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Normality of algebras over commutative rings and the Teichmüller class. III.

Examples

Johannes Huebschmann[1]

Dedicated to Ronnie Brown on the occasion of his 80th birthday

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Abstract We describe various non-trivial examples that illustrate the approach to the "Teichmüller cocycle map" developed elsewhere in terms of crossed 2-fold extensions and generalizations thereof.

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✉ Johannes Huebschmann
Johannes.Huebschmann@math.univ-lille1.fr

[1] CNRS-UMR 8524, Labex CEMPI (ANR-11-LABX-0007-01), USTL, UFR de Mathématiques, 59655 Villeneuve d'Ascq Cedex, France

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20 Introduction

We describe various non-trivial examples that illustrate the approach to the "Teichmüller cocycle map" developed in [21,22] in terms of crossed 2-fold extensions. We recall the classical situation for number fields and show how it extends to rings of integers in number fields. We then construct explicit examples of a non-trivial Teichmüller class that arise in Grothendieck's theory of the Brauer group of a topological space. We finally interpret various group 3-cocycles constructed in C*-algebra theory as variants of the Teichmüller 3-cocycle. We keep the section numbering from [21,22].

21 Number fields

21.1 General remarks

Consider an algebraic number field K (a finite-dimensional extension of the field $\mathbb{Q}$ of rational numbers). Let Q be a finite group of operators on K, let $\mathfrak{k} = K^Q$, and consider the resulting Galois extension $K|\mathfrak{k}$. Let J_K denote the abelian group of *idèles* of $K|\mathfrak{k}$ and C_K that of *idèle classes*, and consider the familiar Q-module extension

$$0 \longrightarrow \mathrm{U}(K) \longrightarrow J_K \longrightarrow C_K \longrightarrow 0 \tag{21.1}$$

[30, (III.2) p. 117]. By the "main theorem of class field theory",

$$\mathrm{H}^2(Q, C_K) \cong \tfrac{1}{[K:\mathfrak{k}]}\mathbb{Z}/\mathbb{Z}$$

[1, §VII.3 Lemma 6 p. 49], [38, RESULT p. 196], [30, (III.6.8) Theorem p. 150], the group $\mathrm{H}^2(Q, C_K)$ has a canonical generator, referred to as the *fundamental class* of the extension $K|\mathfrak{k}$ and written as $u_{K|\mathfrak{k}} \in \mathrm{H}^2(Q, C_K)$. As a side remark we note that, given a group extension $C_K \rightarrowtail W_{K|\mathfrak{k}} \twoheadrightarrow Q$ that represents the class $u_{K|\mathfrak{k}} \in \mathrm{H}^2(Q, C_K)$, the group $W_{K|\mathfrak{k}}$ is referred to as the Weil group of the field extension $K|\mathfrak{k}$ [1, Ch. XV], [38, §11.6 p. 200], [37]. The Weil group is uniquely determined since $\mathrm{H}^1(Q, C_K)$ is zero [1, Ch. XV].

Let m denote the l.c.m. of the local degrees. We summarize the results of [27, Theorem 2, Theorem 3], [29], and others as follows, cf. [1, §VII.4 Theorem 12 and Theorem 14 p. 53], [38, §11.4 Case $r = 3$ p. 199].

Proposition 21.1 (i) *The boundary homomorphism*

$$\delta : \mathrm{H}^2(Q, C_K) \longrightarrow \mathrm{H}^3(Q, \mathrm{U}(K))$$

in the long exact cohomology sequence associated with (21.1) *is surjective, and the group* $\mathrm{H}^3(Q, \mathrm{U}(K))$ *is cyclic of order* $s = \frac{[K:\mathfrak{k}]}{m}$*, generated by the image*

$$t_{K|\mathfrak{k}} = \delta(u_{K|\mathfrak{k}}) \in \mathrm{H}^3(Q, \mathrm{U}(K)).$$

(ii) *The class* $t_{K|\mathfrak{k}}$ *splits in some extension field* L *of* K *that is normal over* $\mathfrak{k}$*, indeed, things may be arranged in such a way that* $L|K$ *is cyclic.*

Under the present circumstances, the eight term exact sequence [22, (17.2)] boils down to the classical five term exact sequence, cf. [22, (17.3)], given, e.g., in [18, p. 130], combined with the canonical isomorphisms

$$\mathrm{H}^2(Q, \mathrm{U}(K)) \cong \mathrm{B}(K|\mathfrak{k}), \quad \mathrm{H}^2(G, \mathrm{U}(L)) \cong \mathrm{B}(L|\mathfrak{k}), \quad \mathrm{H}^2(N, \mathrm{U}(L))^Q \cong \mathrm{B}(L|K)^Q,$$

and the exactness of this sequence entails that $t_{K|\mathfrak{k}}$ is the Teichmüller class associated to some Q-normal crossed product central simple K-algebra having L as maximal commutative subalgebra. In the literature, the generator

$$t_{K|\mathfrak{k}} = \delta(u_{K|\mathfrak{k}}) \in \mathrm{H}^3(Q, \mathrm{U}(K))$$

is referred to as the *Teichmüller* 3*-class* [1, §VII.4 p. 52], [38, §11.4 Case $r = 3$ p. 199], here interpreted as the *obstruction* to the global degree being computed as the l.c.m. of the local degrees.

21.2 Explicit examples

Thus to get examples, all we need is a Galois extension $K|\mathfrak{k}$ having $s > 1$. While, in view of the Hilbert–Speiser Theorem, this is impossible when the Galois group Q is cyclic, for example, the fields $K = \mathbb{Q}(\sqrt{13}, \sqrt{17})$ or $K = \mathbb{Q}(\sqrt{2}, \sqrt{17})$ have as Galois group Q the four group and $s = 2$ [27], see also [38, §11.4 p. 199] and [28, Ch. VIII Exampe 4.5 p. 238].

