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The n-fold reduced bar construction

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Abstract This paper is about a correspondence between monoidal structures in categories and n-fold loop spaces. We developed a new syntactical technique whose role is to substitute the coherence results, which were the main ingredients in the proof that the Segal–Thomason bar construction provides an appropriate simplicial space. The results we present here enable more common categories to enter this delooping machine. For example, such as the category of finite sets with two monoidal structures brought by the disjoint union and Cartesian product.

Keywords Bar construction · Monoidal categories · Infinite loop spaces

Mathematics Subject Classification 18D10 · 57T30 · 55P47 · 55P48

1 Introduction

A correspondence between monoidal structures in categories and loop spaces was initially established by Stasheff in [23]. Since then, a connection of various algebraic structures on a category with onefold, twofold, n-fold, and infinite loop spaces is a subject of many papers (see [3,9,12,14,16,17,22,25], and references therein). The categories in question are usually equipped with one or several monoidal structures,

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and natural transformations providing symmetry, braiding, or some other kind of interchange between these structures. There are two main approaches to the subject. One is operadic and the other is through the Segal–Thomason bar construction, which we simply call *reduced* bar construction, as in [25]. The latter, to which we will keep to throughout the paper, is an approach to the Quillen plus construction and it is the initial step connecting various monoidal categories with loop spaces.

The n-fold reduced bar construction based on an n-fold monoidal category $\mathcal{M}$ is an iteration of a construction of a simplicial object based on a monoid in a category whose monoidal structure is given by finite products. The goal is to obtain a lax functor $\overline{W}\mathcal{M}$ from $(\Delta^{op})^n$, the nth power of the opposite of the simplicial category, to the category *Cat*, of categories and functors, such that

$$\overline{W}\mathcal{M}(k_1, \ldots, k_n) = \mathcal{M}^{k_1 \cdot \ldots \cdot k_n}.$$

The main result of this paper states that for every $n \geq 2$, the n-fold reduced bar construction delivers a certain lax functor. This is what we mean by *correctness* of the reduced bar construction. We prove this result gradually—the cases $n = 2$, $n = 3$ and $n \geq 3$ are dealt with respectively in Theorems 4.5, 6.5 and 8.5.

Following the ideas of [3, Section 2], we show in Sect. 9, that the lax functor $\overline{W}\mathcal{M}$ satisfies some additional conditions. Roughly speaking, some particular arrows of $(\Delta^{op})^n$, which are built out of face maps corresponding to projections, have to be mapped by $\overline{W}\mathcal{M}$ to identities. Such a lax functor is called Segal's in [20].

By applying Street's rectification to $\overline{W}\mathcal{M}$ (see [24]) one obtains a functor V, with the same source and target as $\overline{W}\mathcal{M}$. From [20, Corollary 4.4], when B is the classifying space functor, it follows that $B \circ V$ is a multisimplicial space with some properties guaranteing that, up to group completion (see [18,22]), the realization of this multisimplicial space is an n-fold delooping of $B\mathcal{M}$ (see [20, Theorem 5.1]). A thorough survey of results concerning these matters is given in [20] and the case $n = 2$ is considered separately in [21].

This paper is strongly influenced by [3]. One can find the main ideas followed by us in Sections 0, 1 and 2 of that paper. Also, the reader should consider [10] as an earlier source of these ideas. The notions of two, three and n-fold monoidal categories used in [3] and the corresponding notions used in this paper are compared in Sects. 2, 5 and 7. The case $n = 2$ is studied systematically in Sect. 2.

A definition of n-fold monoidal category is usually inductive as one starts with the 2-category *Cat* whose monoidal structure is given by 2-products. The 0-cells of a 2-category $Mon(Cat)$ are pseudomonoids (or monoids) in *Cat*, i.e., monoidal (or strict monoidal) categories. Then one makes a choice what to consider to be the 1-cells of $Mon(Cat)$, i.e., how strictly they should preserve the monoidal structure. The monoidal structure of $Mon(Cat)$ is again given by 2-products. A pseudomonoid (or a monoid) in $Mon(Cat)$ is a (strict) twofold monoidal category and if we iterate this procedure with the same degree of strictness, we obtain one possible notion of n-fold monoidal category.

Joyal and Street [12], dealt with such a concept having in its basis a certain 2-category of monoidal categories, strong monoidal functors, and monoidal transformations. They showed that such a degree of strictness leads to a sequence of categorial structures starting with monoidal categories, then the braided monoidal categories as the twofold monoidal categories and symmetric monoidal categories as the n-fold monoidal categories for $n \geq 3$.

Balteanu et al. [3], considered a variant of $Mon(Cat)$ consisting of strict monoidal categories, monoidal functors (in which the interchange between multiplicative structures need not be invertible), and monoidal transformations. This was an important advance leading to a definition of n-fold monoidal categories without stabilization at $n = 3$. However, they did not laxify the appropriate interchanges for units, which were treated in their work as strict as possible.

The reason to stop at that notion is probably the impossibility of proving an appropriate coherence result for the completely balanced notion of iterated monoidal categories. Monoidal units usually produce difficulties in coherence results (cf. [13]). The situation brought by diversifying monoidal units in the case of n-fold monoidal structures is very complicated.

The idea of [8,19] was to laxify the interchanges for units as much as the coherence allows. Trimble and the second author showed that a coherence result for pseudocommutative pseudomonoids, for which some structural constraints are invertible, in a 2-category of symmetric monoidal categories, lax symmetric monoidal functors, and monoidal transformations is sufficient for the reduced bar construction.

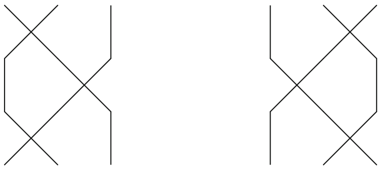
In this paper we consider the variant of $Mon(Cat)$ in which the interchange between multiplicative structures and interchange between units need not be invertible, i.e., a 2-monoidal category of monoidal categories, lax monoidal functors, and monoidal natural transformations. This is the basis used by Aguiar and Mahajan [2], for the definition of the notion of n-fold monoidal category. The possibility of defining n-fold monoidal structures with respect to such a basis is much less explored, perhaps because of difficulties in proving the corresponding coherence results. Such a coherence result usually guarantees commutativity of all the diagrams in n-fold monoidal categories relevant for the reduced bar construction.

Our result is not of the form to prove the coherence and not to worry about the lax conditions. We have developed a syntactical technique whose role is to substitute the coherence results. The correctness of the reduced bar construction is guaranteed by commutativity of certain diagrams. Our main goal is to check this directly.

We consider the two steps that seem to be necessary in the proof of correctness of the reduced bar construction. These steps are roughly sketched below and precisely given in Sects. 4, 6 and 8. It turns out that the definition of n-fold monoidal category given in [2] provides these two steps. We start with checking the correctness of the reduced bar construction based on a twofold monoidal category, i.e., 2-monoidal category of [2], or duoidal category of [4,5]. The first step in this case is trivial, and the second step, which may be simply modified and used for the n-fold case, is more involved.

Then we check the correctness of the reduced bar construction based on a threefold monoidal category, i.e., 3-monoidal category of [2]. We go through two steps that are in spirit the same as in the twofold case. Neither of these steps is now trivial but, as mentioned above, the second is just a modification of the corresponding step in the twofold case. The combinatorial structure of n-fold monoidal categories, defined by iterating this procedure, as it is already shown in [2], stabilizes at $n = 3$. Hence, an n-fold monoidal category, for $n \geq 3$, may be envisaged as a sequence of n monoidal structures in a category, such that every triple of these structures corresponds to a threefold monoidal category. The correctness of the reduced bar construction based on an n-fold monoidal category is obtained as a simple modification of the results mentioned above.

Our techniques are very much syntactical. We rely on a syntactical nature of the simplicial category presented by its generating arrows and equations. These equations are easily turned into rewrite rules, which are useful for some normalization techniques. Also, we try to point out the combinatorial core of the subject. This is the reason why our definition of the reduced bar construction $\overline{W}\mathcal{M}$, although it covers the one of [3], is given in different terms. From a composition of functors involved in the definition of $\overline{W}\mathcal{M}$ we abstract a shuffle of n sequences, whose members are generators of the simplicial category. Then we consider some available transpositions turning this shuffle into one obtained by concatenating these n sequences in a desired order. The first step in the proof of correctness of the reduced bar construction shows that the equations of n-fold monoidal categories suffice to consider any two applications of available transpositions from one shuffle to the other to be equal. This is a consequence of some naturality assumptions in the twofold case. In the n-fold case, for $n \geq 3$, we need some additional equations brought by the assumptions on 1-cells of $Mon_2(Cat)$. Roughly speaking, these equations guarantee that the following two applications of transpositions in our shuffles, which correspond to the Yang–Baxter equation, are equal.



The sequences that constitute a shuffle may be transformed according to the equations of the simplicial category. Let Φ' be the result of such a transformation of a sequence Φ. The second step in the proof of correctness of the reduced bar construction shows that the equations of n-fold monoidal categories suffice to consider the permutation of Φ or of Φ' with a member of another sequence to be equal. All these equations are already present in the twofold case.

Hence, the equations of n-fold monoidal categories guarantee the correctness of the reduced bar construction. On the other hand, these equations are also necessary if one proves the correctness through these two steps. Our work may be characterized as the process of defining the n-fold monoidal categories just from the correctness of the reduced bar construction based on a multiple monoidal structure. We believe there are no further possibilities to laxify the notion of an n-fold monoidal category preserving the correctness of the reduced bar construction.

With respect to the reduced bar construction, our result generalizes all the results mentioned above. It does not involve coherence results whose proofs in the case of n-fold monoidal categories are lengthy and complicated. The two steps of our proofs mentioned above are pretty straightforward. This paper, except for some basic categorial definitions and results needed for Sect. 9, is self-contained.

To conclude, we mention that the interchanges between the monoidal structures required for n-fold monoidal categories are usually brought about by braiding and symmetry. It is pointed out in [8,19] that a bicartesian structure (a category with all finite coproducts and products) brings the desired interchanges but the corresponding coherence result required some unusual properties of such a category—a coproduct of terminal objects should be terminal. Our results show that this coherence is not necessary anymore and that every bicartesian category, for every n, may be conceived as an n-fold monoidal category in $n+1$ different ways. Although a bicartesian category is already ∞-monoidal, since it is symmetric monoidal (in two ways), this fact is interesting—there is a family, indexed by pairs of natural numbers, of reduced bar constructions based on such a category. We discuss these matters in more details at the end of Sect. 9. Also, this gives a positive answer to the second question of [19, Section 8].

2 The twofold monoidal categories

The notion of twofold monoidal category that we use in this paper is defined in [11, Section 4]. It appears in [2, Section 6.1] under the name 2-*monoidal category* and in both [4, Section 2.2] and [5, Section 3] under the name *duoidal category*. The notion appears as the second iterate of the inductive definition mentioned in the introduction. It slightly generalizes the notion of *bimonoidal intermuting category* introduced in [8, Section 12]. The difference between these two notions is that, in bimonoidal intermuting categories, the arrows β and τ from below are required to be isomorphisms. The motivation behind this invertibility requirement is a coherence result in the style of Kelly and Mac Lane (see [13]), which is proved in [8].

Let $Mon(Cat)$ be the 2-category whose 0-cells are the monoidal categories, 1-cells are the monoidal functors, and 2-cells are the monoidal transformations (see [15, XI.2]). The monoidal structure of $Mon(Cat)$ is given by 2-products (see [6, 7.4]).

Definition A *twofold monoidal* category is a pseudomonoid in $Mon(Cat)$.

The unfolded form of this definition is given in Sect. 10 (Appendix). In this paper we are interested in strict monoidal structures and we now give a more symmetric definition of twofold strict monoidal categories. A *twofold strict monoidal* category is a category $\mathcal{M}$ equipped with two strict monoidal structures $\langle \mathcal{M}, \otimes_1, I_1 \rangle$ and $\langle \mathcal{M}, \otimes_2, I_2 \rangle$ together with the arrows $\kappa\colon I_1 \to I_2, \beta\colon I_1 \to I_1 \otimes_2 I_1, \tau\colon I_2 \otimes_1 I_2 \to I_2$, and a natural transformation ι given by the family of arrows

$$\iota_{A,B,C,D}\colon (A \otimes_2 B) \otimes_1 (C \otimes_2 D) \to (A \otimes_1 C) \otimes_2 (B \otimes_1 D),$$

such that the following twelve equations hold:

$$\iota \circ (\mathbf{1} \otimes_1 \iota) = \iota \circ (\iota \otimes_1 \mathbf{1}), \tag{1}$$

$$\iota \circ (\mathbf{1} \otimes_1 \beta) = \mathbf{1}, \tag{2}$$

$$\iota \circ (\beta \otimes_1 \mathbf{1}) = \mathbf{1}, \tag{3}$$

$$\tau \circ (\mathbf{1} \otimes_1 \tau) = \tau \circ (\tau \otimes_1 \mathbf{1}), \tag{4}$$
$$\tau \circ (\mathbf{1} \otimes_1 \kappa) = \mathbf{1}, \tag{5}$$
$$\tau \circ (\kappa \otimes_1 \mathbf{1}) = \mathbf{1}, \tag{6}$$
$$(\mathbf{1} \otimes_2 \iota) \circ \iota = (\iota \otimes_2 \mathbf{1}) \circ \iota, \tag{7}$$
$$(\mathbf{1} \otimes_2 \tau) \circ \iota = \mathbf{1}, \tag{8}$$
$$(\tau \otimes_2 \mathbf{1}) \circ \iota = \mathbf{1}, \tag{9}$$
$$(\mathbf{1} \otimes_2 \beta) \circ \beta = (\beta \otimes_2 \mathbf{1}) \circ \beta, \tag{10}$$
$$(\mathbf{1} \otimes_2 \kappa) \circ \beta = \mathbf{1}, \tag{11}$$
$$(\kappa \otimes_2 \mathbf{1}) \circ \beta = \mathbf{1}. \tag{12}$$

The twofold monoidal categories defined in [3, Definition 1.7] are the twofold strict monoidal categories from above in which, moreover, it is assumed that $I_1 = I_2 = 0$ and $\kappa = \beta = \tau = \mathbf{1}_0$. (The tensors $\otimes_1$ and $\otimes_2$ are denoted in [3] by $\square_1$ and $\square_2$, while the natural transformation ι is denoted by η.) Hence, from our list of 12 equations, the Eqs. (4), (5), (6), (10), (11) and (12) are trivial, (1) is the *internal associativity condition*, (7) is the *external associativity condition*, (8) and (9) make together the *internal unit condition* and (2) and (3) make together the *external unit condition* (see [3, Definition 1.7]).

Also, every braided monoidal category is a twofold monoidal category in our sense. Both monoidal structures of such a twofold monoidal category are the same, and all the ι arrows are obtained by braiding.

3 The reduced bar construction

Here we will only give a definition of the reduced bar construction based on a strict monoidal category. We refer to [19, Section 6] for the complete analysis of this construction.

Let Δ (denoted by Δ^+ in [15]) be the topologist's *simplicial* category defined as in [15, VII.5] for whose arrows we take over the notation used in that book. In order to use geometric dimension, the objects of Δ, which are the nonempty ordinals $\{1, 2, 3, \ldots\}$ are rewritten as $\{0, 1, 2, \ldots\}$. Hence, for $n \geq 1$ and $0 \leq i \leq n$, the source of δ_i^n is $n-1$ and the target is n, while for $n \geq 1$ and $0 \leq i \leq n-1$, the source of σ_i^n is n and the target is $n-1$. When we speak of Δ^{op}, then we denote its arrows $(\delta_i^n)^{op} : n \to n-1$ by d_i^n and $(\sigma_i^n)^{op} : n-1 \to n$ by s_i^n.

The arrows of Δ^{op} satisfy the following *basic equations*:

$$d_j^{n-1} \circ d_l^n = d_{l-1}^{n-1} \circ d_j^n, \quad \text{when} \quad l-1 \geq j,$$
$$s_j^{n+1} \circ s_l^n = s_{l+1}^{n+1} \circ s_j^n, \quad \text{when} \quad l+1 > j,$$
$$d_j^n \circ s_l^n = \begin{cases} s_{l-1}^{n-1} \circ d_j^{n-1}, & \text{when} \quad j \leq l-1, \\ \mathbf{1}, & \text{when} \quad l \in \{j, j-1\}, \\ s_l^{n-1} \circ d_{j-1}^{n-1}, & \text{when} \quad j \geq l+2. \end{cases}$$

These particular equations whose left-hand sides are treated as redexes and the right-hand sides as the corresponding contracta serve to define the normal form (see below). The definition of the natural transformation ω (the ultimate ingredient in our construction) is completely based on this normal form. We use some syntactical techniques in this paper—it is therefore important how we represent the arrows by terms. However, we will never write brackets to denote the association of the binary operation of composition, and appropriate identity arrows could be considered present in a term or deleted from it, if necessary. The following proposition is analogous to [15, VII.5, Proposition 2].

Proposition 3.1 *The category Δ^{op} is generated by the arrows $d_i^n : n \to n-1$ for $n \geq 1$, $0 \leq i \leq n$, and $s_i^n : n-1 \to n$ for $n \geq 1$, $0 \leq i \leq n-1$, subject to the basic equations of Δ^{op}.*

Proof As in the lemma preceding [15, VII.5, Proposition 2], one can prove that every arrow of Δ^{op} has a unique representation of the form $\mathbf{1}$ or

$$s_{l_1} \circ \ldots \circ s_{l_k} \circ d_{j_1} \circ \ldots \circ d_{j_m},$$

(with the superscripts omitted) for $k + m \geq 1$, $l_1 > \cdots > l_k$, $j_1 \geq \cdots \geq j_m$. The basic equations of Δ^{op} (read from the left to the right as reduction rules) suffice to put any composite of d's and s's into the above form (cf. the proof of $S4_{\Box}$ Coherence in [7, Section 3]). □

We call the arrows $\mathbf{1}_n$, d_i^n, and s_i^n *basic arrows* of Δ^{op}. Also, we call the above representation of an arrow f of Δ^{op} the *normal form* of f and we denote it by f^{nf}. This normal form does not completely correspond to the normal form given in the mentioned lema of [15, VII.5]—by varying the directions of the reduction rules corresponding to the first two basic equations of Δ^{op} one may obtain other possible normal forms.

Remark 3.2 If $f_1, \ldots, f_k$ are basic, non-identity arrows of Δ^{op} such that $f_k \circ \ldots \circ f_1$ is defined and not a normal form, then there is $1 \leq i \leq k-1$ such that $f_{i+1} \circ f_i$ is the left hand side of one of the basic equations of Δ^{op}.

