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Monoform objects and localization theory in abelian categories

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Abstract Let $\mathcal{A}$ be an abelian category. In this paper we study monoform objects and atoms introduced by Kanda. We classify full subcategories of $\mathcal{A}$ by means of subclasses of $\mathrm{ASpec}\mathcal{A}$, the atom spectrum of $\mathcal{A}$. We also study the atomical decomposition and localization theory in terms of atoms. As some applications of our results, we study the category Mod-A where A is a fully right bounded noetherian ring.

Keywords Abelian category · Grothendieck category · Serre subcategory · Right noetherian rings

Mathematics Subject Classification 18E10 · 18E15

1 Introduction

Throughout this paper, $\mathcal{A}$ is an abelian category and all subcategories of $\mathcal{A}$ are full and closed under isomorphisms.

The classification of subcategories of a category is an important area in the recent years which has been studied by numerous authors in [2,3,5,9,13] and [14]. The prototype of this subject goes back to 1962 when Gabriel [1] presented a classification of the Serre subcategories of the category of A-modules, where A is a commutative noetherian ring. He showed that there exists a one-to-one correspondence between Serre subcategories of A-Mod and specialization closed subsets of $\mathrm{Spec}A$.

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For arbitrary rings, Storrer [11] defined and studied monoform modules and atoms which play the role of prime ideals. For a commutative noetherian ring, the isomorphism class of indecomposable injective modules are in a one-one correspondence with the prime ideals, whereas in the noncommutative noetherian rings the atoms are. This is one of the reasons why he worked with atoms.

Recently, Kanda [5] has investigated the classification of Serre subcategories of an arbitrary noetherian abelian category $\mathcal{A}$ in terms of monoform objects and atoms. Since abelian categories do not have enough injective objects in general, Kanda showed that these notions can be suitable replacements for indecomposable injective modules. In this paper we get further properties of monoform objects. We will develop Kanda's results for some more general abelian categories.

Section 2 is devoted to monoform objects in an abelian category. Kanda showed that every nonzero noetherian object has a monoform subobject. In Theorem 2.1, we showed how we can construct this subobject. We define atomical decomposition of any subobject of a noetherian object M of $\mathcal{A}$ and show that if M is a noetherian object, then every subobject N has an atomical decomposition which determines the associated atoms of M/N (cf. Propositions 2.5, 2.7). We define maximal atoms and present sufficient conditions, so that an object has a maximal atom in its atom support (cf. Proposition 2.11). In Theorem 2.12, we prove that the maximal atoms can characterize finite length objects.

In Sect. 3, we study the relation between monoform objects and full subcategories of an abelian category. In Theorem 3.8, we show that if $\mathcal{A}$ has a set of generators, there exists a one-to-one correspondence between

$$\mathcal{S} = \{\mathcal{X} \mid \mathcal{X} \text{ is a Serre subcategory of noeth } \mathcal{A}\}$$

and the class of all open subclasses of $\mathrm{ASpec}(\mathcal{A})$.

Section 4 is mainly devoted to localization theory in Grothendieck categories. In Theorem 4.6, we show that if $\mathcal{A}$ is a locally noetherian Grothendieck category, there is a bijection between full subcategories of $\mathcal{A}$ which are closed under subobjects, injective envelopes and direct sums, and subclasses of $\mathrm{ASpec}(\mathcal{A})$. In sequel we study the localization theory in terms of atoms due to Kanda [6]. We show that several results in the classical localization theory of module category can be true in Grothendieck categories in sense of the new version of localization. More precisely, we prove that if α is an atom, then $E(\alpha)_\alpha \cong E(\alpha)$, where $E(\alpha)$ is the injective envelope of α (cf. Proposition 4.8). We also show that in Proposition 4.11 that if $\mathcal{X}$ is a localizing subcategory of $\mathcal{A}$ with the canonical functor $F : \mathcal{A} \to \mathcal{A}/\mathcal{X}$ and the idempotent radical $t_{\mathcal{X}}$ and also if M is an object of $\mathcal{A}$, then $\mathrm{AAss}F(M) = \mathrm{AAss}(M/t_{\mathcal{X}}(M))$. Moreover if $\mathcal{X}$ is stable, then $\mathrm{AAss}F(M) = \mathrm{AAss}(M)\backslash\mathrm{ASupp}\mathcal{X}$. In Proposition 4.13, we prove that if N is a proper subobject of a noetherian object M in $\mathcal{A}$ and α is a minimal atom in $\mathrm{ASupp}(M/N)$ such that $\mathcal{X}_\alpha$ is stable, then the α-component of any atomical decomposition of N is $\eta_M^{-1}(GF(N))$ where $F : \mathcal{A} \to \mathcal{A}_\alpha$ is the canonical functor with the right adjoint functor $G : \mathcal{A}_\alpha \to \mathcal{A}$. This fact implies that the α-component is uniquely determined only by M, N and α.

In Section 5, as applications of our results, we study the fully right bounded noetherian rings.

2 Monoform and uniform objects in abelian categories

Throughout this section $\mathcal{A}$ is an abelian category. We begin this section by some definitions and notions due to Kanda. For more details, we refer the reader to [5].

A nonzero object M in $\mathcal{A}$ is *monoform* if for any nonzero subobject N of M, there exists no common nonzero subobject of M and M/N, which means that there does not exist a nonzero subobject of M which is isomorphic to a subobject of M/N. A nonzero object U in $\mathcal{A}$ is *uniform* if for any nonzero subobjects L and L' of U, we have $L' \cap L \neq 0$; in other words, U is uniform if every nonzero subobject of U is essential in U. We remark that uniform objects had already been known as coirreducible objects (cf. [10, p. 119]).

Let $\mathrm{ASpec}_0\mathcal{A}$ denote the class of all monoform objects of $\mathcal{A}$. Two monoform objects H and H' are *atom-equivalent* if they have a common nonzero subobject. By [5, Proposition 2.8], the atom equivalence establishes an equivalence relation on $\mathrm{ASpec}_0\mathcal{A}$ and so for every monoform object H, equivalence class $\overline{H}$ is called an *atom*, that is

$$\overline{H} = \{G \in \mathrm{ASpec}_0\mathcal{A} | H \text{ and } G \text{ have a common nonzero subobject}\}.$$

The *atom spectrum* $\mathrm{ASpec}\mathcal{A}$ of $\mathcal{A}$ is the quotient class of $\mathrm{ASpec}_0\mathcal{A}$ consisting of all atoms induced by this equivalence relation.

An object M of $\mathcal{A}$ is *noetherian* if every ascending chain of subobjects of M stabilizes.

For an object M of $\mathcal{A}$, the *atom support* of M, denoted by $\mathrm{ASupp}(M)$, is a subclass of $\mathrm{ASpec}\mathcal{A}$ as follows

$$\mathrm{ASupp}M = \{\overline{H} \in \mathrm{ASpec}\mathcal{A} | \text{ there exists } H' \in \overline{H} \text{ which is a subquotient of } M\}.$$

The associated atoms of a module had been studied in [11], but a generalization of that for abelian category was defined in [5].

For an object M, we define the *associated atoms* of M, denoted by $\mathrm{AAss}(M)$, a subclass of $\mathrm{ASupp}(M)$ as follows

$$\mathrm{AAss}M = \{\overline{H} \in \mathrm{ASupp}(M) | \text{ there exists } H' \in \overline{H} \text{ which is a subobject of } M\}.$$

Kanda [5], by a categorical proof, showed that if M is a nonzero noetherian object of $\mathcal{A}$, then it has a monoform subobject. Here, we present a different proof which shows how we can find such a subobject directly.

Theorem 2.1 [5, Theorem 2.9] *Any nonzero noetherian object M has a monoform subobject. In particular M has a filtration*

$$0 = L_0 \subset L_1 \subset \cdots \subset L_n = M$$

such that L/L_{i-1} is a monofrom objects for any $i = 1, \ldots, n$.

