjective. We already knew that R_τ is injective by Proposition 3.4. Therefore, R_τ is an isomorphism, so the map

$$\mathbb{Z}^4 \longrightarrow \mathrm{Pic}(\mathcal{A}_{\mathbb{C}}(1)),$$

is an isomorphism as well. □

Recall the comparison map $\mathrm{Pic}(\mathcal{A}_{\mathbb{C}}(1)) \longrightarrow \mathrm{Pic}(\mathcal{A}(1)$ constructed in Remark 4.3. We can now describe it explicitly.

Proposition 5.7 *Setting* $\tau = 1$ *induces the surjective group homomorphism*

$$\begin{aligned} \mathrm{Pic}(\mathcal{A}_{\mathbb{C}}(1)) \cong \mathbb{Z}^4 &\longrightarrow \mathrm{Pic}(\mathcal{A}(1)) \cong Z^2 \oplus \mathbb{Z}/2 \\ [\Sigma^{a,b}\Omega^c J_{\mathbb{C}}^d] &\longmapsto [\Sigma^a \Omega^c J^d]. \end{aligned}$$

Proof The motivic joker $J_{\mathbb{C}}$ is sent to the classical joker J. Note also that, since $(-)/(\tau - 1)$ forgets the second degree, the Picard element $\Sigma^{a,b}\mathbb{M}_2$ goes to $\Sigma^a\mathbb{F}_2$. Finally, since free $\mathcal{A}_{\mathbb{C}}(1)$-modules are sent to free $\mathcal{A}(1)$-modules, the functor $(-)/(\tau - 1)$ is compatible with the functor Ω in both categories. □

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