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Normality of algebras over commutative rings and the Teichmüller class. II.

Crossed pairs and the relative case

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Dedicated to Ronnie Brown on the occasion of his 80th birthday

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Abstract Using a suitable notion of normal Galois extension of commutative rings, we develop the relative theory of the generalized Teichmüller cocycle map. We interpret the theory in terms of the Deuring embedding problem, construct an eight term exact sequence involving the relative Teichmüller cocycle map and suitable relative versions of generalized Brauer groups and compare the theory with the group cohomology eight term exact sequence involving crossed pairs. We also develop somewhat more sophisticated versions of the ordinary, equivariant and crossed relative Brauer groups and show that the resulting exact sequences behave better with regard to comparison of the theory with group cohomology than do the naive notions of the generalized relative Brauer groups.

Keywords Teichmüller cocycle · Crossed module · Crossed pair · Normal algebra · Crossed product · Deuring embedding problem · Group cohomology · Galois theory of commutative rings · Azumaya algebra · Brauer group · Galois cohomology · Non-commutative Galois theory · Non-abelian cohomology

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11 Introduction

We explore further the approach to the "Teichmüller cocycle map" developed in [14] in terms of crossed 2-fold extensions. For intelligibility, we recall briefly the situation: Let S be a unitary commutative ring, Q a group that acts on S by ring automorphisms via a homomorphism $\kappa_Q\colon Q \to \mathrm{Aut}(S)$, and let R denote the subring of S that consists of the elements of S which are fixed under Q. A *Q-normal* S-algebra consists of a central S-algebra A and a homomorphism $\sigma\colon Q \to \mathrm{Out}(A)$ into the group $\mathrm{Out}(A)$ of outer automorphisms of A that lifts the action of Q on S. With respect to the abelian group $\mathrm{U}(S)$ of invertible elements of S, endowed with the Q-module structure coming from the Q-action on S, the *Teichmüller complex* of (A, σ) associated to a Q-normal S-algebra (A, σ) is a crossed 2-fold extension $\mathrm{e}_{(A,\sigma)}$ starting at $\mathrm{U}(S)$ and ending at Q, and this crossed 2-fold extension represents a class, the *Teichmüller class* of (A, σ), in the third group cohomology group $\mathrm{H}^3(Q, \mathrm{U}(S))$ of Q with coefficients in $\mathrm{U}(S)$.

We now review rapidly the contents of the sections of the present paper. A more detailed introduction for the entire series that consists of [14], the present paper, and [15] can be found in the introduction to [14].

In Section 12 we introduce the concept of a *Q-normal Galois extension of commutative rings*; associated to such a Q-normal Galois extension $T|S$ of commutative rings is a *structure extension* $\mathrm{e}_{(T|S)}\colon N \rightarrowtail G \twoheadrightarrow Q$ of Q by the Galois group $N = \mathrm{Aut}(T|S)$ of $T|S$ and an action $G \to \mathrm{Aut}(T)$ of G on T by ring automorphisms. In Section 13 we associate to a *crossed pair* (e, ψ) with respect to $\mathrm{e}_{(T|S)}$ and $\mathrm{U}(T)$, endowed with the G-module structure coming from the G-action on T, see [13] or Section 13 below for details on the crossed pair concept, a Q-normal crossed product algebra $(A_{\mathrm{e}}, \sigma_\psi)$

which we refer to as a *crossed pair algebra*. The crossed pair algebra $(A_{\mathrm{e}}, \sigma_\psi)$ represents a member of the kernel $\mathrm{XB}(T|S; G, Q)$ of the obvious homomorphism from $\mathrm{XB}(S, Q)$ to $\mathrm{XB}(T, G)$; this homomorphism exists and is unique, in view of the functoriality of the crossed Brauer group. The assignment to (e, ψ) of $(A_{\mathrm{e}}, \sigma_\psi)$ yields a natural homomorphism of abelian groups from the corresponding abelian group $\mathrm{Xpext}(G, N; \mathrm{U}(T))$ of congruence classes of crossed pairs introduced in [13] to the subgroup $\mathrm{XB}(T|S; G, Q)$ of the crossed Brauer group.

Theorem 13.5 below says that *a class* $k \in \mathrm{H}^3(Q, \mathrm{U}(S))$ *is the Teichmüller class of some crossed pair algebra* $(A_{\mathrm{e}}, \sigma_\psi)$ *with respect to the data if and only if* k *is split in* $T|S$ in the sense that, under inflation $\mathrm{H}^3(Q, \mathrm{U}(S)) \to \mathrm{H}^3(G, \mathrm{U}(T))$, the class k goes to zero. In Section 14, given a Q-normal Galois extension $T|S$ of commutative rings, we again focus our attention on the Deuring embedding problem of a central T-algebra into a central S-algebra and establish two somewhat technical results, Theorems 14.9 and 14.10 below; these results entail, in particular that, if a class $k \in \mathrm{H}^3(Q, \mathrm{U}(S))$ goes under inflation to the Teichmüller class in $\mathrm{H}^3(G, \mathrm{U}(T))$ of some G-normal central T-algebra A, then k is itself the Teichmüller class of some Q-normal central S-algebra B in such a way that, when A is an Azumaya T-algebra, B may be taken to be an Azumaya S-algebra. Sections 15 and 16 are preparatory in character.

Given a Q-normal Galois extension $T|S$ of commutative rings with associated structure extension $\mathrm{e}_{(T|S)} \colon \mathrm{Aut}(T|S) \rightarrowtail G \twoheadrightarrow Q$ and G-action on T, we use the notation $\mathrm{EB}(T|S; G, Q)$ for the kernel of the induced homomorphism from $\mathrm{EB}(S, Q)$ to $\mathrm{XB}(T, G)$; the exact sequence (17.2) below involving the Teichmüller map t now yields an extension of the kind

$$\cdots \longrightarrow \mathrm{H}^2(Q, \mathrm{U}(S)) \longrightarrow \mathrm{EB}(T|S; G, Q) \longrightarrow \mathrm{XB}(T|S; G, Q)$$
$$\xrightarrow{\ t\ } \mathrm{H}^3(Q, \mathrm{U}(S)) \xrightarrow{\ \mathrm{inf}\ } \mathrm{H}^3(G, \mathrm{U}(T))$$

of the corresponding classical low degree four term exact sequence by four more terms. We refer to the resulting theory as the *naive relative theory*. In Theorem 18.1 we compare that exact sequence with the eight term exact sequence in the cohomology of the group extension $\mathrm{e}_{(T|S)}$ with coefficients in $\mathrm{U}(T)$ constructed in [13].

Finally, we develop a more sophisticated variant of the relative theory which behaves better with regard to comparison of the theory with group cohomology than does the naive relative theory; see Theorems 18.4–18.6 and 18.8.

The appendix recollects some material from the theory of stably graded symmetric monoidal categories. We keep the section numbering from [14].

12 Normal ring extensions

As in [14], S denotes a commutative ring and $\kappa_Q \colon Q \to \mathrm{Aut}(S)$ an action of a group Q on S. Let $T|S$ be a Galois extension of commutative rings with Galois group $N = \mathrm{Aut}(T|S)$. We refer to $T|S$ as being *Q-normal* when each automorphism $\kappa_Q(q)$ of S, as q ranges over Q, extends to an automorphism of T.

Somewhat more formally, given a Galois extension $T|S$ of commutative rings with Galois group N, denote by $\mathrm{Aut}^S(T)$ the group of those automorphisms of T that map S to itself, let $\mathrm{res} \colon \mathrm{Aut}^S(T) \to \mathrm{Aut}(S)$ denote the obvious restriction map, so that $N = \mathrm{Aut}(T|S)$ is the kernel of res, let G denote the fiber product group $G = \mathrm{Aut}^S(T) \times_{\mathrm{Aut}(S)} Q$ relative to $\kappa_Q\colon Q \to \mathrm{Aut}(S)$, and let $\pi_Q\colon G \to Q$ denote the canonical homomorphism and $i^N\colon N \to G$ the obvious injection. The obvious homomorphism $\kappa_G\colon G \to \mathrm{Aut}^S(T)$ makes the diagram

$$\begin{array}{ccccccc} 1 \longrightarrow & N & \xrightarrow{i^N} & G & \xrightarrow{\pi_Q} & Q \\ & \Big\| & & \kappa_G\Big\downarrow & & \kappa_Q\Big\downarrow \\ 1 \longrightarrow & \mathrm{Aut}(T|S) & \longrightarrow & \mathrm{Aut}^S(T) & \xrightarrow{\mathrm{res}} & \mathrm{Aut}(S) \end{array} \tag{12.1}$$

commutative, where the unlabeled arrow is the obvious homomorphism. This diagram is a special case of a diagram of the kind [14, (3.19)]. The Galois extension $T|S$ of commutative rings is plainly Q-normal if and only if the homomorphism $\pi_Q\colon G \to Q$ is surjective, that is, if and only if the sequence

$$\mathrm{e}_{(T|S)}\colon 1 \longrightarrow N \xrightarrow{i^N} G \xrightarrow{\pi_Q} Q \longrightarrow 1 \tag{12.2}$$

is exact, i.e., an extension of Q by N. Given a Q-normal Galois extension $T|S$ of commutative rings, we refer to the corresponding group extension (12.2) as the *associated structure extension* and to the corresponding homomorphism

$$\kappa_G\colon G \longrightarrow \mathrm{Aut}^S(T)$$

as the *associated structure homomorphism*. It is immediate that a Q-normal Galois extension $T|S$ with structure extension (12.2) and structure homomorphism

$$\kappa_G\colon G \longrightarrow \mathrm{Aut}^S(T),$$

the injection $S \subseteq T$ being denoted by $i\colon S \subseteq T$, yields the morphism

$$(i, \pi_Q)\colon (S, Q, \kappa_Q) \longrightarrow (T, G, \kappa_G) \tag{12.3}$$

in the change of actions category $\mathcal{C}hange$ introduced in [14, Subsection 3.7].

Example 12.1 Let $K|P$ be a Galois extension of algebraic number fields, and denote by G the Galois group of $K|P$. Let Z be a subfield of K that contains P and is a normal extension of P, and let $N = \mathrm{Gal}(K|Z)$ and $Q = \mathrm{Gal}(Z|P)$. Let T, S and R denote the rings of integers in, respectively, K, Z and P. Suppose that $K|Z$ is unramified but that $Z|P$ is ramified. Then $T|S$ is a Q-normal Galois extension of commutative rings but $T|R$ and $S|R$ are not Galois extensions of commutative rings, cf. [14, Example 2.3].

Let (S, Q, κ) and $(\hat{S}, \hat{Q}, \hat{\kappa})$ be objects of the change of actions category $\mathcal{C}hange$ introduced in [14, Subsection 3.7], and let $T|S$ and $\hat{T}|\hat{S}$ be normal Galois extension of commutative rings with respect to Q and $\hat{Q}$, with structure extensions

$$\mathrm{e}_{(T|S)}\colon N \rightarrowtail G \twoheadrightarrow Q, \quad \mathrm{e}_{(\hat{T}|\hat{S})}\colon \hat{N} \rightarrowtail \hat{G} \twoheadrightarrow \hat{Q}$$

and structure homomorphisms $\kappa_G \colon G \to \mathrm{Aut}^S(T)$ and $\hat{\kappa_G} \colon \hat{G} \to \mathrm{Aut}^{\hat{S}}(\hat{T})$, respectively. Then a *morphism*

$$(h, \phi) \colon T|S \longrightarrow \hat{T}|\hat{S}$$

of normal Galois extensions consists of a ring homomorphism $h \colon T \to \hat{T}$ and a group homomorphism $\phi \colon \hat{G} \to G$ such that

(i) $f = h|S$ is a ring homomorphism $S \to \hat{S}$,
(ii) the values of $\phi|\hat{N}$ lie in N, that is, $\phi|\hat{N}$ is a homomorphism $\hat{N} \to N$, and
(iii) $h({}^{\phi(\hat{x})}t) = {}^{\hat{x}}(h(t)),\ \hat{x} \in \hat{G},\ t \in T$.

13 Crossed pair algebras

As before, S denotes a commutative ring and $\kappa_Q \colon Q \to \mathrm{Aut}(S)$ an action of a group Q on S. In this section we use the results of [13] to offer a partial answer to the question as to which classes in $\mathrm{H}^3(Q, \mathrm{U}(S))$ are Teichmüller classes. Our result extends the classical answer of Eilenberg–Mac Lane [5] (reproduced in [11]); later in the paper we shall give a complete answer.

13.1 Crossed pairs

For intelligibility, we recall that notion from [13, p. 152].

Let

$$1 \longrightarrow N \xrightarrow{i^N} G \longrightarrow Q \longrightarrow 1 \tag{13.1}$$

be a group extension and M a G-module; we write the G-action $G \times M \longrightarrow M$ on M as $(x, y) \mapsto {}^x y$, for $x \in G$ and $y \in M$. Further, let $\mathrm{e} \colon M \rightarrowtail \Gamma \overset{\pi_N}{\twoheadrightarrow} N$ be a group extension whose class $[\mathrm{e}] \in \mathrm{H}^2(N, M)$ is fixed under the standard Q-action on $\mathrm{H}^2(N, M)$. Given $x \in G$, we write

$$\ell_x(y) = {}^x y,\ y \in M, \quad i_x(n) = xnx^{-1},\ n \in N.$$

Write $\mathrm{Aut}_G(\mathrm{e})$ for the subgroup of $\mathrm{Aut}(\Gamma) \times G$ that consists of those pairs (α, x) which make the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & \Gamma & \longrightarrow & N & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \ell_x} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle i_x} & & \\ 0 & \longrightarrow & M & \longrightarrow & \Gamma & \longrightarrow & N & \longrightarrow & 1 \end{array}$$

commutative.