Proof We begin by proving that M has a monoform subobject. If M is monoform, there is nothing to prove. So we assume that M is not monoform. Then M has a nonzero subobject N such that M and M/N have a common nonzero subobject. Let Σ be the class of all such subobjects N of M. Since M is noetherian, Σ has a maximal element K. Hence there exist subobjects $K \subsetneq K_1$ and $X \subset M$ and an isomorphism $\theta : K_1/K \to X$. We show that K_1/K is monoform and so is X. Suppose on the contrary that K_1/K is not monoform. Then there exist subobjects $K \subsetneq K_2 \subsetneq K_3 \subseteq K_1$ and $K \subsetneq K_4 \subsetneq K_1$ such that $K_3/K_2 \cong K_4/K$. Thus $\theta(K_4/K) \cong K_3/K_2$ is a common nonzero subobject of M and M/K_2. This implies that K_2 is in Σ which contradicts the maximality of K. Therefore K_1/K is monoform. In order to prove the second claim, if M is monoform, there is nothing to prove, otherwise it has a monoform subobject L_1 such that M/L_1 is a nonzero object. Clearly M/L_1 is noetherian and if it is monoform, then $M = L_2$. If M/L_1 is not monoform, continuing the previous argument, we find an ascending chain of subobject of M

$$L_0 = 0 \subset L_1 \subset L_2 \subset \dots$$

such that each L_i/L_{i-1} is monoform. Now, since M is noetherian, our constructing implies that there exists $n \in \mathbb{N}$ such that $M = L_n$. □

Remark 2.2 If $\mathcal{A}$ has a set of noetherian generators, then the subclass of associated atoms of every nonzero object is non-empty. More precisely, let $\{U_i\}_I$ be a set of generators of $\mathcal{A}$ such that every U_i is noetherian. For every nonzero object M of $\mathcal{A}$, there exists $i \in I$ and a nonzero morphism $f : U_i \to M$. We notice that $\mathrm{Im} f$ is a noetherian subobject of M and so it follows from Theorem 2.1 that $\mathrm{AAss}(\mathrm{Im} f)$ is non-empty. Therefore the claim follows as $\mathrm{AAss}(\mathrm{Im} f) \subseteq \mathrm{AAss}(M)$.

A proper subobject N of M is said to be *irreducible* if for any two subobjects K and L of M the equality $N = K \cap L$ implies that $N = K$ or $N = L$. Clearly, N is an irreducible subobject of M if and only if M/N is uniform. We say that an expression of a subobject N of M as an intersection $N = N_1 \cap \cdots \cap N_n$ is *irredundant* if $\bigcap_{i \neq j} N_i \nsubseteq N_j$ for each $1 \leq j \leq n$. In other words, we cannot omit any N_i, that means $N \neq N_1 \cap \cdots \cap N_{i-1} \cap N_{i+1} \cap \cdots \cap N_n$.

An *irreducible decomposition* of a subobject N of M is $N = N_1 \cap \cdots \cap N_n$ in which N_is are irreducible subobjects of M.

A proper subobject N of an object M is called *atomical* if $\mathrm{AAss}(M/N)$ has just one element. If $\mathrm{AAss}(M/N) = \{\alpha\}$, then N is called α-*atomical*.

Proposition 2.3 *Every irreducible subobject of a noetherian object M is atomical.*

Proof Let N be an irreducible subobject of M. Without loss of generality, assume that $N = 0$ and so M is a uniform object. Let $\alpha_1, \alpha_2 \in \mathrm{AAss}(M)$. Then there exist monoform subobjects H_1 and H_2 of M with $\alpha_i = \overline{H_i}$ for $i = 1, 2$. But since M is uniform, $H_1 \cap H_2 \neq 0$; and hence $\overline{H_1} = \overline{H_2}$. □

Proposition 2.4 *Let α be an atom and N_1 and N_2 be α-atomical subobjects of a noetherian object M. Then $N_1 \cap N_2$ is α-atomical.*

Proof We can embed $M/N_1 \cap N_2$ as a subobject of $M/N_1 \oplus M/N_2$, so that $\mathrm{AAss}(M/N_1 \cap N_2) = \{\alpha\}$. □

Let M be a noetherian object. An *atomical decomposition* of a subobject L of M is obtained by writing L as a finite intersection $L = L_1 \cap \cdots \cap L_n$ of atomical subobjects L_i of M, so that

(i) The decomposition is irredundant.
(ii) $\mathrm{AAss}(M/L_i) \neq \mathrm{AAss}(M/L_j)$ for $i \neq j$.

Proposition 2.5 *Suppose that $L = L_1 \cap \cdots \cap L_n$ is an atomical decomposition of L in a noetherian object M with $\mathrm{AAss}(M/L_i) = \{\alpha_i\}$ for each i. Then $\mathrm{AAss}(M/L) = \{\alpha_1, \ldots, \alpha_n\}$.*

Proof Without loss of generality, we may assume that $L = 0$. The canonical monomorphism $\beta : M \to M/L_1 \oplus \cdots \oplus M/L_n$ implies that $\mathrm{AAss}(M) \subset \{\alpha_1, \ldots, \alpha_n\}$. Conversely, we show that every α_i lies in $\mathrm{AAss}(M)$. Considering $K = \bigcap_{j \neq i} L_j$, we have $K \cap L_i = 0$ and since $0 = L_1 \cap \cdots \cap L_n$ is irredundant, $K \neq 0$. Furthermore, the canonical isomorphism $K \cong (K + L_i)/L_i$ implies that $\mathrm{AAss}(K) = \{\alpha_i\}$. Now, since K is a subobject of M, we deduce that $\alpha_i \in \mathrm{AAss}(M)$. □

Proposition 2.6 [10, Chap.III. Proposition 3.9] *Let M be a noetherian object of $\mathcal{A}$. Then every subobject of M has an irredundant irreducible decomposition.*

Proposition 2.7 *Every subobject L of a noetherian object M of $\mathcal{A}$ has an atomical decomposition. If $L = L_1 \cap \cdots \cap L_m = N_1 \cap \cdots \cap N_n$ are two atomical decompositions of L in M, then $m = n$ and $\bigcup_{i=1}^m \mathrm{AAss}(M/L_i) = \bigcup_{i=1}^n \mathrm{AAss}(M/N_i)$.*

Proof Using Proposition 2.6, the subobject L can be written as an irredundant intersection $L = L_1 \cap \cdots \cap L_n$ of irreducible subobjects L_i of L and using Proposition 2.3, each L_i is atomical. For each $\alpha \in \bigcup_{i=1}^n \mathrm{AAss}(M/L_i)$, let $L(\alpha)$ denote the intersection of all α-atomicals L_i. Then $L(\alpha)$ is α-atomical by Proposition 2.4, and $L = \bigcap_\alpha L(\alpha)$ is the desired atomical decomposition of L. In order to prove unicity, using Proposition 2.5, we have $\bigcup_{i=1}^m \mathrm{AAss}(M/L_i) = \mathrm{AAss}(M/L) = \bigcup_{i=1}^n \mathrm{AAss}(M/N_i)$. Furthermore, since $\mathrm{AAss}(M/L_i) \neq \mathrm{AAss}(M/L_j)$ and $\mathrm{AAss}(M/N_i) \neq \mathrm{AAss}(M/N_j)$ for all $i \neq j$, we have $m = n$. □

Proposition 2.8 *If M is a monoform object, then any nonzero endomorphism of M is injective.*

Proof Let $f : M \to M$ be a nonzero endomorphism of M and let $0 \neq K = \mathrm{Ker} f$. Then $M/K \cong \mathrm{Im} f \subset M$. Since $\mathrm{Im} f \neq 0$, the objects M and M/K have a common nonzero object which contradicts monoformness of M. □

Definition 2.9 An atom α in $\mathrm{ASpec}\mathcal{A}$ is said to be *maximal* if there exists a simple object H of $\mathcal{A}$ such that $\alpha = \overline{H}$. We denote by $\mathrm{m\text{-}ASpec}\mathcal{A}$, the subclass of $\mathrm{ASpec}\mathcal{A}$ consisting of all maximal atoms.

Proposition 2.10 *Let $\mathcal{A}$ admit a set of noetherian generators and let M be an object in $\mathcal{A}$ with $\mathrm{ASupp}(M) = \{\alpha\}$. Then α is maximal.*

Proof Since $\mathcal{A}$ has a set of noetherian generators, $\mathrm{AAss}(M)$ is non-empty and so $\mathrm{AAss}(M) = \{\alpha\}$. Then there exists a monoform subobject H of M with $\alpha = \overline{H}$ and so $\mathrm{ASupp}(H) = \{\alpha\}$. If α is not maximal, then H is not simple and so it has a nonzero proper subobject K such that $\mathrm{ASupp}(H/K) = \mathrm{Ass}(H/K) = \{\alpha\}$. Then H/K has a monoform subobject H_1 such that $\alpha = \overline{H_1}$; and hence there exists a subobject L of H such that $K \subsetneq L$ and $H_1 = L/K$. But since $\alpha = \overline{L} = \overline{H_1}$, the objects L and H_1 have a common nonzero subobject which contradicts monoformness of L. □

A category $\mathcal{A}$ is said to be *locally small* if the class of subobjects of any given object is a set. We now show when a nonzero object in $\mathcal{A}$ has a maximal atom in its atom support.

