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# Principal ideals in mod-$\ell$ Milnor $K$-theory

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**Abstract** Fix a symbol $\underline{a}$ in the mod-$\ell$ Milnor $K$-theory of a field $k$, and a norm variety $X$ for $\underline{a}$. We show that the ideal generated by $\underline{a}$ is the kernel of the $K$-theory map induced by $k \subset k(X)$ and give generators for the annihilator of the ideal. When $\ell = 2$, this was done by Orlov, Vishik and Voevodsky.

**Keywords** Milnor K-theory · Norm variety · Motives

## 1 Introduction

Let $\ell$ be a prime and $k$ a field containing $1/\ell$. Given units $a_1, \dots, a_n \in k^\times$ we can form the Steinberg symbol $\underline{a} = \{a_1, \dots, a_n\}$ in $K_n^M(k)$; we wish to study the ideal $(\underline{a})$ generated by $\underline{a}$ in $K_n^M(k)/\ell$. What is the quotient ring $(K_*^M(k)/\ell)/(\underline{a})$, and what is the annihilator ideal $\operatorname{ann}(\underline{a})$, so that $(\underline{a}) = (K_*^M(k)/\ell)/\operatorname{ann}(\underline{a})$?

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Here is the main result of this paper; it was proven for $\ell = 2$ by Orlov, Vishik and Voevodsky in [12, 2.1].

**Theorem 1.1** *Suppose that* char $k = 0$*, and let* $X$ *be a norm variety for a nontrivial symbol* $\underline{a}$ *in* $K_n^M(k)/\ell$*. Then:*

(a) *the kernel of* $K_*^M(k)/\ell \longrightarrow K_*^M(k(X))/\ell$ *is the ideal of* $K_*^M(k)/\ell$ *generated by* $\underline{a}$*;*
(b) *the annihilator of* $\underline{a}$ *is the ideal of* $K_*^M(k)/\ell$ *generated by the norms*

$$\{N(\alpha) \in K_*^M(k)/\ell \mid \alpha \in K_*^M(k(x)),\ x \text{ a closed point in } X\}.$$

Theorem 1.1 uses the notion of a *norm variety*; see Definition 3.1 below. The existence of norm varieties is due to Rost; the terminology comes from [13] and [6, 1.18].

*Examples 1.2* Theorem 1.1(a) implies that $K_i^M(k)/\ell \longrightarrow K_i^M(k(X))/\ell$ is an injection when $i < n$, that the kernel of $K_n^M(k)/\ell \longrightarrow K_n^M(k(X))/\ell$ is exactly the cyclic subgroup generated by $\underline{a}$ and that the kernel of $K_{n+1}^M(k)/\ell \longrightarrow K_{n+1}^M(k(X))/\ell$ is the subgroup $\underline{a} \cup k^\times$.

The group of units $b$ in $k^\times/k^{\times\ell}$ such that $\{a_1, \dots, a_n, b\} = 0$ in $K_{n+1}^M(k)/\ell$ forms the degree 1 part of the ideal ann$(\underline{a})$. This group, described in Theorem 1.1(b) was originally described by Voevodsky. If $H_{p,q}(X)$ is the motivic homology of a norm variety for $\underline{a}$, $X$, and $k$ has no extensions of degree $\ell$, Voevodsky proved in [13, A.1 and 2.9] that the pushforward $\pi_* : H_{-1,-1}(X) \longrightarrow H_{-1,-1}(\operatorname{Spec} k) = k^\times$ induces an exact sequence

$$1 \longrightarrow \overline{H}_{-1,-1}(X) \xrightarrow{\pi_*} k^\times \xrightarrow{\underline{a}\cup} K_{n+1}^M(k)/\ell. \tag{1.2a}$$

Here $\overline{H}_{p,q}(X)$ denotes the coequalizer of the two projections $H_{p,q}(X \times X) \rightrightarrows H_{p,q}(X)$. Thus the degree 1 part of ann$(\underline{a})$ is $\overline{H}_{-1,-1}(X)$: $\{\underline{a}, b\} = 0$ if and only if $b \in \overline{H}_{-1,-1}(X)$.

When $n = 1$, write $\underline{a} = (a)$ for $a \in k^\times$, and set $E = k(\sqrt[\ell]{a})$. Then $X = \operatorname{Spec}(E)$ is a norm variety for $\underline{a}$. For simplicity, suppose that $k$ contains an $\ell$th root of unity, $\zeta$. The degree 2 part of $(\underline{a})$ is the group of symbols $a \cup b$; under the isomorphism $H^2_{\text{et}}(k, \mathbb{Z}/\ell) \cong {}_\ell\mathrm{Br}(k)$, $a \cup b$ is identified with the class of the cyclic algebra $A_\zeta(a, b)$ in the Brauer group. Theorem 1.1 describes the group of units $b$ for which $A_\zeta(a, b)$ is a matrix algebra, and the class of division algebras (or classes $[A] \in {}_\ell\mathrm{Br}(k)$) which are equivalent to cyclic algebras. In this case, Kummer theory gives the answer: the group of units is the image $N(E^\times)$ of the norm map $E^\times \longrightarrow k^\times$, and the class of division algebras equivalent to cyclic algebras is the class of algebras split by $E$. (See [20, 6.4.8].) In fact, we have the classical exact sequence

$$1 \longrightarrow N(E^\times) \longrightarrow k^\times \xrightarrow{a\cup} H^2_{\text{et}}(k, \mathbb{Z}/\ell) \longrightarrow H^2_{\text{et}}(E, \mathbb{Z}/\ell)^{\mathrm{Gal}(E/k)}. \tag{1.2b}$$

When $n = 1$, Theorem 1.1 states that for every unit $a$ not in $k^{\times\ell}$ there are exact sequences

$$1 \longrightarrow K_i^M(E)_{\mathrm{Gal}(E/k)} \xrightarrow{N} K_i^M(k) \xrightarrow{\cup a} K_{i+1}^M(k)/\ell \longrightarrow (K_{i+1}^M(E)/\ell)^{\mathrm{Gal}(E/k)}; \tag{1.2c}$$

when $i = 1$ this is exactly (1.2b). This follows from Voevodsky's Galois computations [6, 3.2 and 3.6] (*cf.* [14, 5.2 and 6.11]) and the fact that $\ell \cdot K_i^M(k) \subseteq N(K_i^M(E))$.

Theorem 1.1 follows from the more technical Theorem 1.3. We note that the analysis in [12] did not need to worry about roots of unity, as any field of characteristic 0 contains the square roots of unity, and Pfister quadrics always have points of degree 2. For an odd prime $\ell$, the existence of a norm variety with points of degree $\ell$ is established in [13, 1.21] modulo the Norm Principle, proven in [5, 0.3]; see Chapter 10 of [6].

**Theorem 1.3** *Let* $\mathrm{char}\, k = 0$. *Suppose that* $X$ *is a norm variety for a symbol* $\underline{a}$ *in* $K_n^M(k)/\ell$ *containing a point* $x$ *with* $[k(x) : k] = \ell$. *Write* $q = n + i$ *and let* $\widetilde{K}_q^M(k(X))/\ell$ *denote the equalizer of the maps* $K_q^M(k(X))/\ell \rightrightarrows K_q^M(k(X \times X))/\ell$; $\mathfrak{X}$ *denotes the* 0*-coskeleton of* $X$ *(see Definition 3.3).*

(a) *If* $\mu_\ell \subset k^\times$, *there is an exact sequence for all* $i$:

$$\overline{H}_{-i,-i}(X) \xrightarrow{\pi_*} K_i^M(k) \xrightarrow{\underline{a}\cup} K_q^M(k)/\ell \xrightarrow{\iota} \widetilde{K}_q^M(k(X))/\ell \longrightarrow H^{q+1,q-1}(\mathfrak{X}, \mathbb{Z}/\ell).$$

(b) *If* $\mu_\ell \not\subset k^\times$, *set* $e = [k(\zeta) : k]$ *and* $X' = X \times_{k_1} k(\zeta)$, *where* $k_1 = k(\zeta) \cap k(X)$. *If* $\mathfrak{X}'$ *denotes the* 0*-coskeleton of* $X'$ *over* $k(\zeta)$, *then for all* $i$ *there is an exact sequence:*

$$\overline{H}_{-i,-i}(X)[e^{-1}] \xrightarrow{\pi_*} K_i^M(k)[e^{-1}] \xrightarrow{\underline{a}\cup} K_q^M(k)/\ell \xrightarrow{\iota} \widetilde{K}_q^M(k(X))/\ell \longrightarrow H^{q+1,q-1}(\mathfrak{X}', \mathbb{Z}/\ell)^G.$$

*The map* $\iota$ *is induced by the homomorphism* $k \longrightarrow k(X)$, *and* $G = \mathrm{Gal}(k'/k_1)$.