Since it is hard to find truly explicit examples in the literature, we now briefly sketch a construction of such examples. According to classical results due to Albert, Brauer, Hasse, and E. Noether, every member of $\mathrm{B}(K)$ has a cyclic cyclotomic splitting field [10, Satz 4, Satz 5 p. 118], [28, VIII.2 Theorem 2.6 p. 229], [38, 10.5 Step 3. p. 191]. Indeed, the argument in the last reference shows that, given a central simple K-algebra A, there is a cyclic cyclotomic field $L|K$ such that $[L_{\mathcal{P}} : K_p] \equiv 0(m_p)$ for every prime p of K and such that $[L : K] = \text{l.c.m.}(m_p)$. Thus, consider a cyclic cyclotomic extension $L = K(\zeta)$ having Galois group N cyclic of order n (say). Let σ denote a generator of N, let $\eta \in \mathrm{U}(K)$, and consider the cyclic central simple K-algebra $D(\sigma, \eta)$ generated by $L = K(\zeta)$ and some (indeterminate) u subject to the relations

$$u\lambda = \sigma(\lambda)u,\ u^n = \eta,\ \lambda \in L = K(\zeta), \tag{21.2}$$

necessarily a crossed product of N with L relative to the $\mathrm{U}(L)$-valued 2-cocycle of N determined by η. By construction, $D(\sigma, \eta)$ is split by L. Moreover, given $\vartheta \in \mathrm{U}(L)$, the member $\eta_\vartheta = \eta \prod_{j=0}^{n-1} \vartheta^{\sigma^j}$ of K yields the algebra $D(\sigma, \eta_\vartheta)$, and the association $u \longmapsto \vartheta u$ induces an isomorphism

$$D(\sigma, \eta) \longrightarrow D(\sigma, \eta_\vartheta) \tag{21.3}$$

of central K-algebras. The field $L|\mathfrak{k}$ is the composite field $\mathfrak{k}(\zeta)K$. Hence $L|\mathfrak{k}$ is a Galois extension, and the Galois group G of $L|\mathfrak{k}$ is a central extension of $Q = \mathrm{Gal}(K|\mathfrak{k})$ by the cyclic group $N = \mathrm{Gal}(\mathfrak{k}(\zeta)|\mathfrak{k})$ of order n, a split extension if and only if $\mathfrak{k}(\zeta) \cap K = \mathfrak{k}$. The member η of K represents the corresponding cohomology class $[\eta] \in \mathrm{H}^2(N, \mathrm{U}(L))$, and $[\eta] \in \mathrm{H}^2(N, \mathrm{U}(L))^Q$ if and only if, given $x \in Q = \mathrm{Gal}(K|\mathfrak{k})$, there is some $\vartheta_x \in \mathrm{U}(L)$ such that the association $u \longmapsto \vartheta_x u$ induces an automorphism

$$\Theta_x : D(\sigma, \eta) \longrightarrow D(\sigma, \eta) \tag{21.4}$$

of central K-algebras that extends the automorphism $x : K \to K$ over $\mathfrak{k}$.

The sequence

$$0 \longrightarrow \mathrm{H}^2(N, \mathrm{U}(L)) \longrightarrow \mathrm{H}^2(N, J_L) \stackrel{\mathrm{inv}_1}{\longrightarrow} \tfrac{1}{|N|}\mathbb{Z}/\mathbb{Z} \longrightarrow 0 \tag{21.5}$$

is well known to be exact, cf., e.g., [30, III.5.6 Proposition p. 143], and taking Q-invariants, we obtain the injection

$$0 \longrightarrow \mathrm{H}^2(N, \mathrm{U}(L))^Q \longrightarrow \mathrm{H}^2(N, J_L)^Q. \tag{21.6}$$

Given a prime p of K, for each prime $\mathcal{P}$ of L above p, the local extension $L_{\mathcal{P}}|K_p$ is likewise a cyclic cyclotomic extension. From a given system of local invariants in $\mathrm{H}^2(N, J_L)^Q$ that goes to zero under $\mathrm{inv}_1 : \mathrm{H}^2(N, J_L) \to \frac{1}{|N|}\mathbb{Z}/\mathbb{Z}$, at each prime $\mathcal{P}$ of L above p that occurs in that system of local invariants, we can construct an explicit cyclic central K_p-algebra $D(\sigma_{\mathcal{P}}, \eta_{\mathcal{P}})$ defined in terms of a prime element of K_p and, using, e.g., the recipe in the proof of [10, Satz 9 p. 119 ff.], we can then construct a member η of K such that the cyclic algebra $D(\sigma, \eta)$ has the given local invariants. By construction, then, the class of $D(\sigma, \eta)$ in $\mathrm{H}^2(N, \mathrm{U}(L))$ is Q-invariant. Hence $D(\sigma, \eta)$ acquires a Q-normal structure, necessarily non-trivial when its Teichmüller class is non-zero, and the above reasoning classifies those cyclic Q-normal algebras that are non-trivially Q-normal.

22 Rings of integers and beyond

Let R be a regular domain, and let K denote its quotient field. By [3, Theorem 7.2 p. 388], the induced homomorphism $\mathrm{B}(R) \to \mathrm{B}(K)$ between the Brauer groups is a monomorphism. It is known that, furthermore, the canonical map $\mathrm{B}(R) \to \cap_p \mathrm{B}(R_p)$ from the Brauer group $\mathrm{B}(R)$ to the intersection $\cap_p \mathrm{B}(R_p)$ taken over all height one primes p is an isomorphism, cf., e.g., [35, Theorem 9.7 p. 64].