By [15, XI.3, Theorem 1], we may regard *Cat* as a strict monoidal category whose monoidal structure is given by finite products. Let $\mathcal{M}$ be a strict monoidal category, hence a monoid in *Cat*. The *reduced bar construction* (see [25]) based on $\mathcal{M}$ is a functor $\overline{W}\mathcal{M} \colon \Delta^{op} \to Cat$ defined as follows.

$$\begin{aligned}
&\overline{W}\mathcal{M}(n) = \mathcal{M}^n,\\
&\overline{W}\mathcal{M}(d_0^n)(A_1, A_2, \ldots, A_n) = (A_2, \ldots, A_n),\\
&\overline{W}\mathcal{M}(d_n^n)(A_1, \ldots, A_{n-1}, A_n) = (A_1, \ldots, A_{n-1}),
\end{aligned}$$

and for $1 \leq i \leq n-1$ and $0 \leq j \leq n-1$,

$$\overline{W}\mathcal{M}(d_i^n)(A_1, \ldots, A_i, A_{i+1}, \ldots, A_n) = (A_1, \ldots, A_i \otimes A_{i+1}, \ldots, A_n),$$
$$\overline{W}\mathcal{M}(s_j^n)(A_1, \ldots, A_j, A_{j+1}, \ldots, A_{n-1}) = (A_1, \ldots, A_j, I, A_{j+1}, \ldots, A_{n-1}),$$

where $\otimes$ is the tensor and I is the unit of the strict monoidal category $\mathcal{M}$.

We denote by $\overline{W}\mathcal{M}^m : \Delta^{op} \to Cat$ the reduced bar construction based on the mth power of the strict monoidal category $\mathcal{M}$ (which is again a strict monoidal category with the structure defined component-wise). When $\mathcal{M}$ is a twofold strict monoidal category (or an n-fold, in general), then we denote by $\overline{W}\mathcal{M}_i : \Delta^{op} \to Cat$ the reduced bar construction based on the ith monoidal structure of $\mathcal{M}$. By combining these two notations, $\overline{W}\mathcal{M}_i^m : \Delta^{op} \to Cat$ denotes the reduced bar construction based on the mth power of the strict monoidal category whose monoidal structure is the ith monoidal structure of $\mathcal{M}$.

4 The twofold reduced bar construction

We start with a definition of the twofold reduced bar construction based on a twofold strict monoidal category. This construction corresponds to the one given in the proof of [3, Theorem 2.1], save that the latter construction is based on a category that is twofold monoidal in the sense of that paper. Then we switch to an equivalent notion, which is of a combinatorial flavour. Such an approach is more suitable for our techniques, and it highlights the combinatorial core of the subject.

Let $\mathcal{M}$ be a twofold strict monoidal category. By relying on the structure of $\mathcal{M}$, we define a function from objects of $(\Delta^{op})^2$ to objects of *Cat* and a function from arrows of $(\Delta^{op})^2$ to arrows of *Cat*. These two functions are both denoted by $\overline{W}\mathcal{M}$.

Definition The *twofold reduced bar construction* $\overline{W}\mathcal{M}$ is defined on objects of $(\Delta^{op})^2$ as:

$$\overline{W}\mathcal{M}(n, m) = \mathcal{M}^{n \cdot m},$$

and it is defined on arrows of $(\Delta^{op})^2$ in the following manner.

For f an arrow of (Δ^{op}), we have

$$\overline{W}\mathcal{M}(f, \mathbf{1}_m) = \overline{W}\mathcal{M}_1^m(f),$$

where $\overline{W}\mathcal{M}_1^m$ is, according to the notation introduced at the end of Sect. 3, the reduced bar construction based on $\mathcal{M}^m$ with monoidal structure given by $\otimes_1$ and I_1. For example, $\overline{W}\mathcal{M}(d_1^3, \mathbf{1}_2) : \mathcal{M}^6 \to \mathcal{M}^4$ is such that

$$\overline{W}\mathcal{M}(d_1^3, \mathbf{1}_2)(A, B, C, D, E, F) = (A \otimes_1 C, B \otimes_1 D, E, F),$$

while $\overline{W}\mathcal{M}(s_0^3, \mathbf{1}_2) : \mathcal{M}^4 \to \mathcal{M}^6$ is such that

$$\overline{W}\mathcal{M}(s_0^2, \mathbf{1}_2)(A, B, C, D) = (I_1, I_1, A, B, C, D).$$

For g an arrow of (Δ^{op}), we have

$$\overline{W}\mathcal{M}(\mathbf{1}_n, g) = (\overline{W}\mathcal{M}_2(g))^n,$$

where $\overline{W}\mathcal{M}_2$ is, according to the notation introduced at the end of Sect. 3, the reduced bar construction based on the strict monoidal structure given by $\otimes_2$ and I_2 of $\mathcal{M}$. For example, $\overline{W}\mathcal{M}(\mathbf{1}_2, d_1^3) : \mathcal{M}^6 \to \mathcal{M}^4$ is such that

$$\overline{W}\mathcal{M}(\mathbf{1}_2, d_1^3)(A, B, C, D, E, F) = (A \otimes_2 B, C, D \otimes_2 E, F),$$

while $\overline{W}\mathcal{M}(\mathbf{1}_2, s_0^3) : \mathcal{M}^4 \to \mathcal{M}^6$ is such that

$$\overline{W}\mathcal{M}(\mathbf{1}_2, s_0^3)(A, B, C, D) = (I_2, A, B, I_2, C, D).$$

Finally, for $f : n_s \to n_t$ and $g : m_s \to m_t$, ("s" comes from *source* and "t" from *target*) we have

$$\overline{W}\mathcal{M}(f, g) = (\overline{W}\mathcal{M}_2(g))^{n_t} \circ \overline{W}\mathcal{M}_1^{m_s}(f).$$

For example, $\overline{W}\mathcal{M}(d_1^3, s_0^3) : \mathcal{M}^6 \to \mathcal{M}^6$ is such that

$$\overline{W}\mathcal{M}(d_1^3, s_0^3)(A, B, C, D, E, F) = (I_2, A \otimes_1 C, B \otimes_1 D, I_2, E, F)$$

In general, $\overline{W}\mathcal{M}$ is not a functor from $(\Delta^{op})^2$ to *Cat* since it does not preserve composition (it preserves identities). For example, $\overline{W}\mathcal{M}(d_1^2, \mathbf{1}_1) \circ \overline{W}\mathcal{M}(\mathbf{1}_2, d_1^2) : \mathcal{M}^4 \to \mathcal{M}$ is such that

$$(\overline{W}\mathcal{M}(d_1^2, \mathbf{1}_1) \circ \overline{W}\mathcal{M}(\mathbf{1}_2, d_1^2))(A, B, C, D) = (A \otimes_2 B) \otimes_1 (C \otimes_2 D),$$

while $\overline{W}\mathcal{M}(d_1^2, d_1^2) : \mathcal{M}^4 \to \mathcal{M}$ is such that

$$\overline{W}\mathcal{M}(d_1^2, d_1^2)(A, B, C, D) = (A \otimes_1 C) \otimes_2 (B \otimes_1 D).$$

Our goal is to show that $\overline{W}\mathcal{M} : (\Delta^{op})^2 \to Cat$ is a *lax functor* in the sense of [24]. This means that for every composable pair of arrows $e_1 = (f_1, g_1)$ and $e_2 = (f_2, g_2)$ of $(\Delta^{op})^2$, there is a natural transformation

$$\omega_{e_2,e_1} : \overline{W}\mathcal{M}(e_2) \circ \overline{W}\mathcal{M}(e_1) \dot{\to} \overline{W}\mathcal{M}(e_2 \circ e_1),$$

such that the following diagram commutes:

$$\begin{array}{ccccc} & & \overline{W}\mathcal{M}(e_3)\circ\overline{W}\mathcal{M}(e_2)\circ\overline{W}\mathcal{M}(e_1) & & \\ & \swarrow^{\omega_{e_3,e_2}\overline{W}\mathcal{M}(e_1)} & & \searrow^{\overline{W}\mathcal{M}(e_3)\omega_{e_2,e_1}} & \\ \overline{W}\mathcal{M}(e_3\circ e_2)\circ\overline{W}\mathcal{M}(e_1) & & & & \overline{W}\mathcal{M}(e_3)\circ\overline{W}\mathcal{M}(e_2\circ e_1) \\ & \searrow_{\omega_{e_3\circ e_2,e_1}} & & \swarrow_{\omega_{e_3,e_2\circ e_1}} & \\ & & \overline{W}\mathcal{M}(e_3\circ e_2\circ e_1) & & \end{array}$$

The rest of this section is devoted to a proof of laxness of $\overline{W}\mathcal{M}$.

For $k \geq 1$, let $f_k \dots f_1$ be a sequence of basic arrows of Δ^{op} such that the composition $f_k \circ \dots \circ f_1$ is defined. We say that $\Phi = (f_k, 1) \dots (f_1, 1)$ is a sequence of *colour* 1 and we abbreviate the term $f_k \circ \dots \circ f_1$ by $\circ\Phi$. A sequence of *colour* 2 (or of any other colour) is defined in the same manner. We assume that, if necessary, appropriate identities could always be added to, or deleted from sequences of any colour. However, for measuring the *length* of such a sequence, only non-identity members are taken into account.

Let $\Phi = (f_k, 1) \dots (f_1, 1)$ be a sequence of colour 1 and let $\Gamma = (g_l, 2) \dots (g_1, 2)$ be a sequence of colour 2, such that $\circ\Phi : n_s \to n_t$ and $\circ\Gamma : m_s \to m_t$. Let $\mathcal{M}$ be a twofold strict monoidal category. We define a functor

$$\overline{W}\mathcal{M}_{\Gamma\Phi} \colon \mathcal{M}^{n_s \cdot m_s} \to \mathcal{M}^{n_t \cdot m_t}$$

as the following composition

$$(\overline{W}\mathcal{M}_2(g_l))^{n_t} \circ \dots \circ (\overline{W}\mathcal{M}_2(g_q))^{n_t} \circ \overline{W}\mathcal{M}_1^{m_s}(f_k) \circ \dots \circ \overline{W}\mathcal{M}_1^{m_s}(f_1).$$

Let $f = \circ\Phi$ and $g = \circ\Gamma$. Since both $\overline{W}\mathcal{M}_1$ and $\overline{W}\mathcal{M}_2$ are functors, we have that $\overline{W}\mathcal{M}_{\Gamma\Phi} = \overline{W}\mathcal{M}(f, g)$. This fact leads to a *combinatorial* definition of the twofold reduced bar construction $\overline{W}\mathcal{M}$, according to which $\overline{W}\mathcal{M}(f, g)$ could be defined as $\overline{W}\mathcal{M}_{\Gamma\Phi}$ for arbitrary Φ of colour 1 such that $\circ\Phi = f$ and arbitrary Γ of colour 2 such that $\circ\Gamma = g$.

In order to define the natural transformations ω involved in Diagram 4.1, we introduce the following notions. Let Θ be a *shuffle* of Φ and Γ as above. For example, let Φ be $(d_1^2, 1)(d_1^3, 1)$, let Γ be $(d_2^3, 2)(s_0^3, 2)(d_1^3, 2)$, and let Θ be the following shuffle

$$(d_2^3, 2)(d_1^2, 1)(d_1^3, 1)(s_0^3, 2)(d_1^3, 2).$$

In this case, we have that $\circ\Phi \colon 3 \to 1$ and $\circ\Gamma \colon 3 \to 2$.

For every member $(f, 1)$ of Θ, we define its *inner power* to be the target of its right-closest $(g, 2)$ in Θ. We may assume that such $(g, 2)$ exists since we can always add the appropriate identity of colour 2 to the right of $(f, 1)$ in Θ. For every member $(g, 2)$ of Θ, we define its *outer power* to be the target of its right-closest $(f, 1)$ in Θ. For Θ as above, we have that the inner power of $(d_1^2, 1)$ is 3 and the outer power of $(d_2^3, 2)$ is 1.

For a twofold strict monoidal category $\mathcal{M}$ and for an arbitrary shuffle Θ of Φ and Γ, as for the shuffle $\Gamma\Phi$ (obtained by concatenating Γ and Φ), we can define a functor

$$\overline{W}\mathcal{M}_{\Theta} \colon \mathcal{M}^{n_s \cdot m_s} \to \mathcal{M}^{n_t \cdot m_t}$$

Table 1 $\chi(f, g)$ in nontrivial cases

f	g	$\chi(f, g)$
s_j^{n+1}	s_i^{m+1}	$(\mathbf{1}^{j(m+1)}, \mathbf{1}^i, \kappa, \vec{\mathbf{1}})$
$d_j^n, 1 \leq j \leq n-1$	s_i^{m+1}	$(\mathbf{1}^{(j-1)(m+1)}, \mathbf{1}^i, \tau, \vec{\mathbf{1}})$
s_j^{n+1}	$d_i^m, 1 \leq i \leq m-1$	$(\mathbf{1}^{j(m-1)}, \mathbf{1}^{i-1}, \beta, \vec{\mathbf{1}})$
$d_j^n, 1 \leq j \leq n-1$	$d_i^m, 1 \leq i \leq m-1$	$(\mathbf{1}^{(j-1)(m-1)}, \mathbf{1}^{i-1}, \iota, \vec{\mathbf{1}})$

in the following way: replace in Θ every $(f, 1)$ whose inner power is i by $\overline{W}\mathcal{M}_1^i(f)$, and every $(g, 2)$ whose outer power is o by $(\overline{W}\mathcal{M}_2(g))^o$, and insert $\circ$'s. For Θ as above, we have that $\overline{W}\mathcal{M}_\Theta$ is

$$\overline{W}\mathcal{M}_2(d_2^3) \circ \overline{W}\mathcal{M}_1^3(d_1^2) \circ \overline{W}\mathcal{M}_1^3(d_1^3) \circ (\overline{W}\mathcal{M}_2(s_0^3))^3 \circ (\overline{W}\mathcal{M}_2(d_1^3))^3,$$

which gives that $\overline{W}\mathcal{M}_\Theta(A, B, C, D, E, F, G, H, J)$ is the ordered pair

$$(I_2 \otimes_1 I_2 \otimes_1 I_2, ((A \otimes_2 B) \otimes_1 (D \otimes_2 E) \otimes_1 (G \otimes_2 H)) \otimes_2 (C \otimes_1 F \otimes_1 J)).$$

For basic arrows $f : n \to n'$ and $g : m \to m'$ of Δ^{op} we define a natural transformation

$$\chi(f, g) : \overline{W}\mathcal{M}_1^{m'}(f) \circ (\overline{W}\mathcal{M}_2(g))^n \;\dot{\to}\; (\overline{W}\mathcal{M}_2(g))^{n'} \circ \overline{W}\mathcal{M}_1^m(f)$$

to be the identity natural transformation except in the following cases:
Here $\mathbf{1}^n$ denotes the n-tuple of identities and $\vec{\mathbf{1}}$ is a tuple of identities whose length can be easily calculated in all the cases, but we will not write the exact length to avoid overlong expressions.

For $j \geq 0$, let $\Theta_0, \ldots, \Theta_j$ be shuffles of Φ and Γ such that Θ_0 is Θ and Θ_j is $\Gamma\Phi$, and if $j > 0$, then for every $0 \leq i \leq j-1$ we have that for some shuffles Π and Λ and non-identity members $(f, 1)$, $(g, 2)$, the shuffle Θ_i is $\Pi(f, 1)(g, 2)\Lambda$ while Θ_{i+1} is $\Pi(g, 2)(f, 1)\Lambda$. We call $\Theta_0, \ldots, \Theta_j$ a *normalizing path* starting with Θ. Its *length* is j. For example,

$$\Theta_0 = (d_2^3, 2)(d_1^2, 1)(d_1^3, 1)(s_0^3, 2)(d_1^3, 2), \quad \Theta_1 = (d_2^3, 2)(d_1^2, 1)(s_0^3, 2)(d_1^3, 1)(d_1^3, 2),$$
$$\Theta_2 = (d_2^3, 2)(d_1^2, 1)(s_0^3, 2)(d_1^3, 2)(d_1^3, 1), \quad \Theta_3 = (d_2^3, 2)(s_0^3, 2)(d_1^2, 1)(d_1^3, 2)(d_1^3, 1),$$
$$\Theta_4 = (d_2^3, 2)(s_0^3, 2)(d_1^3, 2)(d_1^2, 1)(d_1^3, 1)$$

is a normalizing path of length 4 starting with Θ as in the example given above.

Proposition 4.1 *Every normalizing path starting with Θ has the same length.*

Proof The length of every normalizing path starting with Θ is $\sum_{(f,1)\text{ in }\Theta} k(f,1)$, where $k(f,1)$ is the number of non-identity members of colour 2 to the right of $(f,1)$ in Θ. □

If $\Theta_i = \Pi(f,1)(g,2)\Lambda$ and $\Theta_{i+1} = \Pi(g,2)(f,1)\Lambda$, then

$$\varphi_i = \overline{W}\mathcal{M}_\Pi\ \chi(f,g)\ \overline{W}\mathcal{M}_\Lambda$$

is a natural transformation from $\overline{W}\mathcal{M}_{\Theta_i}$ to $\overline{W}\mathcal{M}_{\Theta_{i+1}}$. (In the case when Π or Λ are single-coloured, we can always add the appropriate identity of the other colour in order to define $\overline{W}\mathcal{M}_\Pi$ and $\overline{W}\mathcal{M}_\Lambda$.) Let

$$\varphi(\Theta_0,\ldots,\Theta_j) = \begin{cases} \varphi_{j-1}\circ\ldots\circ\varphi_0, & \text{when}\quad j\geq 1,\\ \mathbf{1}, & \text{when}\quad j=0.\end{cases}$$

Suppose $\Theta'_0,\ldots,\Theta'_j$ is another normalizing path starting with Θ. Then $\varphi(\Theta'_0,\ldots,\Theta'_j)$ is again a natural transformation from $\overline{W}\mathcal{M}_\Theta$ to $\overline{W}\mathcal{M}_{\Gamma\Phi}$. We can show that these natural transformations are in fact the same.

Theorem 4.2 $\varphi(\Theta_0,\ldots,\Theta_j) = \varphi(\Theta'_0,\ldots,\Theta'_j)$.

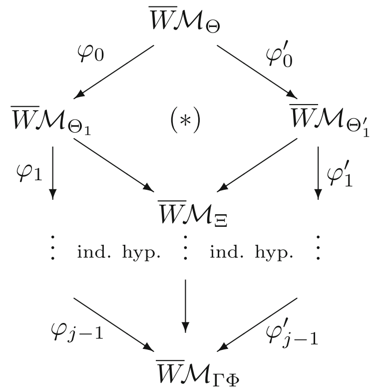
Proof We proceed by induction on $j \geq 0$. If $j = 0$, then $\varphi(\Theta_0) = \varphi(\Theta'_0) = \mathbf{1}$. If $j > 0$ and $\Theta_1 = \Theta'_1$, then we apply the induction hypothesis to the sequences of shuffles $\Theta_1,\ldots,\Theta_j$ and $\Theta'_1,\ldots,\Theta'_j$. Suppose now $\Theta_1 \neq \Theta'_1$ and $\varphi_0 = \overline{W}\mathcal{M}_\Pi\ \chi(f,g)\ \overline{W}\mathcal{M}_\Lambda$, $\varphi'_0 = \overline{W}\mathcal{M}_{\Pi'}\ \chi(f',g')\ \overline{W}\mathcal{M}_{\Lambda'}$. Then either

$$\Theta = \Pi_1(f',1)(g',2)\Pi_2(f,1)(g,2)\Lambda \quad\text{or}\quad \Theta = \Pi(f,1)(g,2)\Lambda_1(f',1)(g',2)\Lambda_2.$$

In the first case, by naturality we have

$$\begin{aligned}&\overline{W}\mathcal{M}_{\Pi_1}\ \chi(f',g')\ \overline{W}\mathcal{M}_{\Pi_2(g,2)(f,1)\Lambda}\circ\overline{W}\mathcal{M}_{\Pi_1(f',1)(g',2)\Pi_2}\ \chi(f,g)\ \overline{W}\mathcal{M}_\Lambda\\ &\quad= \overline{W}\mathcal{M}_{\Pi_1(g',2)(f',1)\Pi_2}\ \chi(f,g)\ \overline{W}\mathcal{M}_\Lambda\circ\overline{W}\mathcal{M}_{\Pi_1}\ \chi(f',g')\ \overline{W}\mathcal{M}_{\Pi_2(f,1)(g,2)\Lambda},\quad (*)\end{aligned}$$

and by applying the induction hypothesis twice we obtain the following commutative diagram, in which Ξ is $\Pi_1(g',2)(f',1)\Pi_2(g,2)(f,1)\Lambda$.