The sequences (1.2a), (1.2b) and (1.2c) begin with an injection. This is often, but not always, the case.

**Question 1.4** *In the situation of Theorem 1.3(a) with* $\mu_\ell \subset k^\times$, *when is* $\pi_*$ *an injection?*

For $i = 0$, the map $\pi_*$ is an injection: $\overline{H}_{0,0}(X) = \mathbb{Z}$, and its image in $K_0(k) = \mathbb{Z}$ is $\ell\mathbb{Z}$. (This observation goes back to [9, 8.7.2].) This calculation shows that the mod-$\ell$ reduction $\overline{H}_{0,0}(X, \mathbb{Z}/\ell) \longrightarrow K_0^M(k)/\ell$ of $\pi_*$ is not always an injection.

The map $\pi_*$ is an injection for $i = 1$ by Eq. (1.2a), and for $n = 1$ by Lemma 2.5 below. However, if $k$ does not contain the $\ell^{th}$ roots of unity, $\pi_*$ need not be an injection even for $i = n = 1$, as the classical Hilbert Theorem 90 can fail; see Example 2.6 below.

Theorem 1.1(b) could be strengthened to only look at norms of elements in $K_1^M(k) = k^\times$ if we knew that the answer to the following question was affirmative:

**Question 1.5** *If $E/F$ is a Galois extension of prime degree, is $K_{n+1}^M(E)$ always generated by symbols $\{a_1, \ldots, a_n, b\}$ with $a_i \in F^\times$ and $b \in E^\times$?*

*It suffices to check the case $n = 1$: is $K_2^M(E)$ is always generated by symbols $\{a, b\}$ with $a \in F^\times$ and $b \in E^\times$?*

If $\ell = 2$, $\ell = 3$ or $k$ is $\ell$-special, this is the case; $K_2^M k(x)$ is generated by symbols $\{a, b\}$ with $a \in k^\times$ and $b \in k(x)^\times$; see [8, Lemma 2], [1, p. 388]. By Becher [2, 1.1], $K_n^M k(x)$ is also generated by symbols $\{\alpha, \beta\}$ with $\alpha \in K_{n-m}^M(k)$, $\beta \in K_m^M(k)$ if $\ell < 2^{m+1}$.

The restriction to prime degree is necessary in Question 1.5. Becher has pointed out in [2, 3.1] that if $E = k(x, y)$ and $F = k(x^\ell, y^\ell)$ then $\{x, y\}$ cannot be written in this form, as the tame symbol $\partial_y : K_2^M(E) \longrightarrow k(x)^\times/k^{\times\ell}$ shows. In this case, $[E : F] = \ell^2$.

*Remark 1.6* Although most of our results work over perfect fields of arbitrary characteristic, the assumption that $k$ has characteristic 0 is needed in two places.

1. To prove that norm varieties exist for symbols of length $n$. This would go through for any perfect field of positive characteristic (by induction on $n$) if we could prove that for symbols of length $n - 1$ over $k$, a norm variety $Y$ exists which satisfies the *Norm Principle* (see [5, 0.3] or [6, 10.17]). The inductive step is given in [6, 10.21].
2. We also need characteristic 0 to show that the symmetric characteristic class $s_d(X)$ of a norm variety is nonzero modulo $\ell^2$. The proof in characteristic 0 is due to Rost (unpublished), and given in Proposition 10.13 of [6], and depends upon the Connor–Floyd theory of equivariant cobordisms on complex $G$-manifolds (as given by Theorem 8.16 in *loc. cit.*) It is possible that a proof in characteristic $p > 0$ could be given along the lines of [13, 5.2], if we assume resolution of singularities.

We will therefore state as many of our results in as much generality as possible, only restricting to characteristic zero when absolutely necessary.

*Remark 1.7* After writing this paper, we discovered that many of our results are in Yagita's paper [21] and in the Merkurjev–Suslin paper [10] when $\mu_\ell \subset k^\times$. (Compare [21, Thm. 10.3] to our 1.3a and [10, 2.1] to our 1.1 and 1.3.) The basic technique in these papers, and in ours, is the same: generalize the ideas in [12], using Rost's norm varieties for $\ell > 2$. Yagita's proof is somewhat sketchy as it predated a clear understanding of norm varieties. Merkurjev and Suslin prove Theorem 1.1(b) when $\mu_\ell \subset k^\times$, but their result does not discuss $K_n^M(k)$ in the absence of roots of unity. Since neither of these results directly addresses the ring structure of $K_*^M(k)/\ell$, nor do they contain the final term $H^{q+1,q-1}$ in our Theorem 0.3, we feel that our exposition should be added to the public record.

*Notation and conventions* We fix a prime $\ell$ and an $\ell$th root of unity $\zeta$. We write $H^{p,q}(Y, \mathbb{Z}/\ell)$ for $H^p_{\text{nis}}(Y, \mathbb{Z}/\ell(q))$.

## 2 Borel–Moore homology

The first term in Theorem 1.3 uses the motivic homology group $H_{-i,-i}(X)$ of a smooth projective variety $X$ (with coefficients in $\mathbb{Z}$). However, it is more useful to think of it as the Borel–Moore homology group $H^{BM}_{-i,-i}(X)$, which is covariant for proper maps between smooth varieties, and contravariant for finite flat maps; see [4, p. 185] or [11, 16.13]. We define $H^{BM}_{-i,-i}(X)$ to be $H^{BM}_{-i,-i}(X,\mathbb{Z})$ if char $k=0$, and $H^{BM}_{-i,-i}(X,\mathbb{Z}[1/p])$ if char $k=p>0$.

Let $X$ be smooth and projective. We then have $H_{-i,-i}(X)=H^{BM}_{-i,-i}(X)$, and more generally $H_{p,q}(X,\mathbb{Z})=H^{BM}_{p,q}(X,\mathbb{Z})$, because the natural map from $M(X)=\mathbb{Z}_{\text{tr}}(X)$ to $M^c(X)$ in **DM** is an isomorphism for smooth projective $X$. Recall from [11, 2.8] that $\mathbb{Z}_{\text{tr}}(X)$ denotes the sheaf with transfers represented by $X$, and Voevodsky's triangulated category **DM** is a localization of the derived category of sheaves with transfers; see for example [11, p. 110]. The motivic homology groups $H_{p,q}(X,\mathbb{Z})$ of $X$ are defined to be $\text{Hom}_{\mathbf{DM}}(\mathbb{Z}(q)[p], M(X))$, while the Borel–Moore homology groups $H^{BM}_{p,q}(X,\mathbb{Z})$ are defined to be $\text{Hom}_{\mathbf{DM}}(\mathbb{Z}(q)[p], M^c(X))$; see [4, p. 185] or [11, 14.17, 16.20].

The case $i=1$ of the following result was proven in [13].

**Proposition 2.1** *Let $X$ be a smooth variety over a perfect field $k$; write $X^{(0)}$ for the closed points of $X$ and $X^{(1)}$ for the dimension 1 points of $X$. If $i\geq 0$, $H^{BM}_{-i,-i}(X)$ is the abelian group generated by symbols $[x,\alpha]$, where $x$ is a closed point of $X$ and $\alpha\in K^M_i(k(x))$, modulo the relations*

(i) $[x,\alpha][x,\alpha']=[x,\alpha+\alpha']$ *and*
(ii) *the image of the tame symbol $K^M_{i+1}(k(y))\longrightarrow\bigoplus K^M_i(k(x))\longrightarrow H^{BM}_{-i,-i}(X)$ is zero for every dimension 1 point $y$ of $X$.*

*That is, we have an exact sequence*

$$\coprod_{y\in X^{(1)}} K^M_{i+1}(k(y)) \xrightarrow{\text{tame}} \coprod_{x\in X^{(0)}} K^M_i(k(x)) \longrightarrow H^{BM}_{-i,-i}(X)\longrightarrow 0.$$

*In addition, $H^{BM}_{-i,-i}(X)$ is isomorphic to $H^{2d+i,d+i}(X,\mathbb{Z})$.*

*Proof* Let $A$ denote the abelian group with generators $[x,\alpha]$ and relations (i) and (ii), described in the Proposition, and set $d=\dim(X)$. We first show that $A$ is isomorphic to $H^d(X,\mathscr{H}^{d+i})$, where $\mathscr{H}^q$ denotes the Zariski sheaf associated to the presheaf $H^{q,d+i}(-,\mathbb{Z})$. For each $q$, $\mathscr{H}^q$ is a homotopy invariant Zariski sheaf, by [11, 24.1]. As such, it has a canonical flasque "Gersten" resolution on each smooth $X$ (given in [11, 24.11]), whose $c^{th}$ term is the coproduct over codimension $c$ points $z$ of $X$ of the skyscraper sheaves $H^{q-c,d+i-c}(k(z))$. Taking $q=d+i$, and recalling that $K^M_i\cong H^{i,i}$ on fields, we see that the stalks of the skyscraper sheaves in the $(d-1)^{st}$ and $d^{th}$ terms are groups $K^M_{i+1}(k(y))$ and $K^M_i(k(x))$. Moreover, the map $K^M_{i+1}(k(y))\longrightarrow K^M_i(k(x))$ is the tame symbol if $x\in\overline{\{y\}}$, and zero otherwise. As $H^d(X,\mathscr{H}^{d+i})$ is obtained by taking global sections and then cohomology, it is isomorphic to $A$.