The homomorphism

$$\beta\colon \Gamma \longrightarrow \mathrm{Aut}_G(\mathrm{e}),\ \beta(y) = (i_y, i^N(\pi_N(y))),\ y \in \Gamma,$$

together with the obvious action of $\mathrm{Aut}_G(\mathrm{e})$ on Γ, yields a crossed module

$$(\Gamma, \mathrm{Aut}_G(\mathrm{e}), \beta)$$

whence, in particular, $\beta(\Gamma)$ is a normal subgroup of $\mathrm{Aut}_G(\mathrm{e})$; we denote by $\mathrm{Out}_G(\mathrm{e})$ the cokernel of β and write the resulting crossed 2-fold extension as

$$\hat{\mathrm{e}}\colon 0 \longrightarrow M^N \longrightarrow \Gamma \xrightarrow{\beta} \mathrm{Aut}_G(\mathrm{e}) \longrightarrow \mathrm{Out}_G(\mathrm{e}) \longrightarrow 1. \tag{13.2}$$

The map $\mathrm{Der}(N, M) \longrightarrow \mathrm{Aut}_G(\mathrm{e})$ given by the association

$$\mathrm{Der}(N, M) \ni d \longmapsto (\alpha_d, 1),\ \alpha_d(y) = (d\pi_N(y))y, \quad y \in \Gamma,$$

is an injective homomorphism; this homomorphism and the obvious map

$$\mathrm{Aut}_G(\mathrm{e}) \longrightarrow G$$

yield the group extension

$$0 \longrightarrow \mathrm{Der}(N, M) \longrightarrow \mathrm{Aut}_G(\mathrm{e}) \longrightarrow G \longrightarrow 1,$$

the map $\mathrm{Aut}_G(\mathrm{e}) \to G$ being surjective, since the class $[\mathrm{e}] \in \mathrm{H}^2(N, M)$ is supposed to be fixed under Q. Further, let $\zeta : M \to \mathrm{Der}(N, M)$ be the homomorphism defined by $(\zeta(m))(n) = m({}^n m)^{-1}$, as m ranges over M and n over N. With these preparations out of the way, the data fit into the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & M^N & = & M^N & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & \Gamma & \xrightarrow{\pi_N} & N & \longrightarrow & 1 \\
 & & \zeta\downarrow & & \beta\downarrow & & i^N\downarrow & & \\
0 & \longrightarrow & \mathrm{Der}(N, M) & \longrightarrow & \mathrm{Aut}_G(\mathrm{e}) & \longrightarrow & G & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{H}^1(N, M) & \longrightarrow & \mathrm{Out}_G(\mathrm{e}) & \longrightarrow & Q & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array} \qquad (13.3)$$

with exact rows and columns. We use the notation

$$\overline{\mathrm{e}}\colon 0 \longrightarrow \mathrm{H}^1(N, M) \longrightarrow \mathrm{Out}_G(\mathrm{e}) \longrightarrow Q \longrightarrow 1$$

for the bottom row extension of (13.3). This extension is the cokernel, in the category of group extensions with abelian kernel, of the morphism (ζ, β, i) of group extensions.

Suppose now that the extension $\overline{\mathrm{e}}$ splits; we then say that e *admits a crossed pair structure*, and we refer to a section $\psi\colon Q \to \mathrm{Out}_G(\mathrm{e})$ of $\overline{\mathrm{e}}$ as a *crossed pair structure on the group extension* $\mathrm{e}\colon M \rightarrowtail \Gamma \stackrel{\pi_N}{\twoheadrightarrow} N$ *with respect to the group extension* (13.1). By definition, a *crossed pair* (e, ψ) *with respect to the group extension* (13.1) *and the G-module M* consists of a group extension $\mathrm{e}\colon M \rightarrowtail \Gamma \twoheadrightarrow N$ whose class $[\mathrm{e}] \in \mathrm{H}^2(N, M)$ is fixed under Q such that the associated extension $\overline{\mathrm{e}}$ splits, together with a section $\psi\colon Q \to \mathrm{Out}_G(\mathrm{e})$ of $\overline{\mathrm{e}}$ [13, p. 152].

Suitable classes of crossed pairs with respect to (13.1) and the G-module M constitute an abelian group $\mathrm{Xpext}(G, N; M)$ [13, Theorem 1]. Moreover, cf. [13, Theorem 2], suitably defined homomorphisms

$$j\colon \mathrm{H}^2(G, M) \longrightarrow \mathrm{Xpext}(G, N; M),\ \ \Delta\colon \mathrm{Xpext}(G, N; M) \longrightarrow \mathrm{H}^3(Q, M^N)$$

yield an extension of the classical five term exact sequence to an eight term exact sequence of the kind

$$\begin{aligned}
0 \longrightarrow & \mathrm{H}^1(Q, M^N) \stackrel{\mathrm{inf}}{\longrightarrow} \mathrm{H}^1(G, M) \stackrel{\mathrm{res}}{\longrightarrow} \mathrm{H}^1(N, M)^Q \stackrel{\Delta}{\longrightarrow} \mathrm{H}^2(Q, M^N) \\
\stackrel{\mathrm{inf}}{\longrightarrow} & \mathrm{H}^2(G, M) \stackrel{j}{\longrightarrow} \mathrm{Xpext}(G, N; M) \stackrel{\Delta}{\longrightarrow} \mathrm{H}^3(Q, M^N) \stackrel{\mathrm{inf}}{\longrightarrow} \mathrm{H}^3(G, M).
\end{aligned} \qquad (13.4)$$

For later reference, we recall the construction of Δ. To this end, given a crossed pair

$$(\mathrm{e}\colon 0 \to M \to \Gamma \to N \to 1,\ \psi\colon Q \to \mathrm{Out}_G(\mathrm{e}))$$

with respect to the group extension (13.1) and the G-module M, let B^ψ denote the fiber product group $\mathrm{Aut}_G(\mathrm{e}) \times_{\mathrm{Out}_G(\mathrm{e})} Q$ with respect to the crossed pair structure map $\psi\colon Q \to \mathrm{Out}_G(\mathrm{e})$ and, furthermore, let $\partial^\psi\colon \Gamma \to B^\psi$ denote the obvious homomorphism; together with the obvious action of B^ψ on Γ induced by the canonical homomorphism $B^\psi \to \mathrm{Aut}_G(\mathrm{e})$, the exact sequence

$$\mathrm{e}_\psi\colon 0 \longrightarrow M^N \longrightarrow \Gamma \xrightarrow{\partial^\psi} B^\psi \longrightarrow Q \longrightarrow 1 \tag{13.5}$$

is a crossed 2-fold extension and hence represents a class in $\mathrm{H}^3(Q, M^N)$. We refer to e_ψ *as the crossed 2-fold extension associated to the crossed pair* (e, ψ). The homomorphism $\Delta\colon \mathrm{Xpext}(G, N; M) \to \mathrm{H}^3(Q, M^N)$ is given by the assignment to a crossed pair (e, ψ) of its associated crossed 2-fold extension e_ψ.

Remark 13.1 By [12, Theorem 1], the association $\mathrm{e} \mapsto \bar{\mathrm{e}}$ yields a conceptual description of the differential $d_2\colon \mathrm{E}_2^{0,2} \to \mathrm{E}_2^{2,1}$ of the Lyndon-Hochschild-Serre spectral sequence $(\mathrm{E}_r^{p,q}, d_r)$ associated with the group extension (13.1) and the G-module M.

Proposition 13.2 *In the special case where the N-action on M is trivial, given a group extension* $\mathrm{e}\colon M \rightarrowtail \Gamma \overset{\pi_N}{\twoheadrightarrow} N$ *that admits a crossed pair structure, crossed pair structures* $\psi\colon Q \to Out_G(\mathrm{e})$ *on the group extension* e *correspond bijectively to actions of G on Γ that turn $i^N \circ \pi_N\colon \Gamma \to G$ into a crossed module in such a way that the canonical homomorphism $G \to B^\psi = Aut_G(\mathrm{e}) \times_{Out_G(\mathrm{e})} Q$ is an isomorphism.* □

Remark 13.3 Given the group extension (13.1), consider a group extension

$$\mathrm{e}\colon 1 \longrightarrow X \longrightarrow K \xrightarrow{\pi_N} N \longrightarrow 1,$$

the group X not necessarily being abelian, let $\phi = i^N \circ \pi_N\colon K \to G$ denote the composite of i and π_N, and let $\mathrm{Aut}(\mathrm{e})$ denote the subgroup of $\mathrm{Aut}(K)$ that consists of the automorphisms of K that map X to itself; such a homomorphism ϕ is referred to in [17] as a *normal homomorphism*. Conjugation in K yields a homomorphism $\beta\colon K \to \mathrm{Aut}(\mathrm{e})$ from K onto a normal subgroup $\beta(K)$ of $\mathrm{Aut}(\mathrm{e})$, and the restriction ζ of β to X, that is, conjugation in K with elements of X, yields a homomorphism $\zeta\colon X \to \mathrm{Aut}(\mathrm{e})$ from X onto a normal subgroup $\zeta(X)$ of $\mathrm{Aut}(\mathrm{e})$ as well; let $\mathrm{can}\colon \mathrm{Aut}(\mathrm{e}) \to \mathrm{Aut}(\mathrm{e})/\zeta(X)$ denote the canonical surjection. A *modular structure on* ϕ is a homomorphism $\theta\colon G \to \mathrm{Aut}(\mathrm{e})/\zeta(X)$ making the diagram

$$
\begin{array}{ccc}
K & \xrightarrow{\ \pi_N\ } & N \\
{\scriptstyle\beta}\downarrow & \overset{\phi}{\searrow} & \downarrow{\scriptstyle i^N} \\
\mathrm{Aut}(\mathrm{e}) & & G \\
{\scriptstyle \mathrm{can}}\downarrow & \overset{\theta}{\swarrow} & \\
\mathrm{Aut}(\mathrm{e})/\zeta(X) & &
\end{array}
$$

commutative [17]. A *pseudo-module* is defined to be a pair (ϕ, θ) that consists of a normal homomorphism ϕ and a modular structure θ on ϕ [17].

Let (ϕ, θ) be a pseudo-module and consider the two abstract kernels $G \to \mathrm{Out}(X)$ and $Q \to \mathrm{Out}(K)$ induced by that pseudo-module. Now, fix an abstract G-kernel structure $\omega\colon G \to \mathrm{Out}(X)$ on X in advance and consider the group $\mathrm{Aut}_G(\mathrm{e})$ that consists of the pairs $(\alpha, x) \in \mathrm{Aut}(\mathrm{e}) \times G$ which make the diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & X & \longrightarrow & K & \longrightarrow & N & \longrightarrow & 1 \\
 & & {\scriptstyle \alpha|X}\downarrow & & {\scriptstyle \alpha}\downarrow & & {\scriptstyle i_x}\downarrow & & \\
1 & \longrightarrow & X & \longrightarrow & K & \longrightarrow & N & \longrightarrow & 1
\end{array}
$$

commutative in such a way that the image of $\alpha|X$ in $\mathrm{Out}(X)$ coincides with the value $\omega(x) \in \mathrm{Out}(X)$. Then the modular structures on ϕ that induce, in particular, the abstract G-kernel structure ω on X are given by homomorphisms

$$\theta\colon G \longrightarrow \mathrm{Aut}_G(\mathrm{e})/\zeta(X).$$

In the special case where X is abelian, an abstract G-kernel structure on X is an ordinary G-module structure, and those modular structures $\theta\colon G \to \mathrm{Aut}_G(\mathrm{e})/\zeta(X)$ correspond bijectively to crossed pair structures $\psi\colon Q \to \mathrm{Out}_G(\mathrm{e})$ on e.

13.2 Crossed pairs and normal algebras

Let $T|S$ be a Q-normal Galois extension of commutative rings, with structure extension

$$\mathrm{e}_{(T|S)}\colon 1 \longrightarrow N \xrightarrow{\ i^N\ } G \longrightarrow Q \longrightarrow 1$$

and structure homomorphism $\kappa_G\colon G \to \mathrm{Aut}^S(T)$; in particular, the group N is finite. Let $(\mathrm{e}\colon \mathrm{U}(T) \rightarrowtail \Gamma \twoheadrightarrow N,\ \psi\colon Q \to \mathrm{Out}_G(\mathrm{e}))$ be a crossed pair with respect to the group extension $\mathrm{e}_{(T|S)}$ and the G-module $\mathrm{U}(T)$. The corresponding crossed 2-fold extension (13.2) now takes the form

$$\hat{\mathrm{e}}\colon 0 \longrightarrow \mathrm{U}(S) \longrightarrow \Gamma \longrightarrow \mathrm{Aut}_G(\mathrm{e}) \longrightarrow \mathrm{Out}_G(\mathrm{e}) \longrightarrow 1.$$

To the crossed pair (e, ψ), we associate a Q-normal S-algebra $(A_{\mathrm{e}}, \sigma_\psi)$ as follows.

The composite $\vartheta\colon \Gamma \to N \to \mathrm{Aut}(T)$ yields an action of Γ on T; let A_{e} denote the crossed product algebra $(T, N, \mathrm{e}, \vartheta)$. Since the group N is finite, A_{e} is an Azumaya S-algebra; this fact also follows from [14, Proposition 5.4(xi)]. Recall that there is an obvious injection $i\colon \Gamma \to \mathrm{U}(A_{\mathrm{e}})$. The following is immediate.

Proposition 13.4 *Setting*

$$^{i_\sharp(\alpha,x)}(ty) = (^x t)(^\alpha y), \tag{13.6}$$

as t ranges over T, y over Γ, and (α, x) over $Aut_G(\mathrm{e})$ ($\subseteq Aut(\Gamma) \times G$), we obtain a morphism

$$(i, i_\sharp)\colon (\Gamma, Aut_G(\mathrm{e}), \beta) \longrightarrow (\mathrm{U}(A_{\mathrm{e}}), Aut(A_{\mathrm{e}}, Q), \partial)$$

of crossed modules which, in turn, induces the morphism

$$\begin{array}{ccccccccccc}
\hat{\mathrm{e}}\colon 0 & \longrightarrow & \mathrm{U}(S) & \longrightarrow & \Gamma & \xrightarrow{\beta} & Aut_G(\mathrm{e}) & \longrightarrow & Out_G(\mathrm{e}) & \longrightarrow & 1 \\
 & & \| & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i_\sharp} & & \downarrow{\scriptstyle i_\flat} & & \\
\mathrm{e}_{(A_{\mathrm{e}},Q)}\colon 0 & \longrightarrow & \mathrm{U}(S) & \longrightarrow & \mathrm{U}(A_{\mathrm{e}}) & \xrightarrow{\partial} & Aut(A_{\mathrm{e}}, Q) & \longrightarrow & Out(A_{\mathrm{e}}, Q) & \longrightarrow & 1
\end{array}$$

of crossed 2-fold extensions, where $i_\flat$ denotes the induced homomorphism.

Given a crossed pair $(\mathrm{e}\colon 0 \to \mathrm{U}(T) \to \Gamma \to N \to 1,\ \psi\colon Q \to \mathrm{Out}_G(\mathrm{e}))$ with respect to the group extension $\mathrm{e}_{(T|S)}$ and the G-module $\mathrm{U}(T)$, let

$$\sigma_\psi = i_\flat \circ \psi\colon Q \longrightarrow \mathrm{Out}_G(\mathrm{e}) \longrightarrow \mathrm{Out}(A_{\mathrm{e}}, Q);$$

it is then obvious that $(A_{\mathrm{e}}, \sigma_\psi)$ is a Q-normal (Azumaya) S-algebra, and we refer to $(A_{\mathrm{e}}, \sigma_\psi)$ as a *Q-normal crossed pair algebra with respect to the Q-normal Galois extension $T|S$* of commutative rings.