Proposition 2.11 (i) *Let M be a nonzero noetherian object. Then* $\mathrm{ASupp}(M)$ *contains a maximal atom. In particular, if $\mathcal{A}$ has a set of noetherian generators, then every nonzero object of $\mathcal{A}$ has a maximal atom in its atom support.*

(ii) *If $\mathcal{A}$ is a locally small category, then every nonzero finitely generated object has a maximal atom in its support. In particular if $\mathcal{A}$ has arbitrary direct sums and a set of finitely generated generators, then every nonzero object of $\mathcal{A}$ has a maximal atom in its support.*

Proof (i) Since M is noetherian, it has a maximal subobject N. Then M/N is a simple object so that $\alpha = \overline{H}$ is a maximal atom. For the second claim, let $\{U_i\}_I$ be a set of noetherian generators of $\mathcal{A}$. For every nonzero object M of $\mathcal{A}$, there exists $i \in I$ and a nonzero morphism $f : U_i \to M$. We notice that $\mathrm{Im} f$ is a noetherian subobject of M. According to the first case, the object $\mathrm{Im} f$ has a maximal atom, and since $\mathrm{ASupp}(\mathrm{Im} f) \subset \mathrm{ASupp}(M)$, the object M has a maximal atom. (ii) If $\mathcal{A}$ is locally small, the class of subobjects of any objects is actually a set. An analogous proof of module category, using Zorn's lemma, concludes that every nonzero finitely generated object has a maximal subobjects. For the second claim, $\mathcal{A}$ has a generator and hence using [10, Chap IV. Proposition 6.6], it is locally small. A similar argument mentioned in (i) implies that any nonzero object of $\mathcal{A}$ contains a nonzero finitely generated subobjects. Now, the result follows by the first part. □

We recall that an object M has *finite length* if it has a *composition series*, which means that there exists a finite chain of subobjects

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

such that each object M_i/M_{i-1} is simple.

An object M of $\mathcal{A}$ is *artinian* if every descending chain of its subobjects stabilizes.

We now show that the maximal atom supports can characterize finite length objects.

Theorem 2.12 *Let M be a nonzero object of $\mathcal{A}$.*

(i) *If M is noetherian, then* $\mathrm{ASupp}(M) \subset m\text{-}\mathrm{ASpec}\mathcal{A}$ *if and only if M has finite length.*

(ii) *If M is an artinian object, then* $\mathrm{ASupp}(M) \subset m\text{-}\mathrm{ASpec}(\mathcal{A})$.

Proof (i) Assume that $\mathrm{ASupp}(M) \subset \text{m-ASpec}\mathcal{A}$. We will construct a composition series for M. As M is a nonzero noetherian object, $\mathrm{AAss}(M)$ is a non-empty set and $\mathrm{AAss}(M) \subset \mathrm{ASupp}(M)$. Given $\alpha_1 \in \mathrm{AAss}(M)$, there exists a monoform subobject M_1 of M such that $\alpha_1 = \overline{M_1}$. Since α_1 is maximal, we may assume that M_1 is simple. If $M/M_1 = 0$, there is nothing to prove and so we assume that $M/M_1 \neq 0$. Then, for this case, we have

$$\mathrm{AAss}(M/M_1) \subset \mathrm{ASupp}(M/M_1) \subset \mathrm{ASupp}(M)$$

and $\mathrm{AAss}(M/M_1)$ is non-empty. Suppose that $\alpha_2 \in \mathrm{AAss}(M/M_1)$. Repeating the previous argument there exists a simple subobject M_2/M_1 of M/M_1 such that $\alpha_2 = \overline{M_2/M_1}$. Continuing this way for M/M_i for $i \geq 1$, we have an ascending chain $M_0 \subset M_1 \subset \ldots$ of subobjects of M such that each M_{i+1}/M_i is simple. As M is noetherian, this chain finally stabilizes and hence there exists a positive integer t such that $M_t = M$. Conversely, suppose that M is of finite length. Then there exists a composition series

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

such that each object M_i/M_{i-1} is simple. For every simple object H, it is clear that $\mathrm{ASupp}(H) = \{\overline{H}\}$, and hence $\mathrm{ASupp}(M) = \{\overline{M_1/M_0}, \ldots, \overline{M_n/M_{n-1}}\} \subset \text{m-ASpec}\mathcal{A}$.

(ii) Let $\alpha \in \mathrm{ASupp}(M)$. Then by the definition there exists a subobject K of M and a monoform subobject H of M/K such that $\alpha = \overline{H}$. Since M is artinian, M/K is artinian; and so H can be considered as a minimal (simple) subobject of M/K. This implies that $\alpha \in \text{m-ASpec}(\mathcal{A})$.

□

3 Monoform objects and their relationship to subcategories

Throughout this section $\mathcal{A}$ is an abelian category.

Let $\mathcal{X}$ be a subcategory of $\mathcal{A}$. We set

$$\langle \mathcal{X} \rangle_{\mathrm{sub}} = \{M \in \mathcal{A} | M \text{ is a subobject of an object of } \mathcal{X}\};$$
$$\langle \mathcal{X} \rangle_{\mathrm{quot}} = \{M \in \mathcal{A} | M \text{ is a quotient object of an object of } \mathcal{X}\};$$

For any subcategories $\mathcal{X}$ and $\mathcal{Y}$ of $\mathcal{A}$, we set the subcategory $\mathcal{X} \star \mathcal{Y}$ by

$$\mathcal{X} \star \mathcal{Y} = \{M \in \mathcal{A} | \text{ there exists an exact sequence } 0 \to L \to M \to N \to 0 \text{ with } L \in \mathcal{X} \text{ and } N \in \mathcal{Y}\}.$$

For any $n \in \mathbb{N}_0$, we set $\mathcal{X}^0 = \{0\}$ and $\mathcal{X}^n = \mathcal{X}^{n-1} \star \mathcal{X}$. In the case where $\mathcal{X}^2 = \mathcal{X}$, we say that $\mathcal{X}$ is *closed under extension*. We also define $\langle \mathcal{X} \rangle_{\mathrm{ext}} = \bigcup_{n \geq 0} \mathcal{X}^n$ as the smallest subcategory of $\mathcal{A}$ containing $\mathcal{X}$ and closed under extension.

A full subcategory $\mathcal{X}$ of $\mathcal{A}$ is called *Serre* if it is closed under taking subobjects, quotients and extensions. For any subcategory $\mathcal{X}$ we denote by $\langle \mathcal{X} \rangle_{\text{Serre}}$ the smallest Serre subcategory of $\mathcal{A}$ containing $\mathcal{X}$.

Definition 3.1 Let $\pi : \text{ASpec}_0\mathcal{A} \twoheadrightarrow \text{ASpec}\mathcal{A}$ be the canonical projection. For any subcategory $\mathcal{X} \subset \mathcal{A}$, we define the subclasses $\text{ASupp}(\mathcal{X})$ and $\text{AAss}(\mathcal{X})$ of $\text{ASpec}\mathcal{A}$ by

$$\text{ASupp}(\mathcal{X}) = \pi(\langle\langle \mathcal{X} \rangle_{\text{sub}}\rangle_{\text{quot}} \cap \text{ASpec}_0(\mathcal{A}));$$
$$\text{AAss}(\mathcal{X}) = \pi(\langle \mathcal{X} \rangle_{\text{sub}} \cap \text{ASpec}_0(\mathcal{A})).$$

It is straightforward by the definition that if $\mathcal{X}$ is a subcategory of $\mathcal{A}$, then

$$\text{ASupp}(\mathcal{X}) = \bigcup_{M \in \mathcal{X}} \text{ASupp}M;$$
$$\text{AAss}(\mathcal{X}) = \bigcup_{M \in \mathcal{X}} \text{AAss}M.$$

From [5] we recall that a subclass Φ of $\text{ASpec}\mathcal{A}$ is *open* if for any $\alpha \in \Phi$, there exists a monoform object H in $\mathcal{A}$ such that $\alpha = \overline{H}$ and $\text{ASupp}(H) \subset \Phi$. We remark that for any object M of $\mathcal{A}$, the subclass $\text{ASupp}(M)$ is open in $\text{ASpec}\mathcal{A}$. Hence $\text{ASupp}(\mathcal{X})$ is open in $\text{ASpec}\mathcal{A}$ for any subcategory $\mathcal{X}$ of $\mathcal{A}$.

The following proposition gives a characterization of maximal atoms in terms of open subclass of $\text{ASpec}\mathcal{A}$.

Proposition 3.2 *Let $\mathcal{A}$ admit a set of noetherian generators. Then $\{\alpha\}$ is an open subclass of* $\text{ASpec}(\mathcal{A})$ *if and only if α is a maximal atom.*

Proof If α is not maximal, then by Proposition 2.10, we have $\text{ASupp}(H) \nsubseteq \{\alpha\}$ for every monoform object H with $\alpha = \overline{H}$, which contradicts the fact that $\{\alpha\}$ is open. Conversely, if α is a maximal atom with a simple object H such that $\alpha = \overline{H}$, then $\text{ASupp}(H) = \{\alpha\}$; and hence $\{\alpha\}$ is open. □

We denote by $\text{noeth}\mathcal{A}$ the subcategory of $\mathcal{A}$ consisting of all noetherian objects.