Let K be an algebraic number field, S its ring of integers, and let r denote the number of embeddings of K into the reals. The Brauer group $\mathrm{B}(S)$ of S is zero when $r = 1$ and isomorphic to a direct product of $r - 1$ copies of the cyclic group with two elements when $r \geq 2$. This is a consequence of a result in [2], see, e.g., [4, (6.49) p. 151]. While a central S-Azumaya algebra representing a non-trivial member of $\mathrm{B}(S)$ need not be representable as an ordinary crossed product with respect to a Galois extension of S, see, e.g., [6] and the literature there, a right H-Galois extension $T|S$ of rings of integers with respect to a general finite-dimensional Hopf algebra H which splits all classes in the Brauer group $\mathrm{B}(S)$ can easily be found [6, Proposition 2.1 p. 246]. The question as to, whether or not, given a finite group Q of operators on K and hence on S, along these lines, Q-normal S-Azumaya algebras arise is a largely unexplored territory. The example [6, Remark 2.6 p. 249] yields a Q-equivariant Q-normal Azumaya algebra for Q the cyclic group with two elements.

Consider now an algebraic number field K, a finite group Q of operators on K, let $\mathfrak{k} = K^Q$, and let S be the ring of integers in K and R that in $\mathfrak{k}$. Consider a field extension $L|K$ such that $K|\mathfrak{k}$ is normal, with Galois group G, let $N = \mathrm{Gal}(L|K)$, so that the Galois groups fit into an extension $N \rightarrowtail G \twoheadrightarrow Q$, and let T denote the ring of integers in L. Let $\mathbb{S}_{L|K}$ denote the finite set of primes of K that ramify in L and let $\mathbb{S}_L$ denote the finite set of primes of L above the primes in $\mathbb{S}_{L|K}$. Inverting the primes in $\mathbb{S}_L$ and those in $\mathbb{S}_{L|K}$ we obtain a Galois extension $T_{\mathbb{S}_L}|S_{\mathbb{S}_{L|K}}$ of commutative rings with Galois group N. Let, furthermore, $\mathbb{S}_{K|\mathfrak{k}}$ denote those primes of $\mathfrak{k}$ such that the primes in $\mathbb{S}_{L|K}$ are exactly the primes above $\mathbb{S}_{K|\mathfrak{k}}$, and let $R_{\mathbb{S}_{K|\mathfrak{k}}}$ denote the corresponding ring that arises from R by inverting the primes in $\mathbb{S}_{K|\mathfrak{k}}$. Then the data constitute a Q-normal Galois extension of commutative rings but, while $R_{\mathbb{S}_{K|\mathfrak{k}}} = S^Q_{\mathbb{S}_{L|K}}$, the ring extension $S_{\mathbb{S}_{L|K}}|R_{\mathbb{S}_{K|\mathfrak{k}}}$ need not be a Galois extension of commutative rings. Recall the exact sequence [22, (18.1)], for $T_{\mathbb{S}_L}|S_{\mathbb{S}_{L|K}}|R_{\mathbb{S}_{K|\mathfrak{k}}}$ as well as for $L|K|\mathfrak{k}$. The inclusions into the quotient fields yield a commutative diagram

$$
\begin{array}{ccccc}
\mathrm{H}^2(G, \mathrm{U}(T_{\mathbb{S}_L})) & \xrightarrow{j} & \mathrm{Xpext}(G, N; \mathrm{U}(T_{\mathbb{S}_L})) & \xrightarrow{\Delta} & \mathrm{H}^3(Q, \mathrm{U}(S_{\mathbb{S}_{L|K}})) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{H}^2(G, \mathrm{U}(L)) & \xrightarrow{j} & \mathrm{H}^2(N, \mathrm{U}(L))^Q & \xrightarrow{\Delta} & \mathrm{H}^3(Q, \mathrm{U}(K)).
\end{array}
\tag{22.1}
$$

Suitably interpreting the constructions in Section 21 above, we can then construct crossed pair extensions that represent members of $\mathrm{Xpext}(G, N; \mathrm{U}(T_{\mathbb{S}_L}))$ whose images in $\mathrm{H}^2(N, \mathrm{U}(L))^Q$ have non-zero values in $\mathrm{H}^3(Q, \mathrm{U}(K))$. Hence the associated crossed pair algebras then have non-zero Teichmüller class in $\mathrm{H}^3(Q, \mathrm{U}(S_{\mathbb{S}_{L|K}}))$. This yields non-trivial examples of Teichmüller classes of normal Azumaya algebras over rings of algebraic numbers with finitely many primes inverted. We intend to give the details at another occasion. The Galois module structure of groups like $\mathrm{U}(S)$ and $\mathrm{Pic}(S)$ is delicate, cf., e.g., [13], and the calculation of the relevant group cohomology groups is not an easy matter. More work is called for in this area.

23 Examples arising in algebraic topology

23.1 General remarks

Let X be a topological space, and let S denote the algebra of continuous complex-valued functions on X. Isomorphism classes of Azumaya S-algebras of rank $n > 1$ correspond bijectively to isomorphism classes of principal $\mathrm{PGL}(n, \mathbb{C})$-bundles.

When X is a finite CW-complex, by a Theorem of Serre [14, Theorem 1.6], the Brauer group $\mathrm{B}(S)$ is canonically isomorphic to the torsion part $\mathrm{H}^3(X)_{\mathrm{tors}}$ of the third integral cohomology group $\mathrm{H}^3(X)$ of X. The isomorphism is realized explicitly as follows: Let $\mathcal{M}ap(X, \mathbb{C})$ denote the sheaf of germs of continuous $\mathbb{C}$-valued functions on X and $\mathcal{M}ap(X, \mathbb{C}^*)$ that of continuous $\mathbb{C}^*$-valued functions on X. The exponential exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{M}ap(X, \mathbb{C}) \longrightarrow \mathcal{M}ap(X, \mathbb{C}^*) \longrightarrow 0$$

of sheaves on X yields an isomorphism $\mathrm{H}^2(X, \mathcal{M}ap(X, \mathbb{C}^*)) \cong \mathrm{H}^3(X)$ of sheaf cohomology groups (valid more generally for paracompact X). The theorem of Serre's just quoted says that, X being a finite CW-complex, the canonical map from the Brauer group $\mathrm{B}(S)$ to $\mathrm{H}^2(X, \mathcal{M}ap(X, \mathbb{C}^*))$ is an isomorphism

$$\mathrm{B}(S) \longrightarrow \mathrm{H}^2(X, \mathcal{M}ap(X, \mathbb{C}^*))_{\mathrm{tors}}.$$