Theorem 13.5 *Let $T|S$ be a Q-normal Galois extension of commutative rings, with structure extension $\mathrm{e}_{(T|S)}\colon N \rightarrowtail G \twoheadrightarrow Q$ and structure homomorphism*

$$\kappa_G\colon G \to Aut^S(T),$$

cf. Section 12 above. Then a class $k \in \mathrm{H}^3(Q, \mathrm{U}(S))$ is the Teichmüller class of some crossed pair algebra $(A_{\mathrm{e}}, \sigma_\psi)$ with respect to the Q-normal Galois extension $T|S$ if and only if k is split in $T|S$ in the sense that k goes to zero under inflation

$$\mathrm{H}^3(Q, \mathrm{U}(S)) \longrightarrow \mathrm{H}^3(G, \mathrm{U}(T)).$$

With $M = \mathrm{U}(T)$ and $M^N = \mathrm{U}(S)$, the theorem is a consequence of the exactness, at $\mathrm{H}^3(Q, \mathrm{U}(S))$, of the sequence (13.4). Indeed, by construction, the homomorphism Δ is given by the assignment to a crossed pair

$$(\mathrm{e}\colon \mathrm{U}(T) \rightarrowtail \Gamma \twoheadrightarrow N,\ \psi\colon Q \to \mathrm{Out}_G(\mathrm{e}))$$

with respect to the group extension $\mathrm{e}_{(T|S)}$ and the G-module $\mathrm{U}(T)$ of the corresponding crossed 2-fold extension (13.5), which now takes the form

$$\mathrm{e}_\psi : 0 \longrightarrow \mathrm{U}(S) \longrightarrow \Gamma \xrightarrow{\partial^\psi} B^\psi \longrightarrow Q \longrightarrow 1.$$

Theorem 13.5 is therefore a consequence of the following, which is again immediate.

Proposition 13.6 *Given a crossed pair* (e, ψ) *with respect to the group extension* $\mathrm{e}_{(T|S)}$ *and the* G*-module* $\mathrm{U}(T)$*, the morphism* $(i, i_\sharp)$ *of crossed modules in Proposition 13.4 above induces a congruence morphism*

$$\begin{array}{ccccccccccc}
\mathrm{e}_\psi : 0 & \longrightarrow & \mathrm{U}(S) & \longrightarrow & \Gamma & \xrightarrow{\partial^\psi} & B^\psi & \longrightarrow & Q & \longrightarrow & 1 \\
 & & \| & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \hat{i}} & & \| & & \\
\mathrm{e}_{(A_\mathrm{e},\sigma_\psi)} : 0 & \longrightarrow & \mathrm{U}(S) & \longrightarrow & \mathrm{U}(A_\mathrm{e}) & \xrightarrow{\partial^{\sigma_\psi}} & B^{\sigma_\psi} & \longrightarrow & Q & \longrightarrow & 1
\end{array}$$

of crossed 2-fold extensions.

Proof of Theorem 13.5. By exactness, it is immediate that the Teichmüller class of any crossed pair algebra $(A_\mathrm{e}, \sigma_\psi)$ with respect to $T|S$ is split in $T|S$. Hence the condition is necessary. To establish sufficiency, consider a class $k \in \mathrm{H}^3(Q, \mathrm{U}(S))$ which is split in $T|S$, that is, goes to zero under inflation

$$\mathrm{H}^3(Q, \mathrm{U}(S)) \longrightarrow \mathrm{H}^3(G, \mathrm{U}(T)).$$

By exactness, k then arises from some crossed pair (e, ψ) with respect to the group extension $\mathrm{e}_{(T|S)}$ and the G-module $\mathrm{U}(T)$, that is,

$$k = [\mathrm{e}_\psi] \in \mathrm{H}^3(Q, \mathrm{U}(S)).$$

By Proposition 13.6, the Teichmüller class of the associated crossed pair algebra $(A_\mathrm{e}, \sigma_\psi)$ with respect to $T|S$ coincides with $[\mathrm{e}_\psi] = k$. □

14 Normal Deuring embedding and Galois descent for Teichmüller classes

As before, S denotes a commutative ring and $\kappa_Q\colon Q \to \mathrm{Aut}(S)$ an action of a group Q on S. Let $T|S$ be a Q-normal Galois extension of commutative rings, with structure extension

$$\mathrm{e}_{(T|S)}\colon 1 \longrightarrow N \xrightarrow{i^N} G \xrightarrow{\pi_Q} Q \longrightarrow 1 \tag{14.1}$$

and structure homomorphism $\kappa_G\colon G \to \mathrm{Aut}^S(T)$, cf. (12.1). In this section, we prove, among others, that if a class $k \in \mathrm{H}^3(Q, \mathrm{U}(S))$ goes under inflation to the Teichmüller class in $\mathrm{H}^3(G, \mathrm{U}(T))$ of some G-normal T-algebra, then k is itself the Teichmüller class of some Q-normal S-algebra. To this end, we reexamine Deuring's embedding problem, cf. [14, Subsection 4.9 and Section 6].

14.1 The definitions

Let A be a central T-algebra, $(C, \sigma_Q\colon Q \to \mathrm{Out}(C))$ a Q-normal S-algebra, and $A \subseteq C$ an embedding of A into C. We refer to the embedding of A into C as a *Q-normal Deuring embedding with respect to $\sigma_Q\colon Q \to \mathrm{Out}(C)$ and* (14.1) if each automorphism $\kappa_G(x)$ of T, as x ranges over G, extends to an automorphism α of C in such a way that

(i) $[\alpha] = \sigma_Q(\pi_Q(x)) \in \mathrm{Out}(C)$, and
(ii) α maps A to itself.

Remark 14.1 In the special case where Q is the trivial group, the group G boils down to the group $N = \mathrm{Aut}(S|R)$ and, since each automorphism α of C that extends some $x \in N$ is required to map A to itself and to map to the trivial element of $\mathrm{Out}(C)$, that automorphism α necessarily extends to an inner automorphism of C that normalizes A; thus the notion of normal Deuring embedding then comes down to the notion of Deuring embedding introduced in [14, Subsection 4.9].

Remark 14.2 Given an embedding of A into C such that A coincides with the centralizer of T in C, an automorphism α of C extending an automorphism $\kappa_G(x)$ of T for $x \in G$ necessarily maps A to itself. Thus, in the definition of a Q-normal Deuring embedding, condition (ii) is then redundant.

For technical reasons, we need a stronger concept of a normal Deuring embedding. We now prepare for this definition.

Let A be a central T-algebra, C a central S-algebra, and suppose the algebra A to be embedded into C. Recall the crossed module $(\mathrm{U}(C), \mathrm{Aut}(C), \partial_C)$ associated to the central S-algebra C, and consider the associated crossed 2-fold extension

$$\mathrm{e}_C\colon 0 \longrightarrow \mathrm{U}(S) \longrightarrow \mathrm{U}(C) \xrightarrow{\partial_C} \mathrm{Aut}(C) \longrightarrow \mathrm{Out}(C) \longrightarrow 1, \tag{14.2}$$

cf. [14, (4.1)]. The normalizer $N^{\mathrm{U}(C)}(A)$ of A in $\mathrm{U}(C)$ and the centralizer $C^{\mathrm{U}(C)}(T)$ of T in $\mathrm{U}(C)$, together with $\mathrm{U}(A)$ and $\mathrm{U}(C)$, constitute an ascending sequence

$$\mathrm{U}(A) \subseteq C^{\mathrm{U}(C)}(T) \subseteq N^{\mathrm{U}(C)}(A) \subseteq \mathrm{U}(C)$$

of groups. When A coincides with the centralizer of T in C, the inclusion $\mathrm{U}(A) \subseteq C^{\mathrm{U}(C)}(T)$ is the identity.

We continue with the general case where A does not necessarily coincide with the centralizer of T in C. Let $\mathrm{Aut}^A(C)$ denote the group of automorphisms of C that map A to itself. The action of $\mathrm{Aut}(C)$ on $\mathrm{U}(C)$ induces an action of $\mathrm{Aut}^A(C)$ on each of the groups $\mathrm{U}(A)$, $C^{\mathrm{U}(C)}(T)$, and $N^{\mathrm{U}(C)}(A)$, and the restrictions of the homomorphism ∂_C together with the actions yield three crossed modules

$$(N^{\mathrm{U}(C)}(A), \mathrm{Aut}^A(C), \partial_C^N), \tag{14.3}$$

$$(C^{\mathrm{U}(C)}(T), \mathrm{Aut}^A(C), \partial_C^T), \tag{14.4}$$

$$(\mathrm{U}(A), \mathrm{Aut}^A(C), \partial_C^A), \tag{14.5}$$

each homomorphism $\partial_C^N, \partial_C^T, \partial_C^A$ being the corresponding restriction of the homomorphism $\partial_C \colon \mathrm{U}(C) \to \mathrm{Aut}(C)$. We write the associated crossed 2-fold extensions as

$$\mathrm{e}_C^A \colon 0 \longrightarrow \mathrm{U}(S) \longrightarrow \mathrm{U}(A) \xrightarrow{\partial_C^A} \mathrm{Aut}^A(C) \longrightarrow \mathrm{Out}(C, A) \longrightarrow 1, \tag{14.6}$$

$$\mathrm{e}_C^T \colon 0 \longrightarrow \mathrm{U}(S) \longrightarrow C^{\mathrm{U}(C)}(T) \xrightarrow{\partial_C^T} \mathrm{Aut}^A(C) \longrightarrow \mathrm{Out}(C, T) \longrightarrow 1, \tag{14.7}$$

$$\mathrm{e}_C^N \colon 0 \longrightarrow \mathrm{U}(S) \longrightarrow N^{\mathrm{U}(C)}(A) \xrightarrow{\partial_C^N} \mathrm{Aut}^A(C) \longrightarrow \mathrm{Out}^A(C) \longrightarrow 1, \tag{14.8}$$

the groups $\mathrm{Out}(C, A)$, $\mathrm{Out}(C, T)$, and $\mathrm{Out}^A(C)$ being defined by exactness. The inclusions $\mathrm{U}(A) \subseteq C^{\mathrm{U}(C)}(T) \subseteq N^{\mathrm{U}(C)}(A)$ induce a commutative diagram

$$\begin{array}{ccccccccccc}
\mathrm{e}_C^A \colon 0 & \longrightarrow & \mathrm{U}(S) & \longrightarrow & \mathrm{U}(A) & \xrightarrow{\partial_C^A} & \mathrm{Aut}^A(C) & \longrightarrow & \mathrm{Out}(C, A) & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \| & & \downarrow & & \\
\mathrm{e}_C^T \colon 0 & \longrightarrow & \mathrm{U}(S) & \longrightarrow & C^{\mathrm{U}(C)}(T) & \xrightarrow{\partial_C^T} & \mathrm{Aut}^A(C) & \longrightarrow & \mathrm{Out}(C, T) & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \| & & \downarrow & & \\
\mathrm{e}_C^N \colon 0 & \longrightarrow & \mathrm{U}(S) & \longrightarrow & N^{\mathrm{U}(C)}(A) & \xrightarrow{\partial_C^N} & \mathrm{Aut}^A(C) & \longrightarrow & \mathrm{Out}^A(C) & \longrightarrow & 1
\end{array}$$

of morphisms of crossed 2-fold extensions and, by diagram chase, the induced homomorphisms $\mathrm{Out}(C, A) \to \mathrm{Out}(C, T)$ and $\mathrm{Out}(C, T) \to \mathrm{Out}^A(C)$ are surjective.

Restriction induces canonical homomorphisms

$$\mathrm{res} \colon \mathrm{Out}(C, A) \longrightarrow \mathrm{Out}^S(A), \ \mathrm{res} \colon \mathrm{Out}(C, T) \longrightarrow \mathrm{Aut}^S(T)$$

(where the notation "res" is slightly abused) in such a way that the diagram

$$\begin{CD} \mathrm{Out}(C,A) @>\mathrm{res}>> \mathrm{Out}^S(A) \\ @VVV @VV\mathrm{res}V \\ \mathrm{Out}(C,T) @>\mathrm{res}>> \mathrm{Aut}^S(T) \end{CD}$$

is commutative. Moreover, the obvious homomorphism $\mathrm{Out}^A(C) \to \mathrm{Out}(C)$ is injective, and we identify $\mathrm{Out}^A(C)$ with its isomorphic image in $\mathrm{Out}(C)$ if need be.

Now, given a homomorphism $\chi_G\colon G \to \mathrm{Out}(C, A)$, its composite with the restriction map $\mathrm{res}\colon \mathrm{Out}(C, A) \to \mathrm{Out}(A)$ yields a G-normal structure on A. However, in order for such a homomorphism to match the other data, in particular the given Q-normal structure $\sigma_Q\colon Q \to \mathrm{Out}(C)$, we must impose further conditions. We now spell out the details.

Let $\partial^A_{C,\sharp}\colon \mathrm{U}(A)/\mathrm{U}(S) \to \mathrm{Aut}^A(C)$ denote the (injective) homomorphism induced by the crossed module structure map ∂^A_C in the crossed module (14.5). The crossed modules (14.3) and (14.5) yield the commutative diagram

$$\begin{array}{ccccccccccc}
 & & & & 1 & & 1 & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathrm{U}(S) & \longrightarrow & \mathrm{U}(A) & \longrightarrow & \mathrm{U}(A)/\mathrm{U}(S) & \longrightarrow & 1 & & \\
 & & \Vert & & \downarrow & & \downarrow{\scriptstyle \partial^A_{C,\sharp}} & & & & \\
0 & \longrightarrow & \mathrm{U}(S) & \longrightarrow & N^{\mathrm{U}(C)}(T) & \xrightarrow{\partial^N_C} & \mathrm{Aut}^A(C) & \longrightarrow & \mathrm{Out}^A(C) & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \Vert & & \\
 & & 1 & \longrightarrow & N^{\mathrm{U}(C)}(A)/\mathrm{U}(A) & \longrightarrow & \mathrm{Out}(C,A) & \longrightarrow & \mathrm{Out}^A(C) & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & 1 & & 1 & & & &
\end{array}$$

with exact rows and columns, the third row being defined by exactness. This third row is an ordinary group extension, and we denote it by

$$\mathrm{e}_{(A,C)}\colon 1 \to N^{\mathrm{U}(C)}(A)/\mathrm{U}(A) \to \mathrm{Out}(C,A) \to \mathrm{Out}^A(C) \to 1. \tag{14.9}$$

We define a *strong Q-normal Deuring embedding* of A into C *with respect to the Q-normal structure $\sigma_Q\colon Q \to \mathrm{Out}(C)$ and the structure extension* (14.1) to consist of an embedding of A into C together with a homomorphism $\chi_G\colon G \to \mathrm{Out}(C, A)$ that is compatible with the other data in the following sense:

– The restriction $\chi_N\colon N \to N^{\mathrm{U}(C)}(A)/\mathrm{U}(A)$ to $N = \mathrm{Aut}(T|S)$ of the homomorphism χ_G turns the embedding of A into C into a strong Deuring embedding relative to the action $\mathrm{Id}\colon N \to \mathrm{Aut}(T|S)$ of N on T in such a way that the diagram

$$\begin{array}{ccccccccc}
\mathrm{e}_{(T|S)}\colon 1 & \longrightarrow & N & \xrightarrow{i^N} & G & \xrightarrow{\pi_Q} & Q & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \chi_N} & & \downarrow{\scriptstyle \chi_G} & & \downarrow{\scriptstyle \sigma_Q} & & \\
\mathrm{e}_{(A,C)}\colon 1 & \longrightarrow & N^{\mathrm{U}(C)}(A)/\mathrm{U}(A) & \longrightarrow & \mathrm{Out}(C,A) & \longrightarrow & \mathrm{Out}^A(C) & \longrightarrow & 1
\end{array} \tag{14.10}$$

is commutative.