Proposition 3.3 *Let $\mathcal{X}$ be a subcategory of $\mathcal{A}$. Then* $\text{ASupp}^{-1}(\text{ASupp}(\mathcal{X})) \cap$ *$\text{noeth}\mathcal{A} = \langle \mathcal{X} \rangle_{Serre} \cap \text{noeth}\mathcal{A}$, where* $\text{ASupp}^{-1}(U) = \{M \in \mathcal{A} | \text{ASupp}(M) \subseteq U\}$ *for any open subclass U of* $\text{ASpec}\mathcal{A}$.

Proof Let $M \in \text{ASupp}^{-1}(\text{ASupp}(\mathcal{X})) \cap \text{noeth}\mathcal{A}$. Then

$$\text{ASupp}(M) \subset \text{ASupp}(\mathcal{X}) = \pi(\langle\langle \mathcal{X} \rangle_{\text{sub}}\rangle_{\text{quot}}) \cap \text{ASpec}_0\mathcal{A}).$$

It follows from [5, Lemma 4.2] that $M \in \langle\langle\langle \mathcal{X} \rangle_{\text{sub}}\rangle_{\text{quot}} \cap \text{ASpec}_0\mathcal{A})\rangle_{\text{Serre}} \subset \langle \mathcal{X} \rangle_{\text{Serre}}$ and hence $M \in \langle \mathcal{X} \rangle_{\text{Serre}} \cap \text{noeth}\mathcal{A}$. For the converse, it is clear that

$\mathcal{X} \subset \mathrm{ASupp}^{-1}(\mathrm{ASupp}(\mathcal{X}))$ and $\mathrm{ASupp}^{-1}(\mathrm{ASupp}(\mathcal{X}))$ is a Serre subcategory of $\mathcal{A}$. Hence, we have

$$\langle \mathcal{X} \rangle_{\mathrm{Serre}} \subset \mathrm{ASupp}^{-1}(\mathrm{ASupp}(\mathcal{X}))$$

so that $\langle \mathcal{X} \rangle_{\mathrm{Serre}} \cap \mathrm{noeth}\mathcal{A} \subset \mathrm{ASupp}^{-1}(\mathrm{ASupp}(\mathcal{X})) \cap \mathrm{noeth}\mathcal{A}$. □

Lemma 3.4 *Let $\mathcal{A}$ admit a set of noetherian generators and let $\mathcal{X}$ be a Serre subcategory of $\mathcal{A}$. Then* $\mathrm{ASupp}(\mathcal{X}) = \mathrm{ASupp}(\mathcal{X} \cap \mathit{noeth}\mathcal{A})$.

Proof Clearly $\mathrm{ASupp}(\mathcal{X} \cap \mathrm{noeth}\mathcal{A}) \subset \mathrm{ASupp}(\mathcal{X})$. Conversely if α is in $\mathrm{ASupp}(\mathcal{X})$, then there exists an object M of $\mathcal{X}$ such that $\alpha \in \mathrm{ASupp}(M)$. Thus there exists a subobject K of M and a monoform subobject H of M/K such that $\alpha = \overline{H}$. Since $\mathcal{X}$ is Serre, H is in $\mathcal{X}$. On the other hand, since $\mathcal{A}$ admits a set of noetherian generator, H contains a noetherian subobject H_1. Hence $\alpha \in \mathrm{ASupp}(H_1) \subset \mathrm{ASupp}(\mathcal{X} \cap \mathrm{noeth}\mathcal{A})$. □

Definition 3.5 Let $U \subset \mathrm{ASpec}(\mathcal{A})$. A subclass Φ_U of $\mathrm{ASpec}(\mathcal{A})$ is defined as

$$\Phi_U = \{\alpha \in \mathrm{ASpec}(\mathcal{A}) |\text{ exists a monoform object } H \text{ with } \alpha = \overline{H} \text{ and } \mathrm{ASupp}(H) \subset U\}.$$

Lemma 3.6 *Let $U \subset \mathrm{ASpec}(\mathcal{A})$. Then Φ_U is the largest open subclass of* $\mathrm{ASpec}(\mathcal{A})$ *contained in U.*

Proof We first show that Φ_U is open. Suppose that $\alpha \in \Phi_U$. Then there exists a monoform object H of $\mathcal{A}$ such that $\alpha = \overline{H}$ and $\mathrm{ASupp}(H) \subset U$. Given $\beta \in \mathrm{ASupp}(H)$, there exist a subobject K of H and a monoform subobject G of H/K such that $\beta = \overline{G}$. Thus $\mathrm{ASupp}(G) \subset \mathrm{ASupp}(H) \subset U$ which implies that $\beta \in \Phi_U$ and so $\mathrm{ASupp}(H) \subset \Phi_U$. Now we show that $\Phi_U \subset U$. Given $\alpha \in \Phi_U$, there exists a monoform object H of $\mathcal{A}$ such that $\alpha = \overline{H}$ and $\mathrm{ASupp}(H) \subset U$ and so $\alpha \in U$. In order to prove the last assertion let Φ be an open subclass of $\mathrm{ASpec}(\mathcal{A})$ such that $\Phi \subset U$ and assume that $\alpha \in \Phi$. Then by the definition, there exists a monoform object H of $\mathcal{A}$ such that $\alpha = \overline{H}$ and $\mathrm{ASupp}H \subset \Phi \subset U$ so that $\alpha \in \Phi_U$. Therefore $\Phi \subset \Phi_U$. □

A corresponding result to Proposition 3.3 can be obtained for the subclasses of $\mathrm{ASpec}\mathcal{A}$.

Proposition 3.7 *Let U be a subclass of* $\mathrm{ASpec}(\mathcal{A})$. *Then* $\mathrm{ASupp}(\mathrm{ASupp}^{-1}(U)) = \Phi_U$.

Proof We first show that $\mathrm{ASupp}^{-1}(U) = \mathrm{ASupp}^{-1}(\Phi_U)$. Since $\Phi_U \subset U$, we have $\mathrm{ASupp}^{-1}(\Phi_U) \subset \mathrm{ASupp}^{-1}(U)$. Conversely, given $M \in \mathrm{ASupp}^{-1}(U)$, we have $\mathrm{ASupp}(M) \subset U$. Since $\mathrm{ASupp}(M)$ is an open subclass of $\mathrm{ASpec}(\mathcal{A})$, the previous lemma forces that $\mathrm{ASupp}(M) \subset \Phi_U$; and hence $M \in \mathrm{ASupp}^{-1}(\Phi_U)$. We now show that

$$\mathrm{ASupp}(\mathrm{ASupp}^{-1}(\Phi_U)) = \Phi_U.$$

Given $\alpha \in \Phi_U$, there exists a monoform object H with $\alpha = \overline{H}$ and $\mathrm{ASupp}(H) \subseteq U$. Since $\mathrm{ASupp}(H)$ is an open subclass of $\mathrm{ASpec}\mathcal{A}$, in view of the definition, we have $\mathrm{ASupp}(H) \subseteq \Phi_U$ which implies that $H \in \mathrm{ASupp}^{-1}(\Phi_U)$. Thus $\mathrm{ASupp}(H) \subseteq \mathrm{ASupp}(\mathrm{ASupp}^{-1}(\Phi_U))$ so that $\alpha \in \mathrm{ASupp}(\mathrm{ASupp}^{-1}(\Phi_U))$. To do the reverse inclusion, if $\alpha \in \mathrm{ASupp}(\mathrm{ASupp}^{-1}(\Phi_U))$, then there exists $M \in \mathrm{ASupp}^{-1}(\Phi_U)$ such that $\alpha \in \mathrm{ASupp}(M)$. Therefore $\mathrm{ASupp}(M) \subseteq \Phi_U$ so that $\alpha \in \Phi_U$. Consequently, we have the following equalities

$$\mathrm{ASupp}(\mathrm{ASupp}^{-1}(U)) = \mathrm{ASupp}(\mathrm{ASupp}^{-1}(\Phi_U)) = \Phi_U.$$

□

The following theorem is a generalization of [5, Theorem 4.4] for non-noetherian abelian categories.

Theorem 3.8 *Let $\mathcal{A}$ admit a set of noetherian generators and let*

$$\mathcal{S} = \{\mathcal{X} \mid \mathcal{X} \text{ is a Serre subcategory of noeth } \mathcal{A}\}.$$

Then the map Θ by the assignment $\mathcal{X} \mapsto \mathrm{ASupp}(\mathcal{X})$ from $\mathcal{S}$ to the class of all open subclasses of $\mathrm{ASpec}(\mathcal{A})$ establishes a one-to-one correspondence with the inverse map Φ by the assignment $U \to \mathrm{ASupp}^{-1}(U) \cap \mathrm{noeth}\mathcal{A}$.