Let $\xi : P \to X$ be a principal $\mathrm{PGL}(n, \mathbb{C})$-bundle and, relative to the adjoint action of $\mathrm{PGL}(n, \mathbb{C})$ on $\mathrm{M}_n(\mathbb{C})$, let ζ denote the associated vector bundle

$$\zeta : E = P \times_{\mathrm{PGL}(n,\mathbb{C})} \mathrm{M}_n(\mathbb{C}) \longrightarrow X$$

on X. The S-module of continuous sections $A = \Gamma(\zeta)$ of ζ acquires the structure of an Azumaya S-algebra in an obvious manner in such a way that the group $\mathrm{U}(A)$ of units of A gets naturally identified with the space of sections of the associated fiber bundle

$$\mathrm{u}_\xi : P \times_{\mathrm{PGL}(n,\mathbb{C})} \mathrm{GL}_n(\mathbb{C}) \longrightarrow X$$

relative to the adjoint action of $\mathrm{PGL}(n, \mathbb{C})$ on $\mathrm{GL}_n(\mathbb{C})$, endowed with the pointwise group structure. Thus the group $\mathrm{U}(A)$ of units of A can be written as the group

$$\overline{\mathscr{G}}_\xi \cong \mathrm{Map}_{\mathrm{PGL}(n,\mathbb{C})}(P, \mathrm{GL}(n, \mathbb{C}))$$

of $\mathrm{PGL}(n, \mathbb{C})$-equivariant maps from P to $\mathrm{GL}(n, \mathbb{C})$, and the group $\overline{\mathscr{G}}_\xi$, in turn, maps canonically onto the group $\mathscr{G}_\xi \cong \mathrm{Map}_{\mathrm{PGL}(n,\mathbb{C})}(P, \mathrm{PGL}(n, \mathbb{C}))$ of gauge transformations of ξ. The group $\mathrm{Aut}(\xi)$ of bundle automorphisms of ξ, i. e., pairs (Φ, φ) of homeomorphisms that make the diagram

$$\begin{array}{ccc} P & \xrightarrow{\Phi} & P \\ {\scriptstyle \xi}\downarrow & & {\scriptstyle \xi}\downarrow \\ X & \xrightarrow{\varphi} & X \end{array} \tag{23.1}$$

commutative, yields, in a canonical way, a subgroup of the group $\mathrm{Aut}(A)$ of ring automorphisms of A, and the assignment to a section of u_ξ of the induced gauge transformation of the kind (23.1) with $\varphi = \mathrm{Id}$ yields a homomorphism

$$\partial : \overline{\mathscr{G}}_\xi \longrightarrow \mathrm{Aut}(\xi).$$

Denote by $\mathrm{Z}_n(\mathbb{C}) \cong \mathbb{C}^*$ the central diagonal subgroup of $\mathrm{GL}_n(\mathbb{C})$. Identifying the kernel of ∂ with the space of sections of the associated bundle

$$P \times_{\mathrm{PGL}(n,\mathbb{C})} \mathrm{Z}_n(\mathbb{C}) \longrightarrow X,$$

necessarily trivial, since $\mathrm{Z}_n(\mathbb{C})$ is the center of $\mathrm{GL}_n(\mathbb{C})$, we see that the kernel of ∂ is canonically isomorphic to the abelian group $\mathrm{U}(S)$ of continuous functions from X to $\mathbb{C}^*$. Denote the group of homeomorphisms of X by $\mathrm{Homeo}(X)$, and let

$$\mathrm{Out}(\xi) \subseteq \mathrm{Homeo}(X)$$

denote the image of $\mathrm{Aut}(\xi)$ in $\mathrm{Homeo}(X)$ under the forgetful map which assigns to a member (Φ, φ) of $\mathrm{Aut}(\xi)$ the second component φ. The group $\mathrm{Out}(\xi)$ is the group of homeomorphisms φ of X such that the induced principal bundle $\varphi^*\xi$ is isomorphic to ξ. Thus the principal bundle ξ determines the crossed 2-fold extension

$$0 \longrightarrow \mathrm{Map}(X, \mathbb{C}^*) \longrightarrow \overline{\mathscr{G}}_\xi \xrightarrow{\partial} \mathrm{Aut}(\xi) \longrightarrow \mathrm{Out}(\xi) \longrightarrow 1. \tag{23.2}$$

Consider a group Q, and suppose that Q acts on X via a homomorphism σ from Q to $\mathrm{Out}(\xi)$. Requiring that the action be via a homomorphism $\sigma : Q \to \mathrm{Out}(\xi)$ is equivalent to requiring that some group Γ that maps onto Q act on the total space P of ξ in such a way that, given $q \in Q$, there exists some $\gamma \in \Gamma$ such that

$$\begin{array}{ccc} P & \xrightarrow{\gamma} & P \\ {\scriptstyle \xi}\downarrow & & {\scriptstyle \xi}\downarrow \\ X & \xrightarrow{q} & X \end{array}$$

is an automorphism of principal $\mathrm{PGL}(n, \mathbb{C})$-bundles. Requiring that Γ act by bundle automorphisms is equivalent to requiring that the Γ-action on P commute with the $\mathrm{PGL}(n, \mathbb{C})$-action. The homomorphism σ then induces the requisite Q-action

$\kappa_Q : Q \to \mathrm{Aut}(S) = \mathrm{Aut}(\mathrm{Map}(X, \mathbb{C}))$ on $S = \mathrm{Map}(X, \mathbb{C})$, and $(A, \sigma) = (\Gamma(\zeta), \sigma)$ is a Q-normal Azumaya S-algebra.