Next, we show that $A$ is isomorphic to $H^{2d+i,d+i}(X,\mathbb{Z})$. To this end, consider the hypercohomology spectral sequence $E_2^{p,q}=H^p(X,\mathscr{H}^q)\Rightarrow H^{p+q,d+i}(X,\mathbb{Z})$.

Since $H^{q,d+i}(k(z)) = 0$ for $q > d+i$, the spectral sequence is zero unless $p \le d$ and $q \le d+i$. From this we deduce that $H^{2d+i,d+i}(X,\mathbb{Z}) \cong H^d(X,\mathscr{H}^{d+i}) \cong A$.

Finally, we show that $H^{BM}_{-i,-i}(X)$ is isomorphic to $H^{2d+i,d+i}(X,\mathbb{Z})$. Suppose first that $i=0$. Then the presentation describes $CH_0(X) \cong H^{2d,d}(X,\mathbb{Z})$, and by [17] we also have $H^{BM}_{0,0}(X) = CH_0(X)$. Thus we may assume that $i>0$.

If $\mathrm{char}(k)=0$, the proof is finished by the duality calculation, which uses Motivic Duality with $d=\dim(X)$ (see [11, 16.24] or [4, 7.1]):

$$\begin{aligned} H^{BM}_{-i,-i}(X,\mathbb{Z}) &= \mathrm{Hom}_{\mathbf{DM}}(\mathbb{Z}, M^c(X)(i)[i]) = \mathrm{Hom}_{\mathbf{DM}}(\mathbb{Z}(d)[2d], M^c(X)(d+i)[2d+i]) \\ &= \mathrm{Hom}_{\mathbf{DM}}(M(X), \mathbb{Z}(d+i)[2d+i]) = H^{2d+i,d+i}(X,\mathbb{Z}). \end{aligned}$$

Now suppose that $k$ is a perfect field of $\mathrm{char}(k)=p>0$. As we show below in Lemma 2.2, $K^M_i(k(x))$ and $K^M_{i+1}(k(y))$ are uniquely p-divisible for $i \ge 1$ (when $x$ is closed in $X$ and $\mathrm{trdeg}_k k(y) = 1$). Thus $A$ must also be uniquely p-divisible. Since $H^{2d+i,d+i}(X,\mathbb{Z}) \cong A$, the duality calculation above goes through with $\mathbb{Z}$ replaced by $\mathbb{Z}[1/p]$, using the characteristic $p$ version of Motivic Duality (see [7, 5.5.14]) and we have $H^{BM}_{-i,-i}(X,\mathbb{Z}[1/p]) \cong H^{2d+i,d+i}(X,\mathbb{Z}[1/p]) \cong H^{2d+i,d+i}(X,\mathbb{Z})$. □

**Lemma 2.2** (Izhboldin) *Let $E$ be a field of transcendence degree $t$ over a perfect field $k$ of characteristic $p$. Then $K^M_m(E)$ is uniquely $p$-divisible for $m>t$.*

*Proof* For any field $E$ of characteristic $p$, the group $K^M_m(E)$ has no $p$-torsion by Izhboldin's Theorem ([20, III.7.8]), and the dlog map $K^M_m(E)/p \longrightarrow \Omega^m_E$ is an injection with image $\nu(m)$; see [20, III.7.7.2]. Since $k$ is perfect, $\Omega^1_k = 0$ and $\Omega^1_E$ is $t$-dimensional, so if $m>t$ then $\Omega^m_E = 0$ and hence $K^M_m(E)/p = 0$. □

*Example 2.3* (i) $H_{-i,-i}(\mathrm{Spec}\, E) = K^M_i(E)$ for every field $E$ over $k$, as is evident from the presentation in Proposition 2.1.

(ii) If $E$ is a finite extension of $k$, the proper pushforward from $K^M_i(E) = H_{-i,-i}(\mathrm{Spec}\, E)$ to $K^M_i(k) = H_{-i,-i}(\mathrm{Spec}\, k)$ is just the norm map $N_{E/k}$; see [20, III. 7.5.3].

(iii) If $\pi : X \longrightarrow \mathrm{Spec}(k)$ is proper, and $x \in X$ is closed, the restriction of the pushforward

$$\pi_* : H_{-i,-i}(X) \longrightarrow H_{-i,-i}(\mathrm{Spec}\, k) = K^M_i(k)$$

to $K^M_i(k(x))$ sends $[x,\alpha]$ to the norm $N_{k(x)/k}(\alpha)$. This follows from (ii) by functoriality of $H_{-i,-i}$ for the composite $\mathrm{Spec}\, k(x) \longrightarrow X \longrightarrow \mathrm{Spec}\, k$, $x \in X$ closed. From the presentation in Proposition 2.1, the map $N_{X/k}$ is completely determined by the formula $\pi_*[x,\alpha] = N_{k(x)/k}(\alpha)$.

In particular, the image of $\pi_*$ is the subgroup of $K^M_i(k)$ generated by the norms $N_{k(x)/k}(\alpha)$ of $\alpha \in k(x)^\times$ as $x$ ranges over the closed points of $X$.

**Lemma 2.4** *Suppose that $\mu_\ell \subset k$. Let $E = k(\sqrt[\ell]{a})$ and write $X = \mathrm{Spec}(E)$, $G = \mathrm{Gal}(E/k)$. Then $X \times X \cong \coprod_G X$.*

*Proof* Since $E$ is a Galois extension, $E \otimes E \cong \prod_G E$; thus $X \times X \cong \mathrm{Spec}(E \otimes E) \cong \coprod_G X$. □

**Lemma 2.5** *Suppose that $\mu_\ell \subset k$ and $a \in k^\times$, and set $E = k(\sqrt[\ell]{a})$, $X = \operatorname{Spec}(E)$. Then $\overline{H}_{-i,-i}(X) \cong K_i^M(E)_{\operatorname{Gal}(E/k)}$, and $\overline{H}_{-i,-i}(X) \longrightarrow K_i^M(k)$ is an injection.*

*Proof* Note that $E/k$ is Galois with group $G$, so $X \times X \cong \coprod_G X$ by Lemma 2.4 and $\overline{H}_{-i,-i}(X) \cong (K_i^M E)_G$ by Example 2.3(i). In this case, $(K_i^M E)_G$ is a subgroup of $K_i^M(k)$ by (1.2c). □

*Example 2.6* If $E/k$ is not Galois, $\overline{H}_{-i,-i}(\operatorname{Spec}(E)) \longrightarrow K_i^M(k)$ need not be an injection, even for $n = 1$. One way to think of this is to realize that the classical Hilbert 90 asserts exactness of $(E \otimes E)^\times \rightrightarrows E^\times \longrightarrow k^\times$, and Hilbert 90 requires $E/k$ to be Galois [19, 6.4.7]. A concrete example is given by $\ell = 3$, $k = \mathbb{Q}$, and $E = \mathbb{Q}(\sqrt[3]{2})$. In this case, $\operatorname{Spec}(E) \times \operatorname{Spec}(E) \cong \operatorname{Spec}(E \times F)$, where $F = E(\sqrt[3]{1})$, and the coequalizer $\overline{H}_{-1,-1}(\operatorname{Spec}(E))$ of $(E \times F)^\times \rightrightarrows E^\times$ does not inject into $\mathbb{Q}^\times$. This shows that $\pi_*$ in Theorem 1.3(a) is not always an injection.

## 3 Norm varieties

Let $\underline{a} = (a_1, \ldots, a_n)$ be a sequence of units in a field $k$ of characteristic not equal to $\ell$.

**Definition 3.1** A field $F$ over $k$ is said to be a *splitting field* for $\underline{a}$ if $\underline{a}$ vanishes in $K_n^M(F)/\ell$. We say that a variety $X$ is a *splitting variety* for $\underline{a}$ if $k(X)$ is a splitting field for $\underline{a}$, i.e., if $\underline{a}$ vanishes in $K_n^M(k(X))/\ell$.

Let $X$ be a splitting variety for $\underline{a}$. We say that $X$ is an *$\ell$-generic* splitting variety for $\underline{a}$ if any splitting field $F$ has a finite extension $E$ of degree prime to $\ell$ with $X(E) \neq \emptyset$.

A *norm variety* for $\underline{a}$ is a smooth projective variety $X$ of dimension $d = \ell^{n-1} - 1$ which is an $\ell$-generic splitting variety for $\underline{a}$. When $\operatorname{char}(k) = 0$, a norm variety for $\underline{a}$ always exists (see [6, 10.16]).