We proceed analogously in the second case. □

By Theorem 4.2, the following definition is correct.

Definition Let $\varphi_\Theta : \overline{W}\mathcal{M}_\Theta \dot{\to} \overline{W}\mathcal{M}_{\Gamma\Phi}$ be $\varphi(\Theta_0, \ldots, \Theta_j)$, for an arbitrary normalizing path $\Theta_0, \ldots, \Theta_j$ starting with Θ.

For every composable pair of arrows $e_1 = (f_1, g_1)$ and $e_2 = (f_2, g_2)$ of $(\Delta^{op})^2$ we define a natural transformation

$$\omega_{e_2,e_1} : \overline{W}\mathcal{M}(e_2) \circ \overline{W}\mathcal{M}(e_1) \dot{\to} \overline{W}\mathcal{M}(e_2 \circ e_1).$$

In order to do this, note that for a sequence H of any colour, $\circ$H denotes a syntactical object, a word of the form $h_k \circ \ldots \circ h_1$. Hence, a sequence H is completely determined by its colour and $\circ$H.

Definition Let Φ_1 and Φ_2 be sequences of colour 1, and let Γ_1 and Γ_2 be sequences of colour 2 such that $\circ\Phi_1$ is f_1^{nf}, $\circ\Phi_2$ is f_2^{nf}, $\circ\Gamma_1$ is g_1^{nf} and $\circ\Gamma_2$ is g_2^{nf}. We define

$$\omega_{e_2,e_1} \quad \text{as} \quad \varphi_{\Gamma_2\Phi_2\Gamma_1\Phi_1}.$$

Note. The source and target of ω_{e_2,e_1} are as desired since

$$\overline{W}\mathcal{M}(e_2) \circ \overline{W}\mathcal{M}(e_1) = \overline{W}\mathcal{M}_{\Gamma_2\Phi_2\Gamma_1\Phi_1}, \quad \text{and}$$
$$\overline{W}\mathcal{M}(e_2 \circ e_1) = \overline{W}\mathcal{M}_{\Gamma_2\Gamma_1\Phi_2\Phi_1}.$$

It remains to prove that our Diagram 4.1 commutes. Let $e_1 = (f_1, g_1), e_2 = (f_2, g_2)$ and $e_3 = (f_3, g_3)$ be such that the composition $e_3 \circ e_2 \circ e_1$ is defined in $(\Delta^{op})^2$. Let Φ_1, Φ_2, Γ_1 and Γ_2 be as above, and let Φ_3 and Γ_3 be sequences of colour 1 and 2 respectively such that $\circ\Phi_3$ is f_3^{nf} and $\circ\Gamma_3$ is g_3^{nf}. In this case, Diagram 4.1 reads

$$
\begin{array}{ccccc}
 & & \overline{W}\mathcal{M}(e_3)\circ\overline{W}\mathcal{M}(e_2)\circ\overline{W}\mathcal{M}(e_1) & & \\
 & \swarrow^{\varphi_{\Gamma_3\Phi_3\Gamma_2\Phi_2}\overline{W}\mathcal{M}_{\Gamma_1\Phi_1}} & & \searrow^{\overline{W}\mathcal{M}_{\Gamma_3\Phi_3}\,\varphi_{\Gamma_2\Phi_2\Gamma_1\Phi_1}} & \\
\overline{W}\mathcal{M}(e_3\circ e_2)\circ\overline{W}\mathcal{M}(e_1) & & & & \overline{W}\mathcal{M}(e_3)\circ\overline{W}\mathcal{M}(e_2\circ e_1) \\
 & \searrow_{\varphi_{\Gamma'\Phi'\Gamma_1\Phi_1}} & & \swarrow_{\varphi_{\Gamma_3\Phi_3\Gamma''\Phi''}} & \\
 & & \overline{W}\mathcal{M}(e_3\circ e_2\circ e_1) & &
\end{array}
$$

where $\circ\Phi'$ is $(f_3\circ f_2)^{\mathrm{nf}}$, $\circ\Gamma'$ is $(g_3\circ g_2)^{\mathrm{nf}}$, $\circ\Phi''$ is $(f_2\circ f_1)^{\mathrm{nf}}$ and $\circ\Gamma''$ is $(g_2\circ g_1)^{\mathrm{nf}}$.

By Theorem 4.2 we have the following commutative diagram

$$
\begin{array}{ccccc}
 & & \overline{W}\mathcal{M}(e_3)\circ\overline{W}\mathcal{M}(e_2)\circ\overline{W}\mathcal{M}(e_1) & & \\
 & \swarrow^{\varphi_{\Gamma_3\Phi_3\Gamma_2\Phi_2}\overline{W}\mathcal{M}_{\Gamma_1\Phi_1}} & \Big\downarrow{\scriptstyle \varphi_{\Gamma_3\Phi_3\Gamma_2\Phi_2\Gamma_1\Phi_1}} & \searrow^{\overline{W}\mathcal{M}_{\Gamma_3\Phi_3}\,\varphi_{\Gamma_2\Phi_2\Gamma_1\Phi_1}} & \\
\overline{W}\mathcal{M}(e_3\circ e_2)\circ\overline{W}\mathcal{M}(e_1) & & & & \overline{W}\mathcal{M}(e_3)\circ\overline{W}\mathcal{M}(e_2\circ e_1) \\
 & \searrow_{\varphi_{\Gamma_3\Gamma_2\Phi_3\Phi_2\Gamma_1\Phi_1}} & & \swarrow_{\varphi_{\Gamma_3\Phi_3\Gamma_2\Gamma_1\Phi_2\Phi_1}} & \\
 & & \overline{W}\mathcal{M}(e_3\circ e_2\circ e_1) & &
\end{array}
$$

Hence, to prove that Diagram 4.1 commutes, it suffices to show that

$$\text{(i)}\quad \varphi_{\Gamma_3\Gamma_2\Phi_3\Phi_2\Gamma_1\Phi_1}=\varphi_{\Gamma'\Phi'\Gamma_1\Phi_1}\quad\text{and}\quad\text{(ii)}\quad \varphi_{\Gamma_3\Phi_3\Gamma_2\Gamma_1\Phi_2\Phi_1}=\varphi_{\Gamma_3\Phi_3\Gamma''\Phi''}.$$

Lemma 4.3 *If Φ and Φ' are sequences of colour 1 such that $\circ\Phi=\circ\Phi'$ is a basic equation of Δ^{op}, and g is a basic arrow of Δ^{op}, then $\varphi_{\Phi(g,2)}=\varphi_{\Phi'(g,2)}$.*

Proof We have the following cases in which we always assume that d^x_y is such that $1\leq y\leq x-1$ (see Table 1). To deal with d^x_0 and d^x_x is trivial. We will give a detailed proof for three cases, first of which is trivial, with the remaining two needing some of the Eqs. 1–6. The rest is done analogously. □

1.1. Suppose $\circ\Phi=\circ\Phi'$ is $d^{n-1}_j\circ d^n_l=d^{n-1}_{l-1}\circ d^n_j$ for $j\leq l-2$.

1.1.1. Suppose g is s^m_i.

We have two normalizing paths. The first one is starting with $\Phi(g,2)$ and it is

$$(d^{n-1}_j,1)(d^n_l,1)(s^m_i,2),\ \ (d^{n-1}_j,1)(s^m_i,2)(d^n_l,1),\ \ (s^m_i,2)(d^{n-1}_j,1)(d^n_l,1).$$

Now we compute $\varphi_{\Phi(g,2)}$, and we note that $\overline{W}\mathcal{M}_{(d^n_l,1)}$ is formally $\overline{W}\mathcal{M}_{(d^n_l,1),(\mathbf{1}_{m-1},2)}$ (we repeatedly use such an abbreviation throughout the paper):

$$\begin{aligned}\varphi_{\Phi(g,2)} &= \big(\chi(d_j^{n-1}, s_i^m)\ \overline{W}\mathcal{M}_{(d_l^n,1)}\big) \circ \big(\overline{W}\mathcal{M}_{(d_j^{n-1},1)}\ \chi(d_l^n, s_i^m)\big)\\ &= \big(\chi(d_j^{n-1}, s_i^m)\ \overline{W}\mathcal{M}_1^{m-1}(d_l^n)\big) \circ \big(\overline{W}\mathcal{M}_1^m(d_j^{n-1})\ \chi(d_l^n, s_i^m)\big)\\ &= \big((\mathbf{1}^{(j-1)m}, \mathbf{1}^i, \tau, \vec{\mathbf{1}})\ \overline{W}\mathcal{M}_1^{m-1}(d_l^n)\big) \circ \big(\overline{W}\mathcal{M}_1^m(d_j^{n-1})\ (\mathbf{1}^{(l-1)m}, \mathbf{1}^i, \tau, \vec{\mathbf{1}})\big)\\ &= \big(\mathbf{1}^{(j-1)m}, \mathbf{1}^i, \tau, \vec{\mathbf{1}}\big) \circ \big(\mathbf{1}^{(l-2)m}, \mathbf{1}^i, \tau, \vec{\mathbf{1}}\big). \qquad (\text{since } l-1 > j)\end{aligned}$$

On the other hand, the second normalizing path starting with $\Phi'(g, 2)$ is

$$(d_{l-1}^{n-1}, 1)(d_j^n, 1)(s_i^m, 2),\ (d_{l-1}^{n-1}, 1)(s_i^m, 2)(d_j^n, 1),\ (s_i^m, 2)(d_{l-1}^{n-1}, 1)(d_j^n, 1),$$

and therefore

$$\begin{aligned}\varphi_{\Phi'(g,2)} &= \big(\chi(d_{l-1}^{n-1}, s_i^m)\ \overline{W}\mathcal{M}_{(d_j^n,1)}\big) \circ \big(\overline{W}\mathcal{M}_{(d_{l-1}^{n-1},1)}\ \chi(d_j^n, s_i^m)\big)\\ &= \big(\chi(d_{l-1}^{n-1}, s_i^m)\ \overline{W}\mathcal{M}_1^{m-1}(d_j^n)\big) \circ \big(\overline{W}\mathcal{M}_1^m(d_{l-1}^{n-1})\ \chi(d_j^n, s_i^m)\big)\\ &= \big((\mathbf{1}^{(l-2)m}, \mathbf{1}^i, \tau, \vec{\mathbf{1}})\ \overline{W}\mathcal{M}_1^{m-1}(d_j^n)\big) \circ \big(\overline{W}\mathcal{M}_1^m(d_{l-1}^{n-1})\ (\mathbf{1}^{(j-1)m}, \mathbf{1}^i, \tau, \vec{\mathbf{1}})\big)\\ &= (\mathbf{1}^{(l-2)m}, \mathbf{1}^i, \tau, \vec{\mathbf{1}}) \circ (\mathbf{1}^{(j-1)m}, \mathbf{1}^i, \tau, \vec{\mathbf{1}}).\end{aligned}$$

Since $j-1 \neq l-2$, we see that these two tuples of arrows are the same, i.e., we have:

$$\varphi_{\Phi(g,2)} = \big(\mathbf{1}^{(j-1)m+i}, \tau, \mathbf{1}^{(l-j-1)m-1}, \tau, \vec{\mathbf{1}}\big) = \varphi_{\Phi'(g,2)}.$$

1.1.2. Suppose g is d_i^m.

$$\varphi_{\Phi(g,2)} = \big(\mathbf{1}^{(j-1)(m-1)+i-1}, \iota, \mathbf{1}^{(l-j-1)(m-1)-1}, \iota, \vec{\mathbf{1}}\big) = \varphi_{\Phi'(g,2)}.$$

1.2. Suppose $\circ\Phi = \circ\Phi'$ is $d_j^{n-1} \circ d_{j+1}^n = d_j^{n-1} \circ d_j^n$.

1.2.1. Suppose g is s_i^m.

The normalizing path starting with $\Phi(g, 2)$ is

$$(d_j^{n-1}, 1)(d_{j+1}^n, 1)(s_i^m, 2),\ (d_j^{n-1}, 1)(s_i^m, 2)(d_{j+1}^n, 1),\ (s_i^m, 2)(d_j^{n-1}, 1)(d_{j+1}^n, 1),$$

and we have

$$\begin{aligned}\varphi_{\Phi(g,2)} &= \big(\chi(d_j^{n-1}, s_i^m)\ \overline{W}\mathcal{M}_{(d_{j+1}^n,1)}\big) \circ \big(\overline{W}\mathcal{M}_{(d_j^{n-1},1)}\ \chi(d_{j+1}^n, s_i^m)\big)\\ &= \big(\chi(d_j^{n-1}, s_i^m)\ \overline{W}\mathcal{M}_1^{m-1}(d_{j+1}^n)\big) \circ \big(\overline{W}\mathcal{M}_1^m(d_j^{n-1})\ \chi(d_{j+1}^n, s_i^m)\big)\\ &= \big((\mathbf{1}^{(j-1)m}, \mathbf{1}^i, \tau, \vec{\mathbf{1}})\ \overline{W}\mathcal{M}_1^{m-1}(d_{j+1}^n)\big) \circ \big(\overline{W}\mathcal{M}_1^m(d_j^{n-1})\ (\mathbf{1}^{jm}, \mathbf{1}^i, \tau, \vec{\mathbf{1}})\big)\\ &= (\mathbf{1}^{(j-1)m}, \mathbf{1}^i, \tau, \vec{\mathbf{1}}) \circ (\mathbf{1}^{(j-1)m}, \mathbf{1}^i, \mathbf{1} \otimes_1 \tau, \vec{\mathbf{1}})\\ &= (\mathbf{1}^{(j-1)m+i}, \tau \circ (\mathbf{1} \otimes_1 \tau), \vec{\mathbf{1}}).\end{aligned}$$

On the other hand, the normalizing path starting with $\Phi'(g, 2)$ is

$$(d_j^{n-1}, 1)(d_j^n, 1)(s_i^m, 2),\ (d_j^{n-1}, 1)(s_i^m, 2)(d_j^n, 1),\ (s_i^m, 2)(d_j^{n-1}, 1)(d_j^n, 1).$$

We now compute $\varphi_{\Phi'(g,2)}$:

$$\begin{aligned}\varphi_{\Phi'(g,2)} &= \big(\chi(d_j^{n-1}, s_i^m)\ \overline{W}\mathcal{M}_{(d_j^n,1)}) \circ (\overline{W}\mathcal{M}_{(d_j^{n-1},1)}\ \chi(d_j^n, s_i^m)\big)\\ &= \big(\chi(d_j^{n-1}, s_i^m)\ \overline{W}\mathcal{M}_1^{m-1}(d_j^n)) \circ (\overline{W}\mathcal{M}_1^m(d_j^{n-1})\ \chi(d_j^n, s_i^m)\big)\\ &= \big((\mathbf{1}^{(j-1)m}, \mathbf{1}^i, \tau, \vec{\mathbf{1}})\ \overline{W}\mathcal{M}_1^{m-1}(d_j^n)) \circ (\overline{W}\mathcal{M}_1^m(d_j^{n-1})\ (\mathbf{1}^{(j-1)m}, \mathbf{1}^i, \tau, \vec{\mathbf{1}})\big)\\ &= (\mathbf{1}^{(j-1)m}, \mathbf{1}^i, \tau, \vec{\mathbf{1}}) \circ (\mathbf{1}^{(j-1)m}, \mathbf{1}^i, \tau \otimes_1 \mathbf{1}, \vec{\mathbf{1}})\\ &= (\mathbf{1}^{(j-1)m+i}, \tau \circ (\tau \otimes_1 \mathbf{1}), \vec{\mathbf{1}}).\end{aligned}$$

Since, by (4), we have that $\tau \circ (\mathbf{1} \otimes_1 \tau) = \tau \circ (\tau \otimes_1 \mathbf{1})$, we conclude that $\varphi_{\Phi(g,2)} = \varphi_{\Phi'(g,2)}$.