Proof Let $\mathcal{X}$ be any Serre subcategory of noeth$\mathcal{A}$. Then we have

$$\begin{aligned}\Phi\Theta(\mathcal{X}) &= \Phi(\mathrm{ASupp}(\mathcal{X}))\\ &= \mathrm{ASupp}^{-1}(\mathrm{ASupp}(\mathcal{X})) \cap \mathrm{noeth}\mathcal{A}\\ &= \mathcal{X} \cap \mathrm{noeth}\mathcal{A} = \mathcal{X}\end{aligned}$$

where the third equality follows from Proposition 3.3. Now, assume that U is an open subclass of ASpec$\mathcal{A}$. Then we have

$$\begin{aligned}\Theta\Phi(U) &= \Theta(\mathrm{ASupp}^{-1}(U) \cap \mathrm{noeth}\mathcal{A})\\ &= \mathrm{ASupp}(\mathrm{ASupp}^{-1}(U) \cap \mathrm{noeth}\mathcal{A})\\ &= \mathrm{ASupp}(\mathrm{ASupp}^{-1}(U)) = U\end{aligned}$$

where the third equality follows from Lemma 3.4 and the last equality is obtained from Proposition 3.7. □

4 Monoform objects in Grothendieck categories

Throughout this section $\mathcal{A}$ is a Grothendieck category. We start this section with a known result in [10].

Proposition 4.1 [10, Chap. V. Proposition 5.9] *Suppose that $N = N_1 \cap \cdots \cap N_n$ is an irredundant irreducible decomposition of a subobject N of a noetherian object M. Then the canonical monomorphism $\beta : M/N \to M/N_1 \oplus \cdots \oplus M/N_n$ is essential.*

Remark 4.2 For any atom $\alpha \in \mathrm{ASpec}\mathcal{A}$, it follows from [5, Lemma 5.8] that $E(H)$ are isomorphic to each other for all monoform objects H with $\alpha = \overline{H}$. So we denote the isomorphism class of all $E(H)$ with $\overline{H} = \alpha$ by $E(\alpha)$. Observe that every monoform object H is uniform and so is $E(H)$. This fact implies that $E(\alpha)$ is an indecomposable injective object.

Corollary 4.3 *Suppose that* $N = N_1 \cap \cdots \cap N_n$ *is an irredundant irreducible decomposition of a subobject* N *of a noetherian object* M *with* $\mathrm{AAss}(M/N_i) = \{\alpha_i\}$ *for each* i. *Then* $E(M/N) = E(\alpha_1) \oplus \cdots \oplus E(\alpha_n)$.

Proof For simplicity, we may assume that $N = 0$. In view of Proposition 4.1, the canonical monomorphism $\beta : M \to M/N_1 \oplus \cdots \oplus M/N_n$ is essential. Thus $E(M/N_1 \oplus \cdots \oplus M/N_n) = E(M/N_1) \oplus \cdots \oplus E(M/N_n)$ is an essential extension of M; and then $E(M) = E(M/N_1) \oplus \cdots \oplus E(M/N_n)$. On the other hand, for each i there exists a monoform subobject H_i of M/N_i such that $\alpha_i = \overline{H_i}$. Since M/N_i is uniform, H_i is essential in M/N_i and then $E(\alpha_i) = E(H_i) = E(M/N_i)$. □

Lemma 4.4 *Let* M *be an object of* $\mathcal{A}$ *and let* N *be an essential subobject of* M. *Then* $\mathrm{AAss}(N) = \mathrm{AAss}(M)$.

Proof It is clear by definition that $\mathrm{AAss}(N) \subset \mathrm{AAss}(M)$. Given $\overline{H} \in \mathrm{AAss}(M)$, there exists $H_1 \in \overline{H}$ such that $H_1 \subset M$. Since N is essential in M, we have $0 \neq H_1 \cap N \subset N$ and we notice that $H_1 \cap N$ is a monoform such that $H_1 \cap N \in \overline{H}$. Then $\overline{H} \in \mathrm{AAss}(N)$. □

A Grothendieck category is called *locally noetherian* if it has a set of noetherian generators.

Lemma 4.5 *If* E *is an indecomposable injective object in a locally noetherian Grothendieck category* $\mathcal{A}$, *then there exists a monoform object* H *such that* $E = E(H)$.

Proof Since $\mathcal{A}$ is locally noetherian, $\mathrm{AAss}(E)$ is a non-empty set and since E is indecomposable, it is uniform and hence there exists an atom $\alpha \in \mathrm{ASpec}\mathcal{A}$ such that $\mathrm{AAss}(E) = \{\alpha\}$. It should also be noted that every subobject of E is essential. Then there exists a essential monoform subobject H of E such that $\alpha = \overline{H}$ and $E = E(H)$. □

Krause [8, Theorem 2.1] proved that over a commutative noetherian ring A, there exists a one-to-one correspondence between all full subcategories of the category of A-modules which are closed under taking submodules, extensions and direct unions, and the set of subsets of $\mathrm{Spec}A$. The following theorem is a slightly generalized version of [8, Theorem 2.1] for locally noetherian Grothendieck categories if we replace submodules, extensions and direct unions in the category of modules by subobjects, injective envelopes and direct sums in a Grothendieck category. We notice that by horseshoe lemma, to be closed under injective envelopes and direct sums implies to be closed under extensions.

Theorem 4.6 *For a locally noetherian Grothendieck category* $\mathcal{A}$, *the map* $\mathcal{X} \to \mathrm{AAss}(\mathcal{X})$ *induces a bijection between full subcategories of* $\mathcal{A}$ *closed under subobjects, injective envelopes and direct sums, and subsets of* $\mathrm{ASpec}(\mathcal{A})$. *The inverse map sends a subset* Φ *of* $\mathrm{ASpec}(\mathcal{A})$ *to* $\mathrm{AAss}^{-1}(\Phi) = \{M \in \mathcal{A} | \ \mathrm{AAss}(M) \subset \Phi\}$.

Proof Let Φ be a subclass of $\mathrm{ASpec}(\mathcal{A})$. It is clear to see that $\mathrm{AAss}(\mathrm{AAss}^{-1}(\Phi)) = \Phi$. Using Lemma 4.4 and [5, Proposition 5.6], it is clear to see that $\mathrm{AAss}^{-1}(\Phi)$ is closed under taking subobjects, injective envelopes and direct sums. Now, assume that $\mathcal{X}$ is a subcategory of $\mathcal{A}$, which is closed under taking suobjects, injective envelopes and direct sums. We assert that $\mathrm{AAss}^{-1}(\mathrm{AAss}(\mathcal{X})) = \mathcal{X}$. It is clear to see that $\mathcal{X} \subset \mathrm{AAss}^{-1}(\mathrm{AAss}(\mathcal{X}))$. Conversely, given $M \in \mathrm{AAss}^{-1}(\mathrm{AAss}(\mathcal{X}))$, we have $\mathrm{AAss}(M) \subset \mathrm{AAss}(\mathcal{X})$. On the other hand, using Matlis theorem (cf. [10, Chap V. Proposition 4.5]) and Lemmas 4.4 and 4.5 we have

$$E(M) = \bigoplus_{\alpha \in \mathrm{AAss}(M)} E(\alpha)^{(I_\alpha)}.$$

Since $\alpha \in \mathrm{AAss}(M) \subset \mathrm{AAss}(\mathcal{X})$, there exist $N \in \mathcal{X}$ and a monoform subobject H of N such that $\alpha = \overline{H}$. Then by the assumption H and $E(H)$ belong to $\mathcal{X}$. It follows from Remark 4.2 that $E(\alpha) = E(H) \in \mathcal{A}$. Thus the assumption implies that $E(M) \in \mathcal{X}$, which forces M in $\mathcal{X}$. □

For a Serre subcategory $\mathcal{X}$ of $\mathcal{A}$, we define the *quotient category* $\mathcal{A}/\mathcal{X}$ in which the objects are those of $\mathcal{A}$ and for objects M and N of $\mathcal{A}$, we have

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{X}}(M, N) = \varinjlim_{(M', N') \in \mathcal{S}_{M,N}} \mathrm{Hom}_{\mathcal{A}}(M', N/N')$$

where $\mathcal{S}_{M,N}$ is a directed set defined by

$$\mathcal{S}_{M,N} = \{(M', N') | M' \subset M, N' \subset N \text{ with } M/M', N' \in \mathcal{X}\}.$$

If $\mathcal{A}$ is a Grothendieck category, then so is $\mathcal{A}/\mathcal{X}$ together with a canonical exact functor $F : \mathcal{A} \to \mathcal{A}/\mathcal{X}$. We refer the reader to [1] or [12, Chap 4] for more details and the basic properties of the quotient category.