For example, $E = k(\sqrt[\ell]{a_1})$ is a splitting field for $\underline{a} = (a_1, \ldots, a_n)$. Since a norm variety $X$ is $\ell$-generic, there is a finite field extension $E'/E$ of degree prime to $\ell$ and an $E'$-point of $X$. The following result, due to Rost, is proven in Chapter 10 of [6].

**Theorem 3.2** *If $\underline{a}$ is a nonzero symbol over $k$ and $\operatorname{char}(k) = 0$, then there exists a norm variety $X$ for $\underline{a}$ having a closed point $x$ with $[k(x) : k] = \ell$.*

We will frequently use the following fact, proven in [13, 1.21] (see [6, 10.13]): if $k$ has characteristic 0 and $n \geq 2$, the symmetric characteristic class $s_d(X)$ of a norm variety $X$ is nonzero modulo $\ell^2$ (i.e., $X$ is a *$\nu_{n-1}$-variety*).

**Definition 3.3** Given a norm variety $X$, let $\mathfrak{X}$ denote its 0-coskeleton, i.e., the simplicial scheme $p \mapsto X^{p+1}$ with the projections $X^{p+1} \longrightarrow X^p$ as face maps and the diagonal inclusions as degeneracies.

For simplicity, we write $\mathbb{L}$ for $\mathbb{Z}_{(\ell)}(1)[2]$, $R$ for $\mathbb{Z}_{(\ell)}$, and $R_{\text{tr}}(\mathfrak{X})$ for $\mathbb{Z}_{tr}(\mathfrak{X})_{(\ell)}$. We also regard $X$ as a Chow motive. Recall [11, 20.1] that Chow motives form a full subcategory of **DM**, and that an idempotent element $e \in CH^{\dim X}(X \times X)$ gives rise to a summand $(X, e)$ of $X$ in this category. Switching factors in $X \times X$ yields the transpose idempotent $e^t$ and a summand $(X, e^t)$.

**Theorem 3.4** *Let $X$ be a norm variety for $\underline{a}$ such that $s_d(X)$ is nonzero modulo $\ell^2$. Then there is a Chow motive $M = (X, e)$ with coefficients $\mathbb{Z}_{(\ell)}$, such that*

(i) *$M = (X, e)$ is a symmetric Chow motive, i.e., $(X, e) = (X, e^t)$;*
(ii) *The projection $X \longrightarrow \mathbb{Z}_{(\ell)}$ factors as $X \longrightarrow (X, e) \longrightarrow \mathbb{Z}_{(\ell)}$, i.e., is zero on $(X, 1 - e)$;*
(iii) *There is a motive $D$ related to the structure map $y : M \longrightarrow R_{\mathrm{tr}}(\mathfrak{X})$ and its twisted dual $y^D$ by two distinguished triangles in* **DM**, *where $b = d/(\ell - 1)$:*

$$D \otimes \mathbb{L}^b \longrightarrow M \xrightarrow{y} R_{\mathrm{tr}}(\mathfrak{X}) \xrightarrow{s} D \otimes \mathbb{L}^b[1], \tag{3.4a}$$

$$R_{\mathrm{tr}}(\mathfrak{X}) \otimes \mathbb{L}^d \xrightarrow{y^D} M \xrightarrow{u} D \xrightarrow{r} R_{\mathrm{tr}}(\mathfrak{X}) \otimes \mathbb{L}^d[1]. \tag{3.4b}$$

*Proof* This is proven carefully in [6, Ch. 5]; the construction is due to Voevodsky [16, pp. 422–428] and appears in Section 1 of [18]. Specifically, $\underline{a}$ determines a motive $A$ by (5.1), Definition 5.5 and 5.13.1 of [6]; by definition, $M = S^{\ell-1}(A)$ and $D = S^{\ell-2}(A)$, where $S^m(A)$ is the $m$th symmetric product of $A$. Part (i) follows from 5.19; part (ii) follows from 5.9; and part (iii) follows from 5.7 of *loc. cit.* □

Although many of our techniques require the field $k$ to contain the $\ell$th roots of unity, we can sometimes remove this restriction using the following observation. Given a norm variety $X$ over a field $k$, let $k_1$ denote the largest subfield of $k(\zeta)$ contained in $k(X)$. Then $X$ is also a norm variety for $\underline{a}$ over $k_1$.

**Lemma 3.5** *Given a nonzero symbol $\underline{a} \in K_*^M(k)/\ell$, let $X$ be a norm variety for $\underline{a}$ over $k$. Then every component $X'$ of $X_{k(\zeta)}$ is a norm variety for $\underline{a}$ over $k(\zeta)$.*

*Proof* Clearly, $X'$ is a splitting variety for $\underline{a}$ of the right dimension. Given a splitting field $F$ of $\underline{a}$ over $k(\zeta)$, there is a prime-to-$\ell$ extension $E$ of $F$ such that $k(\zeta) \subset E$ and such that there exists a map Spec $E \longrightarrow X$ over $k$. By basechange, there is a map Spec $E \otimes_k k(\zeta) \longrightarrow X_{k(\zeta)}$ over $k(\zeta)$. As $k(\zeta) \subset E$, $E \otimes_k k(\zeta)$ is a $\mathrm{Gal}(k(\zeta)/k)$-indexed product of copies of $E$. Since $\mathrm{Gal}(k(\zeta)/k)$ acts transitively on the components of $X_{k(\zeta)}$, each component $X'$ of $X_{k(\zeta)}$ has an $E$-point. Thus $X'$ is a norm variety over $k(\zeta)$. □

*Remark 3.6* $X_{k(\zeta)}$ is a $\mathrm{Gal}(k_1/k)$-indexed coproduct of copies of $X' = X \times_{k_1}$ Spec $k(\zeta)$.

## 4 Reducing to Theorem 1.3 over fields containing $\ell$-th roots

We are now ready to prove Theorem 1.1 assuming Theorem 1.3. Fix a field $k$ of characteristic 0, a symbol $\underline{a}$ and a norm variety $X$ for $\underline{a}$. We first observe that, given Example 2.3(ii), the statement of Theorem 1.1 is equivalent to the exactness of the sequence

$$H_{-i,-i}(X)/\ell \xrightarrow{\pi_*} K_i^M(k)/\ell \xrightarrow{\underline{a}\cup} K_{i+n}^M(k)/\ell \xrightarrow{\iota} K_{i+n}^M(k(X))/\ell. \tag{4.1}$$

As observed in Example 1.2, Theorem 1.1 for $n = 1$ follows from (1.2c) when $\mu_\ell \subset k^\times$.

**Proposition 4.2** *Suppose that Theorem 1.3 holds over k. Then so does Theorem 1.1.*

*Proof* As the equalizer $\tilde{K}^M_{i+n}(k(X))/\ell$ is a subgroup of $K^M_{i+n}(k(X))/\ell$, Theorem 1.3 implies that there is an exact sequence

$$H_{-i,-i}(X)[e^{-1}] \xrightarrow{\pi_*} K^M_i(k)[e^{-1}] \xrightarrow{\underline{a}\cup} K^M_{i+n}(k)/\ell \xrightarrow{\iota} K^M_{i+n}(k(X))/\ell.$$

(If $\mu \subset k^\times$ then $e = 1$). Exactness of (4.1) is immediate. □

Thus we have reduced the proof of Theorem 1.1 to Theorem 1.3. We will now show that proving Theorem 1.3 over fields containing $\ell$th roots of unity suffices.

**Proposition 4.3** *Suppose that Theorem 1.3 holds for all fields of characteristic* 0 *which contain $\ell$th roots of unity. Then Theorem 1.3 holds for all fields of characteristic* 0.