1.2.2. Suppose g is d_i^m.

$$\begin{aligned}\varphi_{\Phi(g,2)} &= (\mathbf{1}^{(j-1)(m-1)+i-1}, \iota \circ (\mathbf{1} \otimes_1 \iota), \vec{\mathbf{1}})\\ &= (\mathbf{1}^{(j-1)(m-1)+i-1}, \iota \circ (\iota \otimes_1 \mathbf{1}), \vec{\mathbf{1}}) = \varphi_{\Phi'(g,2)}, \quad \text{by (1)}.\end{aligned}$$

2. Suppose $\circ\Phi = \circ\Phi'$ is $s_j^{n+1} \circ s_l^n = s_{l+1}^{n+1} \circ s_j^n$ for $j \le l$.

2.1. Suppose g is s_i^m.

$$\varphi_{\Phi(g,2)} = (\mathbf{1}^{jm+i}, \kappa, \mathbf{1}^{(l-j+1)m-1}, \kappa, \vec{\mathbf{1}}) = \varphi_{\Phi'(g,2)}.$$

2.2. Suppose g is d_i^m.

$$\varphi_{\Phi(g,2)} = (\mathbf{1}^{j(m-1)+i-1}, \beta, \mathbf{1}^{(l-j+1)(m-1)-1}, \beta, \vec{\mathbf{1}}) = \varphi_{\Phi'(g,2)}.$$

3.1. Suppose $\circ\Phi = \circ\Phi'$ is $d_j^n \circ s_l^n = s_{l-1}^{n-1} \circ d_j^{n-1}$ for $j \le l-1$.

3.1.1. Suppose g is s_i^m.

$$\varphi_{\Phi(g,2)} = (\mathbf{1}^{(j-1)m+i}, \tau, \mathbf{1}^{(l-j)m-1}, \kappa, \vec{\mathbf{1}}) = \varphi_{\Phi'(g,2)}.$$

3.1.2. Suppose g is d_i^m.

$$\varphi_{\Phi(g,2)} = (\mathbf{1}^{(j-1)(m-1)+i-1}, \iota, \mathbf{1}^{(l-j)(m-1)-1}, \beta, \vec{\mathbf{1}}) = \varphi_{\Phi'(g,2)}.$$

3.2. Suppose $\circ\Phi = \circ\Phi'$ is $d_j^n \circ s_j^n = \mathbf{1}$.

3.2.1. Suppose g is s_i^m.

$$\begin{aligned}\varphi_{\Phi(g,2)} &= (\mathbf{1}^{(j-1)m+i}, \tau \circ (\mathbf{1} \otimes_1 \kappa), \vec{\mathbf{1}})\\ &= (\mathbf{1}^{(j-1)m+i}, \mathbf{1}, \vec{\mathbf{1}}) = \varphi_{\Phi'(g,2)}, \quad \text{by (5)}.\end{aligned}$$

3.2.2. Suppose g is d_i^m.
We now have this normalizing path starting with $\Phi(g,2)$:

$$(d_j^n, 1)(s_j^n, 1)(d_i^m, 2), \quad (d_j^n, 1)(d_i^m, 2)(s_j^n, 1), \quad (d_i^m, 2)(d_j^n, 1)(s_j^n, 1).$$

Since $\varphi_{\Phi'(g,2)} = \varphi_{\mathbf{1}(g,2)} = \vec{\mathbf{1}}$, we ought to compute $\varphi_{\Phi(g,2)}$:

$$\begin{aligned}\varphi_{\Phi(g,2)} &= \big(\chi(d_j^n, d_i^m)\ \overline{W}\mathcal{M}_{(s_j^n,1)}) \circ (\overline{W}\mathcal{M}_{(d_j^n,1)}\ \chi(s_j^n, d_i^m)\big)\\ &= \big(\chi(d_j^n, d_i^m)\ \overline{W}\mathcal{M}_1^m(s_j^n)) \circ (\overline{W}\mathcal{M}_1^{m-1}(d_j^n)\ \chi(s_j^n, d_i^m)\big)\\ &= \big((\mathbf{1}^{(j-1)(m-1)}, \mathbf{1}^{i-1}, \iota, \vec{\mathbf{1}})\overline{W}\mathcal{M}_1^m(s_j^n))\\ &\quad \circ (\overline{W}\mathcal{M}_1^{m-1}(d_j^n)(\mathbf{1}^{j(m-1)}, \mathbf{1}^{i-1}, \beta, \vec{\mathbf{1}})\big)\\ &= (\mathbf{1}^{(j-1)(m-1)}, \mathbf{1}^{i-1}, \iota, \vec{\mathbf{1}}) \circ (\mathbf{1}^{(j-1)(m-1)}, \mathbf{1}^{i-1}, \mathbf{1} \otimes_1 \beta, \vec{\mathbf{1}})\\ &= (\mathbf{1}^{(j-1)(m-1)+i-1}, \iota \circ (\mathbf{1} \otimes_1 \beta), \vec{\mathbf{1}})\\ &\overset{(2)}{=} (\mathbf{1}^{(j-1)(m-1)+i-1}, \mathbf{1}, \vec{\mathbf{1}}) = \varphi_{\Phi'(g,2)}.\end{aligned}$$

3.3. Suppose $\circ\Phi = \circ\Phi'$ is $d_j^n \circ s_{j-1}^n = \mathbf{1}$.
3.3.1. Suppose g is s_i^m.

$$\begin{aligned}\varphi_{\Phi(g,2)} &= (\mathbf{1}^{(j-1)m+i}, \tau \circ (\kappa \otimes_1 \mathbf{1}), \vec{\mathbf{1}})\\ &= (\mathbf{1}^{(j-1)m+i}, \mathbf{1}, \vec{\mathbf{1}}) = \varphi_{\Phi'(g,2)}, \quad \text{by (6)}.\end{aligned}$$

3.3.2. Suppose g is d_i^m.

$$\begin{aligned}\varphi_{\Phi(g,2)} &= (\mathbf{1}^{(j-1)(m-1)+i-1}, \iota \circ (\beta \otimes_1 \mathbf{1}), \vec{\mathbf{1}})\\ &= (\mathbf{1}^{(j-1)(m-1)+i-1}, \mathbf{1}, \vec{\mathbf{1}}) = \varphi_{\Phi'(g,2)}, \quad \text{by (3)}.\end{aligned}$$

3.4. Suppose $\circ\Phi = \circ\Phi'$ is $d_j^n \circ s_l^n = s_l^{n-1} \circ d_{j-1}^{n-1}$ for $j \geq l+2$.
3.4.1. Suppose g is s_i^m.

$$\varphi_{\Phi(g,2)} = (\mathbf{1}^{lm+i}, \kappa, \mathbf{1}^{(j-l-1)m-1}, \tau, \vec{\mathbf{1}}) = \varphi_{\Phi'(g,2)}.$$

3.4.2. Suppose g is d_i^m.

$$\varphi_{\Phi(g,2)} = (\mathbf{1}^{l(m-1)+i-1}, \beta, \mathbf{1}^{(j-l-1)(m-1)-1}, \iota, \vec{\mathbf{1}}) = \varphi_{\Phi'(g,2)}.$$

□

Lemma 4.4 *If* Ψ *and* Ψ' *are sequences of colour 1 such that* $(\circ\Psi)^{\mathrm{nf}}$ *is* $\circ\Psi'$*, and* g *is a basic arrow of* Δ^{op}*, then* $\varphi_{\Psi(g,2)} = \varphi_{\Psi'(g,2)}$.

Proof Let $\mu(\Psi)$ be a "distance" from $\circ\Psi$ to $(\circ\Psi)^{\mathrm{nf}}$. For example, $\mu(\Psi)$ can be defined as the ordered pair

$$(n, m),$$

where n is the number of subsequences of Ψ that are of the form $(d, 1)(s, 1)$, i.e., s precedes d looking from the right to the left (not necessary immediately) in Ψ, and m is the number of subsequences of Ψ of the form $(s_i, 1)(s_j, 1)$ with $i \leq j$, or $(d_i, 1)(d_j, 1)$ with $i < j$. Suppose that our set of "distances" is lexicographically ordered.

We proceed by induction on $\mu(\Psi)$. If $\mu(\Psi) = (0, 0)$, then $\Psi = \Psi'$ and we are done. If $\mu(\Psi) > (0, 0)$, then, by Remark 3.2, Ψ must be of the form $\Psi_2\Phi\Psi_1$, where $\circ\Phi = \circ\Phi'$ is a basic equation of Δ^{op}. Then we have

$$\begin{aligned}
\varphi_{\Psi_2\Phi\Psi_1(g,2)} &= \varphi_{\Psi_2(g,2)}\overline{W}\mathcal{M}_{\Phi\Psi_1} \circ \overline{W}\mathcal{M}_{\Psi_2}\,\varphi_{\Phi(g,2)}\overline{W}\mathcal{M}_{\Psi_1} \circ \overline{W}\mathcal{M}_{\Psi_2\Phi}\,\varphi_{\Psi_1(g,2)}, \\
&\qquad\qquad\qquad \text{(by Theorem 4.2)} \\
&= \varphi_{\Psi_2(g,2)}\overline{W}\mathcal{M}_{\Phi\Psi_1} \circ \overline{W}\mathcal{M}_{\Psi_2}\,\varphi_{\Phi'(g,2)}\overline{W}\mathcal{M}_{\Psi_1} \circ \overline{W}\mathcal{M}_{\Psi_2\Phi'}\,\varphi_{\Psi_1(g,2)}, \\
&\qquad\qquad\qquad \text{(by Lemma 4.3 and functoriality of } \overline{W}\mathcal{M}_1\text{)} \\
&= \varphi_{\Psi_2\Phi'\Psi_1(g,2)}, \quad \text{(by Theorem 4.2)} \\
&= \varphi_{\Psi'(g,2)}. \quad \text{(by the ind. hyp. since} \mu(\Psi_2\Phi'\Psi_1) < \mu(\Psi_2\Phi\Psi_1))
\end{aligned}$$

□

We can prove now (i) by induction on the length of Γ_1 where in the induction step we use Lemma 4.4. We can prove (ii) in a dual manner using the Eqs. 7–12 for the proof of a lemma dual to Lemma 4.3. So, we have:

Theorem 4.5 *The twofold reduced bar construction* $\overline{W}\mathcal{M}$*, together with the natural transformations* ω*, makes a lax functor from* $(\Delta^{op})^2$ *to* Cat.

5 The threefold monoidal categories

The notion of threefold monoidal category that we use in this paper is defined in [2, Section 7.1] under the name 3-*monoidal category*. In order to define this notion we first define what the arrows between the twofold monoidal categories are.

Definition A *twofold monoidal functor* between twofold monoidal categories $\mathcal{C}$ and $\mathcal{D}$ is a 5-tuple $\langle F, \sigma^1, \delta^1, \sigma^2, \delta^2\rangle$, where for $i \in \{1, 2\}$,

$$\sigma^i_{A,B} \colon FA \otimes^{\mathcal{D}}_i FB \to F(A \otimes^{\mathcal{C}}_i B) \quad \text{and} \quad \delta^i \colon I^{\mathcal{D}}_i \to FI^{\mathcal{C}}_i$$

are arrows of $\mathcal{D}$ natural in A and B, such that $\langle F, \sigma^1, \delta^1\rangle$ and $\langle F, \sigma^2, \delta^2\rangle$ are monoidal functors between, respectively, the first and the second monoidal structures of $\mathcal{C}$ and $\mathcal{D}$. Moreover, the structure brought by the arrows κ, β, τ and ι is preserved, which means that the following four diagrams commute (with the superscripts $\mathcal{C}$ and $\mathcal{D}$ omitted):

$$\begin{array}{ccc} I_1 & \xrightarrow{\kappa} & I_2 \\ {\scriptstyle \delta^1}\downarrow & & \downarrow{\scriptstyle \delta^2} \\ FI_1 & \xrightarrow[F\kappa]{} & FI_2 \end{array}$$

$$\begin{array}{ccc} I_1 & \xrightarrow{\beta} & I_1 \otimes_2 I_1 \\ & & \downarrow{\scriptstyle \delta^1 \otimes_2 \delta^1} \\ {\scriptstyle \delta^1}\downarrow & & FI_1 \otimes_2 FI_1 \\ & & \downarrow{\scriptstyle \sigma^2} \\ FI_1 & \xrightarrow[F\beta]{} & F(I_1 \otimes_2 I_1) \end{array} \qquad \begin{array}{ccc} I_2 \otimes_1 I_2 & \xrightarrow{\tau} & I_2 \\ {\scriptstyle \delta^2 \otimes_1 \delta^2}\downarrow & & \\ FI_2 \otimes_1 FI_2 & & \downarrow{\scriptstyle \delta^2} \\ {\scriptstyle \sigma^1}\downarrow & & \\ F(I_2 \otimes_1 I_2) & \xrightarrow[F\tau]{} & FI_2 \end{array}$$

$$\begin{array}{ccc} (FA \otimes_2 FB) \otimes_1 (FC \otimes_2 FD) & \xrightarrow{\iota} & (FA \otimes_1 FC) \otimes_2 (FB \otimes_1 FD) \\ {\scriptstyle \sigma^2 \otimes_1 \sigma^2}\downarrow & & \downarrow{\scriptstyle \sigma^1 \otimes_2 \sigma^1} \\ F(A \otimes_2 B) \otimes_1 F(C \otimes_2 D) & & F(A \otimes_1 C) \otimes_2 F(B \otimes_1 D) \\ {\scriptstyle \sigma^1}\downarrow & & \downarrow{\scriptstyle \sigma^2} \\ F((A \otimes_2 B) \otimes_1 (C \otimes_2 D)) & \xrightarrow[F\iota]{} & F((A \otimes_1 C) \otimes_2 (B \otimes_1 D)) \end{array}$$

Let $Mon_2(Cat)$ be the 2-category whose 0-cells are the twofold monoidal categories, 1-cells are the twofold monoidal functors, and 2-cells are the *twofold monoidal transformations*, i.e., monoidal transformations with respect to both the structures. The monoidal structure of $Mon_2(Cat)$ is yet again given by 2-products.

Definition A *threefold monoidal* category is a pseudomonoid in $Mon_2(Cat)$.

Hence, a threefold monoidal category consists of the following:

1. a twofold monoidal category $\mathcal{M}$,
2. twofold monoidal functors $\otimes_3 \colon \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ and $I_3 \colon 1 \to \mathcal{M}$,
3. twofold monoidal transformations α_3, ρ_3, and λ_3 such that the structure $\langle \mathcal{M}, \otimes_3, I_3, \alpha_3, \rho_3, \lambda_3\rangle$ satisfies the pseudomonoid conditions.

In an unfolded form, this means that a threefold monoidal category is a category $\mathcal{M}$ equipped with three monoidal structures $\mathcal{M}_1 = \langle \mathcal{M}, \otimes_1, I_1\rangle$, $\mathcal{M}_2 = \langle \mathcal{M}, \otimes_2, I_2\rangle$, and $\mathcal{M}_3 = \langle \mathcal{M}, \otimes_3, I_3\rangle$ such that

[1-2] $\mathcal{M}_1$, $\mathcal{M}_2$, $\kappa : I_1 \to I_2$, $\beta : I_1 \to I_1 \otimes_2 I_1$, $\tau : I_2 \otimes_1 I_2 \to I_2$, and $\iota : (A \otimes_2 B) \otimes_1 (C \otimes_2 D) \to (A \otimes_1 C) \otimes_2 (B \otimes_1 D)$,

[2-3] $\mathcal{M}_2$, $\mathcal{M}_3$, $\kappa' : I_2 \to I_3$, $\beta' : I_2 \to I_2 \otimes_3 I_2$, $\tau' : I_3 \otimes_2 I_3 \to I_3$, and $\iota' : (A \otimes_3 B) \otimes_2 (C \otimes_3 D) \to (A \otimes_2 C) \otimes_3 (B \otimes_2 D)$,

[1-3] $\mathcal{M}_1$, $\mathcal{M}_3$, $\kappa'' : I_1 \to I_3$, $\beta'' : I_1 \to I_1 \otimes_3 I_1$, $\tau'' : I_3 \otimes_1 I_3 \to I_3$ and $\iota'' : (A \otimes_3 B) \otimes_1 (C \otimes_3 D) \to (A \otimes_1 C) \otimes_3 (B \otimes_1 D)$,

are twofold monoidal and, moreover, the following equations hold:

$$\kappa' \circ \kappa = \kappa'', \tag{13}$$

$$\beta' \circ \kappa = (\kappa \otimes_3 \kappa) \circ \beta'', \tag{14}$$

$$\tau' \circ (\kappa'' \otimes_2 \kappa'') \circ \beta = \kappa'', \tag{15}$$

$$\iota' \circ (\beta'' \otimes_2 \beta'') \circ \beta = (\beta \otimes_3 \beta) \circ \beta'', \tag{16}$$

$$\kappa' \circ \tau = \tau'' \circ (\kappa' \otimes_1 \kappa'), \tag{17}$$

$$\beta' \circ \tau = (\tau \otimes_3 \tau) \circ \iota'' \circ (\beta' \otimes_1 \beta'), \tag{18}$$

$$\tau' \circ (\tau'' \otimes_2 \tau'') \circ \iota = \tau'' \circ (\tau' \otimes_1 \tau'), \tag{19}$$

$$\iota' \circ (\iota'' \otimes_2 \iota'') \circ \iota = (\iota \otimes_3 \iota) \circ \iota'' \circ (\iota' \otimes_1 \iota'). \tag{20}$$

Note. The last eight equations represent the four commutative diagrams given above, with F replaced by the twofold monoidal functors I_3 and $\otimes_3$.

As in the case of twofold monoidal categories, we are interested only in *threefold strict monoidal* categories, i.e., when the structures $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}_3$ are strict monoidal.

The threefold monoidal categories defined in [3, Definition 1.7] are the threefold strict monoidal categories from above in which, moreover, it is assumed that $I_1 = I_2 = I_3 = 0$ and all the κ's, β's and τ's are $\mathbf{1}_0$. Hence, from the above list of eight equations, the Eqs. (13), (14), (15), and (17) are trivial, (16), (18) and (19 are variants of internal and external unit conditions, while the Eq. (20) corresponds to the *big hexagonal interchange diagram* (see [3, Definition 1.7]).

6 The threefold reduced bar construction

As in the twofold case, we start with a definition of the threefold reduced bar construction based on a threefold strict monoidal category. Again, this construction corresponds to the one given in the proof of [3, Theorem 2.1], save that the latter construction is based on a category that is threefold monoidal in the sense of that paper.

For a threefold strict monoidal category $\mathcal{M}$, we define functions $\overline{W}\mathcal{M}$ from objects and arrows of $(\Delta^{op})^3$ to objects and arrows of *Cat* in the following manner.

Definition The *threefold reduced bar construction* $\overline{W}\mathcal{M}$ is defined on objects of $(\Delta^{op})^3$ as:

$$\overline{W}\mathcal{M}(n, m, p) = \mathcal{M}^{n \cdot m \cdot p},$$

and for arrows $f : n_s \to n_t$, $g : m_s \to m_t$ and $h : p_s \to p_t$ of Δ^{op}, we define $\overline{W}\mathcal{M}(f, g, h)$ as the composition

$$(\overline{W}\mathcal{M}_3(h))^{n_t \cdot m_t} \circ (\overline{W}\mathcal{M}_2^{p_s}(g))^{n_t} \circ \overline{W}\mathcal{M}_1^{m_s \cdot p_s}(f).$$

For example, $\overline{W}\mathcal{M}(d_1^2, s_1^2, d_1^2) : \mathcal{M}^4 \to \mathcal{M}^2$ is defined as the composition

$$(\overline{W}\mathcal{M}_3(d_1^2))^2 \circ \overline{W}\mathcal{M}_2^2(s_1^2) \circ \overline{W}\mathcal{M}_1^2(d_1^2),$$

and for an object (A, B, C, D) of $\mathcal{M}^4$ we have that

$$\overline{W}\mathcal{M}(d_1^2, s_1^2, d_1^2)(A, B, C, D) = ((A \otimes_1 C) \otimes_3 (B \otimes_1 D), I_2 \otimes_3 I_2).$$

As in the twofold case, $\overline{W}\mathcal{M}$ need not be a functor from $(\Delta^{op})^3$ to *Cat*, and our goal is to prove that it is a lax functor. This means that for every composable pair of arrows $e_1 = (f_1, g_1, h_1)$ and $e_2 = (f_2, g_2, h_2)$ of $(\Delta^{op})^3$, there is a natural transformation

$$\omega_{e_2, e_1} : \overline{W}\mathcal{M}(e_2) \circ \overline{W}\mathcal{M}(e_1) \dot{\to} \overline{W}\mathcal{M}(e_2 \circ e_1),$$

such that Diagram 4.1 commutes.