23.2 Explicit examples involving metacyclic groups

Consider a metacyclic group G given by a presentation

$$G(r, s, t, f) = \langle x, y;\ y^r = 1,\ x^s = y^f,\ xyx^{-1} = y^t \rangle \tag{23.3}$$

where

$$s > 1, \quad r > 1, \quad t^s \equiv 1 \mod r, \quad tf \equiv f \mod r,$$

so that, in particular, the numbers $\frac{t^s - 1}{r}$ and $\frac{(t-1)f}{r}$ are positive integers. The group G is an extension

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1 \tag{23.4}$$

of the cyclic group $Q = C_s$ of order s by the cyclic group $N = C_r$ of order r generated by y. The upshot of the present subsection is an explicit Q-normal crossed pair algebra having a ring of the kind $S = \mathrm{Map}(X, \mathbb{C})$ for some topological space X as its center and having non-zero Teichmüller class in $\mathrm{H}^3(Q, \mathrm{U}(S))$, to be given as (23.25) below.

Suppose that the g.c.d. $(\frac{t^s-1}{r}, r)$ is non-trivial, let $\ell > 1$ denote a non-trivial divisor of $(\frac{t^s-1}{r}, r)$, let $C_{\ell r}$ denote the cyclic group of order ℓr, let v denote a generator of $C_{\ell r}$, and let C_ℓ denote the cyclic subgroup of $C_{\ell r}$ of order ℓ generated by v^r. The assignment to v of y yields a group extension

$$\mathrm{e}_{\ell r} : 0 \longrightarrow C_\ell \longrightarrow C_{\ell r} \longrightarrow C_r \longrightarrow 1 \tag{23.5}$$

representing the generator of $\mathrm{H}^2(C_r, \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell$.

Since $\frac{t^s-1}{\ell r}$ is an integer, the association $Q \times C_{\ell r} \longrightarrow C_{\ell r}$ given by $(x, v) \longmapsto v^t$ yields an action of the group $Q = C_s$ on $C_{\ell r} = \langle v; v^{\ell r} = 1 \rangle$. The induced action

$$G \times C_{\ell r} \longrightarrow C_{\ell r} \tag{23.6}$$

of G on $C_{\ell r}$ via the projection $G \to C_s$ and the obvious homomorphism $\partial : C_{\ell r} \to G$ then constitute a crossed module.

Proposition 23.1 *With respect to a suitable choice of the isomorphism*

$$\mathrm{H}^3(C_s, \mathbb{Z}/\ell) \cong \mathbb{Z}/(\ell, s),$$

the resulting associated crossed 2-fold extension

$$\mathrm{e}^2 : 0 \longrightarrow C_\ell \longrightarrow C_{\ell r} \longrightarrow G \longrightarrow C_s \longrightarrow 1 \tag{23.7}$$

represents the class in $\mathrm{H}^3(C_s, \mathbb{Z}/\ell)$ *that corresponds to* $\frac{(t-1)f}{r} \mod (\ell, s)$.

Proof Let F_x denote the free group on x, let $\mathbb{Z}C_s\langle b\rangle$ denote the free C_s-module on a single generator b, view $\mathbb{Z}C_s\langle b\rangle$ as an F_x-group via the canonical projection from F_x to C_s, define the morphism $\partial : \mathbb{Z}C_s\langle b\rangle \to F_x$ of F_x-groups by $\partial(b) = x^s$, and note that $\partial : \mathbb{Z}C_s\langle b\rangle \to F_x$ is the free crossed module associated to the presentation $\langle x; x^s\rangle$ of the group C_s. Using the familiar notation $\mathrm{I}C_s \subseteq \mathbb{Z}C_s$ for the augmentation ideal of C_s, consider the associated crossed 2-fold extension

$$0 \longrightarrow \mathrm{I}C_s\langle b\rangle \longrightarrow \mathbb{Z}C_s\langle b\rangle \xrightarrow{\partial} F_x \longrightarrow C_s \longrightarrow 1, \tag{23.8}$$

and lift the identity of C_s to a morphism

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathrm{I}C_s\langle b\rangle & \longrightarrow & \mathbb{Z}C_s\langle b\rangle & \xrightarrow{\partial} & F_x & \longrightarrow & C_s & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} & & \| & & \\ 0 & \longrightarrow & C_\ell & \longrightarrow & C_{\ell r} & \xrightarrow{\partial} & G & \longrightarrow & C_s & \longrightarrow & 1 \end{array} \tag{23.9}$$

of crossed 2-fold extensions as follows: With an abuse of the notation x, let $\alpha_0(x) = x$, and let $\alpha_1(b) = v^f$. Then

$$\alpha_1((x-1)b) = {}^x(v^f)v^{-f} = v^{(t-1)f} = (v^r)^{\frac{(t-1)f}{r}} \in C_\ell \subseteq C_{\ell r}.$$

Consequently $\alpha_2((x-1)b) = (v^r)^{\frac{(t-1)f}{r}} \in C_\ell \subseteq C_{\ell r}$, whence α_2 represents the member of $\mathrm{H}^3(C_s, \mathbb{Z}/\ell) \cong \mathbb{Z}/(\ell, s)$ that corresponds to $\frac{(t-1)f}{r} \mod (\ell, s)$. □

Remark 23.2 It is immediate that, for a suitable choice of the parameters, $\frac{(t-1)f}{r}$ is non-trivial modulo (ℓ, s). For example, as in the situation of [20, Theorem E], suppose that p is a prime that divides r, s, $\frac{t^s-1}{r}$ and f, but that it does not divide $\frac{(t-1)f}{r}$. Then, with $\ell = p$, the 2-cocycle α_2 and hence the crossed 2-fold extension e^2 represent a generator of $\mathrm{H}^3(C_s, \mathbb{Z}/p) \cong \mathbb{Z}/p$. Indeed, for a suitable choice of the data, in the notation of [20, Theorem E], this class is that written there as $\omega_x c_x$.