*Proof* Let $k$ be any field of characteristic 0 not containing an $\ell^{th}$ root of unity, $\zeta$. Set $q = n + i$, $k' = k(\zeta)$, $k_1 = k' \cap k(X)$, $e = [k' : k]$ and $G = \mathrm{Gal}(k'/k_1)$, as in the statement of Theorem 1.3(b). By Lemma 3.5 and Remark 3.6, the component $X' = X \times_{k_1} \mathrm{Spec}(k')$ of $X_{k'}$ is a norm variety for $\underline{a}$ over $k'$. The action of $G$ on $k'$ induces actions of $G$ on $X'$ and its 0-skeleton $\mathfrak{X}'$, and induces the last map in Theorem 1.3(b):

$$\tilde{K}^M_q(k(X))/\ell \xrightarrow{j} (\tilde{K}^M_q(k'(X'))/\ell)^G \xrightarrow{\partial} H^{q+1,q-1}(\mathfrak{X}')^G.$$

Since $e$ is prime to $\ell$, inverting $e$ in the exact sequence of Theorem 1.3 for $k'$ yields the exact sequence forming the bottom row of the following diagram, in which the downward arrows are base change maps and the upward arrows are the norm maps.

$$\begin{array}{ccccccccc}
H_{-i,-i}(X)[e^{-1}] & \xrightarrow{\pi_*} & K^M_i(k)[e^{-1}] & \xrightarrow{\underline{a}\cup} & K^M_q(k)/\ell & \xrightarrow{\iota} & \tilde{K}^M_q(k(X))/\ell & \xrightarrow{\partial j} & H^{q+1,q-1}(\mathfrak{X}')^G \\
N\uparrow & & N\uparrow\,\downarrow & & N\uparrow\,\downarrow & & \downarrow j & & \downarrow \\
H_{-i,-i}(X')[e^{-1}] & \xrightarrow{\pi'_*} & K^M_i(k')[e^{-1}] & \xrightarrow{\underline{a}\cup} & K^M_q(k')/\ell & \xrightarrow{\iota} & \tilde{K}^M_q(k'(X'))/\ell & \xrightarrow{\partial} & H^{q+1,q-1}(\mathfrak{X}')
\end{array}$$

As each $K$-group is covariantly functorial, the diagram with the downward set of arrows commutes; the diagram with the upward set of arrows commutes by naturality and the projection formula [20, III.7.5.2]. The downward map $K^M_*(k) \longrightarrow K^M_*(k')$, followed by the norm map, is multiplication by $e = [k' : k]$. A diagram chase now shows that the top row of the diagram is exact. □

*Remark 4.4* The map $j$ is also injective in the above diagram. To see this, note that (by the projection formula) the norm $K^M_q(k'(X'))/\ell \longrightarrow K^M_q(k(X))/\ell$ induces a map $\tilde{N}$ from $\tilde{K}^M_q(k(X'))/\ell$ to $\tilde{K}^M_q(k(X))/\ell$, and the composition $\tilde{N}\, j$ is multiplication by $[k' : k_1]$, not $e$. Note that $\tilde{N}$ does not commute with the norm $K^M_q(k')/\ell \longrightarrow K^M_q(k)/\ell$ unless $k = k_1$.

## 5 The exact sequence

In this section and the next, we assume that our field $k$ contains an $\ell$th root of unity, $\zeta$. As before, we fix a symbol $\underline{a}$ and a norm variety $X$ for $\underline{a}$, writing $\mathfrak{X}$ for the 0-coskeleton of $X$.

Given a complex $\mathcal{F}^\bullet$ of étale sheaves, let $\mathcal{H}^q = \mathcal{H}^q_{\text{nis}}(\mathcal{F}^\bullet)$ denote the Nisnevich sheaf associated to the presheaf $H^q_{\text{et}}(-,\mathcal{F}^\bullet)$. If $\mathcal{F}$ is a locally constant étale sheaf (such as $\mu_\ell^{\otimes i}$), $\mathcal{H}^q(\mathcal{F})$ is a Nisnevich sheaf with transfers, by [11, 6.11, 6.21 and 13.1].

**Lemma 5.1** *If $\mathcal{F}$ is a sheaf, $H^0(\mathfrak{X},\mathcal{H}^q)$ is the equalizer of $H^0(X,\mathcal{H}^q) \rightrightarrows H^0(X\times X,\mathcal{H}^q)$.*

*Proof* This is the definition of $H^0$ on a simplicial scheme; see [3, 5.2.2]. Alternatively, it follows from the spectral sequence $E_1^{p,q} = H^q(X^{p+1},\mathcal{F}) \Rightarrow H^{p+q}(\mathfrak{X},\mathcal{F})$ for the cohomology of a sheaf on a simplicial scheme. □

*Remark 5.2* The Nisnevich sheaves $\mathcal{H}^q(\mu_\ell^{\otimes q})$ are homotopy invariant sheaves with transfers, by [11, 24.1]. By [11, 11.1], if $X$ is smooth then $H^0(X,\mathcal{H}^q(\mu_\ell^{\otimes q}))$—and hence $H^0(\mathfrak{X},\mathcal{H}^q(\mu_\ell^{\otimes q}))$—injects into $\mathcal{H}^q(\mu_\ell^{\otimes q})(\operatorname{Spec} k(X)) = H^q_{\text{et}}(k(X),\mu_\ell^{\otimes q}) \cong K_q^M(k(X))/\ell$.

**Proposition 5.3** *If $\mu_\ell \subset k^\times$, there is a distinguished triangle in* **DM** *for each $q \geq 0$:*

$$\mathbb{Z}/\ell(q-1) \xrightarrow{\zeta} \mathbb{Z}/\ell(q) \longrightarrow \mathcal{H}^q(\mu_\ell^{\otimes q})[-q] \longrightarrow .$$

*Proof* For any Nisnevich complex $C$ and any $q$ we have a distinguished triangle

$$\tau^{\leq q-1}C \longrightarrow \tau^{\leq q}C \longrightarrow H^q(C)[-q] \longrightarrow .$$

Now let $C$ be the total direct image $R\pi_*\mu_\ell^{\otimes q}$, where $\pi : \mathbf{Sm}_{\text{et}} \longrightarrow \mathbf{Sm}_{\text{nis}}$, so $H^*_{\text{nis}}(X,C) = H^*_{\text{et}}(X,\mu_\ell^{\otimes q})$. Since $\mu_\ell \subset k^\times$, multiplication by $\zeta$ induces an isomorphism $\mu_\ell^{\otimes q-1} \cong \mu_\ell^{\otimes q}$. Thus we have an isomorphism $\cup\zeta : R\pi_*\mu_\ell^{\otimes q-1} \xrightarrow{\simeq} C$. In this case, the triangle reads:

$$\tau^{\leq q-1}R\pi_*(\mu_\ell^{\otimes q-1}) \xrightarrow{\zeta} \tau^{\leq q}R\pi_*(\mu_\ell^{\otimes q}) \longrightarrow \mathcal{H}^q(\mu_\ell^{\otimes q})[-q] \longrightarrow .$$

By the Beilinson–Lichtenbaum conjecture (which has now been proven; see [16, 6.17] or [6, Thm. B]), $\mathbb{Z}/\ell(q) \cong \tau^{\leq q}C$ and $\mathbb{Z}/\ell(q-1) \cong \tau^{\leq q-1}R\pi_*\mu_\ell^{\otimes q-1} \cong \tau^{\leq q-1}C$. Combining these facts yields the distinguished triangle in question. □

Let $\tilde{\mathfrak{X}}$ denote the simplicial cone of $\mathfrak{X} \longrightarrow \operatorname{Spec} k$. As a consequence of the Beilinson–Lichtenbaum conjectures, Voevodsky observed that

**Lemma 5.4** *If $X$ is smooth, the map $H^{p,q}(k,\mathbb{Z}/\ell) \longrightarrow H^{p,q}(\mathfrak{X},\mathbb{Z}/\ell)$ is an isomorphism if $p \leq q$ and an injection if $p = q+1$. That is, $H^{p,q}(\tilde{\mathfrak{X}},\mathbb{Z}/\ell) = 0$ if $p \leq q+1$.*

*Proof* See [14, 6.9 and 7.3] or [6, 1.37]. □

**Proposition 5.5** *If $\mu_\ell \subset k^\times$, there is a natural five-term exact sequence:*

$$0 \longrightarrow H^{q,q-1}(\mathfrak{X}, \mathbb{Z}/\ell) \xrightarrow{\zeta} K_q^M(k)/\ell \longrightarrow H^0(\mathfrak{X}, \mathcal{H}^q(\mu_\ell^{\otimes q})) \xrightarrow{\partial} H^{q+1,q-1}(\mathfrak{X}, \mathbb{Z}/\ell).$$

*Proof* Apply $H^q(\mathfrak{X}, -)$ to the distinguished triangle in Proposition 5.3. Using the fact that $H^q(\mathfrak{X}, C[j]) = H^{q+j}(\mathfrak{X}, C)$ and writing $\mathcal{H}^q$ for $\mathcal{H}^q(\mu_\ell^{\otimes q})$, we get

$$\begin{aligned} &H^{-1}(\mathfrak{X}, \mathcal{H}^q) \xrightarrow{\partial} H^{q,q-1}(\mathfrak{X}, \mathbb{Z}/\ell) \xrightarrow{\zeta} \\ &H^{q,q}(\mathfrak{X}, \mathbb{Z}/\ell) \longrightarrow H^0(\mathfrak{X}, \mathcal{H}^q) \xrightarrow{\partial} H^{q+1,q-1}(\mathfrak{X}, \mathbb{Z}/\ell). \end{aligned}$$

The first term ($H^{-1}$) is 0 because the coefficients are a sheaf. By Lemma 5.4 with $p = q$, the third term is $H^{q,q}(k, \mathbb{Z}/\ell) = K_q^M(k)/\ell$ [11, Theorem 5.1]. □