Recall from [6] that $\mathrm{ASpec}\mathcal{A}$ can be regarded as a partially order set together with a specialization order $\leq$ as follows: for any atoms α and β in $\mathrm{ASpec}\mathcal{A}$, we have $\alpha \leq \beta$ if and only if for any open subclass Φ of $\mathrm{ASpec}\mathcal{A}$ satisfying $\alpha \in \Phi$, we have $\beta \in \Phi$. For more materials about $\leq$, we refer the reader to [6,7].

Remark 4.7 If α is a maximal atom, then α is maximal in $\mathrm{ASpec}\mathcal{A}$ under the order $\leq$. To be more precise, assume that H is a simple object with $\alpha = \overline{H}$ and $\beta \in \mathrm{ASpec}\mathcal{A}$ such that $\alpha \leq \beta$. Then, since $\alpha \in \mathrm{ASupp}(H) = \{\alpha\}$, the definition implies that $\beta \in \mathrm{ASupp}(H)$ and so $\beta = \alpha$. According [6, Remark 4.5] these two concepts of maximality do not coincide in general. More precisely, assume that k is a field, regard $k[x]$ as a graded ring with $\deg x = 1$ and consider the locally noetherian Grothendieck category $\mathcal{A} = \mathrm{GrMod}\, k[x]$ of graded $k[x]$-modules. Then $H = k[x]$ is a monoform object of $\mathcal{A}$ such that $\{\overline{H}\}$ is not open subset of $\mathrm{ASpec}\mathcal{A}$; and hence using Proposition 3.2, the atom $\overline{H}$ is not maximal; but it is maximal under the order $\leq$. In later we show that under some conditions they are the same.

A Serre subcategory $\mathcal{X}$ of the Grothendieck category $\mathcal{A}$ is called *localizing* if the canonical functor $F : \mathcal{A} \to \mathcal{A}/\mathcal{X}$ has a right adjoint functor $G : \mathcal{A}/\mathcal{X} \to \mathcal{A}$. The functors F and G induce a functorial morphism $\eta_{(-)} : \mathrm{Id}_{\mathcal{A}}(-) \to GF(-)$ such that for any object M of $\mathcal{A}$, the objects $\mathrm{Ker}\eta_M$ and $\mathrm{Coker}\eta_M$ belong to $\mathcal{X}$.

If $\mathcal{X}$ is a localizing subcategory of $\mathcal{A}$, then in view of [7, Proposition 5.4], the map $\mathrm{ASpec}\mathcal{A}\backslash\mathrm{ASupp}\mathcal{X} \to \mathrm{ASpec}(\mathcal{A}/\mathcal{X})$ given by $\overline{H} \mapsto \overline{F(H)}$ is a homeomorphism. The inverse map is given by $\overline{H'} \mapsto \overline{G(H')}$. Thus we may identify the atoms in quotient category $\mathcal{A}/\mathcal{X}$ with those in $\mathrm{ASpec}\mathcal{A}\backslash\mathrm{ASupp}\mathcal{X}$.

For any atom α in $\mathrm{ASpec}\mathcal{A}$, the closure of α is as follows

$$\overline{\{\alpha\}} = \{\beta \in \mathrm{ASpec}\mathcal{A} | \beta \leq \alpha\}.$$

In this case we have an open subset $\Phi_\alpha = \mathrm{ASpec}\mathcal{A} \setminus \overline{\{\alpha\}}$ and so we can define a localizing subcategory $\mathcal{X}_\alpha$ of $\mathcal{A}$ by $\mathcal{X}_\alpha = \mathrm{ASupp}^{-1}\Phi_\alpha$. We now define the quotient category $\mathcal{A}_\alpha$ of $\mathcal{A}$ induced by $\mathcal{X}_\alpha$ which is $\mathcal{A}_\alpha = \mathcal{A}/\mathcal{X}_\alpha$. In this case the canonical functor F is denoted by $(-)_\alpha$.

Proposition 4.8 *For any atom α in* $\mathrm{ASpec}\mathcal{A}$*, the indecomposable injective $E(\alpha)$ is isomorphic to $GF(E(\alpha))$, where $F : \mathcal{A} \to \mathcal{A}_\alpha$ is the canonical functor with its right adjoint functor $G : \mathcal{A}_\alpha \to \mathcal{A}$.*

Proof There is a natural morphism $\eta_{E(\alpha)} : E(\alpha) \to GF(E(\alpha))$. Clearly $\mathrm{Ker}\ \eta_{E(\alpha)}$ is in $\mathcal{X}_\alpha$, while $\mathrm{AAss}(\mathrm{Ker}\eta_{E(\alpha)}) \subseteq \{\alpha\}$. This implies that $\mathrm{Ker}\eta_{E(\alpha)} = 0$. It then follows from [7, Proposition 4.8] that $E(\alpha)$ is an essential subobject of $GF(E(\alpha))$. In view of [7, Proposition 4.11], $F(E(\alpha))$ is an injective object in $\mathcal{A}_\alpha$ and also in view of [7, Proposition 4.10], $GF(E(\alpha))$ is an injective object in $\mathcal{A}$. Therefore we have $E(\alpha) \cong GF(E(\alpha))$. □

A functor $t : \mathcal{A} \to \mathcal{A}$ is said to be *preradical* if it is a subfunctor of the identity functor on $\mathcal{A}$, in other words, if t assigns to any object M a subobject $t(M)$ and every morphism $M \to N$ in $\mathcal{A}$ induces a morphism $t(M) \to t(N)$ by restriction. A preradical t is called *idempotent* if $t^2 = t$ and is called *radical* if $t(M/t(M)) = 0$ for every object M of $\mathcal{A}$.

Remark 4.9 If $\mathcal{X}$ is a localizing subcategory of $\mathcal{A}$, then according to [10, Chap VI. Proposition 2.3], $\mathcal{A}$ admits an idempotent radical $t_{\mathcal{X}}$ such that for any object M, the subobject $t_{\mathcal{X}}(M)$ is the largest subobject of M belonging to $\mathcal{X}$. We notice that if $F : \mathcal{A} \to \mathcal{A}/\mathcal{X}$ is the canonical functor with the right adjoint functor $G : \mathcal{A}/\mathcal{X} \to \mathcal{A}$, then $t_{\mathcal{X}}(M)$ is the kernel of the natural morphism $\eta_M : M \to GF(M)$.

From [1], a localizing subcategory $\mathcal{X}$ of the Grothendieck category $\mathcal{A}$ is called *stable* if the injective envelope in $\mathcal{A}$ of any object of $\mathcal{X}$ is also an object of $\mathcal{X}$. Further a Grothendieck category is said to be *locally stable* if any its localizing subcategory is stable.

Remark 4.10 According to [1, p. 428, Proposition 10], if A is a commutative noetherian ring, then Mod-A is a locally stable category. To be more precise, assume that $\mathcal{X}$ is a localizing subcategory of the category of R-modules and $M \in \mathcal{X}$. Then $R/\mathfrak{p} \in \mathcal{X}$

for any $\mathfrak{p} \in \mathrm{Supp}_R(M)$. On the other hand it follows from [10, Chap VII, Proposition 5.3] that $\mathrm{Supp}_R(M) = \mathrm{Supp}_R(E(M))$. In order to prove $E(M) \in \mathcal{X}$, it suffices to show that $N \in \mathcal{X}$ for any finitely generated submodule N of $E(M)$. But this follows by considering a finite filtration of submodules $0 = N_0 \subset N_1 \subset \cdots \subset N_n = N$ where each N_i/N_{i-1} is isomorphic to $R/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \mathrm{Supp}(M)$.

Proposition 4.11 *Let $\mathcal{X}$ be a localizing subcategory of $\mathcal{A}$ with the canonical functor $F : \mathcal{A} \to \mathcal{A}/\mathcal{X}$ and the idempotent radical $t_{\mathcal{X}}$. If M is an object of $\mathcal{A}$, then the following conditions hold:*

(i) $\mathrm{AAss}F(M) = \mathrm{AAss}(M/t_{\mathcal{X}}(M))$.
(ii) *If $\mathcal{X}$ is stable, then* $\mathrm{AAss}F(M) = \mathrm{AAss}(M)\backslash ASupp\mathcal{X}$.