– The composite

$$G \xrightarrow{\chi_G} \mathrm{Out}(C, A) \xrightarrow{\mathrm{res}} \mathrm{Aut}^S(T) \tag{14.11}$$

coincides with $\kappa_G\colon G \to \mathrm{Aut}^S(T)$.

Remark 14.3 In the special case where Q is the trivial group, this notion of strong normal Deuring embedding comes down to the notion of strong Deuring embedding introduced in [14, Subsection 4.9].

Given a strong Q-normal Deuring embedding $(A \subseteq C, \chi_G)$ with respect to the Q-normal structure $\sigma_Q\colon Q \to \mathrm{Out}(C)$ and to the group extension (14.1), the composite of χ_G with the restriction map $\mathrm{res}\colon \mathrm{Out}(C, A) \to \mathrm{Out}(A)$ yields a G-normal structure

$$\sigma_G\colon G \longrightarrow \mathrm{Out}(A) \tag{14.12}$$

on A relative to the action $\kappa_G\colon G \to \mathrm{Aut}^S(T)$ of G on T; we refer to this structure as being *associated to the strong Q-normal Deuring embedding*.

14.2 Discussion of the notion of normal Deuring embedding

Recall that G denotes the fiber product group $\mathrm{Aut}^S(T) \times_{\mathrm{Aut}(S)} Q$ relative to the action $\kappa_Q\colon Q \to \mathrm{Aut}(S)$ of Q on S, that $\kappa_G\colon G \to \mathrm{Aut}^S(T)$ is the associated obvious homomorphism, and that κ_G, restricted to N, boils down to the identity $N \to \mathrm{Aut}(T|S)$, cf. (12.1) above.

Let A be a central T-algebra, consider an embedding of A into a central S-algebra C, and let $\sigma_Q\colon Q \to \mathrm{Out}(C)$ be a Q-normal structure on C. Consider the fiber product group $B^{A,\sigma_Q} = \mathrm{Aut}^A(C) \times_{\mathrm{Out}(C)} Q$ relative to the Q-normal structure σ_Q on C. The following is immediate.

Proposition 14.4 *Abstract nonsense identifies the kernel of the canonical homomorphism $B^{A,\sigma_Q} \to Q$ with the normal subgroup $\mathrm{IAut}^A(C)$ of $Aut^A(C)$ that consists of the inner automorphisms of C that map A to itself. Consequently the data determine a crossed module $(N^{\mathrm{U}(C)}(A), B^{A,\sigma_Q}, \partial^{A,\sigma_Q})$, the requisite action of B^{A,σ_Q} on $N^{\mathrm{U}(C)}(A)$ being induced from the canonical homomorphism $B^{A,\sigma_Q} \to Aut^A(C)$, in such a way that the sequence*

$$0 \longrightarrow \mathrm{U}(S) \longrightarrow N^{\mathrm{U}(C)}(A) \xrightarrow{\partial^{A,\sigma_Q}} B^{A,\sigma_Q} \longrightarrow Q \tag{14.13}$$

is exact. □

Since $G = \mathrm{Aut}^S(T) \times_{\mathrm{Aut}(S)} Q$ (relative to the action $\kappa_Q\colon Q \to \mathrm{Aut}(S)$ of Q on S), and since the composite $Q \xrightarrow{\sigma_Q} \mathrm{Out}(C) \xrightarrow{\mathrm{res}} \mathrm{Aut}(S)$ coincides with the structure map $\kappa_Q\colon Q \to \mathrm{Aut}(S)$, by abstract nonsense, the combined homomorphism

$$B^{A,\sigma_Q} \xrightarrow{\mathrm{can}} \mathrm{Aut}^A(C) \xrightarrow{\mathrm{res}} \mathrm{Aut}^S(T)$$

and the canonical homomorphism $\mathrm{can}\colon B^{A,\sigma_Q} \to Q$ induce a homomorphism

$$\pi_G\colon B^{A,\sigma_Q} = \mathrm{Aut}^A(C) \times_{\mathrm{Out}(C)} Q \longrightarrow \mathrm{Aut}^S(T) \times_{\mathrm{Aut}(S)} Q = G. \tag{14.14}$$

The following is again immediate.

Proposition 14.5 *The embedding of A into C is a Q-normal Deuring embedding with respect to the Q-normal structure $\sigma_Q\colon Q \to Out(C)$ on C and the group extension* (14.1) *if and only if the homomorphism $\pi_G\colon B^{A,\sigma_Q} \to G$ is surjective.* □

Whether or not the homomorphism π_G is surjective, we now determine the kernel of π_G. To this end, let $\mathrm{Aut}^A(C|T)$ denote the subgroup of $\mathrm{Aut}^A(C)$ that consists of the automorphisms in $\mathrm{Aut}^A(C)$ that are the identity on T. Since T coincides with the center of A, restriction induces a homomorphism from $\mathrm{Aut}^A(C)$ to $\mathrm{Aut}(T)$, and since S coincides with the center of C, the values of this restriction map lie in the subgroup $\mathrm{Aut}^S(T)$ of $\mathrm{Aut}(T)$ that consists of the automorphisms of T which map S to itself. Thus, all told, restriction yields an exact sequence

$$1 \longrightarrow \mathrm{Aut}^A(C|T) \longrightarrow \mathrm{Aut}^A(C) \xrightarrow{\mathrm{res}} \mathrm{Aut}^S(T) \tag{14.15}$$

of groups.

Consider the fiber product groups

$$B^{A,\kappa_G} = \mathrm{Aut}^A(C) \times_{\mathrm{Aut}^S(T)} G,\ B^{A,\kappa_Q} = \mathrm{Aut}^A(C) \times_{\mathrm{Aut}(S)} Q,$$

relative to the homomorphisms $\kappa_G\colon G \to \mathrm{Aut}^S(T)$ and $\kappa_Q\colon Q \to \mathrm{Aut}(S)$, respectively, and let $\mathrm{can}\colon B^{A,\kappa_G} \to G$ denote the canonical homomorphism. Since G is the fiber product group $\mathrm{Aut}^S(T) \times_{\mathrm{Aut}(S)} Q$ with respect to the homomorphism

$$\kappa_Q\colon Q \longrightarrow \mathrm{Aut}(S),$$

by abstract nonsense, the canonical homomorphism from B^{A,κ_G} to B^{A,κ_Q} is an isomorphism. Moreover, the exact sequence (14.15) induces an exact sequence

$$1 \longrightarrow \mathrm{Aut}^A(C|T) \longrightarrow B^{A,\kappa_G} \xrightarrow{\text{can}} G \tag{14.16}$$

of groups in such a way that

$$\begin{array}{ccccccc}
1 & \longrightarrow & \mathrm{Aut}^A(C|T) & \longrightarrow & B^{A,\kappa_G} & \xrightarrow{\text{can}} & G \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle \kappa_G} \\
1 & \longrightarrow & \mathrm{Aut}^A(C|T) & \longrightarrow & \mathrm{Aut}^A(C) & \xrightarrow{\text{res}} & \mathrm{Aut}^S(T)
\end{array}$$

is a commutative diagram with exact rows.

Abstract nonsense yields a canonical homomorphism

$$\mathrm{Aut}^A(C) \times_{\mathrm{Out}(C)} Q = B^{A,\sigma_Q} \longrightarrow B^{A,\kappa_Q} = \mathrm{Aut}^A(C) \times_{\mathrm{Aut}(S)} Q$$

and hence a canonical homomorphism $B^{A,\sigma_Q} \to B^{A,\kappa_G}$ whose composite

$$B^{A,\sigma_Q} \to B^{A,\kappa_G} \xrightarrow{\text{can}} G$$

with $\text{can}\colon B^{A,\kappa_G} \to G$ coincides with $\pi_G\colon B^{A,\sigma_Q} \to G$.

Proposition 14.6 (i) The homomorphism $B^{A,\sigma_Q} \to B^{A,\kappa_G}$ is injective.
(ii) Under the identification of B^{A,σ_Q} with its isomorphic image in the group B^{A,κ_G}, the group $\mathrm{Aut}^A(C|T)$ being identified with its isomorphic image in B^{A,κ_G} via (14.16), the kernel of $\pi_G\colon B^{A,\sigma_Q} \to G$ gets identified with the normal subgroup of $\mathrm{Aut}^A(C|T)$ that consists of the automorphisms in $\mathrm{Aut}^A(C|T)$ that are inner automorphisms of C.
(iii) Consequently the canonical homomorphism from the centralizer $C^{\mathrm{U}(C)}(T)$ of T in $\mathrm{U}(C)$ to $\mathrm{Aut}^A(C|T)$ yields a surjective homomorphism

$$C^{\mathrm{U}(C)}(T) \longrightarrow \ker(\pi_G\colon B^{A,\sigma_Q} \to G).$$

Proof Since the canonical homomorphism $B^{A,\kappa_G} \to B^{A,\kappa_Q}$ is an isomorphism, the right-hand square in the the commutative diagram

$$\begin{array}{ccccc}
B^{A,\sigma_Q} & \longrightarrow & B^{A,\kappa_G} & \xrightarrow{\text{can}} & Q \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \kappa_Q} \\
\mathrm{Aut}^A(C) & = & \mathrm{Aut}^A(C) & \xrightarrow{\text{res}} & \mathrm{Aut}(S)
\end{array}$$

is a pull back square, and hence inspection of the diagram reveals that the homomorphism $B^{A,\sigma_Q} \to B^{A,\kappa_G}$ is injective. This establishes (i).

To justify (ii), we note first that the kernel of $\mathrm{Aut}^A(C) \to \mathrm{Out}(C)$ is the normal subgroup $\mathrm{IAut}^A(C)$ of $\mathrm{Aut}^A(C)$ that consists of the inner automorphisms of C that map A to itself. Since the group B^{A,σ_Q} is the fiber product group $B^{A,\sigma_Q} = \mathrm{Aut}^A(C) \times_{\mathrm{Out}(C)} Q$, abstract nonsense identifies the kernel of the canonical homomorphism $B^{A,\sigma_Q} \to Q$ with $\mathrm{IAut}^A(C)$, and it is immediate that $\ker(\pi_G)$ is a subgroup of $\mathrm{IAut}^A(C) = \ker(B^{A,\sigma_Q} \to Q)$. On the other hand, B^{A,σ_Q} being identified with the corresponding subgroup of B^{A,κ_G}, the kernel of $\pi_G \colon B^{A,\sigma_Q} \to G$ gets identified with the intersection $B^{A,\sigma_Q} \cap \mathrm{Aut}^A(C|T) \subseteq B^{A,\kappa_G}$ and hence with the intersection

$$\mathrm{IAut}^A(C) \cap \mathrm{Aut}^A(C|T) \subseteq B^{A,\kappa_G}.$$

Consequently the kernel of the homomorphism π_G gets identified with the normal subgroup of $\mathrm{Aut}^A(C|T)$ that consists of the automorphisms in $\mathrm{Aut}^A(C|T)$ that are inner automorphisms of C.

Finally, statement (iii) is an immediate consequence of (ii). □

Proposition 14.7 *Suppose that the embedding of A into C is a Q-normal Deuring embedding with respect to the Q-normal structure $\sigma_Q \colon Q \to Out(C)$ on C and the group extension* (14.1).

(i) *The surjective homomorphism* (14.14) *yields a crossed 2-fold extension*

$$\mathrm{e}^{A,T}_{(C,\sigma_Q)} \colon 0 \longrightarrow \mathrm{U}(S) \longrightarrow C^{\mathrm{U}(C)}(T) \xrightarrow{\partial^{A,T,\sigma_Q}} B^{A,\sigma_Q} \xrightarrow{\pi_G} G \longrightarrow 1. \quad (14.17)$$

(ii) *The values of the Q-normal structure $\sigma_Q \colon Q \to Out(C)$ on C lie in the subgroup $Out^A(C)(= \mathrm{coker}(\partial^N_C \colon N^{\mathrm{U}(C)}(A) \longrightarrow Aut^A(C))$, cf.* (14.8)).

Proof Statement (i) is an immediate consequence of Propositions 14.5 and 14.6 (iii). Moreover, the diagram

$$\begin{array}{ccccc} N^{\mathrm{U}(C)}(A) & \xrightarrow{\partial^{A,\sigma_Q}} & B^{A,\sigma_Q} & \longrightarrow & Q \\ \| & & \downarrow & & \downarrow \sigma_Q \\ N^{\mathrm{U}(C)}(A) & \xrightarrow{\partial^N_C} & \mathrm{Aut}^A(C) & \xrightarrow{\mathrm{can}} & \mathrm{Out}(C) \end{array}$$

is commutative and, in view of Proposition 14.5, the canonical homomorphism from B^{A,σ_Q} to Q is surjective. Consequently the values of $\sigma_Q \colon Q \to \mathrm{Out}(C)$ on C lie in the subgroup $\mathrm{Out}^A(C)(= \mathrm{coker}(\partial^N_C \colon N^{\mathrm{U}(C)}(A) \longrightarrow \mathrm{Aut}^A(C)))$. □

Given a Q-normal Deuring embedding of A into C with respect to the Q-normal structure $\sigma_Q \colon Q \to \mathrm{Out}(C)$ on C and the group extension (14.1), in view of Proposition 14.7(ii), let

$$\mathrm{e}^{A}_{(C,\sigma_Q)} \colon 0 \longrightarrow \mathrm{U}(S) \longrightarrow N^{\mathrm{U}(C)}(A) \xrightarrow{\partial^{A,\sigma_Q}} B^{A,\sigma_Q} \longrightarrow Q \longrightarrow 1 \quad (14.18)$$

denote the associated crossed 2-fold extension induced from (14.8) via the Q-normal structure $\sigma_Q\colon Q \to \mathrm{Out}^A(C)$ on C; the underlying sequence of groups and homomorphisms plainly coincides with (14.13). Recall that the Teichmüller complex $\mathrm{e}_{(C,\sigma_Q)}$ of the kind [14, (4.7)] associated to the Q-normal S-algebra (C, σ_Q) is the crossed 2-fold extension

$$\mathrm{e}_{(C,\sigma_Q)}\colon 0 \longrightarrow \mathrm{U}(S) \longrightarrow \mathrm{U}(C) \xrightarrow{\partial^{\sigma_Q}} B^{\sigma_Q} \longrightarrow Q \longrightarrow 1 \qquad (14.19)$$

induced from (14.2) via the Q-normal structure $\sigma_Q\colon Q \to \mathrm{Out}(C)$ on C. The following is again immediate.