We use coloured sequences and their shuffles in order to define such natural transformations ω. Let Φ, Γ, and H be sequences of colour 1, 2, and 3, respectively, such that $\circ\Phi : n_s \to n_t$, $\circ\Gamma : m_s \to m_t$, and $\circ\mathrm{H} : p_s \to p_t$. Let Θ be a shuffle of these three sequences. For example, let Φ be $(d_2^2, 1)(d_1^3, 1)$, let Γ be $(d_1^2, 2)$, let H be $(s_1^3, 3)(s_1^2, 3)$, and let Θ be the following shuffle

$$(d_2^2, 1)(s_1^3, 3)(d_1^3, 1)(d_1^2, 2)(s_1^2, 3).$$

For every member $(f, 1)$ of Θ, we define its *inner power* to be the product of the targets of its right-closest $(g, 2)$ and right-closest $(h, 3)$ in Θ. We may assume again that such $(g, 2)$ and $(h, 3)$ exist since we can always add an identity of colour 2 and an identity of colour 3 to the right of $(f, 1)$ in Θ. For every member $(g, 2)$ of Θ, we define its *inner power* to be the target of its right-closest $(h, 3)$ in Θ, and we define its *outer power* to be the target of its right-closest $(f, 1)$ in Θ. For every member $(h, 3)$ of Θ, we define its *outer power* to be the product of the targets of its right-closest $(f, 1)$ and right-closest $(g, 2)$ in Θ. For Θ as above, for example, we have that the outer power of $(s_1^2, 3)$ is 6.

Let $\mathcal{M}$ be a threefold strict monoidal category. We define a functor

$$\overline{W}\mathcal{M}_\Theta : \mathcal{M}^{n_s \cdot m_s \cdot p_s} \to \mathcal{M}^{n_t \cdot m_t \cdot p_t}$$

in the following way: replace in Θ each $(f, 1)$ whose inner power is i by $\overline{W}\mathcal{M}_1^i(f)$, every $(g, 2)$ whose inner power is i and outer power is o by $(\overline{W}\mathcal{M}_2^i(g))^o$ and every $(h, 3)$ whose outer power is o by $(\overline{W}\mathcal{M}_3(h))^o$, and insert $\circ$'s. For Θ as above, we have

Table 2 $\chi_w^{1,2}$ in nontrivial cases

f	g	$\chi_w^{1,2}(f,g)$
s_j^{n+1}	s_i^{m+1}	$(\mathbf{1}^{j(m+1)w}, \underbrace{\mathbf{1}^{iw}, \kappa^w, \vec{\mathbf{1}}}_{(m+1)w}, \vec{\mathbf{1}})$
$d_j^n, 1 \leq j \leq n-1$	s_i^{m+1}	$(\mathbf{1}^{(j-1)(m+1)w}, \underbrace{\mathbf{1}^{iw}, \tau^w, \vec{\mathbf{1}}}_{(m+1)w}, \vec{\mathbf{1}})$
s_j^{n+1}	$d_i^m, 1 \leq i \leq m-1$	$(\mathbf{1}^{j(m-1)w}, \underbrace{\mathbf{1}^{(i-1)w}, \beta^w, \vec{\mathbf{1}}}_{(m-1)w}, \vec{\mathbf{1}})$
$d_j^n, 1 \leq j \leq n-1$	$d_i^m, 1 \leq i \leq m-1$	$(\mathbf{1}^{(j-1)(m-1)w}, \underbrace{\mathbf{1}^{(i-1)w}, \iota^w, \vec{\mathbf{1}}}_{(m-1)w}, \vec{\mathbf{1}})$

that $\overline{W}\mathcal{M}_\Theta$ is

$$\overline{W}\mathcal{M}_1^3(d_2^2) \circ (\overline{W}\mathcal{M}_3(s_1^3))^2 \circ \overline{W}\mathcal{M}_1^2(d_1^3) \circ (\overline{W}\mathcal{M}_2^2(d_1^2))^3 \circ (\overline{W}\mathcal{M}_3(s_1^2))^6,$$

which gives that $\overline{W}\mathcal{M}_\Theta(A, B, C, D, E, F)$ is the 3-tuple

$$((A \otimes_2 B) \otimes_1 (C \otimes_2 D), I_3, (I_3 \otimes_2 I_3) \otimes_1 (I_3 \otimes_2 I_3)).$$

It is easy to see that for arrows f, g and h of Δ^{op}, we have that

$$\overline{W}\mathcal{M}(f, g, h) = \overline{W}\mathcal{M}_{\mathrm{H}\Gamma\Phi},$$

for arbitrary Φ of colour 1, Γ of colour 2 and H of colour 3, such that $\circ\Phi = f$, $\circ\Gamma = g$ and $\circ\mathrm{H} = h$. This may serve as a *combinatorial* definition of the threefold reduced bar construction $\overline{W}\mathcal{M}$ (cf. the combinatorial definition of the twofold reduced bar construction given in Sect. 4).

For basic arrows $f : n \to n'$, $g : m \to m'$ of Δ^{op} and $w \geq 0$ we define a natural transformation

$$\chi_w^{1,2}(f, g) \colon \overline{W}\mathcal{M}_1^{m'w}(f) \circ (\overline{W}\mathcal{M}_2^w(g))^n \stackrel{\cdot}{\to} (\overline{W}\mathcal{M}_2^w(g))^{n'} \circ \overline{W}\mathcal{M}_1^{mw}(f)$$

to be the identity natural transformation except in the following cases:

In order to simplify some calculations and improve the presentation of the paper, we introduce the following formal operation of *multiplication* (always from the right) of tuples representing the natural transformations by 0-1 matrices having in each column exactly one entry equal to 1 and all the other entries equal to 0, which is derived from the standard multiplication of matrices. For example,

$$(\mathbf{1}, \kappa, \mathbf{1}) \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} = (\mathbf{1}, \kappa^2, \mathbf{1}^2, \mathbf{1}^2, \kappa^2, \mathbf{1}^2).$$

Table 3 $\chi_u^{2,3}$ in nontrivial cases

g	h	$\chi_u^{2,3}(g,h)$
s_i^{m+1}	s_k^{p+1}	$((\underbrace{\mathbf{1}^{i(p+1)}, \mathbf{1}^k, \kappa', \vec{\mathbf{1}}}_{(m+1)(p+1)})^u)$
$d_i^m, 1 \leq i \leq m-1$	s_k^{p+1}	$((\underbrace{\mathbf{1}^{(i-1)(p+1)}, \mathbf{1}^k, \tau', \vec{\mathbf{1}}}_{(m-1)(p+1)})^u)$
s_i^{m+1}	$d_k^p, 1 \leq k \leq p-1$	$((\underbrace{\mathbf{1}^{i(p-1)}, \mathbf{1}^{k-1}, \beta', \vec{\mathbf{1}}}_{(m+1)(p-1)})^u)$
$d_i^m, 1 \leq i \leq m-1$	$d_k^p, 1 \leq k \leq p-1$	$((\underbrace{\mathbf{1}^{(i-1)(p-1)}, \mathbf{1}^{k-1}, \iota', \vec{\mathbf{1}}}_{(m-1)(p-1)})^u)$

Note that the tuples of the third column of Table 2 are obtained as a result of multiplication of the tuples in the third column of Table 1 by the matrix

$$I_{n'} \otimes I_{m'} \otimes (\underbrace{1, \ldots, 1}_{w}),$$

where I_k is the $k \times k$ identity matrix and $\otimes$ is the *Kronecker product* of matrices.

For basic arrows $g: m \to m'$, $h: p \to p'$ of Δ^{op} and $u \geq 0$ we define a natural transformation

$$\chi_u^{2,3}(g,h)\colon (\overline{W}\mathcal{M}_2^{p'}(g))^u \circ (\overline{W}\mathcal{M}_3(h))^{um} \stackrel{\cdot}{\to} (\overline{W}\mathcal{M}_3(h))^{um'} \circ (\overline{W}\mathcal{M}_2^{p}(g))^u$$

to be the identity natural transformation except in the following cases (Table 3): Note that the tuples of the third column of this table are obtained as a result of multiplication of the tuples in the third column of Table 1 (where m is replaced by p, n is replaced by m, i is replaced by k, j is replaced by i, and κ, β, τ, and ι are replaced by κ', β', τ', and ι') by the matrix

$$(\underbrace{1, \ldots, 1}_{u}) \otimes I_{m'} \otimes I_{p'}.$$

Finally, for basic arrows $f: n \to n'$, $h: p \to p'$ of Δ^{op} and $v \geq 0$ we define a natural transformation

$$\chi_v^{1,3}(f,h)\colon \overline{W}\mathcal{M}_1^{vp'}(f) \circ (\overline{W}\mathcal{M}_3(h))^{nv} \stackrel{\cdot}{\to} (\overline{W}\mathcal{M}_3(h))^{n'v} \circ \overline{W}\mathcal{M}_1^{vp}(f)$$

to be the identity natural transformation except in the following cases (Table 4): As in the previous cases, the tuples of the third column of this table are obtained as a result of multiplication of the tuples in the third column of Table 1 (with some

Table 4 $\chi_v^{1,3}$ in nontrivial cases

f	h	$\chi_v^{1,3}(f,h)$
s_j^{n+1}	s_k^{p+1}	$(\mathbf{1}^{jv(p+1)}, (\underbrace{\mathbf{1}^k, \kappa'', \vec{\mathbf{1}}}_{p+1})^v, \vec{\mathbf{1}})$
$d_j^n, 1 \le j \le n-1$	s_k^{p+1}	$(\mathbf{1}^{(j-1)v(p+1)}, (\underbrace{\mathbf{1}^k, \tau'', \vec{\mathbf{1}}}_{p+1})^v, \vec{\mathbf{1}})$
s_j^{n+1}	$d_k^p, 1 \le k \le p-1$	$(\mathbf{1}^{jv(p-1)}, (\underbrace{\mathbf{1}^{k-1}, \beta'', \vec{\mathbf{1}}}_{p-1})^v, \vec{\mathbf{1}})$
$d_j^n, 1 \le j \le n-1$	$d_k^p, 1 \le k \le p-1$	$(\mathbf{1}^{(j-1)v(p-1)}, (\underbrace{\mathbf{1}^{k-1}, \iota'', \vec{\mathbf{1}}}_{p-1})^v, \vec{\mathbf{1}})$

necessary replacements) by a certain matrix, in this case that matrix is

$$I_{n'} \otimes (\underbrace{1, \ldots, 1}_{v}) \otimes I_{p'}.$$

Lemma 6.1 *For basic arrows $f : n \to n'$, $g : m \to m'$, and $h : p \to p'$ of Δ^{op} the following diagram commutes*:

$$\begin{array}{ccc}
 & \overline{W}\mathcal{M}_{(f,1)(g,2)(h,3)} & \\
\scriptstyle \chi_{p'}^{1,2}(f,g)(\overline{W}\mathcal{M}_3(h))^{nm} \swarrow & & \searrow \scriptstyle \overline{W}\mathcal{M}_1^{p'm'}(f)\chi_n^{2,3}(g,h) \\
\overline{W}\mathcal{M}_{(g,2)(f,1)(h,3)} & & \overline{W}\mathcal{M}_{(f,1)(h,3)(g,2)} \\
\scriptstyle (\overline{W}\mathcal{M}_2^{p'}(g))^{n'}\chi_m^{1,3}(f,h) \downarrow & & \downarrow \scriptstyle \chi_{m'}^{1,3}(f,h)(\overline{W}\mathcal{M}_2^{p}(g))^{n} \\
\overline{W}\mathcal{M}_{(g,2)(h,3)(f,1)} & & \overline{W}\mathcal{M}_{(h,3)(f,1)(g,2)} \\
\scriptstyle \chi_{n'}^{2,3}(g,h)\overline{W}\mathcal{M}_1^{pm}(f) \searrow & & \swarrow \scriptstyle (\overline{W}\mathcal{M}_3(h))^{n'm'}\chi_p^{1,2}(f,g) \\
 & \overline{W}\mathcal{M}_{(h,3)(g,2)(f,1)} &
\end{array}$$

Proof Consider the following table in which d_y^x is such that $0 < y < x$.

This gives a list of all nontrivial cases for f, g, and h. In this table we point out the component of the two $n' \cdot m' \cdot p'$-tuples of arrows, representing the left-hand side and the right-hand side of the above diagram, where we use one of the Eqs. (13)–(20). In all the other components, the left-hand side is equal to the right-hand side by simple categorial arguments.

As an illustration of these arguments, here we give a proof for one of the cases from the table, namely when $f = d_j^n$, $g = s_i^{m+1}$, and $h = d_k^p$. At the left hand side of the diagram we have the following

$$\chi_{p-1}^{1,2}(d_j^n, s_i^{m+1})(\overline{W}\mathcal{M}_3(d_k^p))^{nm} = (\mathbf{1}^{(j-1)(m+1)(p-1)}, \underbrace{\mathbf{1}^{i(p-1)}, \tau^{p-1}, \vec{\mathbf{1}}}_{(m+1)(p-1)}, \vec{\mathbf{1}}), \qquad \text{(L1)}$$

f	g	h	Component	Equations
s_j^{n+1}	s_i^{m+1}	s_k^{p+1}	$j(m+1)(p+1)+i(p+1)+k+1$	(13)
s_j^{n+1}	s_i^{m+1}	d_k^p	$j(m+1)(p-1)+i(p-1)+k$	(14)
s_j^{n+1}	d_i^m	s_k^{p+1}	$j(m-1)(p+1)+(i-1)(p+1)+k+1$	(15)
s_j^{n+1}	d_i^m	d_k^p	$j(m-1)(p-1)+(i-1)(p-1)+k$	(16)
d_j^n	s_i^{m+1}	s_k^{p+1}	$(j-1)(m+1)(p+1)+i(p+1)+k+1$	(17)
d_j^n	s_i^{m+1}	d_k^p	$(j-1)(m+1)(p-1)+i(p-1)+k$	(18)
d_j^n	d_i^m	s_k^{p+1}	$(j-1)(m-1)(p+1)+(i-1)(p+1)+k+1$	(19)
d_j^n	d_i^m	d_k^p	$(j-1)(m-1)(p-1)+(i-1)(p-1)+k$	(20)

$$
\begin{aligned}
&(\overline{W}\mathcal{M}_2^{p-1}(s_i^{m+1}))^{n-1}\chi_m^{1,3}(d_j^n, d_k^p)\\
&\quad = (\overline{W}\mathcal{M}_2^{p-1}(s_i^{m+1}))^{n-1}\ (\mathbf{1}^{(j-1)m(p-1)}, (\underbrace{\mathbf{1}^{k-1}, \iota'', \vec{\mathbf{1}}}_{p-1})^m, \vec{\mathbf{1}})\\
&\quad = (\mathbf{1}^{(j-1)(m+1)(p-1)}, (\underbrace{\mathbf{1}^{k-1}, \iota'', \vec{\mathbf{1}}}_{p-1})^i, \mathbf{1}^{p-1}, (\underbrace{\mathbf{1}^{k-1}, \iota'', \vec{\mathbf{1}}}_{p-1})^{m-i}, \vec{\mathbf{1}}),
\end{aligned} \tag{L2}
$$

$$
\chi_{n-1}^{2,3}(s_i^{m+1}, d_k^p)\overline{W}\mathcal{M}_1^{mp}(d_j^n) = ((\underbrace{\mathbf{1}^{i(p-1)}, \mathbf{1}^{k-1}, \beta', \vec{\mathbf{1}}}_{(m+1)(p-1)})^{n-1}), \tag{L3}
$$

while at the right hand side we have:

$$
\begin{aligned}
&\overline{W}\mathcal{M}_1^{(m+1)(p-1)}(d_j^n)\chi_n^{2,3}(s_i^{m+1}, d_k^p) = \overline{W}\mathcal{M}_1^{(m+1)(p-1)}(d_j^n)((\underbrace{\mathbf{1}^{i(p-1)}, \mathbf{1}^{k-1}, \beta', \vec{\mathbf{1}}}_{(m+1)(p-1)})^n)\\
&\quad = ((\underbrace{\mathbf{1}^{i(p-1)}, \mathbf{1}^{k-1}, \beta', \vec{\mathbf{1}}}_{(m+1)(p-1)})^{j-1}, \underbrace{\mathbf{1}^{i(p-1)+k-1}, \beta' \otimes_1 \beta', \vec{\mathbf{1}}}_{(m+1)(p-1)}, (\underbrace{\mathbf{1}^{i(p-1)}, \mathbf{1}^{k-1}, \beta', \vec{\mathbf{1}}}_{(m+1)(p-1)})^{n-j-1}),
\end{aligned} \tag{D1}
$$

$$
\chi_{m+1}^{1,3}(d_j^n, d_k^p)(\overline{W}\mathcal{M}_2^p(s_i^{m+1}))^n = (\mathbf{1}^{(j-1)(m+1)(p-1)}, (\underbrace{\mathbf{1}^{k-1}, \iota'', \vec{\mathbf{1}}}_{p-1})^{m+1}, \vec{\mathbf{1}}), \tag{D2}
$$

$$
\begin{aligned}
&(\overline{W}\mathcal{M}_3(d_k^p))^{(n-1)(m-1)}\chi_p^{1,2}(d_j^n, s_i^{m+1})\\
&\quad = (\overline{W}\mathcal{M}_3(d_k^p))^{(n-1)(m-1)} \circ (\mathbf{1}^{(j-1)(m+1)p}, \underbrace{\mathbf{1}^{ip}, \tau^p, \vec{\mathbf{1}}}_{(m+1)p}, \vec{\mathbf{1}})\\
&\quad = (\mathbf{1}^{(j-1)(m+1)(p-1)}, \underbrace{\mathbf{1}^{i(p-1)}, \tau^{k-1}, \tau \otimes_3 \tau, \tau^{p-1-k}, \vec{\mathbf{1}}}_{(m+1)(p-1)}, \vec{\mathbf{1}}). \qquad \text{(D3)}
\end{aligned}
$$

Now we take a look at all entries that are not equal to **1** (non-identities). For example, in (L1) the non-identities are at positions

$$(j-1)(m+1)(p+1) + i(p-1) + l, \quad \text{for } 1 \le l \le p-1,$$

and those entries are equal to τ. By comparing the non-identities for (D1), (D2), (D3), (L1), (L2), and (L3), we get that the only difference is at position $(j-1)(m+1)(p+1)+i(p-1)+k$, where we have that $\beta' \circ \mathbf{1} \circ \tau$ must be equal to $(\tau \otimes_3 \tau) \circ \iota'' \circ (\beta' \otimes_1 \beta')$, which is exactly our Eq. (18). □

Let $\Theta_0, \ldots, \Theta_j$ for $j \ge 0$ be shuffles of Φ, Γ, and H such that $\Theta_0 = \Theta$ and $\Theta_j = \mathrm{H}\Gamma\Phi$, and if $j > 0$, then for every $0 \le i \le j-1$ we have that $\Theta_i = \Pi(f,1)(g,2)\Lambda$ and $\Theta_{i+1} = \Pi(g,2)(f,1)\Lambda$, or $\Theta_i = \Pi(g,2)(h,3)\Lambda$ and $\Theta_{i+1} = \Pi(h,3)(g,2)\Lambda$, or $\Theta_i = \Pi(f,1)(h,3)\Lambda$ and $\Theta_{i+1} = \Pi(h,3)(f,1)\Lambda$. We call $\Theta_0, \ldots, \Theta_j$ a *normalizing path* starting with Θ. Its *length* is j and Proposition 4.1 still holds.