Since $\mathrm{H}^2(C_s, \mathbb{C}^*)$ is trivial, the homomorphism $\mathrm{H}^3(C_s, \mathbb{Z}/\ell) \to \mathrm{H}^3(C_s, \mathbb{C}^*)$ induced by the canonical injection $C_\ell \to \mathbb{C}^*$ is injective whence, when α_2 represents a non-trivial cohomology class, the composite of α_2 with the canonical injection $C_\ell \to \mathbb{C}^*$ yields a non-trivial cohomology class in $\mathrm{H}^3(C_s, \mathbb{C}^*) \cong \mathbb{Z}/s$. To construct a crossed 2-fold extension representing that cohomology class, let $\widehat{C}_{\ell r}$ denote the universal group characterized by the requirement that the diagram

$$\begin{array}{ccccccccc} \mathrm{e}_{\ell r} : 0 & \longrightarrow & C_\ell & \longrightarrow & C_{\ell r} & \longrightarrow & C_r & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ \mathrm{e}^* : 0 & \longrightarrow & \mathbb{C}^* & \longrightarrow & \widehat{C}_{\ell r} & \longrightarrow & C_r & \longrightarrow & 1 \end{array} \tag{23.10}$$

be commutative with exact rows. The G-action (23.6) on $C_{\ell r}$ and the trivial G-action on $\mathbb{C}^*$ combine to a G-action

$$G \times \widehat{C}_{\ell r} \longrightarrow \widehat{C}_{\ell r} \tag{23.11}$$

on $\widehat{C}_{\ell r}$ that turns the obvious map $\widehat{\partial} : \widehat{C}_{\ell r} \to G$ into a crossed module. The cohomology class under discussion is represented by the resulting crossed 2-fold extension

$$\mathrm{e}_2^* : 0 \longrightarrow \mathbb{C}^* \longrightarrow \widehat{C}_{\ell r} \xrightarrow{\widehat{\partial}} G \longrightarrow C_s \longrightarrow 1. \tag{23.12}$$

Since the group $\mathbb{C}^*$ is a divisible abelian group, the bottom row extension e^* in (23.10) splits in the category of abelian groups. However, when the class represented by α_2 is non-trivial, such a splitting cannot be compatible with the G-module structures.

Under the present circumstances, since the action of N on $\mathbb{C}^*$ is trivial, the bottom diagram of what corresponds to [22, (13.3)], with $M = \mathbb{C}^*$, takes the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}(N, \mathbb{C}^*) & \longrightarrow & \mathrm{Aut}_G(\mathrm{e}^*) & \longrightarrow & G & \longrightarrow & 1 \\ & & \cong\big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & \mathrm{H}^1(N, \mathbb{C}^*) & \longrightarrow & \mathrm{Out}_G(\mathrm{e}^*) & \longrightarrow & Q & \longrightarrow & 1 \end{array} \tag{23.13}$$

with exact rows and, cf. [22, Proposition 13.2], the G-action (23.11) (turning $\widehat{C}_{\ell r}$ together with the canonical homomorphism $\widehat{\partial} : \widehat{C}_{\ell r} \to G$ into a G-crossed module) determines and is determined by a crossed pair structure $\psi : Q \to \mathrm{Out}_G(\mathrm{e}^*)$ on the group extension e^*. The resulting crossed pair

$$(\mathrm{e}^* : \mathbb{C}^* \rightarrowtail \widehat{C}_{\ell r} \twoheadrightarrow C_r, \psi : Q \to \mathrm{Out}_G(\mathrm{e}^*))$$

represents a non-trivial class

$$[(\mathrm{e}^*, \psi)] \in \mathrm{Xpext}(G, N; \mathbb{C}^*) \tag{23.14}$$

in the group $\mathrm{Xpext}(G, N; \mathbb{C}^*)$ of crossed pair extensions with respect to the group extension (23.4) and the (trivial) G-module $\mathbb{C}^*$, cf. [19, Theorem 1] and [22, Subsection 13.1] for these notions and, cf. [19, Theorem 2] or [22, Subsection 13.1], the homomorphism

$$\Delta : \mathrm{Xpext}(G, N; \mathbb{C}^*) \longrightarrow \mathrm{H}^3(Q, \mathbb{C}^*)$$

sends the class (23.14) to $[\mathrm{e}_2^*] \in \mathrm{H}^3(Q, \mathbb{C}^*) \cong \mathbb{Z}/s$.

Let $\pi : \widetilde{X} \to X$ be a regular covering projection having the group $N = C_r$ as deck transformation group, and let $S = \mathrm{Map}(X, \mathbb{C})$ and $T = \mathrm{Map}(\widetilde{X}, \mathbb{C})$. Then $T|S$ is a Galois extension of commutative rings with Galois group N, cf. [21, Example 2.4]. Suppose that $\widetilde{X}$ is endowed with a G-action that extends the N-action. Then the quotient group $Q = G/N$ acts on X in an obvious manner, and $T|S$ is a Q-normal Galois extension of commutative rings, with structure extension (23.4) and structure homomorphism $\kappa_G : G \to \mathrm{Aut}^S(T)$. By construction,

$$\mathrm{U}(S) = \mathrm{Map}(X, \mathbb{C}^*), \quad \mathrm{U}(T) = \mathrm{Map}(\widetilde{X}, \mathbb{C}^*).$$

Since the groups N, G, and Q are finite, the homomorphisms

$$\mathrm{H}^*(N,\mathbb{C}^*) \longrightarrow \mathrm{H}^*(N,\mathrm{U}(T)), \tag{23.15}$$
$$\mathrm{H}^*(Q,\mathbb{C}^*) \longrightarrow \mathrm{H}^*(Q,\mathrm{U}(S)), \tag{23.16}$$
$$\mathrm{Xpext}(G,N;\mathbb{C}^*) \longrightarrow \mathrm{Xpext}(G,N;\mathrm{U}(T)), \tag{23.17}$$

induced by the canonical injections $\mathbb{C}^* \to \mathrm{Map}(\widetilde{X},\mathbb{C}^*)$ and $\mathbb{C}^* \to \mathrm{Map}(X,\mathbb{C}^*)$ (induced by the assignments to a member of $\mathbb{C}^*$ of the associated constant maps), respectively, are isomorphisms, the third homomorphism being an isomorphism in view of the naturality of the exact sequence [19, (1.9)] (spelled out as the top sequence in the diagram in [22, Theorem 18.8]).