Proposition 14.8 *Suppose that the embedding of A into C is a Q-normal Deuring embedding with respect to the Q-normal structure $\sigma_Q\colon Q \to Out(C)$ on C and the group extension* (14.1).
(i) *The inclusion maps $N^{\mathrm{U}(C)}(A) \to \mathrm{U}(C)$ and $B^{A,\sigma_Q} \to B^{\sigma_Q}$ yield a congruence*

$$\begin{array}{ccccccccccc}
\mathrm{e}^A_{(C,\sigma_Q)}\colon 0 & \longrightarrow & \mathrm{U}(S) & \longrightarrow & N^{\mathrm{U}(C)}(A) & \xrightarrow{\partial^{A,\sigma_Q}} & B^{A,\sigma_Q} & \longrightarrow & Q & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \| & & \\
\mathrm{e}_{(C,\sigma_Q)}\colon 0 & \longrightarrow & \mathrm{U}(S) & \longrightarrow & \mathrm{U}(C) & \xrightarrow{\partial^{\sigma_Q}} & B^{\sigma_Q} & \longrightarrow & Q & \longrightarrow & 1
\end{array} \qquad (14.20)$$

of crossed 2-fold extensions from the crossed 2-fold extension (14.18) *to the crossed 2-fold extension* (14.19).
(ii) *The injection $C^{\mathrm{U}(C)}(T) \to N^{\mathrm{U}(C)}(A)$ yields the morphism*

$$\begin{array}{ccccccccccc}
\mathrm{e}^{A,T}_{(C,\sigma_Q)}\colon 0 & \longrightarrow & \mathrm{U}(S) & \longrightarrow & C^{\mathrm{U}(C)}(T) & \xrightarrow{\partial^{A,T,\sigma_Q}} & B^{A,\sigma_Q} & \longrightarrow & G & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \| & & \downarrow{\scriptstyle \pi_Q} & & \\
\mathrm{e}^A_{(C,\sigma_Q)}\colon 0 & \longrightarrow & \mathrm{U}(S) & \longrightarrow & N^{\mathrm{U}(C)}(A) & \xrightarrow{\partial^{A,\sigma_Q}} & B^{A,\sigma_Q} & \longrightarrow & Q & \longrightarrow & 1
\end{array} \qquad (14.21)$$

of crossed 2-fold extensions from the crossed 2-fold extension (14.17) *to the crossed 2-fold extension* (14.18). □

14.3 Results related with the two notions of normal Deuring embedding

Theorem 14.9 *Let A be a central T-algebra, C a central S-algebra, and $A \subseteq C$ an embedding of A into C having the property that A coincides with the centralizer of T in C. Furthermore, let $\sigma_Q\colon Q \to Out(C)$ be a Q-normal structure on C, and suppose that the embedding of A into C is a Q-normal Deuring embedding with respect to σ_Q and the group extension* (14.1). *Then the data determine a unique homomorphism $\chi_G\colon G \to Out(C, A)$ that turns the given Q-normal Deuring embedding of A into C*

into a strong Q-normal Deuring embedding of A into C with respect to the given data in such a way that, relative to the associated G-normal structure

$$\sigma_G \colon G \xrightarrow{\chi_G} Out(C, A) \xrightarrow{res} Out(A)$$

on A, cf. (14.12)*,*

$$[\mathrm{e}_{(A,\sigma_G)}] = \inf[\mathrm{e}_{(C,\sigma_Q)}] \in \mathrm{H}^3(G, \mathrm{U}(T)).$$

Proof Recall that the Teichmüller complex $\mathrm{e}_{(C,\sigma_Q)}$ of the Q-normal S-algebra (C, σ_Q), spelled out above as (14.19), represents the Teichmüller class

$$[\mathrm{e}_{(C,\sigma_Q)}] \in \mathrm{H}^3(Q, \mathrm{U}(S))$$

of the Q-normal central S-algebra (C, σ_Q).

Suppose that the embedding of A into C is a Q-normal Deuring embedding with respect to the Q-normal structure $\sigma_Q \colon Q \to \mathrm{Out}(C)$ on C and the group extension (14.1). By Proposition 14.8(i), the crossed 2-fold extension $\mathrm{e}^A_{(C,\sigma_Q)}$, cf. (14.18), is available and is congruent to $\mathrm{e}_{(C,\sigma_Q)}$, whence

$$[\mathrm{e}_{(C,\sigma_Q)}] = [\mathrm{e}^A_{(C,\sigma_Q)}] \in \mathrm{H}^3(Q, \mathrm{U}(S)).$$

Moreover, by Proposition 14.8(ii), the crossed 2-fold extension (14.17) is available and, since the centralizer of A in C coincides with T, the inclusion $\mathrm{U}(A) \subseteq C^{\mathrm{U}(C)}(T)$ identifies the group $\mathrm{U}(A)$ of invertible elements of A with the centralizer $C^{\mathrm{U}(C)}(T)$ of T in $\mathrm{U}(C)$. Hence the crossed 2-fold extension (14.17) has the form

$$\mathrm{e}^{A,T}_{(C,\sigma_Q)} \colon 0 \longrightarrow \mathrm{U}(S) \longrightarrow \mathrm{U}(A) \xrightarrow{\partial^{A,T,\sigma_Q}} B^{A,\sigma_Q} \longrightarrow G \longrightarrow 1, \quad (14.22)$$

and the injection $\iota \colon \mathrm{U}(A) \to N^{\mathrm{U}(C)}(A)$ induces the morphism (14.21) of crossed 2-fold extensions in Proposition 14.8(ii); this is a morphism of crossed 2-fold extensions of the kind $(1, \iota, 1, \pi_Q) \colon \mathrm{e}^{A,T}_{(C,\sigma_Q)} \to \mathrm{e}^A_{(C,\sigma_Q)}$.

Denote by $i \colon \mathrm{U}(S) \to \mathrm{U}(T)$ the inclusion. The canonical homomorphism

$$B^{A,\sigma_Q} = \mathrm{Aut}^A(C) \times_{\mathrm{Out}(C)} Q \longrightarrow \mathrm{Aut}^A(C)$$

induces a morphism

$$(\mathrm{Id}, \cdot) \colon (\mathrm{U}(A), B^{A,\sigma_Q}, \partial^{A,T,\sigma_Q}) \longrightarrow (\mathrm{U}(A), \mathrm{Aut}^A(C), \partial^A_C)$$

of crossed modules and hence a homomorphism $\chi_G\colon G \to \mathrm{Out}(C, A)$ such that

$$
\begin{array}{ccccccccccc}
\mathrm{e}^{A,T}_{(C,\sigma_Q)}\colon 0 & \longrightarrow & \mathrm{U}(S) & \longrightarrow & \mathrm{U}(A) & \xrightarrow{\partial^{A,T,\sigma_Q}} & B^{A,\sigma_Q} & \longrightarrow & G & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle i} & & \| & & \downarrow & & \downarrow{\scriptstyle \chi_G} & & \\
\mathrm{e}_A\colon 0 & \longrightarrow & \mathrm{U}(T) & \longrightarrow & \mathrm{U}(A) & \xrightarrow{\partial^A_C} & \mathrm{Aut}^A(C) & \longrightarrow & \mathrm{Out}(C, A) & \longrightarrow & 1
\end{array}
$$

is a morphism of crossed 2-fold extensions from (14.22) to [14, (4.1)]. The homomorphism χ_G turns the given Q-normal Deuring embedding of C into A into a strong Q-normal Deuring embedding of C into A with respect to the given data.

The G-normal structure $\sigma_G\colon G \overset{\chi_G}{\to} \mathrm{Out}(C, A) \overset{\mathrm{res}}{\to} \mathrm{Out}(A)$ associated to the strong Q-normal Deuring embedding, in turn, induces a morphism

$$
\begin{array}{ccccccccccc}
\mathrm{e}^{A,T}_{(C,\sigma_Q)}\colon 0 & \longrightarrow & \mathrm{U}(S) & \longrightarrow & \mathrm{U}(A) & \xrightarrow{\partial^{A,T,\sigma_Q}} & B^{A,\sigma_Q} & \longrightarrow & G & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle i} & & \| & & \downarrow & & \| & & \\
\mathrm{e}_{(A,\sigma_G)}\colon 0 & \longrightarrow & \mathrm{U}(T) & \longrightarrow & \mathrm{U}(A) & \xrightarrow{\partial^{\sigma_G}} & B^{\sigma_G} & \longrightarrow & G & \longrightarrow & 1
\end{array}
$$

of crossed 2-fold extensions from (14.22) to the corresponding crossed 2-fold extension $\mathrm{e}_{(A,\sigma_G)}$ of the kind [14, (4.7)]. Consequently $[\mathrm{e}_{(A,\sigma_G)}] = \inf[\mathrm{e}_{(C,\sigma_Q)}]$. □

Theorem 14.9 has a converse; this converse sort of a characterizes the Teichmüller classes in $\mathrm{H}^3(Q, \mathrm{U}(S))$.

Theorem 14.10 *Let $k \in \mathrm{H}^3(Q, \mathrm{U}(S))$, let A be a central T-algebra, and let*

$$\sigma_G\colon G \longrightarrow \mathit{Out}(A)$$

be a G-normal structure on A relative to the action $\kappa_G\colon G \to \mathit{Aut}^S(T)$ of G on T. Suppose that

$$\inf(k) = [\mathrm{e}_{(A,\sigma_G)}] \in \mathrm{H}^3(G, \mathrm{U}(T)).$$

Then there is a Q-normal S-central crossed product algebra

$$(C, \sigma_Q) = ((A, N, \mathrm{e}, \vartheta), \sigma_Q)$$

related with the other data in the following way.

- *The Q-normal algebra $(C, \sigma_Q) = ((A, N, \mathrm{e}, \vartheta), \sigma_Q)$ has Teichmüller class k;*
- *once the Q-normal algebra $((A, N, \mathrm{e}, \vartheta), \sigma_Q)$ has been fixed, the data determine a homomorphism $\chi_G\colon G \to \mathit{Out}(C, A)$ that turns the obvious embedding of A into $(A, N, \mathrm{e}, \vartheta)$ into a strong Q-normal Deuring embedding with respect to*

$$\sigma_Q\colon Q \longrightarrow Out(A, N, \mathrm{e}, \vartheta)$$

and the group extension (14.1);
– *the associated G-normal structure*

$$G \xrightarrow{\chi_G} Out(C, A) \xrightarrow{res} Out(A)$$

on A, cf. (14.12), and the given G-normal structure $\sigma_G\colon G \to Out(A)$ on A coincide.

Complement 14.11 *In the situation of Theorem 14.10, if A is an Azumaya T-algebra, the algebra $(A, N, \mathrm{e}, \vartheta)$ is an Azumaya S-algebra.*

Remark 14.12 In the special case where $\inf(k) = 0$, the argument to be given comes down to that given for the statement of Theorem 13.5, and this theorem is in fact a special case of Theorem 14.10.

Proof of Theorem 14.10. For convenience, we split the reasoning into Propositions 14.13–14.15 below.

Consider a G-normal central T-algebra (A, σ_G), and denote by $\sigma_N\colon N \to \mathrm{Out}(A)$ the restriction of $\sigma_G\colon G \to \mathrm{Out}(A)$ to N so that (A, σ_N) is an N-normal central T-algebra. The obvious unlabeled vertical arrow and the injection i^N turn

$$\begin{array}{ccccccccccc}
\mathrm{e}_{(A,\sigma_N)}\colon 0 & \longrightarrow & \mathrm{U}(T) & \longrightarrow & \mathrm{U}(A) & \xrightarrow{\partial^{\sigma_N}} & B^{\sigma_N} & \longrightarrow & N & \longrightarrow & 1 \\
 & & \| & & \| & & \downarrow & & \downarrow{\scriptstyle i^N} & & \\
\mathrm{e}_{(A,\sigma_G)}\colon 0 & \longrightarrow & \mathrm{U}(T) & \longrightarrow & \mathrm{U}(A) & \xrightarrow{\partial^{\sigma_G}} & B^{\sigma_G} & \longrightarrow & G & \longrightarrow & 1
\end{array}$$

into a commutative diagram having as its rows the (exact) Teichmüller complexes $\mathrm{e}_{(A,\sigma_N)}$ and $\mathrm{e}_{(A,\sigma_G)}$ of (A, σ_N) and (A, σ_G), respectively. Consequently the combined homomorphism

$$B^{\sigma_G} \longrightarrow G \xrightarrow{\pi_Q} Q$$

yields a group extension

$$1 \longrightarrow B^{\sigma_N} \longrightarrow B^{\sigma_G} \longrightarrow Q \longrightarrow 1. \tag{14.23}$$

Let

$$\hat{\mathrm{e}}\colon 0 \longrightarrow \mathrm{U}(T) \longrightarrow \mathrm{U}(A) \longrightarrow \mathrm{U}(A)/\mathrm{U}(T) \longrightarrow 1 \tag{14.24}$$

and

$$1 \longrightarrow \mathrm{U}(A)/\mathrm{U}(T) \xrightarrow{\phi} B^{\sigma_N} \longrightarrow N \longrightarrow 1$$

be the obvious group extensions so that splicing them yields the Teichmüller complex

$$\mathrm{e}_{(A,\sigma_N)}\colon 0 \longrightarrow \mathrm{U}(T) \longrightarrow \mathrm{U}(A) \longrightarrow B^{\sigma_N} \longrightarrow N \longrightarrow 1$$

of (A, σ_N). We denote the resulting morphism

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{U}(A)/\mathrm{U}(T) & \longrightarrow & B^{\sigma_G} & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow \phi & & \| & & \downarrow \pi_Q & & \\ 1 & \longrightarrow & B^{\sigma_N} & \longrightarrow & B^{\sigma_G} & \longrightarrow & Q & \longrightarrow & 1 \end{array} \tag{14.25}$$

of group extensions by Φ.