**Corollary 5.6** *Theorem 1.3 holds for $n = 1$.*

*Proof* By Proposition 4.3, we may assume $\zeta \in k$ so that $X = \mathrm{Spec}(E)$, $E = k(\sqrt[\ell]{a})$ and $X \times X = \coprod_G X$ (by Lemma 2.4), where $G = \mathrm{Gal}(E/k)$. By Lemma 5.1 with $\mathcal{F}$ being $\mu_\ell^{\otimes q}$, $H^0(\mathfrak{X}, \mathcal{H}^q)$ is the equalizer of $H^q(X, \mu_\ell^{\otimes q}) \rightrightarrows \prod_G H^q(X, \mu_\ell^{\otimes q})$, i.e., $H^q(X, \mu_\ell^{\otimes q})^G$. Since $H^q(X, \mu_\ell^{\otimes q})$ is $K_q^M(E)/\ell$, we have $H^0(\mathfrak{X}, \mathcal{H}^q) \cong (K_q^M(E)/\ell)^G$. Proposition 5.5 yields exactness of

$$K_q^M(k)/\ell \longrightarrow (K_q^M(E)/\ell)^G \xrightarrow{\partial} H^{q+1,q-1}(\mathfrak{X}, \mathbb{Z}/\ell).$$

Now combine this with the exact sequence (1.2c), using Lemma 2.5 to identify $\overline{H}_{-i,-i}(X)$. □

Our next goal, achieved in Corollary 5.8, is to connect the first map in Proposition 5.5 to the cup product with $\underline{a}$. We assume that $n \geq 2$, so that $d = \dim(X) > 0$ and $s_d(X)$ is defined.

**Proposition 5.7** *Let $X$ be a norm variety for $\underline{a}$ such that $s_d(X) \not\equiv 0 \pmod{\ell^2}$. For $i \geq 0$, there is a four-term exact sequence*

$$\overline{H}_{-i,-i}(X)_{(\ell)} \xrightarrow{\pi_*} K_i^M(k)_{(\ell)} \xrightarrow{r^*} H^{i+2d+1,i+d}(D, \mathbb{Z}_{(\ell)}) \longrightarrow 0.$$

*Suppose in addition that $X$ has a point of degree $\ell$. Then the following sequence is exact:*

$$\overline{H}_{-i,-i}(X) \xrightarrow{\pi_*} K_i^M(k) \xrightarrow{r^*} H^{i+2d+1,i+d}(D, \mathbb{Z}_{(\ell)}) \longrightarrow 0.$$

*Proof* Let $M$, $D$ and $\mathbb{L}$ be as in Theorem 3.4. Since $H^{p,q}(M[1]) = H^{p-1,q}(M)$, applying $H^{i+2d+1,i+d}(-,\mathbb{Z}_{(\ell)})$ to the distinguished triangle in (3.4b) gives us the exact sequence

$$H^{i+2d,i+d}(M,\mathbb{Z}_{(\ell)}) \longrightarrow H^{i+2d,i+d}(\mathfrak{X}\otimes\mathbb{L}^d) \xrightarrow{r^*} H^{i+2d+1,i+d}(D,\mathbb{Z}_{(\ell)}) \xrightarrow{u^*} H^{i+2d+1,i+d}(M,\mathbb{Z}_{(\ell)})$$

where for brevity we have written $H^{p,q}(\mathfrak{X}\otimes\mathbb{L}^d)$ for $\mathrm{Hom}_{\mathbf{DM}}(R_{\mathrm{tr}}(\mathfrak{X})\otimes\mathbb{L}^d,\mathbb{Z}_{(\ell)}(q)[p])$. We will show that this may be rewritten as the 4-term sequence of the proposition.

Because $M$ is a direct summand of $X$, $H^{p,q}(M,\mathbb{Z}_{(\ell)})$ is a summand of $H^{p,q}(X,\mathbb{Z}_{(\ell)})$, which vanishes whenever $p-q > \dim(X)$; see [11, 3.6]. Hence the last term $H^{i+2d+1,i+d}(M,\mathbb{Z}_{(\ell)})$ vanishes. Similarly, the first term, $H^{i+2d,i+d}(M,\mathbb{Z}_{(\ell)})$, is a summand of $H^{i+2d,i+d}(X,\mathbb{Z}_{(\ell)})$, which we showed to be isomorphic to $H_{-i,-i}(X,\mathbb{Z}_{(\ell)})$ if $i \geq 0$, in the proof of Proposition 2.1. Therefore we may replace the first term by $H_{-i,-i}(X,\mathbb{Z}_{(\ell)})$. Since $X \longrightarrow \mathrm{Spec}(k)$ factors through $\mathfrak{X}$, the map $\pi_* : H_{-i,-i}(X,\mathbb{Z}_{(\ell)}) \longrightarrow H_{-i,-i}(k,\mathbb{Z}_{(\ell)}) = K_i^M(k)_{(\ell)}$ factors through the coequalizer $\overline{H}_{-i,-i}(X,\mathbb{Z}_{(\ell)})$ of the two projections from $H_{-i,-i}(X\times X,\mathbb{Z}_{(\ell)})$. We also know that

$$\begin{aligned} H^{i+2d,i+d}(\mathfrak{X}\otimes\mathbb{L}^d) &= \mathrm{Hom}_{\mathbf{DM}}(\mathfrak{X}\otimes\mathbb{L}^d,\mathbb{Z}_{(\ell)}(i+d)[i+2d]) = \mathrm{Hom}_{\mathbf{DM}}(\mathfrak{X},\mathbb{Z}_{(\ell)}(i)[i]) \\ &= H^{i,i}(\mathfrak{X},\mathbb{Z}_{(\ell)}) \cong H^{i,i}(\mathrm{Spec}\,k,\mathbb{Z}_{(\ell)}) \cong K_i^M(k)\otimes\mathbb{Z}_{(\ell)} = K_i^M(k)_{(\ell)}, \end{aligned}$$

where the last two isomorphisms follow from Lemma 5.4 and the Nestorenko–Suslin–Totaro Theorem [11, 5.1]. Thus we have constructed an exact sequence

$$\overline{H}_{-i,-i}(X,\mathbb{Z}_{(\ell)}) \xrightarrow{\pi_*} K_i^M(k)_{(\ell)} \xrightarrow{r^*} H^{i+2d+1,i+d}(D,\mathbb{Z}_{(\ell)}) \longrightarrow 0.$$

When $X$ has a point $x$ of degree $\ell$ over $k$, every element $\alpha$ of $K_i^M(k)$ has $\ell\alpha = \pi_*([x,\alpha])$, so the cokernel of $\pi_* : H_{-i,-i}(X) \longrightarrow H_{-i,-i}(k) = K_i^M(k)$ has exponent $\ell$, and is the same as the cokernel of $\overline{H}_{-i,-i}(X,\mathbb{Z}_{(\ell)}) \longrightarrow K_i^M(k)_{(\ell)}$. Thus we can replace the first two terms of the exact sequence with these to get the desired sequence. □

**Corollary 5.8** *If $\mu_\ell \subset k^\times$, there are maps $\alpha_i : H^{i+2d+1,i+d}(D,\mathbb{Z}_{(\ell)}) \longrightarrow H^{n+i,n+i-1}(\mathfrak{X},\mathbb{Z}/\ell)$ for all $i$ so that $\underline{a}\cup : K_i^M(k)/\ell \longrightarrow K_{n+i}^M(k)/\ell$ (the cup product with $\underline{a}$) factors as*

$$K_i^M(k)/\ell \xrightarrow{r^*} H^{i+2d+1,i+d}(D,\mathbb{Z}_{(\ell)}) \xrightarrow{\alpha_i} H^{n+i,n+i-1}(\mathfrak{X},\mathbb{Z}/\ell) \xhookrightarrow{\zeta} K_{n+i}^M(k)/\ell.$$

*Proof* Set $q = n+i$. For each closed point $x$ of $X$, the diagram

$$\begin{array}{ccc} K_i^M(k(x))/\ell & \xrightarrow{N} & K_i^M(k)/\ell \\ \downarrow{\scriptstyle \underline{a}\cup = 0} & & \downarrow{\scriptstyle \underline{a}\cup} \\ K_q^M(k(x))/\ell & \xrightarrow{N} & K_q^M(k)/\ell \end{array}$$

commutes by the projection formula [20, III.7.5.2]. Thus the map $H_{-i,-i}(X) \longrightarrow K_q^M(k)/\ell$ is zero, since by Proposition 2.1 it is induced by the maps

$$K_i^M(k(x))/\ell \xrightarrow{N} K_i^M(k)/\ell \xrightarrow{\underline{a}\cup} K_q^M(k)/\ell.$$

By Proposition 5.7, the cup product factors through the quotient $H^{i+2d+1,i+d}(D, \mathbb{Z}_{(\ell)})$ of $K_i^M(k)/\ell$. It remains to show that the image $\underline{a}K_i^M(k)$ of the cup product lands in the subgroup $H^{q,q-1}(\mathfrak{X}, \mathbb{Z}/\ell)$ of $K_q^M(k)/\ell$. Since $H^0(X, \mathcal{H}^q(\mu_\ell^{\otimes q}))$ is a subgroup of $K_q^M(k(X))/\ell$ (by Remark 5.2), it suffices by Proposition 5.5 to show that $\underline{a}K_i^M(k)$ vanishes in $K_q^M(k(X))/\ell$. This is so because $k(X)$ splits $\underline{a}$. □

In Corollary 5.12, we will show that the map $\alpha_i$ is an isomorphism. The inverse of $\alpha_i$ will be constructed using the cohomology operations $Q_i$ constructed in [15, p. 51]. Each $Q_i$ has bidegree $(2\ell^i - 1, \ell^i - 1)$; see *loc. cit.* or [6, 13.3] for a summary of their properties. Thus the composite $Q = Q_{n-1}Q_{n-2}\cdots Q_0$ has bidegree $(2b\ell - n + 2, b\ell - n + 1)$, where $b = d/(\ell - 1) = \ell^{n-2} + \cdots + \ell + 1$.