If $\Theta_i = \Pi(f,1)(g,2)\Lambda$ and $\Theta_{i+1} = \Pi(g,2)(f,1)\Lambda$, then for w being the target of the leftmost member of Λ of colour 3 we have that

$$\varphi_i = \overline{W}\mathcal{M}_\Pi\ \chi_w^{1,2}(f,g)\ \overline{W}\mathcal{M}_\Lambda$$

is a natural transformation from $\overline{W}\mathcal{M}_{\Theta_i}$ to $\overline{W}\mathcal{M}_{\Theta_{i+1}}$. We define φ_i analogously in the other two possibilities for the pair Θ_i, Θ_{i+1} relying on $\chi_u^{2,3}(g,h)$ or $\chi_v^{1,3}(f,h)$, for u being the target of the leftmost member of Λ of colour 1 and v being the target of the leftmost member of Λ of colour 2. We define $\varphi(\Theta_0, \ldots, \Theta_k)$ as in the twofold case and for $\Theta'_0, \ldots, \Theta'_k$ being another normalizing path starting with Θ, we can show the following.

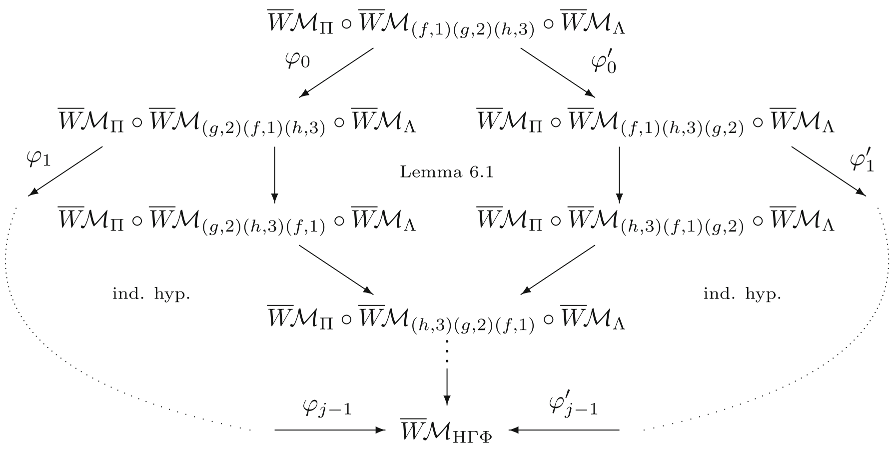
Theorem 6.2 $\varphi(\Theta_0, \ldots, \Theta_j) = \varphi(\Theta'_0, \ldots, \Theta'_j)$.

Proof We proceed by induction on $j \ge 0$. If $j = 0$, then $\varphi(\Theta_0) = \varphi(\Theta'_0) = \mathbf{1}$.

If $j > 0$, then we are either in the situation as in the proof of Theorem 4.2 and we proceed analogously, or for some basic arrows $f: n \to n'$, $g: m \to m'$, and $h: p \to p'$ of Δ^{op} we have that

$$\varphi_0 = \overline{W}\mathcal{M}_\Pi\ \chi_{p'}^{1,2}(f,g)\ \overline{W}\mathcal{M}_{(h,3)\Lambda} \quad \text{and} \quad \varphi'_0 = \overline{W}\mathcal{M}_{\Pi(f,1)}\ \chi_n^{2,3}(g,h)\ \overline{W}\mathcal{M}_\Lambda.$$

In the latter case, we use Lemma 6.1 and the induction hypothesis twice to obtain the following commutative diagram.



□

By Theorem 6.2, the following definition is correct.

Definition Let $\varphi_\Theta : \overline{W}\mathcal{M}_\Theta \dot{\to} \overline{W}\mathcal{M}_{\mathrm{H}\Gamma\Phi}$ be $\varphi(\Theta_0, \ldots, \Theta_j)$, for an arbitrary normalizing path $\Theta_0, \ldots, \Theta_j$ starting with Θ.

We are ready to define a natural transformation

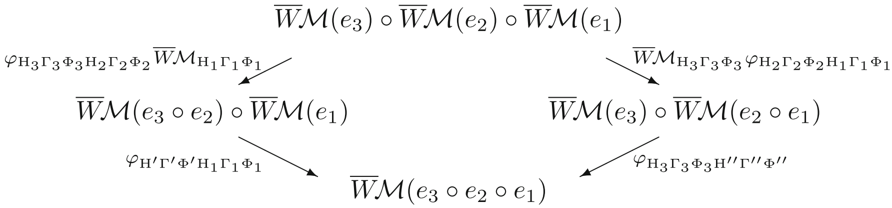
$$\omega_{e_2,e_1} : \overline{W}\mathcal{M}(e_2) \circ \overline{W}\mathcal{M}(e_1) \dot{\to} \overline{W}\mathcal{M}(e_2 \circ e_1),$$

for every composable pair of arrows $e_1 = (f_1, g_1, h_1)$ and $e_2 = (f_2, g_2, h_2)$ of $(\Delta^{op})^3$.

Definition Let Φ_1 and Φ_2 be sequences of colour 1, let Γ_1 and Γ_2 be sequences of colour 2, and let H_1 and H_2 be sequences of colour 3, such that $\circ\Phi_1$ is f_1^{nf}, $\circ\Phi_2$ is f_2^{nf}, $\circ\Gamma_1$ is g_1^{nf}, $\circ\Gamma_2$ is g_2^{nf}, $\circ\mathrm{H}_1$ is h_1^{nf}, and $\circ\mathrm{H}_2$ is h_2^{nf}. We define

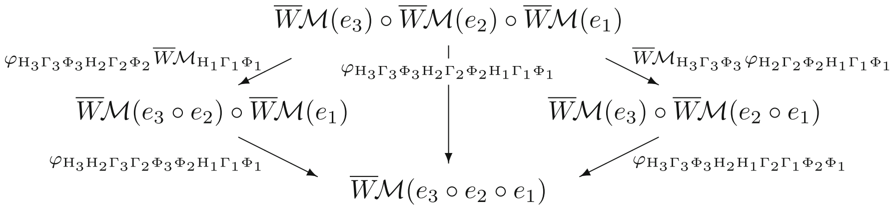
$$\omega_{e_2,e_1} \quad \text{as} \quad \varphi_{\mathrm{H}_2\Gamma_2\Phi_2\mathrm{H}_1\Gamma_1\Phi_1}.$$

It remains to prove that our Diagram 4.1 commutes. Let $e_1 = (f_1, g_1, h_1)$, $e_2 = (f_2, g_2, h_2)$ and $e_3 = (f_3, g_3, h_3)$ be such that the composition $e_3 \circ e_2 \circ e_1$ is defined in $(\Delta^{op})^3$. Let Φ_1, Φ_2, Γ_1, Γ_2, H_1 and H_2 be as above, and let Φ_3, Γ_3 and H_3 be sequences of colour 1, 2 and 3 respectively such that $\circ\Phi_3$ is f_3^{nf}, $\circ\Gamma_3$ is g_3^{nf} and $\circ\mathrm{H}_3$ is h_3^{nf}. In this case, Diagram 4.1 reads



where $\circ\Phi'$ is $(f_3 \circ f_2)^{\mathrm{nf}}$, $\circ\Gamma'$ is $(g_3 \circ g_2)^{\mathrm{nf}}$, $\circ\mathrm{H}'$ is $(h_3 \circ h_2)^{\mathrm{nf}}$, $\circ\Phi''$ is $(f_2 \circ f_1)^{\mathrm{nf}}$, $\circ\Gamma''$ is $(g_2 \circ g_1)^{\mathrm{nf}}$ and $\circ\mathrm{H}''$ is $(h_2 \circ h_1)^{\mathrm{nf}}$.

By Theorem 6.2 we have the following commutative diagram



Hence, to prove that Diagram 4.1 commutes, it suffices to show that

$$
\begin{aligned}
&\text{(i)} \quad \varphi_{H_3H_2\Gamma_3\Gamma_2\Phi_3\Phi_2H_1\Gamma_1\Phi_1} = \varphi_{H'\Gamma'\Phi'H_1\Gamma_1\Phi_1} \quad \text{and}\\
&\text{(ii)} \quad \varphi_{H_3\Gamma_3\Phi_3H_2H_1\Gamma_2\Gamma_1\Phi_2\Phi_1} = \varphi_{H_3\Gamma_3\Phi_3H''\Gamma''\Phi''}.
\end{aligned}
$$

To prove (i) and (ii) we use the same arguments as in the twofold case. Let x, y, and z be three different elements of the set $\{1, 2, 3\}$ such that $x < y$. Note that the position of $(\mathbf{1}_q, z)$ in the two shuffles of the lemma below is irrelevant; $(\mathbf{1}_q, z)$ serves just to keep φ correctly defined and to introduce the parameter q.

Lemma 6.3 *If Φ and Φ' are sequences of colour x such that $\circ\Phi = \circ\Phi'$ is a basic equation of Δ^{op}, and g is a basic arrow of Δ^{op}, then for every $q \geq 0$ we have that $\varphi_{\Phi(g,y)(\mathbf{1}_q,z)} = \varphi_{\Phi'(g,y)(\mathbf{1}_q,z)}$.*

Proof Suppose the target of $\circ\Phi$ is n' and the target of g is m'. If $x = 1$, $y = 2$, and $z = 3$, then we proceed as in Lemma 4.3 with all the cases modified so that the tuples representing the natural transformations are multiplied by the matrix $I_{n'} \otimes I_{m'} \otimes (\underbrace{1, \ldots, 1}_{q})$. For example, Case 1.1.1 now reads

$$
\varphi_{\Phi(g,2)(\mathbf{1}_q,3)} = (\mathbf{1}^{(j-1)mq}, \mathbf{1}^{iq}, \tau^q, \mathbf{1}^{((l-j-1)m-1)q}, \tau^q, \vec{\mathbf{1}}) = \varphi_{\Phi'(g,2)(\mathbf{1}_q,3)}.
$$

If $x = 2$, $y = 3$, and $z = 1$, we again proceed as in Lemma 4.3 with all the cases modified so that κ, β, τ, and ι are replaced by κ', β', τ', and ι', and the tuples representing the natural transformations are multiplied by the matrix $(\underbrace{1, \ldots, 1}_{q}) \otimes I_{n'} \otimes I_{m'}$. For example, Case 1.1.1 now reads

$$\varphi_{\Phi(g,3)(\mathbf{1}_q,1)} = ((\mathbf{1}^{(j-1)m}, \mathbf{1}^i, \tau', \mathbf{1}^{(l-j-1)m-1}, \tau', \vec{\mathbf{1}})^q) = \varphi_{\Phi'(g,3)(\mathbf{1}_q,1)}.$$

If $x = 1$, $y = 3$, and $z = 2$, we modify all the cases of Lemma 4.3 so that κ, β, τ, and ι are replaced by κ'', β'', τ'', and ι'', and the tuples representing the natural transformations are multiplied by the matrix $I_{n'} \otimes (\underbrace{1, \ldots, 1}_{q}) \otimes I_{m'}$. For example, Case 1.1.1 now reads

$$\begin{aligned}\varphi_{\Phi(g,3)(\mathbf{1}_q,2)} &= (\mathbf{1}^{(j-1)mq}, (\mathbf{1}^i, \tau'', \mathbf{1}^{m-i-1})^q, \mathbf{1}^{(l-j-2)mq}, (\mathbf{1}^i, \tau'', \mathbf{1}^{m-i-1})^q, \vec{\mathbf{1}}) \\ &= \varphi_{\Phi'(g,3)(\mathbf{1}_q,2)}.\end{aligned}$$

□

By relying on Lemma 6.3, we can prove a lemma analogous to Lemma 4.4 and this suffices for the proof of (i) by induction on the sum of lengths of H_1 and Γ_1. We can prove (ii) in a dual manner. Hence, we have:

Theorem 6.5 *The threefold reduced bar construction $\overline{W}\mathcal{M}$, together with the natural transformations ω, makes a lax functor from $(\Delta^{op})^3$ to* Cat.

7 The *n*-fold monoidal categories

The notion of n-fold monoidal category that we use in this paper is defined in [2, Section 7.6] under the name *n-monoidal category*. Before we define the notion of $(n+1)$-fold monoidal category, for $n \geq 3$, we first define what the arrows between the n-fold monoidal categories are. For this inductive definition we assume that an n-fold monoidal category, for $n \geq 3$, is a category $\mathcal{M}$ equipped with n monoidal structures $\mathcal{M}_1 = \langle \mathcal{M}, \otimes_1, I_1 \rangle, \ldots, \mathcal{M}_n = \langle \mathcal{M}, \otimes_n, I_n \rangle$ such that for every $1 \leq k < l < m \leq n$, the category $\mathcal{M}$ with the structures $\mathcal{M}_k, \mathcal{M}_l$ and $\mathcal{M}_m$ is threefold monoidal. Hence, for every $1 \leq k < l \leq n$, the category $\mathcal{M}$ with the structures $\mathcal{M}_k$ and $\mathcal{M}_l$ is twofold monoidal. We denote by $\kappa_{k,l} \colon I_k \to I_l$, $\beta_{k,l} \colon I_k \to I_k \otimes_l I_k$, $\tau_{k,l} \colon I_l \otimes_k I_l \to I_l$ and

$$\iota_{k,l} \colon (A \otimes_l B) \otimes_k (C \otimes_l D) \to (A \otimes_k C) \otimes_l (B \otimes_k D)$$

the required arrows of $\mathcal{M}$.

Definition An *n-fold monoidal functor* between two n-fold monoidal categories $\mathcal{C}$ and $\mathcal{D}$ is a $(2n+1)$-tuple $\langle F, \sigma^1, \delta^1, \ldots, \sigma^n, \delta^n \rangle$, where for $k \in \{1, \ldots, n\}$,

$$\sigma^k_{A,B} \colon FA \otimes^{\mathcal{D}}_k FB \to F\left(A \otimes^{\mathcal{C}}_k B\right) \quad \text{and} \quad \delta^k \colon I^{\mathcal{D}}_k \to FI^{\mathcal{C}}_k$$

are arrows of $\mathcal{D}$ natural in A and B, such that $\langle F, \sigma^k, \delta^k \rangle$ is a monoidal functor between the kth monoidal structures of $\mathcal{C}$ and $\mathcal{D}$. Moreover, for every $1 \leq k < l \leq n$, the following four diagrams commute (with the superscripts $\mathcal{C}$ and $\mathcal{D}$ omitted):

$$\begin{CD} I_k @>\kappa_{k,l}>> I_l \\ @V\delta^kVV @VV\delta^lV \\ FI_k @>>F\kappa_{k,l}> FI_l \end{CD}$$

$$\begin{CD} I_k @>\beta_{k,l}>> I_k \otimes_l I_k \\ @. @VV\delta^k \otimes_l \delta^kV \\ @V\delta^kVV FI_k \otimes_l FI_k \\ @. @VV\sigma^lV \\ FI_k @>>F\beta_{k,l}> F(I_k \otimes_l I_k) \end{CD} \qquad \begin{CD} I_l \otimes_k I_l @>\tau_{k,l}>> I_l \\ @V\delta^l \otimes_k \delta^lVV @. \\ FI_l \otimes_k FI_l @. @VV\delta^lV \\ @V\sigma^kVV @. \\ F(I_l \otimes_k I_l) @>>F\tau_{k,l}> FI_l \end{CD}$$

$$\begin{CD} (FA \otimes_l FB) \otimes_k (FC \otimes_l FD) @>\iota_{k,l}>> (FA \otimes_k FC) \otimes_l (FB \otimes_k FD) \\ @V\sigma^l \otimes_k \sigma^lVV @VV\sigma^k \otimes_l \sigma^kV \\ F(A \otimes_l B) \otimes_k F(C \otimes_l D) @. F(A \otimes_k C) \otimes_l F(B \otimes_k D) \\ @V\sigma^kVV @VV\sigma^lV \\ F((A \otimes_l B) \otimes_k (C \otimes_l D)) @>>F\iota_{k,l}> F((A \otimes_k C) \otimes_l (B \otimes_k D)) \end{CD}$$

Let $Mon_n(Cat)$ be the 2-category whose 0-cells are the n-fold monoidal categories, 1-cells are the n-fold monoidal functors, and 2-cells are the *n-fold monoidal transformations*, i.e., monoidal transformations with respect to all n structures. The monoidal structure of $Mon_n(Cat)$ is again given by 2-products.

Definition An $(n+1)$-*fold monoidal* category is a pseudomonoid in $Mon_n(Cat)$.

By this inductive definition, it is clear that an n-fold monoidal category satisfies the assumptions given above, which we may take as an unfolded form of this definition. As in the case of twofold and threefold monoidal categories, we are only interested in *n-fold strict monoidal* categories, i.e., when the structures $\mathcal{M}_1, \ldots, \mathcal{M}_n$ are strict monoidal.

The n-fold monoidal categories defined in [3, Definition 1.7] are the n-fold strict monoidal categories from above in which, moreover, it is assumed that $I_1 = \cdots = I_n = 0$, and all the κ, β and τ arrows are replaced by the identity $\mathbf{1}_0$. Also, for every n, a symmetric monoidal category is n-fold monoidal with all n monoidal structures being the same.

On the other hand, it is not true that every n-fold strict monoidal category in our sense is an n-fold monoidal in the sense of [3]. It is not only the case that the difference would appear in arrows that involve the units. The arrows of the form

$$A \otimes_i B \to A \otimes_j B \quad \text{and} \quad A \otimes_i B \to B \otimes_j A,$$

for $i < j$ (see [3, Remark 1.4]), show that the axiomatization of n-fold monoidal categories given in [3] leads to a non-conservative extension of its fragment without units. These arrows are not presumed by our definition. Hence, the categories would be different in their unit-free fragments too.

8 The n-fold reduced bar construction

In Sects. 4 and 6, we have defined the n-fold reduced bar construction for $n = 2$ and $n = 3$. We define, in the same manner, the n-fold reduced bar construction for arbitrary $n \geq 3$. This construction corresponds to the one given in the proof of [3, Theorem 2.1], save that the latter construction is based on a category that is n-fold monoidal in the sense of that paper.

For an n-fold strict monoidal category $\mathcal{M}$, we define functions $\overline{W}\mathcal{M}$ from objects and arrows of $(\Delta^{op})^n$ to objects and arrows of *Cat* in the following manner.