Consider the Teichmüller complex

$$\mathrm{e}_{(A,\sigma_G)}\colon 0 \longrightarrow \mathrm{U}(T) \longrightarrow \mathrm{U}(A) \xrightarrow{\partial^{\sigma_G}} B^{\sigma_G} \longrightarrow G \longrightarrow 1$$

associated to the given G-normal structure $\sigma_G\colon G \to \mathrm{Out}(A)$ on A, cf. [14, (4.7)]. Since $\mathrm{U}(T)$ is a central subgroup of $\mathrm{U}(A)$, the group extension $\hat{\mathrm{e}}$ spelled out above as (14.24) is a central extension and, as noted in Proposition 13.2, G-crossed pair structures on $\hat{\mathrm{e}}$ are equivalent to B^{σ_G}-actions on $\mathrm{U}(A)$ that turn $\mathrm{U}(A) \to B^{\sigma_G}$ into a crossed module. Thus the action of B^{σ_G} on $\mathrm{U}(A)$ that results from the given G-normal structure $\sigma_G\colon G \to \mathrm{Out}(A)$ via the associated crossed 2-fold extension $\mathrm{e}_{(A,\sigma_G)}$ induces a crossed pair structure $\hat{\psi}\colon G \to \mathrm{Out}_{B^{\sigma_G}}(\hat{\mathrm{e}})$ on $\hat{\mathrm{e}}$ with respect to the group extension $\mathrm{U}(A)/\mathrm{U}(T) \rightarrowtail B^{\sigma_G} \twoheadrightarrow G$ and the G-module $\mathrm{U}(T)$. Then the canonical homomorphism $\hat{\gamma}\colon \mathrm{Aut}_{B^{\sigma_G}}(\hat{\mathrm{e}}) \longrightarrow \mathrm{Aut}(A)$ yields a morphism

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{U}(T) & \longrightarrow & \mathrm{U}(A) & \longrightarrow & \mathrm{Aut}_{B^{\sigma_G}}(\hat{\mathrm{e}}) & \longrightarrow & \mathrm{Out}_{B^{\sigma_G}}(\hat{\mathrm{e}}) & \longrightarrow & 1 \\ & & \| & & \| & & \downarrow \hat{\gamma} & & \downarrow \hat{\gamma}_\sharp & & \\ 0 & \longrightarrow & \mathrm{U}(T) & \longrightarrow & \mathrm{U}(A) & \longrightarrow & \mathrm{Aut}(A) & \longrightarrow & \mathrm{Out}(A) & \longrightarrow & 1 \end{array}$$

of crossed 2-fold extensions such that the composite

$$G \xrightarrow{\hat{\psi}} \mathrm{Out}_{B^{\sigma_G}}(\hat{\mathrm{e}}) \xrightarrow{\hat{\gamma}_\sharp} \mathrm{Out}(A) \tag{14.26}$$

coincides with $\sigma_G\colon G \to \mathrm{Out}(A)$.

Proposition 14.13 *Let $k \in \mathrm{H}^3(Q, \mathrm{U}(S))$, let (A, σ_G) be a G-normal central T-algebra, and suppose that $\inf(k) = [\mathrm{e}_{(A,\sigma_G)}] \in \mathrm{H}^3(G, \mathrm{U}(T))$. Then there is a group extension*

$$\tilde{\mathrm{e}}\colon 0 \longrightarrow \mathrm{U}(T) \longrightarrow \Gamma \longrightarrow B^{\sigma_N} \longrightarrow 1$$

together with a crossed pair structure $\tilde{\psi}\colon Q \to Out_{B^{\sigma_G}}(\tilde{\mathrm{e}})$ on $\tilde{\mathrm{e}}$ with respect to the group extension (14.23) *and the B^{σ_G}-module* $\mathrm{U}(T)$*, the requisite module structure*

being induced by the map $B^{\sigma_G} \to G$ *in* $\mathrm{e}_{(A,\sigma_G)}$, *related with the other data in the following way, where* $B^{\tilde\psi}$ *denotes the fiber product group* $Aut_{B^{\sigma_G}}(\tilde{\mathrm{e}}) \times_{Out_{B^{\sigma_G}}(\tilde{\mathrm{e}})} Q$ *with respect to* $\tilde\psi\colon Q \to Out_{B^{\sigma_G}}(\tilde{\mathrm{e}})$.
(i) *The crossed 2-fold extension*

$$\mathrm{e}_{\tilde\psi}\colon 0 \longrightarrow \mathrm{U}(S) \longrightarrow \Gamma \longrightarrow B^{\tilde\psi} \longrightarrow Q \longrightarrow 1$$

associated to the crossed pair $(\tilde{\mathrm{e}}, \tilde\psi)$, *cf.* (13.5), *represents* k.
(ii) *Relative to the obvious actions of the group* $Aut_{B^{\sigma_G}}(\tilde{\mathrm{e}})$ ($\subseteq Aut(\Gamma) \times B^{\sigma_G}$) *on the groups* $\mathrm{U}(T), \mathrm{U}(A), \mathrm{U}(A)/\mathrm{U}(T), \Gamma, B^{\sigma_N}$ *and* N, *the extension group* Γ *in* $\tilde{\mathrm{e}}$ *fits into a commutative diagram of* $Aut_{B^{\sigma_G}}(\tilde{\mathrm{e}})$*-groups with exact rows and columns as follows:*

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
\hat{\mathrm{e}}\colon 0 & \longrightarrow & \mathrm{U}(T) & \longrightarrow & \mathrm{U}(A) & \longrightarrow & \mathrm{U}(A)/\mathrm{U}(T) & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow \phi & & \\
\tilde{\mathrm{e}}\colon 0 & \longrightarrow & \mathrm{U}(T) & \longrightarrow & \Gamma & \longrightarrow & B^{\sigma_N} & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & N & = & N & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array} \qquad (14.27)$$

Proof By [13, Theorem 2], the morphism (14.25) of group extensions induces a morphism for the corresponding eight term exact sequences in group cohomology constructed in [13]. In particular, Φ induces the commutative diagram

$$\begin{array}{ccccccc}
\mathrm{H}^2(B^{\sigma_G}, \mathrm{U}(T)) & \longrightarrow & \mathrm{Xpext}(B^{\sigma_G}, B^{\sigma_N}; \mathrm{U}(T)) & \xrightarrow{\Delta} & \mathrm{H}^3(Q, \mathrm{U}(S)) & \longrightarrow & \mathrm{H}^3(B^{\sigma_G}, \mathrm{U}(T)) \\
\| & & \downarrow \Phi^* & & \downarrow \mathrm{inf} & & \| \\
\mathrm{H}^2(B^{\sigma_G}, \mathrm{U}(T)) & \longrightarrow & \mathrm{Xpext}(B^{\sigma_G}, \mathrm{U}(A)/\mathrm{U}(T); \mathrm{U}(T)) & \xrightarrow{\Delta} & \mathrm{H}^3(G, \mathrm{U}(T)) & \longrightarrow & \mathrm{H}^3(B^{\sigma_G}, \mathrm{U}(T)).
\end{array}$$

By the construction of Δ, cf. Subsection 13.1 above or [13, Subsection 1.2],

$$\Delta[(\hat{\mathrm{e}}, \hat\psi)] = [\mathrm{e}_{(A,\sigma_G)}],$$

and so, by exactness, $\mathrm{inf}(k) = [\mathrm{e}_{(A,\sigma_G)}]$ goes to zero in $\mathrm{H}^3(B^{\sigma_G}, \mathrm{U}(T))$. Therefore k goes to zero in $\mathrm{H}^3(B^{\sigma_G}, \mathrm{U}(T))$, and hence there is a group extension

$$\tilde{\mathrm{e}}\colon 0 \longrightarrow \mathrm{U}(T) \longrightarrow \Gamma \longrightarrow B^{\sigma_N} \longrightarrow 1$$

of the asserted kind together with a crossed pair structure $\tilde{\psi}\colon Q \to \mathrm{Out}_{B^{\sigma_G}}(\tilde{\mathrm{e}})$ on $\tilde{\mathrm{e}}$ with respect to the group extension (14.23) and the B^{σ_G}-module $\mathrm{U}(T)$ whose B^{σ_G}-module structure is induced by the projection $B^{\sigma_G} \to G$ in $\mathrm{e}_{(A,\sigma_G)}$ so that

$$\Delta[(\tilde{\mathrm{e}}, \tilde{\psi})] = k \in \mathrm{H}^3(Q, \mathrm{U}(S));$$

moreover, making a suitable choice of $(\tilde{\mathrm{e}}, \tilde{\psi})$ by means of some diagram chase if need be, we can arrange for $[(\tilde{\mathrm{e}}, \tilde{\psi})]$ to go to $[(\hat{\mathrm{e}}, \hat{\psi})]$ in the sense that

$$\Phi^*[(\tilde{\mathrm{e}}, \tilde{\psi})] = [(\hat{\mathrm{e}}, \hat{\psi})] \in \mathrm{Xpext}(B^{\sigma_G}, \mathrm{U}(A)/\mathrm{U}(T); \mathrm{U}(T)).$$

The crossed pair $(\tilde{\mathrm{e}}, \tilde{\psi})$ has the asserted properties. For $\Delta[(\tilde{\mathrm{e}}, \tilde{\psi})] = [\mathrm{e}_{\tilde{\psi}}]$ by definition, and so assertion (i) holds. Moreover, since $\Phi^*[(\tilde{\mathrm{e}}, \tilde{\psi})] = [(\hat{\mathrm{e}}, \hat{\psi})]$, assertion (ii) holds as well. The details are as follows, cf. [13, Subsection 2.2].

Since $\Phi^*[(\tilde{\mathrm{e}}, \tilde{\psi})] = [(\hat{\mathrm{e}}, \hat{\psi})]$, we may identify $(\hat{\mathrm{e}}, \hat{\psi})$ with the induced crossed pair $(\tilde{\mathrm{e}}\Phi, \tilde{\psi}^{\Phi})$, cf. [13]. Recall that $\tilde{\mathrm{e}}\Phi$ is the group extension induced from $\tilde{\mathrm{e}}$ via the injective homomorphism $\phi\colon \mathrm{U}(A)/\mathrm{U}(T) \to B^{\sigma_N}$ and let $\mathrm{U} = \ker(\Gamma \longrightarrow N)$; since ϕ identifies $\mathrm{U}(A)/\mathrm{U}(T)$ with the kernel of $B^{\sigma_N} \to N$, we can write the induced group extension $\tilde{\mathrm{e}}\Phi$ as

$$\tilde{\mathrm{e}}\Phi\colon 0 \longrightarrow \mathrm{U}(T) \longrightarrow \mathrm{U} \longrightarrow \mathrm{U}(A)/\mathrm{U}(T) \longrightarrow 1.$$

To explain the induced crossed pair structure $\tilde{\psi}^{\Phi}\colon G \to \mathrm{Out}_{B^{\sigma_G}}(\tilde{\mathrm{e}}\Phi)$, we note first that the injection $\mathrm{U} \to \Gamma$ induces a morphism

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{U}(S) & \longrightarrow & \mathrm{U} & \longrightarrow & \mathrm{Aut}_{B^{\sigma_G}}(\tilde{\mathrm{e}}) & \longrightarrow & \mathrm{Out}_{B^{\sigma_G}}(\tilde{\mathrm{e}}) \times_Q G & \longrightarrow & 1 \\
 & & \Vert & & \downarrow & & \Vert & & \downarrow & & \\
0 & \longrightarrow & \mathrm{U}(S) & \longrightarrow & \Gamma & \longrightarrow & \mathrm{Aut}_{B^{\sigma_G}}(\tilde{\mathrm{e}}) & \longrightarrow & \mathrm{Out}_{B^{\sigma_G}}(\tilde{\mathrm{e}}) & \longrightarrow & 1
\end{array}$$

of crossed 2-fold extensions. Moreover, restriction of the operators on Γ to U yields a homomorphism

$$\mathrm{res}\colon \mathrm{Aut}_{B^{\sigma_G}}(\tilde{\mathrm{e}}) \longrightarrow \mathrm{Aut}_{B^{\sigma_G}}(\tilde{\mathrm{e}}\Phi),$$

and this homomorphism, in turn, yields a morphism

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{U}(S) & \longrightarrow & \mathrm{U} & \longrightarrow & \mathrm{Aut}_{B^{\sigma_G}}(\tilde{\mathrm{e}}) & \longrightarrow & \mathrm{Out}_{B^{\sigma_G}}(\tilde{\mathrm{e}}) \times_Q G & \longrightarrow & 1 \\
 & & \downarrow & & \Vert & & \downarrow \mathrm{res} & & \downarrow \mathrm{res}_\flat & & \\
0 & \longrightarrow & \mathrm{U}(T) & \longrightarrow & \mathrm{U} & \longrightarrow & \mathrm{Aut}_{B^{\sigma_G}}(\tilde{\mathrm{e}}\Phi) & \longrightarrow & \mathrm{Out}_{B^{\sigma_G}}(\tilde{\mathrm{e}}\Phi) & \longrightarrow & 1
\end{array}$$

of crossed 2-fold extensions. The crossed pair structure $\tilde\psi^{\Phi}\colon G \to \mathrm{Out}_{B^{\sigma_G}}(\tilde{\mathrm{e}}\Phi)$ is the composite

$$G \xrightarrow{\tilde\psi_G} \mathrm{Out}_{B^{\sigma_G}}(\tilde{\mathrm{e}}) \times_Q G \xrightarrow{\mathrm{res}_\flat} \mathrm{Out}_{B^{\sigma_G}}(\tilde{\mathrm{e}}\Phi)$$

of $\mathrm{res}_\flat$ with the canonical lift of the crossed pair structure $\tilde\psi\colon Q \to \mathrm{Out}_{B^{\sigma_G}}(\tilde{\mathrm{e}})$ on $\tilde{\mathrm{e}}$ to a homomorphism $\tilde\psi_G\colon G \to \mathrm{Out}_{B^{\sigma_G}}(\tilde{\mathrm{e}}) \times_Q G$; see [13, Propositions 2.3 and 2.4]. The identity $\Phi^*[(\tilde{\mathrm{e}}, \tilde\psi)] = [(\hat{\mathrm{e}}, \hat\psi)]$ means that the two crossed pairs $(\hat{\mathrm{e}}, \hat\psi)$ and $(\tilde{\mathrm{e}}\Phi, \tilde\psi^{\Phi})$ are congruent as crossed pairs. Thus we may take U to be $\mathrm{U}(A)$ such that the following hold:

- The injection $\mathrm{U}(A) \to \Gamma$ induces a morphism $\hat{\mathrm{e}} \to \tilde{\mathrm{e}}$ of group extensions whose restriction to $\mathrm{U}(T)$ is the identity, as displayed in diagram (14.27) above, and
- the crossed pair structure $\hat\psi\colon G \to \mathrm{Out}_{B^{\sigma_G}}(\hat{\mathrm{e}})$ on $\hat{\mathrm{e}}$ is the composite

$$G \xrightarrow{\tilde\psi_G} \mathrm{Out}_{B^{\sigma_G}}(\tilde{\mathrm{e}}) \xrightarrow{\mathrm{res}_\sharp} \mathrm{Out}_{B^{\sigma_G}}(\hat{\mathrm{e}}) \tag{14.28}$$

  of $\tilde\psi_G$ with the homomorphism $\mathrm{res}_\sharp\colon \mathrm{Out}_{B^{\sigma_G}}(\tilde{\mathrm{e}}) \times_Q G \to \mathrm{Out}_{B^{\sigma_G}}(\hat{\mathrm{e}})$ induced by the obvious restriction homomorphism $\mathrm{res}\colon \mathrm{Aut}_{B^{\sigma_G}}(\tilde{\mathrm{e}}) \longrightarrow \mathrm{Aut}_{B^{\sigma_G}}(\hat{\mathrm{e}})$.