**Definition 5.9** Define the $\mathbb{Z}$-graded ring $\mathbb{H}^*(k)$ by

$$\mathbb{H}^i(-) = \bigoplus_{s\in\mathbb{Z}} H^{i+s,s}(-, \mathbb{Z}/\ell).$$

In particular, $\mathbb{H}^0(k) \cong K_*^M(k)/\ell$. The cohomology operation $Q$ maps $\mathbb{H}^i(Y)$ to $\mathbb{H}^{i+b\ell+1}(Y)$. Note that $\mathbb{H}^i(\tilde{\mathfrak{X}}) = 0$ for $i \le 1$, by Lemma 5.4.

Now the operations $Q_j$ vanish on each $K_p^M(k)/\ell = H^{p,p}(k, \mathbb{Z}/\ell)$, because $H^{p,q}(k, \mathbb{Z}/\ell) = 0$ for $p > q$. Since the $Q_j$ are derivations for $\ell$ odd ([6, 13.10]), this means that $\mathbb{H}^*(Y)$ is a graded $K_*^M(k)/\ell$-module for each $Y$, and each $Q_j : \mathbb{H}^i(Y) \longrightarrow \mathbb{H}^{i+\ell^j}(Y)$ is a $K_*^M(k)/\ell$-module homomorphism. Thus $Q : \mathbb{H}^i \longrightarrow \mathbb{H}^{i+b\ell+1}$ is also a $K_*^M(k)/\ell$-module homomorphism.

**Lemma 5.10** *Let $X$ be a norm variety over a field of characteristic 0, and let $\mathfrak{X}$ be its 0-coskeleton. Then the map $Q : \mathbb{H}^1(\mathfrak{X}) \longrightarrow \mathbb{H}^{b\ell+2}(\mathfrak{X})$ is an injection.*

*Proof* Since $H^{p,q}(\operatorname{Spec} k, \mathbb{Z}/\ell) = 0$ for $p > q$, we have $\mathbb{H}^i(\operatorname{Spec} k) = 0$ for $i > 0$. This yields isomorphisms $\mathbb{H}^i(\mathfrak{X}) \xrightarrow{\cong} \mathbb{H}^{i+1}(\tilde{\mathfrak{X}})$ for all $i > 0$. In particular, $\mathbb{H}^1(\mathfrak{X}) \cong \mathbb{H}^2(\tilde{\mathfrak{X}})$. Thus it suffices to show that $Q$ is injective on $\mathbb{H}^2(\tilde{\mathfrak{X}})$. Setting $a(j) = 2 + \frac{\ell^j - 1}{\ell - 1}$, $Q_{j-1}\cdots Q_0$ maps $\mathbb{H}^2(\tilde{\mathfrak{X}})$ to $\mathbb{H}^{a(j)}(\tilde{\mathfrak{X}})$. In particular it suffices to show that $Q_j$ is injective on $\mathbb{H}^{a(j)}(\tilde{\mathfrak{X}})$ for all $0 \le j \le n - 1$. Because $X$ is a norm variety, we know from [14, 3.2] (or [6, 10.14]) and [6, 13.20] that the Margolis sequence is exact for each $Q_j$, $j < n$:

$$\mathbb{H}^{a(j)-\ell^j}(\tilde{\mathfrak{X}}) \xrightarrow{Q_j} \mathbb{H}^{a(j)}(\tilde{\mathfrak{X}}) \xrightarrow{Q_j} \mathbb{H}^{a(j)+\ell^j}(\tilde{\mathfrak{X}}).$$

By Lemma 5.4, the left term is zero because $a(j) - \ell^j \le 1$. The result follows. □

Since $X$ is a splitting variety, $\underline{a}$ vanishes in $K_n^M(k(X))/\ell$. By Remark 5.2, $\underline{a}$ vanishes in $H^0(X, \mathcal{H}^n(\mu_\ell^{\otimes n}))$. It follows from Proposition 5.5 (or [16, 6.5]) that there is a unique element $\delta$ in $H^{n,n-1}(\mathfrak{X}, \mathbb{Z}/\ell)$ whose image in $K_n^M(k)/\ell$ is $\underline{a}$.

In the following proposition, $\zeta$ is the map defined in Proposition 5.5, $\alpha$ is the direct sum of the maps $\alpha_i$ defined in Corollary 5.8, and the maps $r^*$, $s^*$ are given in Theorem 3.4.

**Proposition 5.11** *If $s_d(X) \not\equiv 0 \pmod{\ell^2}$, the following diagram commutes up to sign, and the top composite is multiplication by $\underline{a}$.*

$$\begin{array}{ccccc} \mathbb{H}^0(\mathfrak{X}) & \xrightarrow{\delta\cup} & \mathbb{H}^1(\mathfrak{X}) & \xhookrightarrow{\zeta} & K_*^M(k)/\ell \\ {\scriptstyle r^*}\downarrow & \overset{\alpha}{\nearrow} & \downarrow{\scriptstyle Q} & & \\ \mathbb{H}^{d+1}(D) & \xrightarrow{s^*} & \mathbb{H}^{b\ell+2}(\mathfrak{X}) & & \end{array}$$

*Proof* Note that all maps in the diagram are (right) module maps over the ring $K_*^M(k)/\ell \cong \mathbb{H}^0(\mathfrak{X})$. This is clear for multiplication by $\delta$, and we have already seen that the cohomology operation $Q$ is also a $\mathbb{H}^0(\mathfrak{X})$-module map. Finally, the maps $r^*$ and $s^*$ are also $\mathbb{H}^0(\mathfrak{X})$-module maps, since they come from morphisms in **DM**; see (3.4a) and (3.4b).

The top row sends $x \in \mathbb{H}^0(\mathfrak{X})$ to $\zeta(\delta \cup x) = \underline{a} \cup x$; since $\zeta$ is an injection (by Proposition 5.5), and $\underline{a} \cup x = \zeta \circ \alpha^* r^*(x)$ (by Corollary 5.8), the upper triangle commutes: $\delta \cup x = \alpha^* r^*(x)$.

We will show that $s^* r^*(1) = (-1)^{n-1} Q(\delta)$. By linearity for $\mathbb{H}^0(\mathfrak{X})$, it will follow that $s^* r^*(x) = (-1)^{n-1} Q(\delta \cup x)$ for all $x \in \mathbb{H}^0(\mathfrak{X})$. Since $r^*$ is surjective by Proposition 5.7, the result will follow.