Definition The *n-fold reduced bar construction* $\overline{W}\mathcal{M}$ is defined on objects of $(\Delta^{op})^n$ as:

$$\overline{W}\mathcal{M}\,(k_1, \dots, k_n) = \mathcal{M}^{k_1\cdot\dots\cdot k_n},$$

and for arrows $f_k : s_k \to t_k$, $1 \leq k \leq n$, of Δ^{op}, we define $\overline{W}\mathcal{M}(f_1, \dots, f_n)$ as the composition

$$(\overline{W}\mathcal{M}_n(f_n))^{t_1\cdot\dots\cdot t_{n-1}} \circ \dots \circ (\overline{W}\mathcal{M}_k^{s_{k+1}\cdot\dots\cdot s_n}(f_k))^{t_1\cdot\dots\cdot t_{k-1}} \circ \dots \circ \overline{W}\mathcal{M}_1^{s_2\cdot\dots\cdot s_n}(f_1).$$

For example, for $\mathcal{M}$ being a fourfold strict monoidal category, the functor $\overline{W}\mathcal{M}(d_1^2, s_1^2, d_1^2, s_0^2) : \mathcal{M}^4 \to \mathcal{M}^4$ is defined as the composition

$$(\overline{W}\mathcal{M}_4(s_0^2))^2(\overline{W}\mathcal{M}_3(d_1^2))^2 \circ \overline{W}\mathcal{M}_2^2(s_1^2) \circ \overline{W}\mathcal{M}_1^2(d_1^2),$$

and for an object (A, B, C, D) of $\mathcal{M}^4$ we have that

$$\overline{W}\mathcal{M}(d_1^2, s_1^2, d_1^2, s_0^2)(A, B, C, D) = (I_4, (A \otimes_1 C) \otimes_3 (B \otimes_1 D), I_4, I_2 \otimes_3 I_2).$$

In order to prove that $\overline{W}\mathcal{M}$ is a lax functor, for every composable pair of arrows e_1 and e_2 of $(\Delta^{op})^n$, we have to define a natural transformation

$$\omega_{e_2,e_1} : \overline{W}\mathcal{M}(e_2) \circ \overline{W}\mathcal{M}(e_1) \stackrel{\cdot}{\to} \overline{W}\mathcal{M}(e_2 \circ e_1),$$

such that Diagram 4.1 commutes. For this we use again coloured sequences and their shuffles.

Let $\Phi_1, \dots, \Phi_n$ be sequences of colours $1, \dots, n$, respectively and let Θ be a shuffle of these n sequences. For every member (f, k) of Θ, we define its *inner power* and its *outer power* to be

$$\prod_{k<l\leq n} t_l \quad \text{and} \quad \prod_{1\leq l<k} t_l,$$

respectively, where t_l is the target of its right-closest member of Θ of colour l (again with adding appropriate identities if necessary). We assume that the empty product is 1. This definition is in accordance with the corresponding definitions for two and threefold cases; the difference is that the powers fixed to be 1 (like, for example, the outer power of $(f, 1)$) are not mentioned there.

Let $\mathcal{M}$ be an n-fold strict monoidal category and let our sequences be such that for every $1 \leq k \leq n$, $\circ\Phi_k : s_k \to t_k$. We define a functor

$$\overline{W}\mathcal{M}_\Theta : \mathcal{M}^{s_1 \cdot \ldots \cdot s_n} \to \mathcal{M}^{t_1 \cdot \ldots \cdot t_n}$$

in the following way: replace in Θ every (f, k) whose inner power is i and outer power is o by $(\overline{W}\mathcal{M}^i_k(f))^o$, and insert $\circ$'s.

It is easy to see that for arrows f_k, $1 \leq k \leq n$, of Δ^{op}, we have that

$$\overline{W}\mathcal{M}(f_1, \ldots, f_n) = \overline{W}\mathcal{M}_{\Phi_n \ldots \Phi_1},$$

for arbitrary sequences Φ_k of colour k, $1 \leq k \leq n$, such that $\circ\Phi_k = f_k$. This may serve as an alternative (*combinatorial*) definition of the n-fold reduced bar construction $\overline{W}\mathcal{M}$.

We define the natural transformations ω following the lines of Sects. 4 and 6. In order to compare some notions needed for this definition with the corresponding notions introduced in Sects. 4 and 6, we use the symbol n for an object of Δ^{op}. To prevent ambiguities, we introduce a new symbol $\dot{n}$, and assume that our category $\mathcal{M}$ is $\dot{n}$-fold strict monoidal and that $\overline{W}\mathcal{M}$ is the $\dot{n}$-fold reduced bar construction. This includes just a few occurrences of $\dot{n}$ ending with Lemma 8.1, when we return to the standard notation.

For basic arrows $f : n \to n'$ and $g : m \to m'$ of Δ^{op}, for k, l such that $0 \leq k < l \leq \dot{n}$, and $u, v, w \geq 0$, we define a natural transformation

$$\chi^{k,l}_{u,v,w}(f, g) : (\overline{W}\mathcal{M}^{vm'w}_k(f))^u \circ (\overline{W}\mathcal{M}^w_l(g))^{unv} \dot{\to} (\overline{W}\mathcal{M}^w_l(g))^{un'v} \circ (\overline{W}\mathcal{M}^{vmw}_k(f))^u$$

to be the identity natural transformation except in the following cases:

f	g	$\chi^{k,l}_{u,v,w}(f, g)$
s^{n+1}_j	s^{m+1}_i	$((\mathbf{1}^{j(m+1)vw}, (\underbrace{\mathbf{1}^{iw}, \kappa^w_{k,l}, \vec{\mathbf{1}}}_{(m+1)w})^v, \vec{\mathbf{1}})^u)$
d^n_j, $1 \leq j \leq n-1$	s^{m+1}_i	$((\mathbf{1}^{(j-1)(m+1)vw}, (\underbrace{\mathbf{1}^{iw}, \tau^w_{k,l}, \vec{\mathbf{1}}}_{(m+1)w})^v, \vec{\mathbf{1}})^u)$
s^{n+1}_j	d^m_i, $1 \leq i \leq m-1$	$((\mathbf{1}^{j(m-1)vw}, (\underbrace{\mathbf{1}^{(i-1)w}, \beta^w_{k,l}, \vec{\mathbf{1}}}_{(m-1)w})^v, \vec{\mathbf{1}})^u)$
d^n_j, $1 \leq j \leq n-1$	d^m_i, $1 \leq i \leq m-1$	$((\mathbf{1}^{(j-1)(m-1)vw}, (\underbrace{\mathbf{1}^{(i-1)w}, \iota^w_{k,l}, \vec{\mathbf{1}}}_{(m-1)w})^v, \vec{\mathbf{1}})^u)$

Note that the tuples of the third column of the table above are obtained as a result of multiplication of the tuples in the third column of Table 1 (where κ, β, τ, and ι are replaced by $\kappa_{k,l}$, $\beta_{k,l}$, $\tau_{k,l}$, and $\iota_{k,l}$) by the matrix

$$\underbrace{(1,\dots,1)}_{u}\otimes I_{n'}\otimes\underbrace{(1,\dots,1)}_{v}\otimes I_{m'}\otimes\underbrace{(1,\dots,1)}_{w}.$$

For the following lemma, which is analogous to Lemma 6.1, we assume that $f: n\to n'$, $g: m\to m'$, and $h: p\to p'$ are basic arrows of Δ^{op}, that $1\leq a<b<c\leq \dot{n}$, that Λ is a shuffle of sequences of colours $1,\dots,\dot{n}$ with only identity arrows in it, and that

$$u=\prod_{1\leq l<a} t_l,\quad v_1=\prod_{a<l<b} t_l,\quad v_2=\prod_{b<l<c} t_l,\quad w=\prod_{b<l\leq\dot{n}} t_l,$$

where t_l is the target of the leftmost member of Λ of colour l.

For example, if $\dot{n}=7$, $a=2$, $b=4$, $c=5$, and

$$\Lambda=(\mathbf{1}_2,1)(\mathbf{1}_n,2)(\mathbf{1}_3,3)(\mathbf{1}_m,4)(\mathbf{1}_p,5)(\mathbf{1}_5,6)(\mathbf{1}_4,7),$$

then $u=2$, $v_1=3$, $v_2=1$, and $w=20$.

Lemma 8.1 *The following diagram commutes*:

$$\begin{array}{ccc}
 & \overline{W}\mathcal{M}_{(f,a)(g,b)(h,c)\Lambda} & \\
\swarrow\ \chi^{a,b}_{u,v_1,v_2p'w}(f,g)\overline{W}\mathcal{M}_{(h,c)\Lambda} & & \overline{W}\mathcal{M}_{(f,a)\Lambda}\,\chi^{b,c}_{unv_1,v_2,w}(g,h)\ \searrow \\
\overline{W}\mathcal{M}_{(g,b)(f,a)(h,c)\Lambda} & & \overline{W}\mathcal{M}_{(f,a)(h,c)(g,b)\Lambda} \\
\overline{W}\mathcal{M}_{(g,b)\Lambda}\,\chi^{a,c}_{u,v_1mv_2,w}(f,h)\ \downarrow & & \downarrow\ \chi^{a,c}_{u,v_1m'v_2,w}(f,h)\overline{W}\mathcal{M}_{(g,b)\Lambda} \\
\overline{W}\mathcal{M}_{(g,b)(h,c)(f,a)\Lambda} & & \overline{W}\mathcal{M}_{(h,c)(f,a)(g,b)\Lambda} \\
\searrow\ \chi^{b,c}_{un'v_1,v_2,w}(g,h)\overline{W}\mathcal{M}_{(f,a)\Lambda} & & \overline{W}\mathcal{M}_{(h,c)\Lambda}\,\chi^{a,b}_{u,v_1,v_2pw}(f,g)\ \swarrow \\
 & \overline{W}\mathcal{M}_{(h,c)(g,b)(f,a)\Lambda} &
\end{array}$$

Proof The tuples representing the natural transformations of the left-hand side and the right-hand side of this diagram are obtained by multiplying the corresponding tuples of the diagram in Lemma 6.1 (where κ, κ', and κ'' are replaced by $\kappa_{a,b}$, $\kappa_{b,c}$, and $\kappa_{a,c}$, etc.) by the matrix

$$\underbrace{(1,\dots,1)}_{u}\otimes I_{n'}\otimes\underbrace{(1,\dots,1)}_{v_1}\otimes I_{m'}\underbrace{(1,\dots,1)}_{v_2}\otimes I_{p'}\otimes\underbrace{(1,\dots,1)}_{w}.$$

Hence, Lemma 6.1 directly implies this lemma. □

Let $\Theta_0, \ldots, \Theta_j$, for $j \geq 0$, be shuffles of $\Phi_1, \ldots, \Phi_n$ such that $\Theta_0 = \Theta$ and $\Theta_j = \Phi_n \ldots \Phi_1$, and if $j > 0$, then for every $0 \leq i \leq j-1$ we have that for some $1 \leq k < l \leq n$, $\Theta_i = \Pi(f,k)(g,l)\Lambda$ and $\Theta_{i+1} = \Pi(g,l)(f,k)\Lambda$. We call $\Theta_0, \ldots, \Theta_j$ a *normalizing path* starting with Θ. Its *length* is j and Proposition 4.1 still holds.

For u, v, and w being respectively

$$\prod_{1\leq z<k} t_z, \quad \prod_{k<z<l} t_z, \quad \prod_{l<z\leq n} t_z,$$

where t_z is the target of the leftmost member of Λ of colour z, we have that

$$\varphi_i = \overline{W}\mathcal{M}_\Pi\ \chi^{k,l}_{u,v,w}(f,g)\ \overline{W}\mathcal{M}_\Lambda,$$

is a natural transformation from $\overline{W}\mathcal{M}_{\Theta_i}$ to $\overline{W}\mathcal{M}_{\Theta_{i+1}}$. We define $\varphi(\Theta_0, \ldots, \Theta_j)$ as in the twofold case and for $\Theta'_0, \ldots, \Theta'_j$ being another normalizing path starting with Θ, the following theorem is proved in the same manner as Theorem 6.2, relying on Lemma 8.1 instead of Lemma 6.1.

Theorem 8.2 $\varphi(\Theta_0, \ldots, \Theta_j) = \varphi(\Theta'_0, \ldots, \Theta'_j)$.

By Theorem 8.2, the following definition is correct.

Definition Let $\varphi_\Theta : \overline{W}\mathcal{M}_\Theta \dot{\to} \overline{W}\mathcal{M}_{\Phi_n\ldots\Phi_1}$ be $\varphi(\Theta_0, \ldots, \Theta_j)$, for an arbitrary normalizing path $\Theta_0, \ldots, \Theta_j$ starting with Θ.

We are ready to define a natural transformation

$$\omega_{e_2,e_1} : \overline{W}\mathcal{M}(e_2) \circ \overline{W}\mathcal{M}(e_1) \dot{\to} \overline{W}\mathcal{M}(e_2 \circ e_1),$$

for every composable pair of arrows $e_1 = (f_1^1, \ldots, f_n^1)$ and $e_2 = (f_1^2, \ldots, f_n^2)$ of $(\Delta^{op})^n$.

Definition Let Φ_k^1 and Φ_k^2, for $1 \leq k \leq n$, be sequences of colour k, such that $\circ\Phi_k^1$ is $(f_k^1)^{\text{nf}}$ and $\circ\Phi_k^2$ is $(f_k^2)^{\text{nf}}$. We define

$$\omega_{e_2,e_1} \quad \text{as} \quad \varphi_{\Phi_n^2\ldots\Phi_1^2\Phi_n^1\ldots\Phi_1^1}.$$

It remains to prove that our Diagram 4.1 commutes. Let $e_1 = (f_1^1, \ldots, f_n^1)$, $e_2 = (f_1^2, \ldots, f_n^2)$ and $e_3 = (f_1^3, \ldots, f_n^3)$ be such that the composition $e_3 \circ e_2 \circ e_1$ is defined in $(\Delta^{op})^n$. Let Φ_k^1, Φ_k^2 and Φ_k^3, for $1 \leq k \leq n$, be sequences of colour k, such that $\circ\Phi_k^1$ is $(f_k^1)^{\text{nf}}$, $\circ\Phi_k^2$ is $(f_k^2)^{\text{nf}}$ and $\circ\Phi_k^3$ is $(f_k^3)^{\text{nf}}$. In this case, Diagram 4.1 reads

$$\begin{array}{ccccc} & & \overline{W}\mathcal{M}(e_3)\circ\overline{W}\mathcal{M}(e_2)\circ\overline{W}\mathcal{M}(e_1) & & \\ & \swarrow^{\varphi_{\Phi^3_n\ldots\Phi^3_1\Phi^2_n\ldots\Phi^2_1}\overline{W}\mathcal{M}_{\Phi^1_n\ldots\Phi^1_1}} & & \searrow^{\overline{W}\mathcal{M}_{\Phi^3_n\ldots\Phi^3_1}\varphi_{\Phi^2_n\ldots\Phi^2_1\Phi^1_n\ldots\Phi^1_1}} & \\ \overline{W}\mathcal{M}(e_3\circ e_2)\circ\overline{W}\mathcal{M}(e_1) & & & & \overline{W}\mathcal{M}(e_3)\circ\overline{W}\mathcal{M}(e_2\circ e_1) \\ & \searrow_{\varphi_{\Phi'_n\ldots\Phi'_1\Phi^1_n\ldots\Phi^1_1}} & & \swarrow_{\varphi_{\Phi^3_n\ldots\Phi^3_1\Phi''_n\ldots\Phi''_1}} & \\ & & \overline{W}\mathcal{M}(e_3\circ e_2\circ e_1) & & \end{array}$$

where $\circ\Phi'_k$ is $(f^3_k\circ f^2_k)^{\text{nf}}$ and $\circ\Phi''_k$ is $(f^2_k\circ f^1_k)^{\text{nf}}$.

By Theorem 8.2 we have the following commutative diagram

$$\begin{array}{ccccc} & & \overline{W}\mathcal{M}(e_3)\circ\overline{W}\mathcal{M}(e_2)\circ\overline{W}\mathcal{M}(e_1) & & \\ & \swarrow^{\varphi_{\Phi^3_n\ldots\Phi^3_1\Phi^2_n\ldots\Phi^2_1}\overline{W}\mathcal{M}_{\Phi^1_n\ldots\Phi^1_1}} & \Big\downarrow{\scriptstyle \varphi_{\Phi^3_n\ldots\Phi^3_1\Phi^2_n\ldots\Phi^2_1\Phi^1_n\ldots\Phi^1_1}} & \searrow^{\overline{W}\mathcal{M}_{\Phi^3_n\ldots\Phi^3_1}\varphi_{\Phi^2_n\ldots\Phi^2_1\Phi^1_n\ldots\Phi^1_1}} & \\ \overline{W}\mathcal{M}(e_3\circ e_2)\circ\overline{W}\mathcal{M}(e_1) & & & & \overline{W}\mathcal{M}(e_3)\circ\overline{W}\mathcal{M}(e_2\circ e_1) \\ & \searrow_{\varphi_{\Phi^3_n\Phi^2_n\ldots\Phi^3_1\Phi^2_1\Phi^1_n\ldots\Phi^1_1}} & & \swarrow_{\varphi_{\Phi^3_n\ldots\Phi^3_1\Phi^2_n\Phi^1_n\ldots\Phi^2_1\Phi^1_1}} & \\ & & \overline{W}\mathcal{M}(e_3\circ e_2\circ e_1) & & \end{array}$$

Hence, to prove that Diagram 4.1 commutes, it suffices to show that

$$\text{(i)}\quad \varphi_{\Phi^3_n\Phi^2_n\ldots\Phi^3_1\Phi^2_1\Phi^1_n\ldots\Phi^1_1} = \varphi_{\Phi'_n\ldots\Phi'_1\Phi^1_n\ldots\Phi^1_1} \quad \text{and}$$
$$\text{(ii)}\quad \varphi_{\Phi^3_n\ldots\Phi^3_1\Phi^2_n\Phi^1_n\ldots\Phi^2_1\Phi^1_1} = \varphi_{\Phi^3_n\ldots\Phi^3_1\Phi''_n\ldots\Phi''_1}.$$

To prove (i) and (ii) we use the same arguments as in the twofold case. Let Λ be a shuffle of sequences of colours $1,\ldots,n$ with only identity arrows in it. Let $1\leq k<l\leq n$ and let u, v, and w be respectively

$$\prod_{1\leq z<k} t_z,\quad \prod_{k<z<l} t_z,\quad \prod_{l<z\leq n} t_z,$$

where t_z is the target of the leftmost member of Λ of colour z.

Lemma 8.3 *If Φ and Φ' are sequences of colour k such that $\circ\Phi=\circ\Phi'$ is a basic equation of Δ^{op}, and g is a basic arrow of Δ^{op}, then we have that $\varphi_{\Phi(g,l)\Lambda}=\varphi_{\Phi'(g,l)\Lambda}$.*

Proof Suppose the target of $\circ\Phi$ is n' and the target of g is m'. We proceed as in Lemma 4.3 with all the cases modified so that κ, β, τ, and ι are replaced by $\kappa_{k,l}$, $\beta_{k,l}$, $\tau_{k,l}$, and $\iota_{k,l}$, and the tuples representing the natural transformations are multiplied by the matrix

$$\underbrace{(1,\ldots,1)}_{u}\otimes I_{n'}\otimes\underbrace{(1,\ldots,1)}_{v}\otimes I_{m'}\otimes\underbrace{(1,\ldots,1)}_{w}.$$

So, for example, Case 1.1.1 now reads

$$
\begin{aligned}
&\varphi_{\Phi(g,l)\Lambda}\\
&\quad = ((\mathbf{1}^{(j-1)mvw}, (\mathbf{1}^{iw}, \tau^{w}_{k,l}, \mathbf{1}^{(m-i-1)w})^{v}, \mathbf{1}^{(l-j-2)mvw}, (\mathbf{1}^{iw}, \tau^{w}_{k,l}, \mathbf{1}^{(m-i-1)w})^{v}, \vec{\mathbf{1}})^{u})\\
&\quad = \varphi_{\Phi'(g,l)\Lambda}.
\end{aligned}
$$

□

By relying on Lemma 8.3, we can prove a lemma analogous to Lemma 4.4 and this suffices for the proof of (i) by induction on the sum of lengths of $\Phi^1_n, \ldots, \Phi^1_2$. We can prove (ii) in a dual manner. So, for every $n \geq 2$ we have:

Theorem 8.5 *The n-fold reduced bar construction $\overline{W}\mathcal{M}$, together with the natural transformations ω, makes a lax functor from $(\Delta^{op})^n$ to* Cat.