For later reference, we now give an explicit description of a representative of the image of (23.14) under (23.17). To this end, let C_T denote the universal group characterized by the requirement that the diagram

$$\begin{array}{ccccccccc}
\mathrm{e}^*: 0 & \longrightarrow & \mathbb{C}^* & \longrightarrow & \widehat{C}_{\ell r} & \longrightarrow & C_r & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
\mathrm{e}_T: 0 & \longrightarrow & \mathrm{U}(T) & \longrightarrow & C_T & \longrightarrow & C_r & \longrightarrow & 1
\end{array} \tag{23.18}$$

be commutative with exact rows. The G-action (23.11) on $\widehat{C}_{\ell r}$ and the G-action on $\mathrm{U}(T)$ combine to a G-action

$$G \times C_T \longrightarrow C_T \tag{23.19}$$

on C_T. With $M = \mathrm{U}(T)$, diagram [22, (13.3)] takes the form

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathrm{U}(S) & = & \mathrm{U}(S) & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{U}(T) & \longrightarrow & C_T & \longrightarrow & N & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Der}(N,\mathrm{U}(T)) & \longrightarrow & \mathrm{Aut}_G(\mathrm{e}_T) & \longrightarrow & G & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{H}^1(N,\mathrm{U}(T)) & \longrightarrow & \mathrm{Out}_G(\mathrm{e}_T) & \longrightarrow & Q & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & & ,
\end{array} \tag{23.20}$$

with exact rows and columns, the G-action (23.19) on C_T induces a section

$$\Psi_T : G \longrightarrow \mathrm{Aut}_G(\mathrm{e}_T)$$

for the third row group extension in (23.20), and this section, in turn, induces a section $\psi_T : Q \to \mathrm{Out}_G(\mathrm{e}_T)$ for the bottom row extension in (23.20) in such a way that (e_T, ψ_T) is a crossed pair. By construction, then, the image

$$\Delta[(\mathrm{e}_T, \psi_T)] \in \mathrm{H}^3(Q, \mathrm{U}(S))$$

of the class

$$[(\mathrm{e}_T, \psi_T)] \in \mathrm{Xpext}(G, N; \mathrm{U}(T)) \tag{23.21}$$

is represented by the crossed 2-fold extension that arises as the top row of the commutative diagram

$$\begin{array}{ccccccccccc} \mathrm{e}^2_T : 0 & \longrightarrow & \mathrm{U}(S) & \longrightarrow & C_T & \xrightarrow{\partial_T} & B^{\psi_T} & \longrightarrow & Q & \longrightarrow & 1 \\ & & \| & & \| & & \downarrow & & \downarrow{\scriptstyle \psi_T} & & \\ 0 & \longrightarrow & \mathrm{U}(S) & \longrightarrow & C_T & \longrightarrow & \mathrm{Aut}_G(\mathrm{e}_T) & \longrightarrow & \mathrm{Out}_Q(\mathrm{e}_T) & \longrightarrow & 1, \end{array}$$

the group B^{ψ_T} being characterized by the requirement that the right-hand square be a pull back square.

The naturality of the constructions entails that the diagram

$$\begin{array}{ccc} \mathrm{Xpext}(G, N; \mathbb{C}^*) & \xrightarrow{\Delta} & \mathrm{H}^3(Q, \mathbb{C}^*) \\ \cong\downarrow & & \cong\downarrow \\ \mathrm{Xpext}(G, N; \mathrm{U}(T)) & \xrightarrow{\Delta} & \mathrm{H}^3(Q, \mathrm{U}(S)) \end{array} \tag{23.22}$$

is commutative. In the case at hand the commutativity of (23.22) is an immediate consequence of the observation that the above homomorphism $\Psi_T : G \to \mathrm{Aut}_G(\mathrm{e}_T)$ induces a homomorphism $G \to B^{\psi_T}$ which makes the diagram

$$\begin{array}{ccccccccccc} \mathrm{e}^*_2 : 0 & \longrightarrow & \mathbb{C}^* & \longrightarrow & \widehat{C}_{\ell r} & \xrightarrow{\widehat{\partial}} & G & \longrightarrow & Q & \longrightarrow & 1 \\ & & \| & & \| & & \downarrow & & \downarrow{\scriptstyle \psi_T} & & \\ \mathrm{e}^2_T : 0 & \longrightarrow & \mathrm{U}(S) & \longrightarrow & C_T & \xrightarrow{\partial_T} & B^{\psi_T} & \longrightarrow & Q & \longrightarrow & 1 \end{array} \tag{23.23}$$

commutative. This commutativity, in turn, implies that (i) the class (23.21) yields a non-trivial class in the group $\mathrm{Xpext}(G, N; \mathrm{U}(T))$ of crossed pair extensions with respect to (23.4) and $\mathrm{U}(T)$ and that (ii) this class goes under Δ to the image in $\mathrm{H}^3(Q, \mathrm{U}(S))$ of the class $[\mathrm{e}_2] \in \mathrm{H}^3(Q, \mathbb{C}^*) \cong \mathbb{Z}/s$, non-trivial for suitable choices of the parameters, cf. Remark 23.2 above.