The morphism $\hat{\mathrm{e}} \to \tilde{\mathrm{e}}$ of group extensions yields the commutative diagram (14.27) and, by construction, this is a commutative diagram of $\mathrm{Aut}_{B^{\sigma_G}}(\tilde{\mathrm{e}})$-groups. □

We continue the proof of Theorem 14.10. Maintaining the hypotheses of Proposition 14.13, we write

$$\mathrm{e}\colon 1 \longrightarrow \mathrm{U}(A) \xrightarrow{j} \Gamma \longrightarrow N \longrightarrow 1$$

for the group extension that arises as the middle column of diagram (14.27) and denote by $\vartheta\colon \Gamma \to \mathrm{Aut}(A)$ the combined homomorphism

$$\Gamma \longrightarrow B^{\sigma_N} \longrightarrow \mathrm{Aut}(A).$$

Consider the crossed product algebra $(A, N, \mathrm{e}, \vartheta)$. By construction

$$(A, N, \mathrm{e}, \vartheta) = A^t\Gamma / \langle a - j(a), a \in \mathrm{U}(A)\rangle,$$

cf. [14, Section 5]. By [14, Proposition 5.3(iv)], since $T|S$ is a Galois extension of commutative rings with Galois group N, the group Γ now gets identified with the normalizer $N^{\mathrm{U}(A,N,\mathrm{e},\vartheta)}(A)$ of A in the crossed product algebra $(A, N, \mathrm{e}, \vartheta)$.

Recall the notation B^{σ_G} for the fiber product group $\mathrm{Aut}(A) \times_{\mathrm{Out}(A)} G$ with respect to the given G-normal structure $\sigma_G\colon G \to \mathrm{Out}(A)$ on A, cf. [14, Subsection 4.4]. Furthermore, recall from Subsection 13.1 above that $\mathrm{Aut}_{B^{\sigma_G}}(\tilde{\mathrm{e}})$ denotes the subgroup of $\mathrm{Aut}(\Gamma) \times B^{\sigma_G}$ that consists of the pairs (α, x) which render the diagram

$$\begin{array}{ccccccccc} \tilde{\mathrm{e}}\colon 0 & \longrightarrow & \mathrm{U}(T) & \longrightarrow & \Gamma & \longrightarrow & B^{\sigma_N} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \ell_x} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle i_x} & & \\ \tilde{\mathrm{e}}\colon 0 & \longrightarrow & \mathrm{U}(T) & \longrightarrow & \Gamma & \longrightarrow & B^{\sigma_N} & \longrightarrow & 1 \end{array}$$

commutative; here, given $x \in B^{\sigma_G}$, the notation $i_x\colon B^{\sigma_N} \to B^{\sigma_N}$ refers to conjugation by $x \in B^{\sigma_G}$ and $\ell_x\colon \mathrm{U}(T) \to \mathrm{U}(T)$ to the canonical action of B^{σ_G} on $\mathrm{U}(T)$ (recall that T denotes the center of A) induced from the action of B^{σ_G} on A and hence on $\mathrm{U}(T)$ via the canonical homomorphism $B^{\sigma_G} \to \mathrm{Aut}(A)$.

Proposition 14.14 *Setting*

$$^{(\alpha,x)}(ay) = {}^{x}a^{\alpha}y,\ a \in A,\ y \in \Gamma, \tag{14.29}$$

where $(\alpha, x) \in Aut_{B^{\sigma_G}}(\tilde{\mathrm{e}})$ $(\subseteq Aut(\Gamma) \times B^{\sigma_G} \subseteq Aut(\Gamma) \times Aut(A) \times G)$, *we obtain a homomorphism*

$$\gamma\colon Aut_{B^{\sigma_G}}(\tilde{\mathrm{e}}) \longrightarrow Aut^{A}(A, N, \mathrm{e}, \vartheta),$$

and this homomorphism, in turn, yields morphisms

$$\begin{array}{ccccccccc} 0 \to & \mathrm{U}(S) & \longrightarrow & \Gamma & \longrightarrow & Aut_{B^{\sigma_G}}(\tilde{\mathrm{e}}) & \longrightarrow & Out_{B^{\sigma_G}}(\tilde{\mathrm{e}}) & \longrightarrow 1 \\ & \| & & \| & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma_\sharp} & \\ 0 \to & \mathrm{U}(S) & \to & N^{\mathrm{U}(A,N,\mathrm{e},\vartheta)}(A) & \to & Aut^{A}(A, N, \mathrm{e}, \vartheta) & \to & Out^{A}(A, N, \mathrm{e}, \vartheta) & \to 1 \end{array}$$

and

$$\begin{array}{ccccccccc} 0 \to & \mathrm{U}(S) & \to & \mathrm{U}(A) & \longrightarrow & Aut_{B^{\sigma_G}}(\tilde{\mathrm{e}}) & \longrightarrow & Out_{B^{\sigma_G}}(\tilde{\mathrm{e}}) \times_Q G & \longrightarrow 1 \\ & \| & & \| & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma_\flat} & \\ 0 \to & \mathrm{U}(S) & \to & \mathrm{U}(A) & \to & Aut^{A}(A, N, \mathrm{e}, \vartheta) & \to & Out((A, N, \mathrm{e}, \vartheta), A) & \to 1 \end{array}$$

of crossed 2-fold extensions. Furthermore, the homomorphisms $\gamma_\sharp$, $\gamma_\flat$, *and the obvious unlabeled homomorphisms render the diagram*

$$\begin{array}{ccc} Out_{B^{\sigma_G}}(\tilde{\mathrm{e}}) \times_Q G & \longrightarrow & Out_{B^{\sigma_G}}(\tilde{\mathrm{e}}) \\ \downarrow{\scriptstyle \gamma_\flat} & & \downarrow{\scriptstyle \gamma_\sharp} \\ Out((A, N, \mathrm{e}, \vartheta), A) & \longrightarrow & Out^{A}(A, N, \mathrm{e}, \vartheta) \end{array} \tag{14.30}$$

commutative.

Proof The left A-module that underlies the twisted group ring $A^t\Gamma$ is the free A-module having Γ as an A-basis, whence it is manifest that (14.29) yields an action of the group $\mathrm{Aut}_{B^{\sigma_G}}(\tilde{\mathrm{e}})$ on that left A-module.

Next we show that the $\mathrm{Aut}_{B^{\sigma G}}(\tilde{\mathrm{e}})$-action on the left A-module that underlies the twisted group ring $A^t\Gamma$ is compatible with the multiplicative structure of $A^t\Gamma$. To this end, consider the crossed module $(\Gamma, \mathrm{Aut}_{B^{\sigma G}}(\tilde{\mathrm{e}}), \beta)$, cf. the middle columns of the commutative diagram (13.3) above. Since $\beta\colon \Gamma \to \mathrm{Aut}_{B^{\sigma G}}(\tilde{\mathrm{e}})$ is a morphism of $\mathrm{Aut}_{B^{\sigma G}}(\tilde{\mathrm{e}})$-groups, given $y \in \Gamma$ and $(\alpha, x) \in \mathrm{Aut}_{B^{\sigma G}}(\tilde{\mathrm{e}})$,

$$\beta({}^{(\alpha,x)}y) = (\alpha, x)\beta(y)(\alpha, x)^{-1} \in \mathrm{Aut}_{B^{\sigma G}}(\tilde{\mathrm{e}}).$$

Let can: $\mathrm{Aut}_{B^{\sigma G}}(\tilde{\mathrm{e}}) \to \mathrm{Aut}(A)$ denote the canonical homomorphism. It is now manifest that the action $\vartheta\colon \Gamma \to \mathrm{Aut}(A)$ of Γ on A factors through β, that is, ϑ coincides with the combined homomorphism

$$\Gamma \xrightarrow{\beta} \mathrm{Aut}_{B^{\sigma G}}(\tilde{\mathrm{e}}) \xrightarrow{\mathrm{can}} \mathrm{Aut}(A).$$

Hence, given $(\alpha, x) \in \mathrm{Aut}_{B^{\sigma G}}(\tilde{\mathrm{e}})$ $(\subseteq \mathrm{Aut}(\Gamma) \times B^{\sigma G})$, $b \in A$, and $y \in \Gamma$,

$$ {}^{x\vartheta(y)x^{-1}}b = {}^{\vartheta({}^{\alpha}y)}b. \tag{14.31}$$

Thus, given $(\alpha, x) \in \mathrm{Aut}_{B^{\sigma G}}(\tilde{\mathrm{e}})$, $y \in \Gamma$, $a \in A$, in view of (14.31) we conclude

$$ {}^{(\alpha,x)}(ya) = {}^{(\alpha,x)}({}^{\vartheta(y)}a\, y) = ({}^{x\vartheta(y)x^{-1}x}a)^{\alpha}y = ({}^{\vartheta({}^{\alpha}y)x}a)^{\alpha}y = {}^{\alpha}y\,{}^{x}a.$$

Consequently (14.29) yields an action of $\mathrm{Aut}_{B^{\sigma G}}(\tilde{\mathrm{e}})$ on the algebra $A^t\Gamma$.

Finally, to show that the action of $\mathrm{Aut}_{B^{\sigma G}}(\tilde{\mathrm{e}})$ on the algebra $A^t\Gamma$ preserves the two-sided ideal $< a - j(a), a \in \mathrm{U}(A) >$ in $A^t\Gamma$, let $a \in \mathrm{U}(A)$ and $(\alpha, x) \in \mathrm{Aut}_{B^{\sigma G}}(\tilde{\mathrm{e}})$. In view of Proposition 14.13(ii), $j({}^{x}a) = {}^{\alpha}(j(a))$, whence

$$ {}^{(\alpha,x)}(a - j(a)) = ({}^{x}a - j({}^{x}a)).$$

□

With respect to the crossed pair structure $\tilde{\psi}\colon Q \to \mathrm{Out}_{B^{\sigma G}}(\tilde{\mathrm{e}})$ on $\tilde{\mathrm{e}}$, the fiber product group $B^{\tilde{\psi}} = \mathrm{Aut}_{B^{\sigma G}}(\tilde{\mathrm{e}}) \times_{\mathrm{Out}_{B^{\sigma G}}(\tilde{\mathrm{e}})} Q$ is defined. As before, we denote by $\tilde{\psi}_G\colon G \to \mathrm{Out}_{B^{\sigma G}}(\tilde{\mathrm{e}}) \times_Q G$ the canonical lift, into the fiber product group with respect to the surjection $\pi_Q\colon G \to Q$, of the crossed pair structure $\tilde{\psi}\colon Q \to \mathrm{Out}_{B^{\sigma G}}(\tilde{\mathrm{e}})$ on $\tilde{\mathrm{e}}$. Define $\chi_G\colon G \to \mathrm{Out}((A, N, \mathrm{e}, \vartheta), A)$ to be the combined homomorphism

$$\chi_G\colon G \xrightarrow{\tilde{\psi}_G} \mathrm{Out}_{B^{\sigma G}}(\tilde{\mathrm{e}}) \xrightarrow{\gamma_\flat} \mathrm{Out}((A, N, \mathrm{e}, \vartheta), A). \tag{14.32}$$

Moreover, the composite homomorphism

$$\sigma_Q\colon Q \xrightarrow{\tilde{\psi}} \mathrm{Out}_{B^{\sigma G}}(\tilde{\mathrm{e}}) \xrightarrow{\gamma_\sharp} \mathrm{Out}^A(A, N, \mathrm{e}, \vartheta)$$

yields a Q-normal structure $\sigma_Q\colon Q \to \mathrm{Out}^A(A, N, \mathrm{e}, \vartheta)$ on the central S-algebra $(A, N, \mathrm{e}, \vartheta)$. Denote by $i\colon \Gamma \to \mathrm{U}(A, N, \mathrm{e}, \vartheta)$ the inclusion and by $\tilde{\gamma}$ the combined homomorphism

$$\tilde{\gamma}\colon B^{\tilde{\psi}} \xrightarrow{\text{can}} \mathrm{Aut}_{B^{\sigma_G}}(\tilde{\mathrm{e}}) \xrightarrow{\gamma} \mathrm{Aut}^A(A, N, \mathrm{e}, \vartheta). \tag{14.33}$$