We see, by analyzing this result, that the conditions imposed by the definition of n-fold monoidal categories are not only sufficient, but they are also necessary to prove the correctness of the n-fold reduced bar construction. If one proves this through the steps established by our Theorem 8.2 and Lemmata analogous to Lemma 8.3, then all the combinatorial structure of n-fold monoidal categories is used.

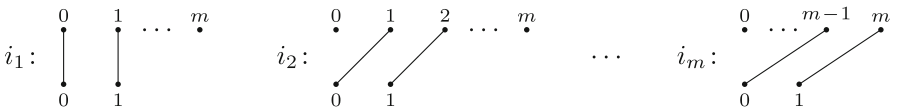
Since every n-fold monoidal category in the sense of [3] is an n-fold strict monoidal category in our sense, Theorem 8.5 gives an alternative proof for [3, Theorem 2.1]. Every braided strict monoidal category is a twofold monoidal category in the sense of [3] and every symmetric strict monoidal category is an ∞-monoidal category in the sense of [3]. Hence, our Theorem 8.5 covers all the related results concerning these categories. Also, the correctness of the reduced bar construction of [19, Lemma 7.1] follows from this theorem.

9 Delooping

This section, which is inspired by [3, Section 2], explains how to use Theorem 8.5 for delooping of classifying spaces of n-fold monoidal categories. Theorem 2.2 of [3] says that the group completion of the nerve of an n-fold monoidal category is an n-fold loop space. It is an easy corollary of a generalization of [22, Proposition 1.5] and [3, Theorem 2.1].

A formulation of a generalization of [22, Proposition 1.5] is given in [3, paragraph preceding Theorem 2.1]. This seems to be a folklore result amongst the experts, but we couldn't find written proof, or a precise formulation of it. The note [20] is prepared to rectify that. We sketch a delooping procedure based on the results of this note.

For $m \geq 1$, consider the arrows $i_1, \ldots, i_m : m \to 1$ of Δ^{op} given by the following diagrams.



These arrows are related to projections, which is explained in [21, Section 2] and [20, Section 3].

We use the following notation in the sequel. For functors $F_i : \mathcal{A} \to \mathcal{B}_i$, $1 \leq i \leq m$, let $\langle F_1, \ldots, F_m \rangle : \mathcal{A} \to \mathcal{B}_1 \times \cdots \times \mathcal{B}_m$ be the functor obtained by the Cartesian structure of *Cat*.

Let $\overline{W}\mathcal{M}$ be the n-fold reduced bar construction for $n \geq 2$. It is easy to verify that for every $l \in \{0, \ldots, n-1\}$ and every $k \geq 0$, the functor $W : \Delta^{op} \to Cat$ defined as

$$\overline{W}\mathcal{M}(\underbrace{1, \ldots, 1}_{l}, \_, k, \ldots, k)$$

is such that

$$\langle W(i_1), \ldots, W(i_m) \rangle : W(m) \to (W(1))^m$$

is the identity. This means that $\overline{W}\mathcal{M}$ is Segal's lax functor according to [20, Definition 4.2].

Let V be a rectification of $\overline{W}\mathcal{M}$ obtained by [24, Theorem 2], and let $B : Cat \to Top$ be the *classifying space* functor, i.e., the composition $| \, | \circ N$, where $N : Cat \to Top^{\Delta^{op}}$ is the *nerve* functor, and $| \, | : Top^{\Delta^{op}} \to Top$ is the standard *geometric realization* functor. By [20, Corollary 4.4], $B \circ V$ is a multisimplicial space such that for X being the simplicial space defined as

$$(B \circ V)(\underbrace{1, \ldots, 1}_{l}, \_, k, \ldots, k),$$

the map

$$\langle X(i_1), \ldots, X(i_m) \rangle : X(m) \to (X(1))^m$$

is a homotopy equivalence.

By applying [20, Lemma 3.1] to the simplicial space $(B \circ V)(1, \ldots, 1, \_)$, we obtain a homotopy associative H-space structure on $(B \circ V)(1, \ldots, 1)$. The following theorem (in which $| \, |$ denotes the standard geometric realization of multisimplicial spaces) is taken over from [20, Theorem 5.1].

Theorem 8.6 *If* $(B \circ V)(1, \ldots, 1)$*, with respect to the above* H*-space structure is grouplike, then* $B\mathcal{M} \simeq \Omega^n |B \circ V|$.

Hence, up to group completion, the realization $|B \circ V|$ of the multisimplicial space $B \circ V$ is an n-fold delooping of the classifying space $B\mathcal{M}$ of $\mathcal{M}$.

Bicartesian categories, i.e., categories with all finite coproducts and products may serve as examples of n-fold monoidal categories that are not n-fold monoidal in the sense of [3]. If we denote the nullary and binary coproducts of a bicartesian category by 0 and $+$, and nullary and binary products by 1 and $\times$, then the unique arrows

$$\kappa: 0 \to 1, \quad \beta: 0 \to 0 \times 0, \quad \tau: 1+1 \to 1$$

of this category together with the arrows

$$\iota_{A,B,C,D}: (A \times B) + (C \times D) \to (A + C) \times (B + D),$$

which are canonical in the coproduct–product structures, guarantee that such a category may be conceived as a twofold monoidal with the first monoidal structure given by $+$ and 0, and the second given by $\times$ and 1. Furthermore, such a category may be conceived as an n-fold monoidal category in $n+1$ different ways by taking first $0 \leq k \leq n$ monoidal structures to be given by the symmetric monoidal structure brought by $+$ and 0, and the remaining $n-k$ monoidal structures to be given by the symmetric monoidal structure brought by $\times$ and 1.

As a consequence of this fact there is a family, indexed by pairs of natural numbers, of reduced bar constructions based on a bicartesian category (strictified in both monoidal structures). This is related to Adams' remark on E_∞ ring spaces given in [1, §2.7] where the bicartesian category *FinSet* of finite sets and functions, with disjoint union as $+$ and Cartesian product as $\times$, is mentioned. According to Segal [22, §2],"most fundamental Γ-space" arises from this category under disjoint union.

By applying our results, it is possible to combine the disjoint union and Cartesian product in the category *FinSet* to obtain various multisimplicial spaces. Since we have the initial (and a terminal) object in *FinSet*, its classifying space is contractible and all the other realizations of simplicial sets in question are path-connected. Hence, the induced H-space structures are grouplike, and there is no need for group completion when one starts to deloop *FinSet* with respect to the disjoint union and then continue to deloop it with respect to Cartesian product. However, all these deloopings are contractible.

Since the notion of n-fold monoidal category is equationally presented, there are n-fold monoidal categories freely generated by sets of objects. We believe that delooping of classifying spaces of such categories deserves particular attention. Also, some other examples of n-fold monoidal categories from the literature (e.g. [2, Sections 6.4 and 7.3]) could be interesting from the point of view of delooping.

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10 Appendix

By the definition given in Sect. 2, a twofold monoidal category consists of the following:

1. a monoidal category $\langle \mathcal{M}, \otimes_1, I_1, \alpha_1, \rho_1, \lambda_1 \rangle$ (here α_1, ρ_1, and λ_1, respectively, denote associativity, right and left identity natural isomorphisms),
2. monoidal functors $\otimes_2 \colon \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ and $I_2 \colon 1 \to \mathcal{M}$,
3. monoidal transformations α_2, ρ_2, and λ_2 such that $\langle \mathcal{M}, \otimes_2, I_2, \alpha_2, \rho_2, \lambda_2 \rangle$ satisfies the pseudomonoid conditions (i.e., the equations of a monoidal category).

That $\otimes_2$ is a monoidal functor means that there is a natural transformation ι given by the family of arrows

$$\iota_{A,B,C,D} \colon (A \otimes_2 B) \otimes_1 (C \otimes_2 D) \to (A \otimes_1 C) \otimes_2 (B \otimes_1 D),$$

and an arrow $\beta \colon I_1 \to I_1 \otimes_2 I_1$ such that the following three diagrams commute:

$$\begin{CD}
(A \otimes_1 (B \otimes_1 C)) \otimes_2 (D \otimes_1 (E \otimes_1 F)) @>{\alpha_1 \otimes_2 \alpha_1}>> ((A \otimes_1 B) \otimes_1 C) \otimes_2 ((D \otimes_1 E) \otimes_1 F)\\
@A{\iota}AA @AA{\iota}A\\
(A \otimes_2 D) \otimes_1 ((B \otimes_1 C) \otimes_2 (E \otimes_1 F)) @. (1) \qquad ((A \otimes_1 B) \otimes_2 (D \otimes_1 E)) \otimes_1 (C \otimes_2 F)\\
@A{\mathbf{1} \otimes_1 \iota}AA @AA{\iota \otimes_1 \mathbf{1}}A\\
(A \otimes_2 D) \otimes_1 ((B \otimes_2 E) \otimes_1 (C \otimes_2 F)) @>>{\alpha_1}> ((A \otimes_2 D) \otimes_1 (B \otimes_2 E)) \otimes_1 (C \otimes_2 F)
\end{CD}$$

$$\begin{CD}
(A \otimes_2 B) \otimes_1 (I_1 \otimes_2 I_1) @>{\iota}>> (A \otimes_1 I_1) \otimes_2 (B \otimes_1 I_1)\\
@A{\mathbf{1} \otimes_1 \beta}AA (2) @VV{\rho_1 \otimes_2 \rho_1}V\\
(A \otimes_2 B) \otimes_1 I_1 @>>{\rho_1}> A \otimes_2 B
\end{CD}$$

$$\begin{CD}
(I_1 \otimes_2 I_1) \otimes_1 (A \otimes_2 B) @>{\iota}>> (I_1 \otimes_1 A) \otimes_2 (I_1 \otimes_1 B)\\
@A{\beta \otimes_1 \mathbf{1}}AA (3) @VV{\lambda_1 \otimes_2 \lambda_1}V\\
I_1 \otimes_1 (A \otimes_2 B) @>>{\lambda_1}> A \otimes_2 B
\end{CD}$$

That I_2 is a monoidal functor means that there are arrows $\tau : I_2 \otimes_1 I_2 \to I_2$ and $\kappa \colon I_1 \to I_2$ such that the following diagrams commute:

$$\begin{array}{ccc}
I_2 \otimes_1 (I_2 \otimes_1 I_2) & \xrightarrow{\alpha_1} & (I_2 \otimes_1 I_2) \otimes_1 I_2\\
\mathbf{1} \otimes_1 \tau \downarrow & (4) & \downarrow \tau \otimes_1 \mathbf{1}\\
I_2 \otimes_1 I_2 & & I_2 \otimes_1 I_2\\
& \tau \searrow \quad \swarrow \tau & \\
& I_2 &
\end{array}$$

$$\begin{array}{ccc}
I_2 \otimes_1 I_1 & \xrightarrow{\mathbf{1} \otimes_1 \kappa} & I_2 \otimes_1 I_2\\
& \rho_1 \searrow \quad (5) \quad \swarrow \tau & \\
& I_2 &
\end{array}
\qquad
\begin{array}{ccc}
I_1 \otimes_1 I_2 & \xrightarrow{\kappa \otimes_1 \mathbf{1}} & I_2 \otimes_1 I_2\\
& \lambda_1 \searrow \quad (6) \quad \swarrow \tau & \\
& I_2 &
\end{array}$$

Note that the unusual numbering of the following diagrams is due to our wish to dualize the first six diagrams in some way, which can clearly be seen from the list of 12 equations at the end of Sect. 2. That α_2 is a monoidal transformation means that the following diagrams commute:

$$\begin{array}{ccc}
(A \otimes_2 (B \otimes_2 C)) \otimes_1 (D \otimes_2 (E \otimes_2 F)) & \xrightarrow{\alpha_2 \otimes_1 \alpha_2} & ((A \otimes_2 B) \otimes_2 C) \otimes_1 ((D \otimes_2 E) \otimes_2 F) \\
\iota \downarrow & & \downarrow \iota \\
(A \otimes_1 D) \otimes_2 ((B \otimes_2 C) \otimes_1 (E \otimes_2 F)) & (7) & ((A \otimes_2 B) \otimes_1 (D \otimes_2 E)) \otimes_2 (C \otimes_1 F) \\
\mathbf{1} \otimes_2 \iota \downarrow & & \downarrow \iota \otimes_2 \mathbf{1} \\
(A \otimes_1 D) \otimes_2 ((B \otimes_1 E) \otimes_2 (C \otimes_1 F)) & \xrightarrow[\alpha_2]{} & ((A \otimes_1 D) \otimes_2 (B \otimes_1 E)) \otimes_2 (C \otimes_1 F)
\end{array}$$

$$\begin{array}{ccccc}
I_1 \otimes_2 (I_1 \otimes_2 I_1) & & \xrightarrow{\alpha_2} & & (I_1 \otimes_2 I_1) \otimes_2 I_1 \\
\mathbf{1} \otimes_2 \beta \uparrow & & (10) & & \uparrow \beta \otimes_2 \mathbf{1} \\
I_1 \otimes_2 I_1 & & & & I_1 \otimes_2 I_1 \\
& \nwarrow \beta & & \beta \nearrow & \\
& & I_1 & &
\end{array}$$

That ρ_2 is a monoidal transformation means that the following diagrams commute:

$$\begin{array}{ccc}
(A \otimes_2 I_2) \otimes_1 (B \otimes_2 I_2) & \xrightarrow{\iota} & (A \otimes_1 B) \otimes_2 (I_2 \otimes_1 I_2) \\
\rho_2 \otimes_1 \rho_2 \downarrow & (8) & \downarrow \mathbf{1} \otimes_2 \tau \\
A \otimes_1 B & \xleftarrow[\rho_2]{} & (A \otimes_1 B) \otimes_2 I_2
\end{array}$$

$$\begin{array}{ccc}
I_1 \otimes_2 I_1 & \xrightarrow{\mathbf{1} \otimes_2 \kappa} & I_1 \otimes_2 I_2 \\
\beta \nwarrow & (11) & \swarrow \rho_2 \\
& I_1 &
\end{array}$$

Finally, that λ_2 is a monoidal transformation means that the following diagrams commute:

$$\begin{array}{ccc}
(I_2 \otimes_2 A) \otimes_1 (I_2 \otimes_2 B) & \xrightarrow{\iota} & (I_2 \otimes_1 I_2) \otimes_2 (A \otimes_1 B) \\
\lambda_2 \otimes_1 \lambda_2 \downarrow & (9) & \downarrow \tau \otimes_2 \mathbf{1} \\
A \otimes_1 B & \xleftarrow[\lambda_2]{} & I_2 \otimes_2 (A \otimes_1 B)
\end{array}$$

$$\begin{array}{ccc}
I_1 \otimes_2 I_1 & \xrightarrow{\kappa \otimes_2 \mathbf{1}} & I_2 \otimes_2 I_1 \\
\beta \nwarrow & (12) & \swarrow \lambda_2 \\
& I_1 &
\end{array}$$

References

1. Adams, J.F.: Infinite Loop Spaces. Annals of Mathematics Studies, vol. 90. Princeton University Press, Princeton (1978)
2. Aguiar, M., Mahajan, S.: Monoidal Functors, Species and Hopf Algebras. CRM Monograph Series, vol. 29. American Mathematical Society, Providence (2010)
3. Balteanu, C., Fiedorowicz, Z., Schwänzl, R., Vogt, R.: Iterated monoidal categories. Adv. Math. **176**, 277–349 (2003)
4. Batanin, M., Markl, M.: Centers and homotopy centers in enriched monoidal categories. Adv. Math. **230**, 1811–1858 (2012)
5. Booker, T., Street, R.: Tannaka duality and convolution for duoidal categories. Theory Appl. Categ. **28**, 166–205 (2013)
6. Borceux, F.: Handbook of Categorical Algebra I. Cambridge University Press, Cambridge (1994)
7. Došen, K., Petrić, Z.: Coherence for modalities. J. Pure Appl. Algebra **215**, 1606–1628 (2011). arXiv:0809.2494
8. Došen, K., Petrić, Z.: Intermutation. Appl. Categ. Struct. **20**, 43–95 (2012). arXiv:math/0701325
9. Dunn, G.: E_n-monoidal categories and their group completions. J. Pure Appl. Algebra **95**, 27–39 (1994)
10. Fiedorowicz, Z.: Operads and iterated monoidal categories. https://people.math.osu.edu/fiedorowicz.1/ (1995)
11. Garner, R.: Understanding the small object argument. Appl. Categ. Struct. **17**, 247–285 (2009)
12. Joyal, A., Street, R.: Braided tensor categories. Adv. Math. **102**, 20–78 (1993)
13. Kelly, G.M., Mac Lane, S.: Coherence in closed categories. J. Pure Appl. Algebra **1**, 97–140, 219 (1971)
14. Mac Lane, S.: Natural associativity and commutativity. Rice Univ. Stud. Pap. Math. **49**, 28–46 (1963)
15. Mac Lane, S.: Categories for the Working Mathematician. Springer, Berlin (1971) (expanded second edition, 1998)
16. May, J.P.: E_∞-spaces, group completions, and permutative categories. In: Segal, G. (ed.) New Developments in Topology. London Mathematical Society Lecture Note Series, vol. 11, pp. 61–93. Cambridge University Press, Cambridge (1974)
17. McClure, J.E., Smith, J.H.: Operads and cosimplicial objects: an introduction. In: Greenlees, J.P.C. (ed.) Axiomatic, Enriched and Motivic Homotopy Theory, pp. 133–171. Kluwer Academic Publishers, Dordreht (2002)
18. McDuff, D., Segal, G.: Homology fibrations and the “group-completion” theorem. Inventiones mathematicae **31**, 279–284 (1976)
19. Petrić, Z., Trimble, T.: Symmetric bimonoidal intermuting categories and $\omega \times \omega$ reduced bar constructions. Appl. Categ. Struct. **22**, 467–499 (2014). arXiv:0906.2954
20. Petrić, Z.: Segal’s multisimplicial spaces. Publications de l’ Institut Mathematique **97**(111), 11–21 (2015). arXiv:1407.3914
21. Petrić, Z.: Monoids, Segal’s condition and bisimplicial spaces. In: Jokanović, D. (ed.) Matematička Konferencija Republike Srpske, Zbornik radova I, pp. 7–16 (2015). arXiv:1505.05010
22. Segal, G.: Categories and cohomology theories. Topology **13**, 293–312 (1974)
23. Stasheff, J.D.: Homotopy associativity of H-spaces, I, II. Trans. Am. Math. Soc. **108**, 275–292, 293–312 (1963)
24. Street, R.: Two constructions on lax functors. Cahiers de topologie et géométrie différentielle **13**, 217–264 (1972)
25. Thomason, R.W.: Homotopy colimits in the category of small categories. Math. Proc. Camb. Philos. Soc. **85**(91), 91–109 (1979)