We need to recall the definition of $\phi_V(\mu)$ from [16, p. 413] and [6, 5.10]. Given an element $\mu$ in $H^{2b+1,b}(\mathfrak{X}, \mathbb{Z}/\ell)$, form the triangle $A \longrightarrow \mathfrak{X} \xrightarrow{\mu} \mathfrak{X}(b)[2b+1]$ and set $S = S^{\ell-2}A$. Since $H^{2b\ell+2,b\ell}(\mathfrak{X}, \mathbb{Z}/\ell) \cong \mathrm{Hom}_{\mathbf{DM}}(R_{\mathrm{tr}}(\mathfrak{X}), R_{\mathrm{tr}}(\mathfrak{X})(b\ell)[b\ell+2])$, to define $\phi_V(\mu)$ it suffices to assign it a map $R_{tr}(\mathfrak{X}) \longrightarrow R_{tr}(\mathfrak{X})(b\ell)[b\ell+2]$. We define $\phi_V(\mu)$ to be represented by the composition

$$R_{\mathrm{tr}}(\mathfrak{X}) \xrightarrow{s} S(b)[2b+1] \xrightarrow{r\otimes 1} R_{\mathrm{tr}}(\mathfrak{X})(b\ell)[b\ell+2].$$

When $\mu = Q_{n-2} \cdots Q_0(\delta)$, we get the distinguished triangles (3.4a) and (3.4b) with $D = S$. Thus the composition $s^* \circ r^*$ in the above diagram is multiplication by the element $\phi_V(\mu)$. By [16, Thm. 3.8] (cf. [6, Cor. 6.33]), $\phi_V$ agrees with $Q_0 P^b$, where $P^b$ is the reduced power operation. In addition, since $\mu$ is annihilated by the $Q_i$ with $i \le n-2$ we have

$$s^* r^*(1) = Q_0 P^b(\mu) = (-1)^{n-1} Q_{n-1}(\mu) = (-1)^{n-1} Q(\delta);$$

see [16, p. 427] or [6, 5.14]. This shows that the bottom right triangle commutes in the above diagram. □

*Remark* In the proof of Proposition 5.11, we have cited Definition 5.10, Corollary 6.33 and Lemma 5.14 from the book [6]. These are slightly improved versions of Lemma 3.2 and (5.2), Theorem 3.8 and Lemma 5.13 in Voevodsky's paper [16]. Note that [16, 5.13] is missing several minus signs.

**Corollary 5.12** *In Proposition 5.11, $Q$ and $\alpha$ are isomorphisms, and the maps $r^*$ and $\delta \cup -$ are surjections.*

*Proof* From Proposition 5.7, we see that $r^*$ is surjective. By [6, 4.16], $s^*$ is an isomorphism (because $d+1 > d$), and $Q$ is an injection by Lemma 5.10. The results follows from a diagram chase. □

Note that $H^{q,q-1}(\mathfrak{X}) = 0$ for $q < n$, because by Corollary 5.12 this is a quotient of $H^{q-n,q-n}(\mathfrak{X})$. Recall that $\widetilde{K}^M_q(k(x))/\ell$ is the equalizer of the two maps

$$\iota_1, \iota_2 : K^M_q(k(X))/\ell \rightrightarrows K^M_q(k(X \times X))/\ell.$$

The following result was proved for $n = 1$ in Corollary 5.6, and will be proved for $n \geq 2$ in the next section.

**Proposition 5.13** $H^0(\mathfrak{X}, \mathcal{H}^q(\mu_\ell^{\otimes q})) \cong \tilde{K}^M_q(k(X))/\ell.$

We are now ready to prove Theorem 1.3 when $n \geq 2$.

*Proof of Theorem 1.3* Putting Proposition 5.5 for $q = n + i$ and Proposition 5.7 together, we get that the rows are exact in the following diagram, where $H^{p,q}(-)$ denotes $H^{p,q}(-, \mathbb{Z}/\ell)$.

$$\begin{array}{ccccccc}
 & & \overline{H}_{-i,-i}(X) & \longrightarrow & K^M_i(k) & \xrightarrow{r^*} & H^{i+2d+1,i+d}(D, \mathbb{Z}_{(\ell)}) \\
 & & & & \downarrow{\scriptstyle a\cup} & \searrow{\scriptstyle \delta\cup} & \downarrow{\scriptstyle \alpha} \\
H^{q+1,q-1}(\mathfrak{X}) & \longleftarrow & H^0(\mathfrak{X}, \mathcal{H}^q(\mu_\ell^{\otimes q})) & \longleftarrow & K^M_q(k)/\ell & \xleftarrow{\zeta} & H^{q,q-1}(\mathfrak{X})
\end{array}$$

From Corollary 5.12 we can conclude that the five-term sequence indicated by the dotted arrow is exact:

$$\overline{H}_{-i,-i}(\mathfrak{X}) \longrightarrow K^M_i(k) \xrightarrow{a\cup} K^M_q(k)/\ell \longrightarrow H^0(\mathfrak{X}, \mathcal{H}^q(\mu_\ell^{\otimes q})) \longrightarrow H^{q+1,q-1}(\mathfrak{X}). \tag{5.1a}$$

Theorem 1.3 now follows from Proposition 5.13. □

## 6 The fourth term

Let $\iota_1, \iota_2$ be the two inclusions $k(X) \hookrightarrow k(X \times X)$ induced by the projections $X \times X \longrightarrow X$. To finish the proof of Theorem 1.3, we need to prove Proposition 5.13 for $n \geq 2$.

**Lemma 6.1** *Fix $n \geq 2$. Let $E_i$ be the equalizer of the morphisms $p_i$ and $p'_i$ in the following diagram. Then in the commutative diagram*

$$\begin{array}{ccccc}
H^0(\mathfrak{X},\mathcal{H}^q) & \hookrightarrow & H^0(X,\mathcal{H}^q) & \rightrightarrows & H^0(X\times X,\mathcal{H}^q)\\
\downarrow & & \downarrow & & \downarrow\\
E_0 & \hookrightarrow & K_q^M(k(X))/\ell & \overset{p_0}{\underset{p_0'}{\rightrightarrows}} & K_q^M(k(X\times X))/\ell\\
\downarrow & & \downarrow & & \downarrow\\
E_1 & \hookrightarrow & \bigoplus_{x\in X^{(1)}} K_{q-1}^M(k(x))/\ell & \overset{p_1}{\underset{p_1'}{\rightrightarrows}} & \bigoplus_{y\in(X\times X)^{(1)}} K_{q-1}^M(k(y))/\ell
\end{array}$$

*the first row and all of the columns are exact.*

*Proof* Exactness of the first row (i.e., that $H^0(\mathfrak{X},\mathcal{H}^q)$ is the equalizer) is immediate from Lemma 5.1. The two right-hand columns are exact, as they are obtained from the Gersten resolutions for $\mathcal{H}^q$. The homomorphisms which are known to be injective are denoted $\hookrightarrow$. By an elementary diagram chase, the left-hand column is also exact. □

In order to prove Proposition 5.13 it thus suffices to show that $E_1 \cong 0$ in Lemma 6.1.

**Lemma 6.2** *If* $n \geq 2$, $E_1 = \ker p_1 = \ker p_1'$.

*Proof* Since $n > 1$, we have $\dim X = \ell^{n-1} - 1 \geq 1$. For any point $x \in X^{(1)}$ the summand indexed by $x$ is mapped by $p_1$ and $p_1'$ to the summands indexed by the generic points of $x \times X$ and $X \times x$, respectively. Since these points (and hence the summands) are distinct, the images of $p_1$ and $p_1'$ intersect in 0. It follows that their equalizer is $\ker(p_1) = \ker(p_1')$, as asserted. □

**Proposition 6.3** *If* $X$ *is a smooth variety of dimension* $\geq 1$, *then* $p_1$ *is injective.*

*Proof* For each $x \in X^{(1)}$, let $y_x$ denote a generic point of $x \times X$; since $X$ is smooth, $x \times X$ is reduced. We will show that the composition of $p_1$ with the projection $\pi_x$ onto $K_{q-1}^M(k(y_x))/\ell$,

$$\bigoplus_{x\in X^{(1)}} K_{q-1}^M(k(x))/\ell \xrightarrow{p_1} \bigoplus_{y\in(X\times X)^{(1)}} K_{q-1}^M(k(y))/\ell \xrightarrow{\pi_x} K_{q-1}^M(k(y_x))/\ell,$$

is an injection on the $x$-summand; since $\pi_x p_1$ is zero on all the other summands of the left term, it will follow that $p_1$ is an injection.

Fix $x$ and write $F$ for $k(X)$; as $X$ is smooth, the function field of $x \times X$ is a finite product of fields. Choosing an affine neighborhood Spec $R$ of $x$, $x$ is given by a height 1 prime ideal $\mathfrak{p}$ of $R$: $k(x) = \mathrm{frac}(R/\mathfrak{p})$ and $F = \mathrm{frac}(R)$. Note that $k(x)\otimes R$ is a regular ring because $X$ is smooth over $k$. The kernel $\mathfrak{m}$ of the multiplication map

$$k(x) \otimes R \longrightarrow k(x) \otimes k(x) \xrightarrow{\mu} k(x)$$

is a maximal ideal of $k(x)\otimes R$, and the localization $R' = (k(x)\otimes R)_{\mathfrak{m}}$ at $\mathfrak{m}$ is a regular local ring with residue field $k(x)$ and fraction field $k(y_x)$. Choose a regular sequence

$r_1, \ldots, r_d$ generating the maximal ideal of $R'$; by iterated use of [20, III.7.3], there is a *specialization map*

$$K_*^M(k(y_x)) \xrightarrow{\lambda} K_*^M(k(x))$$

which is a left inverse to the component $p_1^x : K_*^M(k(x)) \longrightarrow K_*^M(k(y_x))$ of $p_1$. □

Proposition 5.13 now follows for $n \geq 2$, since norm varieties are smooth by definition. This completes the proof of Theorem 1.3.

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