Recall the homomorphism [22, (18.2)], of the kind

$$\mathrm{cpa} : \mathrm{Xpext}(G, N; \mathrm{U}(T)) \longrightarrow \mathrm{XB}(T|S; G, Q).$$

This homomorphism fits into the commutative diagram

$$\begin{array}{ccc} \mathrm{H}^1(Q, \mathrm{H}^1(N, \mathrm{U}(T))) & \xrightarrow{\mathrm{d}_2} & \mathrm{H}^3(Q, \mathrm{U}(S)) \\ {\scriptstyle\alpha}\downarrow & & \| \\ \mathrm{Xpext}(G, N; \mathrm{U}(T)) & \xrightarrow{\Delta} & \mathrm{H}^3(Q, \mathrm{U}(S)) \\ {\scriptstyle\mathrm{cpa}}\downarrow & & \| \\ \mathrm{XB}(T|S; G, Q) & \xrightarrow{t} & \mathrm{H}^3(Q, \mathrm{U}(S)). \end{array} \tag{23.24}$$

Here the upper square is part of the diagram in [19, Subsection 1.4], and the lower square results from [22, Theorem 18.1]. Consequently the Q-normal crossed pair algebra

$$(A_{\mathrm{e}_T}, \sigma_{\psi_T}) \tag{23.25}$$

with respect to the Q-normal Galois extension $T|S$ of commutative rings that arises from the crossed pair (e_T, ψ_T) with respect to (23.4) and $\mathrm{U}(T)$ via the construction in [22, Subsection 13.2] has non-zero Teichmüller class in $\mathrm{H}^3(Q, \mathrm{U}(S))$.

To realize this kind of example concretely, consider a faithful unitary representation E of complex dimension n of the metacyclic group G [9, §47 p. 335]. Things can be arranged in such a way that the unitary G-representation yields an action of G on the unit sphere $S^{2n-1} \subseteq \mathbb{C}^n$ so that the restriction of the action to $N = \mathbb{C}_r$ is free but, apart from trivial cases, the G-action itself will not be free. Thus we may take $\widetilde{X} = S^{2n-1}$ and $X = S^{2n-1}/C_r$ (a lens space) and carry out the above construction.

Remark 23.3 The above observation that the bottom row in (23.10) splits in the category of abelian groups translates, in view of the exactness of the sequence [19, (1.9)] (spelled out as the top sequence in the diagram in [22, Theorem 18.8]) to the fact that the above homomorphism α is an isomorphism.

Remark 23.4 Apart from trivial cases, while C_T acquires a G-action, this action does not turn the obvious map $C_T \to G$ into a crossed module since the action of N on the kernel $\mathrm{U}(T)$ of $C_T \to G$ is non-trivial when N is non-trivial. Thus we cannot get away with the crossed pair concept, more general than that of a crossed module.

24 Examples arising from C*-dynamical systems

The group 3-cocycle in [36, II 3.1 p. 147] with values in the group of units of the center of a von Neumann algebra is an instance of a Teichmüller cocycle in the von Neumann algebra context. The aim of this 3-cocycle was indeed to explore a crossed product construction formally of the same kind as the crossed product in [21, Section 5]. In

[25,26], V. Jones pushed these ideas further and showed that, for an arbitrary discrete group, such a 3-cohomology class can be realized on a *hyperfinite* factor. To our knowledge, the relationship with the Teichmüller cocycle was not observed in the literature, however.

Given a topological space X, the results of [11,12] are nowadays well known to establish an isomorphism

$$\delta : \mathrm{B}(X) \longrightarrow \check{\mathrm{H}}^3(X, \mathbb{Z}) \tag{24.1}$$

between the Brauer group $\mathrm{B}(X)$ of Morita equivalence classes $[A]$ in the sense of Rieffel of continuous-trace C*-algebras A having spectrum X and the third Čech-cohomology group $\check{\mathrm{H}}^3(X, \mathbb{Z})$, see, e.g., [7,34]. The continuous-trace C*-algebras A having spectrum X can be characterized as the C*-algebras which are locally Morita equivalent to the commutative algebra $C_0(X)$ of continuous complex-valued functions on X that vanish at infinity, and the Dixmier–Douady class is the obstruction to building a global equivalence with $C_0(X)$ from the local equivalences.

Let now Q denote a group and suppose that Q acts on X and hence on $C_0(X)$. Given a continuous-trace C*-algebras A having spectrum X, just as before, we define a Q-normal structure on A to be a homomorphism $\kappa : Q \to \mathrm{Out}(A) = \mathrm{Aut}(A)/\mathrm{Inn}(A)$. Then, with a suitable definition of the group $\mathrm{U}(A)$ of units of A, the Teichmüller class in $\mathrm{H}^3(Q, \mathrm{U}(A))$ is defined, just as before in the ordinary algebraic case. Since the algebraic theory developed in [21,22] involves only the objects themselves but does not involve any cocycles, it is now a laborious but most likely rather straightforward endeavor to extend that theory to the C*-algebra case.

In [7], the theory of C*-algebra Brauer groups was extended so that group actions can be accomodated, and a corresponding equivariant Brauer group was defined. In [7, Lemma 4.6], even a version of a Teichmüller cocycle shows up (but the authors did not recognize that the cocycle they constructed is a kind of Teichmüller cocycle).

In this area there are presumably many examples of a non-trivial Teichmüller class to be found and new phenomena are lurking behind. See also [8,31–33,39].

25 Complements

Other explicit examples of a non-trivial Teichmüller cocycle can be found in [5] and [23].

Remark 25.1 In [16], the Teichmüller cocycle serves as a crucial means for building a Galois theory of skew fields. It is worthwhile noting that, in "non-commutative Galois theory", a counterexample in [40, p. 141] serves as well as a counterexample in [15, p. 558], [17, p. 298], and [24, §VI.11 p. 147].

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