Proof (i) Using [7, Proposition 5.6], we have $\mathrm{AAss}GF(M) = \mathrm{AAss}F(M)$ and $\mathrm{AAss}F(M) \subseteq \mathrm{ASupp}F(M) = \mathrm{ASupp}(M)\backslash\mathrm{ASupp}(\mathcal{X})$. In view of Remark 4.9, there exists a short exact sequence

$$0 \to M/t_{\mathcal{X}}(M) \to GF(M) \to \mathrm{Coker}\eta_M \to 0$$

in $\mathcal{A}$ where $t_{\mathcal{X}}(M)$ and $\mathrm{Coker}\eta_M$ belong to $\mathcal{X}$. It follows from [7, Proposition 4.8] that $M/t_{\mathcal{X}}(M)$ is essential in $GF(M)$ so that $\mathrm{AAss}(M/t_{\mathcal{X}}(M)) = \mathrm{AAss}GF(M) = \mathrm{AAss}F(M)$. (ii) Observe that the morphism η_M has a factorization $M \overset{\theta}{\to} M/t_{\mathcal{X}}(M) \overset{\gamma}{\to} GF(M)$, where θ is epic and γ is monic. Let $\alpha = \overline{H}$ be an arbitrary atom in $\mathrm{AAss}(M/t_{\mathcal{X}}(M))$ such that H is a subobject of $M/t_{\mathcal{X}}(M)$ with the monomorphism $\iota : H \to M/t_{\mathcal{X}}(M)$. We notice that since $M/t_{\mathcal{X}}(M)$ does not contain any nonzero subobject in $\mathcal{X}$, we deduce that $\alpha \notin \mathrm{ASupp}\mathcal{X}$. We now have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & t_{\mathcal{X}}(M) & \longrightarrow & M & \xrightarrow{\theta} & M/t_{\mathcal{X}}(M) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow{\scriptstyle\kappa} & & \uparrow{\scriptstyle\iota} & & \\
0 & \longrightarrow & t_{\mathcal{X}}(K) & \longrightarrow & K & \xrightarrow[\beta]{} & H & \longrightarrow & 0.
\end{array}$$

where the right square is the pullback of θ and ι. We notice that since ι is monic, κ is also monic by [10, Chap IV. Proposition 5.1] and K is not in $\mathcal{X}$ as H is not in $\mathcal{X}$. We show that $\alpha \in \mathrm{AAss}(K)$; and hence the proof completes as K is a subobject of M. If $t_{\mathcal{X}}(K)$ is an essential subobject of K, then $E(K) = E(t_{\mathcal{X}}(K))$ is in $\mathcal{X}$ because $\mathcal{X}$ is stable; and hence K is in $\mathcal{X}$ which is a contradiction. Therefore $t_{\mathcal{X}}(K)$ is not essential in K and so there exists a nonzero subobject Y of K such that $Y \cap t_{\mathcal{X}}(K) = 0$. This implies that Y is isomorphic to a subobject of H; and hence $\alpha = \overline{Y} \in \mathrm{AAss}(K)$. The inclusion $\mathrm{AAss}(M)\backslash\mathrm{ASupp}(\mathcal{X}) \subseteq \mathrm{AAss}(F(M))$ holds by [7, Proposition 5.6]. □

Given a nonzero object M of $\mathcal{A}$, we say that an atom $\alpha \in \mathrm{ASupp}(M)$ is *minimal* in $\mathrm{ASupp}(M)$, if it is minimal under the order $\leq$. We denote by $\mathrm{MinASupp}(M)$, the set of all minimal atoms in $\mathrm{Supp}M$.

Proposition 4.12 *Let $\mathcal{A}$ be a locally noetherian Grothendieck category and let M be a nonzero noetherian object of $\mathcal{A}$. If $\alpha \in MinASupp(M)$ such that $\mathcal{X}_\alpha$ is stable, then* $\alpha \in \mathrm{AAss}(M)$.

Proof Let α be in $\mathrm{MinASupp}(M)$. It follows from [6, Lemma 5.16 and Proposition 6.6] that $\mathrm{AAss}(M_\alpha) = \mathrm{ASupp}(M_\alpha) = \{\alpha\}$. Now, applying Proposition 4.11 to $F(-) = (-)_\alpha$, we have $\alpha \in \mathrm{AAss}(M)$. □

If $\mathcal{A}$ is a locally stable locally noetherian Grothendieck category, then two concept of maximality of atoms coincide. Because, if α is an atom which is maximal in $\mathrm{ASpec}\mathcal{A}$ under the order $\leq$, then there exists a noetherian monoform object H such that $\alpha = \overline{H}$. If H is not simple, then it contains a nonzero maximal subobject H_1 and so $G = H/H_1$ is simple. Since $\mathcal{A}$ is locally stable, it follows form Proposition 4.12 that $\mathrm{MinASupp}(H) = \{\alpha\}$. Then $\overline{G} \in \mathrm{ASupp}(H)$ forces that $\alpha \leq \overline{G}$. Now, the maximality of α implies that $\alpha = \overline{G}$; and hence G and H have a common nonzero subobject which is a contradiction.

Proposition 4.13 *Let $\mathcal{A}$ be a locally noetherian Grothendieck category and let N be a proper subobject of a noetherian object M in $\mathcal{A}$. If α is a minimal atom in $\mathrm{ASupp}(M/N)$ such that $\mathcal{X}_\alpha$ is stable, then the α-component of any atomical decomposition of N is $\eta_M^{-1}(GF(N))$ where $F : \mathcal{A} \to \mathcal{A}_\alpha$ is the canonical functor with the right adjoint functor $G : \mathcal{A}_\alpha \to \mathcal{A}$. Therefore the α-component is uniquely determined only by M, N and α.*

Proof Suppose that $N = \bigcap_{i=1}^n N_i$ is an atomical decomposition of N with $\mathrm{AAss}M/N_i = \{\alpha_i\}$. The canonical monomorphism $M/N \rightarrowtail \bigoplus_{i=1}^n M/N_i$ implies $\alpha \in \mathrm{ASupp}(M/N_i)$ for some $1 \leq i \leq n$. On the other hand, since $\mathrm{ASupp}(M/N_i) \subseteq \mathrm{ASupp}(M/N)$, using Proposition 4.12, we have $\alpha \in \mathrm{AAss}(M/N_i) = \{\alpha_i\}$. Renumbering, we may assume that $\alpha = \alpha_1$. The same reasoning deduces that $\alpha \notin \mathrm{ASupp}M/N_i$ for any $i = 2, \dots, n$. Thus $M_\alpha = N_{i\alpha}$ for any $i = 2, \dots, n$ so that $N_\alpha = N_{1\alpha}$. Using the natural transformation $\eta_{(-)} : \mathrm{id}_{(-)} \to GF$ and noting that F is exact and G is left exact, we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N_1 & \longrightarrow & M & \longrightarrow & M/N_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \eta_{N_1}} & & \downarrow{\scriptstyle \eta_M} & & \downarrow{\scriptstyle \eta_{M/N_1}} & & \\
0 & \longrightarrow & GF(N_1) & \longrightarrow & GF(M) & \longrightarrow & GF(M/N_1) & &
\end{array}$$

We assert that η_{M/N_1} is a monomorphism, otherwise $\alpha \in \mathrm{AAss}(\mathrm{Ker}\eta_{M/N_1})$ which is impossible as $F(\eta_{M/N_1})$ is an isomorphism. Now, it is straightforward to show that the left square is pullback so that we conclude $N_1 = \eta_M^{-1}(GF(N_1))$. Therefore we have

$$N_1 = \eta_M^{-1}(GF(N_1)) = \eta_M^{-1}(G(N_{1\alpha})) = \eta_M^{-1}(G(N_\alpha)) = \eta_M^{-1}(GF(N)).$$

□

5 In the case right noetherian rings

In this section we assume that A is a noncommutative right noetherian ring and Mod-A denotes the category of all right A-modules. Our intention of this section is to extend a result due to Krause [8, Theorem 2.1] for fully right bounded noetherian rings.

Proposition 5.1 *Let A be a fully right bounded noetherian ring. The map $\mathcal{X} \to \mathrm{Ass}(\mathcal{X}) = \bigcup_{M\in\mathcal{X}} \mathrm{Ass}(M)$ induces a bijection between the full subcategories of Mod-A closed under taking submodules, injective envelopes and direct sums, and the set of subsets of* $\mathrm{Spec}(A)$. *The inverse map sends a subset U of* $\mathrm{Spec}(A)$ *to* $\mathrm{Ass}^{-1}(U) = \{M \in \mathcal{A} | \ \mathrm{Ass}(M) \subseteq U\}$.

Proof It is known that Mod-A is a locally noetherian Grothendieck category. Using [10, Chap VII. Theorem 2.1], there is a bijection $\Phi : \mathcal{E}(A) \to \mathrm{Spec}(A)$, given by $E \mapsto \mathrm{ass}(E)$, between $\mathcal{E}(A)$ the set of isomorphism class of indecomposable injective right A-modules and $\mathrm{Spec}(A)$ the set of prime ideals. On the other hand, by [5, Theorem 5.11], there is a bijection $\Psi : \mathrm{ASpec}(A) \to \mathcal{E}(A)$, given by $\alpha \mapsto E(\alpha)$, between $\mathrm{ASpec}(A)$ the set of atoms of Mod-A and $\mathcal{E}(A)$ the set of isomorphism class of indecomposable injective right A-modules. Therefore $\Theta = \Phi\Psi : \mathrm{ASpec}(A) \to \mathrm{Spec}(A)$ is bijective. For any subcategory $\mathcal{X}$ of Mod-A which is closed under taking submodules, injective envelopes and direct sums, and any subset U of $\mathrm{Spec}(A)$, using Theorem 4.6, it suffices to show that $\Theta(\mathrm{AAss}(\mathcal{X})) = \mathrm{Ass}(\mathcal{X})$ and $\mathrm{AAss}^{-1}(\Theta^{-1}(U)) = \mathrm{Ass}^{-1}(U)$. For the first equality, let $\Theta(\alpha) \in \Theta(\mathrm{AAss}(\mathcal{X}))$ where $\alpha \in \mathrm{AAss}(\mathcal{X})$. We hence have $\Theta(\alpha) = \Phi(E(\alpha)) = \mathfrak{p}$ where $\mathrm{Ass}(E(\alpha)) = \{\mathfrak{p}\}$. Since $\alpha \in \mathrm{AAss}(\mathcal{X})$, there exist $M \in \mathcal{X}$ and monoform submodule H of M such that $\alpha = \overline{H}$ and $\alpha \in \mathrm{AAss}(M)$. Hence $E(H) = E(\alpha)$ is a direct summand of $E(M)$ which forces $\Theta(\alpha) = \mathfrak{p} \in \mathrm{Ass}(E(M)) = \mathrm{Ass}(M)$. Conversely assume that $\mathfrak{p} \in \mathrm{Ass}\mathcal{X}$ and so there exists a module $M \in \mathcal{X}$ such that $\mathfrak{p} \in \mathrm{Ass}(M)$. We may assume that M is noetherian, and so according to Proposition 2.6 and Corollary 4.3, we have $E(M) = E(\alpha_1) \oplus \cdots \oplus E(\alpha_n)$ where $\mathrm{AAss}(M) = \{\alpha_1, \ldots, \alpha_n\}$. Then there exists $1 \leq i \leq n$ such that $\mathrm{Ass}(E(\alpha_i)) = \{\mathfrak{p}\}$; hence $\Theta(\alpha_i) = \mathfrak{p}$. The second equality follows easily by a similar argument using in the first one. □

Corollary 5.2 *Let A be a fully right bounded noetherian ring and let $\mathcal{X}$ be a subcategory of Mod-A which is closed under taking submodules, injective envelopes and direct sums. Then $M \in \mathcal{X}$ if and only if $A/\mathfrak{p} \in \mathcal{X}$ for all $\mathfrak{p} \in$* $\mathrm{Ass}(M)$.

Proof By virtue of Proposition 5.1 we have $\mathcal{X} = \mathrm{Ass}^{-1}(\mathrm{Ass}(\mathcal{X}))$. Hence $M \in \mathcal{X}$ if and only if $\mathrm{Ass}(M) \subseteq \mathrm{Ass}(\mathcal{X})$ if and only if $\{\mathfrak{p}\} = \mathrm{Ass}(R/\mathfrak{p}) \subseteq \mathrm{Ass}(\mathcal{X})$ for all $\mathfrak{p} \in \mathrm{Ass}(M)$ if and only if $R/\mathfrak{p} \in \mathcal{X}$ for all $\mathfrak{p} \in \mathrm{Ass}(M)$. □

Krause [8, Theorem 2.1] considered the condition to be closed under extensions on full subcategories while in Corollary 4.7, we have a stronger condition that the full subcategories are closed under injective envelopes. It is easy by horseshoe lemma to see that to be closed under injective envelopes and direct sums implies to be closed under extensions in our desired subcategories. The theme of proof of [8, Theorem 2.1] with the mentioned conditions turns to [8, Lemma 2.4] which states $E(A/\mathfrak{p}) \in$

$\mathcal{X}$ if $A/\mathfrak{p} \in \mathcal{X}$ where the subcategory $\mathcal{X}$ of A-Mod has the same condition as [8, Theorem 2.1]. However the proof of Krause [8, Lemma 2.4], does not work if A is not commutative. In the rest of this section we slightly settle [8, Lemma 2.4] for noncommutative noetherian rings.

For every prime ideal $\mathfrak{p}$ of A we consider the multiplicatively closed subset of A

$$S(\mathfrak{p}) = \{s \in A \mid s + \mathfrak{p} \text{ is regular in } A/\mathfrak{p}\}.$$

A multiplicatively closed subset S of A is a *right Ore set* if for every $a \in A$ and $s \in S$, there exist $b \in A$ and $t \in S$ such that $at = sb$. The *left Ore sets* can be defined similarly. Finally, S is said to be an *Ore set* if it is both right and left Ore set.

Lemma 5.3 *Let $\mathfrak{p}$ be a prime ideal of A and let $S(\mathfrak{p})$ be a right Ore set in A. Then each element of $S(\mathfrak{p})$ acts as a nonzero-divisor on the right A-module $E(A/\mathfrak{p})$.*

Proof See [4, Theorem 3.2]. □

Lemma 5.4 *Let $\mathfrak{p}$ be a prime ideal of A and let $S(\mathfrak{p})$ be a right Ore set in A. Then $E(A/\mathfrak{p})$ is a right $A[S^{-1}]$-module.*

Proof As A is a right noetherian, it satisfies ACC on right annihilators and so by using [10, Chap II. Proposition 1.5], $S(\mathfrak{p})$ is a right denominator set. Thus $A[S^{-1}]$ exists. On the other hand, according to Lemma 5.3, the module $E(A/\mathfrak{p})$ is $S(\mathfrak{p})$-torsion free. Now, the result is obtained by [10, Chap II. Proposition 3.7]. □

A two-sided ideal $\mathfrak{a}$ has *Artin-Rees property* if for every right ideal $\mathfrak{b}$ of A and positive integer n, there exists $h(n) > 0$ such that $\mathfrak{a}^{h(n)} \cap \mathfrak{b} \subseteq \mathfrak{b}\mathfrak{a}^n$.

Proposition 5.5 *Let $\mathfrak{p}$ be a prime ideal of A, let $S(\mathfrak{p})$ be an Ore set in A and let $\mathfrak{p}$ have Artin-Rees property. Then $E(A/\mathfrak{p})$ is obtained from $A/\mathfrak{p}$ by taking submodules, extensions and direct unions.*

Proof There is a decomposition $E(A/\mathfrak{p}) = \oplus E_i$ into indecomposable injective modules E_i with $\mathrm{Ass}(E_i) = \{\mathfrak{p}\}$ for each i. Since $\mathfrak{p}$ has Artin-Rees property, it follows from [10, Chap VII. Theorem 4.4] that $E_i = \bigcup_{n\geq 1}(0 :_{E_i} \mathfrak{p}^n)$ for each i; hence $E(A/\mathfrak{p}) = \bigcup_{n\geq 1}(0 :_{E(A/\mathfrak{p})} \mathfrak{p}^n)$. For every integer $n > 0$, consider the right A-submodule $I_n = (0 :_{E(A/\mathfrak{p})} \mathfrak{p}^n)$ of $E(A/\mathfrak{p})$. Then each factor I_{n+1}/I_n is a right $A/\mathfrak{p}$-module. On the other hand, in view of Lemma 5.4, $E(A/\mathfrak{p})$ is a right $A[S(\mathfrak{p})^{-1}]$-module and so is I_{n+1}/I_n. Thus I_{n+1}/I_n is a right $Q = A[S(\mathfrak{p})^{-1}]/\mathfrak{p}A[S(\mathfrak{p})^{-1}]$-module. It follows from [4, Theorem 2.1] that Q is the classical right quotient ring of the prime ring $A/\mathfrak{p}$; and hence according to [10, Chap II. Proposition 2.6], Q is a simple ring. In view of [10, Chap I. Proposition 7.8], the ring Q is semi-simple. Hence [10, Chap I. Proposition 7.7] implies that I_{n+1}/I_n is a projective right Q-module so that it is a direct summand of a free right Q-module. On the other hand, since $S(\mathfrak{p})$ is an Ore set in A, the set $X = \{s + \mathfrak{p} | s \in S(\mathfrak{p})\}$ is an Ore set in $A/\mathfrak{p}$; and hence it is clear to see that Q is a direct union of the form

$$Q = \bigcup_{s\in S(\mathfrak{p})} (s + \mathfrak{p})^{-1}(A/\mathfrak{p}).$$

We further notice that for each $s \in S(\mathfrak{p})$, there exists an isomorphism $(s+\mathfrak{p})^{-1}(A/\mathfrak{p}) \cong A/\mathfrak{p}$ of right $A/\mathfrak{p}$-modules and so it is an isomorphism of right A-modules. Then Q is obtained from $A/\mathfrak{p}$ by taking the direct unions; hence I_n/I_{n+1} is obtained from $A/\mathfrak{p}$ because it is a submodule of a direct sum of Q. Now using the extensions condition and an easy induction, I_n is obtained from $A/\mathfrak{p}$. □

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