Proposition 14.15 *Write $C = (A, N, \mathrm{e}, \vartheta)$. The homomorphisms $\sigma_Q, \kappa_G, \chi_G, \sigma_G, i$, and $\tilde{\gamma}$ match in the following sense.*
(i) *The homomorphisms σ_Q and χ_G yield a commutative diagram*

$$\begin{array}{lcccccccc} \mathrm{e}_{(T|S)}\colon 1 & \longrightarrow & N & \longrightarrow & G & \xrightarrow{\pi_Q} & Q & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \chi_N} & & \downarrow{\scriptstyle \chi_G} & & \downarrow{\scriptstyle \chi_Q} & & \\ \mathrm{e}_{(A,C)}\colon 1 & \longrightarrow & N^{\mathrm{U}(C)}(A)/\mathrm{U}(A) & \longrightarrow & Out(C, A) & \longrightarrow & Out^A(C) & \longrightarrow & 1 \end{array} \tag{14.34}$$

with exact rows.
(ii) *The composite homomorphism*

$$G \xrightarrow{\chi_G} Out((A, N, \mathrm{e}, \vartheta), A) \xrightarrow{res} Out(A) \tag{14.35}$$

coincides with $\sigma_G\colon G \to Out(A)$.
(iii) *The two homomorphisms i and $\tilde{\gamma}$ yield a morphism of crossed 2-fold extensions*

$$\begin{array}{ccccccccccc} 1 & \longrightarrow & \mathrm{U}(S) & \longrightarrow & \Gamma & \longrightarrow & B^{\tilde{\psi}} & \longrightarrow & Q & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \tilde{\gamma}} & & \downarrow{\scriptstyle \sigma_Q} & & \\ 1 & \longrightarrow & \mathrm{U}(S) & \longrightarrow & \mathrm{U}(A, N, \mathrm{e}, \vartheta) & \longrightarrow & Aut^A(A, N, \mathrm{e}, \vartheta) & \longrightarrow & Out^A(A, N, \mathrm{e}, \vartheta) & \longrightarrow & 1 \end{array}$$

whence $(i, \tilde{\gamma})$ induces a congruence $(1, i, \cdot, 1)\colon \mathrm{e}_{\tilde{\psi}} \longrightarrow \mathrm{e}_{((A,N,\mathrm{e},\vartheta),\sigma_Q)}$ of crossed 2-fold extensions.
(iv) *The homomorphism $\chi_N\colon N \to N^{\mathrm{U}(C)}(A)/\mathrm{U}(A)$ turns the embedding of A into $C = (A, N, \mathrm{e}, \vartheta)$ into a strong N-normal Deuring embedding with respect to*

$$\mathrm{Id}\colon N \longrightarrow Aut(T|S).$$

Proof (i) It is obvious that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\pi_Q} & Q \\ \downarrow{\scriptstyle \tilde{\psi}_G} & & \downarrow{\scriptstyle \tilde{\psi}} \\ \mathrm{Out}_{B^{\sigma_G}}(\tilde{\mathrm{e}}) \times_Q G & \longrightarrow & \mathrm{Out}_{B^{\sigma_G}}(\tilde{\mathrm{e}}) \end{array}$$

is commutative. Combining this diagram with the commutative diagram (14.30), we obtain the right-hand square of (14.34). Since the lower row of that diagram is exact, the homomorphisms χ_G and σ_Q induce the requisite homomorphism

$$\chi_N\colon N \to N^{\mathrm{U}(C)}/\mathrm{U}(A).$$

(ii) Consider the diagram

$$\begin{array}{ccccc} G & \xrightarrow{\tilde\psi_G} & \mathrm{Out}_{B^{\sigma_G}}(\tilde{\mathrm{e}}) \times_Q G & \xrightarrow{\gamma_\flat} & \mathrm{Out}((A, N, \mathrm{e}, \vartheta), A) \\ & \searrow^{\hat\psi} & \downarrow \mathrm{res}_\sharp & & \downarrow \mathrm{res} \\ & & \mathrm{Out}_{B^{\sigma_G}}(\hat{\mathrm{e}}) & \xrightarrow{\gamma_\sharp} & \mathrm{Out}(A). \end{array}$$

The right-hand square is commutative in an obvious manner. The left-hand triangle is commutative since, as noted earlier, the composite (14.28) coincides with $\hat\psi$. The upper row yields the homomorphism $\chi_G\colon G \to \mathrm{Out}((A, N, \mathrm{e}, \vartheta), A)$, by the very definition (14.32) of χ_G.

As noted above, the composite (14.26), viz. $G \xrightarrow{\hat\psi} \mathrm{Out}_{B^{\sigma_G}}(\hat{\mathrm{e}}) \xrightarrow{\hat\gamma_\sharp} \mathrm{Out}(A)$, yields the given G-normal structure $\sigma_G\colon G \to \mathrm{Out}(A)$ on A. Consequently (14.35) coincides with the structure map $\sigma_G\colon G \to \mathrm{Out}(A)$ as asserted.
(iii) This is obvious.
(iv) Consider the commutative diagram

$$\begin{array}{ccccccccc} \mathrm{e}_{(T|S)}\colon 1 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & 1 \\ & & \downarrow \chi_N & & \downarrow \chi_G & & \downarrow \sigma_Q & & \\ \mathrm{e}_{(A,C)}\colon 1 & \longrightarrow & N^{\mathrm{U}(C)}(A)/\mathrm{U}(A) & \longrightarrow & \mathrm{Out}(C, A) & \longrightarrow & \mathrm{Out}^A(C) & \longrightarrow & 1 \\ & & \downarrow \eta_\flat & & \downarrow \mathrm{res} & & \downarrow \mathrm{res} & & \\ 1 & \longrightarrow & \mathrm{Aut}(T|S) & \longrightarrow & \mathrm{Aut}^S(T) & \xrightarrow{\mathrm{res}} & \mathrm{Aut}(S). & & \end{array}$$

By construction, the outer-most diagram coincides with the commutative diagram (12.1), and the left-most column is the composite [14, (4.9)], with N substituted for Q and $\mathrm{Aut}(T|S)$ for $\mathrm{Aut}(S)$. Consequently the composite

$$\eta_\flat \circ \chi_N\colon N \longrightarrow \mathrm{Aut}(T|S)$$

is the identity. Since $T|S$ is a Galois extension of commutative rings with Galois group N, by [14, Proposition 5.3(ii)], the algebra A coincides with the centralizer of T in $C = (A, N, \mathrm{e}, \vartheta)$ whence, by [14, Proposition 4.11(iii)], the homomorphism $\eta_\flat$ is injective. Consequently $\eta_\flat$ and χ_N are isomorphisms, and $\chi_N\colon N \to N^{\mathrm{U}(C)}(A)/\mathrm{U}(A)$ turns the embedding of A into C into a strong N-normal Deuring embedding with respect to $\mathrm{Id}\colon N \to \mathrm{Aut}(T|S)$. □

We can now complete the proof of Proposition 14.13: Since the structure homomorphism $\kappa_G\colon G \to \mathrm{Out}(A)$ is a G-normal structure relative to the action

$$\kappa_G\colon G \longrightarrow \mathrm{Aut}^S(T)$$

of G on T, by definition, the composite homomorphism

$$G \stackrel{\sigma_G}{\longrightarrow} \mathrm{Out}(A) \stackrel{\mathrm{res}}{\longrightarrow} \mathrm{Aut}^S(T)$$

coincides with $\kappa_G\colon G \to \mathrm{Aut}^S(T)$; since, by Proposition 14.15(ii), the homomorphism (14.35) coincides with $\sigma_G\colon G \to \mathrm{Out}(A)$, we conclude that the composite

$$G \stackrel{\chi_G}{\longrightarrow} \mathrm{Out}((A, N, \mathrm{e}, \vartheta), A) \stackrel{\mathrm{res}}{\longrightarrow} \mathrm{Aut}^S(T)$$

coincides with κ_G, cf. (14.11).

By Proposition 14.15(i), the diagram (14.34) is commutative, and by Proposition 14.15(iv), the homomorphism $\chi_N\colon N \to N^{\mathrm{U}(C)}(A)/\mathrm{U}(A)$ turns the embedding of A into C into a strong N-normal Deuring embedding with respect to

$$\mathrm{Id}\colon N \longrightarrow \mathrm{Aut}(T|S).$$

Consequently, cf. (14.32), the homomorphism

$$\chi_G\colon G \longrightarrow \mathrm{Out}((A, N, \mathrm{e}, \vartheta), A)$$

turns the embedding of A into $(A, N, \mathrm{e}, \vartheta)$ into a strong Q-normal Deuring embedding with respect to the Q-normal structure $\sigma_Q\colon Q \to \mathrm{Out}(A, N, \mathrm{e}, \vartheta)$ on $(A, N, \mathrm{e}, \vartheta)$ and the structure extension (14.1).

Proposition 14.15(ii) says that the G-normal structure $G \to \mathrm{Out}(A)$ on A associated to the strong Q-normal Deuring embedding, cf. (14.12), coincides with the given G-normal structure $\sigma_G\colon G \to \mathrm{Out}(A)$ on A.

Propositions 14.13 (i) and 14.15(iii) together entail that the Q-normal S-algebra $((A, N, \mathrm{e}, \vartheta), \sigma_Q)$ has Teichmüller class k as asserted since the crossed 2-fold extension $\mathrm{e}_{\tilde{\psi}}$ represents k.

The proof of Theorem 14.10 is now complete. □

Proof of Complement 14.11. This follows from [14, Proposition 5.4 (xi)]. □

Recall that $B^{\tilde{\psi}}$ denotes the fiber product group $B^{\tilde{\psi}} = \mathrm{Aut}_{B^{\sigma_G}}(\tilde{\mathrm{e}}) \times_{\mathrm{Out}_{B^{\sigma_G}}(\tilde{\mathrm{e}})} Q$ with respect to the crossed pair structure $\tilde{\psi}\colon Q \to \mathrm{Out}_{B^{\sigma_G}}(\tilde{\mathrm{e}})$ on $\tilde{\mathrm{e}}$, and that, likewise, B^{A,σ_Q} denotes the fiber product group

$$B^{A,\sigma_Q} = \mathrm{Aut}^A(A, N, \mathrm{e}, \vartheta) \times_{\mathrm{Out}(A,N,\mathrm{e},\vartheta)} Q$$

with respect to the Q-normal structure $\sigma_Q\colon Q \to \mathrm{Out}(A, N, \mathrm{e}, \vartheta)$ on $(A, N, \mathrm{e}, \vartheta)$.

Complement 14.16 *The canonical homomorphism* $B^{\tilde\psi} \longrightarrow B^{A,\sigma_Q}$ *induced by the action* $\tilde\gamma\colon B^{\tilde\psi} \longrightarrow Aut^A(A, N, \mathrm{e}, \vartheta)$ *of* $B^{\tilde\psi}$ *on the crossed product algebra* $(A, N, \mathrm{e}, \vartheta)$*, cf.* (14.33) *above, and the surjection* $B^{\tilde\psi} \longrightarrow Q$ *is an isomorphism.*

Proof The homomorphism $B^{\tilde\psi} \to B^{A,\sigma_Q}$ makes the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{U}(S) & \longrightarrow & \Gamma & \longrightarrow & B^{\tilde\psi} & \longrightarrow & Q & \longrightarrow & 1 \\
 & & \| & & \| & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathrm{U}(S) & \longrightarrow & N^{\mathrm{U}(A,N,\mathrm{e},\vartheta)}(A) & \longrightarrow & B^{A,\sigma_Q} & \longrightarrow & Q & \longrightarrow & 1
\end{array}$$

commutative whence the homomorphism $B^{\tilde\psi} \to B^{A,\sigma_Q}$ is an isomorphism. □

15 Behavior of the crossed Brauer group under Q-normal Galois extensions

Consider a Q-normal Galois extension $T|S$ of commutative rings, with structure extension $\mathrm{e}_{(T|S)}\colon N \rightarrowtail G \overset{\pi_Q}{\twoheadrightarrow} Q$ and structure homomorphism $\kappa_G\colon G \to \mathrm{Aut}^S(T)$, cf. Section 12 above, and denote the injection of S into T by $i\colon S \to T$. Then the abelian group $\mathrm{XB}(T|S; G, Q)$ is defined relative to the associated morphism

$$(i, \pi_Q)\colon (S, Q, \kappa_Q) \longrightarrow (T, G, \kappa_G)$$

in the change of actions category $\mathcal{C}hange$, cf. (12.3) above.

Theorem 15.1 *The sequence*

$$\mathrm{XB}(T|S; G, Q) \xrightarrow{t} \mathrm{H}^3(Q, \mathrm{U}(S)) \xrightarrow{\inf} \mathrm{H}^3(G, \mathrm{U}(T))$$

is exact and, furthermore, natural in the data. Moreover, each class in the image of t *is also the Teichmüller class of some crossed pair algebra.*

Proof The naturality of the constructions entails that $\inf \circ t = 0$. Moreover, by Theorem 13.5, $\ker(\inf) \subset \mathrm{im}(t)$, and each class in the image of t comes from some crossed pair algebra. □

Let $\mathrm{Pic}(T|S)$ denote the kernel of the homomorphism $\mathrm{Pic}(S) \to \mathrm{Pic}(T)$ induced by $i\colon S \to T$. Our next aim is to construct a homomorphism from $\mathrm{H}^1(Q, \mathrm{Pic}(T|S))$ to $\mathrm{XB}(T|S; G, Q)$. To this end, view T as an S-module in the obvious way and let $A = \mathrm{End}_S(T)$. Now, given an automorphism α of A so that $\alpha|S$ is the identity, as above we can turn T into a new A-module ${}^{\alpha}T$ be means of α, and $J(\alpha) = \mathrm{Hom}_A({}^{\alpha}T, T)$ is a faithful finitely generated projective rank one S-module; since $A \otimes T$ is a matrix algebra, $J(\alpha)$ represents a member of $\mathrm{Pic}(T|S)$, and the association $\alpha \mapsto [J(\alpha)]$ yields a homomorphism $\mathrm{Aut}(A|S) \to \mathrm{Pic}(T|S)$ which we claim to be surjective. In order to justify this claim, we first observe that the obvious map $j\colon T^t N \to A$, as explained in [14, Section 2], is an isomorphism, since $T|S$ is a Galois extension of commutative rings with Galois group N. Now, given a derivation $d\colon N \to \mathrm{U}(T)$, define the automorphism α_d of $T^t N$ by

$$\alpha_d(tn) = d(n)tn,\ t \in T,\ n \in N.$$

Then

$$\mathrm{Der}(N, \mathrm{U}(T)) \longrightarrow \mathrm{Aut}(T^t N|S),\ d \longmapsto \alpha_d,$$

is a homomorphism, and $[J(\alpha_d)] \in \mathrm{Pic}(T|S)$ is the image of $[d] \in \mathrm{H}^1(N, \mathrm{U}(T))$ under the standard isomorphism $\mathrm{H}^1(N, \mathrm{U}(T)) \to \mathrm{Pic}(T|S)$ (with N and T substituted for Q and S, respectively, this is, e.g., a consequence of the exactness of [14, (10.1)] at the second term). Hence the homomorphism $\mathrm{Aut}(A|S) \to \mathrm{Pic}(T|S)$ is surjective as asserted. Consequently the obvious homomorphism from $\mathrm{Aut}(A|S)$ to $\mathrm{Aut}(A, Q)$ fits into a commutative diagram

$$\begin{array}{ccccc} \mathrm{Aut}(A|S) & \longrightarrow & \mathrm{Pic}(T|S) & \xrightarrow{\cong} & \mathrm{Out}(A|S) \\ \downarrow & & \downarrow & & \\ \mathrm{Aut}(A, Q) & \longrightarrow & \mathrm{Out}(A, Q) & & \end{array}$$

where the horizontal maps are surjective. Since the G-action on T and that on N induce a canonical section $\sigma_0\colon Q \to \mathrm{Out}(A, Q)$, the canonical homomorphism from $\mathrm{Out}(A, Q)$ to Q is surjective as well. Consequently the sequence

$$0 \longrightarrow \mathrm{Pic}(T|S) \longrightarrow \mathrm{Out}(A, Q) \longrightarrow Q \longrightarrow 1$$

is exact. Now, given a derivation $d\colon Q \to \mathrm{Pic}(T|S)$, define the homomorphism

$$\sigma_d\colon Q \longrightarrow \mathrm{Out}(A, Q)$$

by $\sigma(q) = d(q)\sigma_0(q)$, as q ranges over Q. Then (A, σ_d) is a Q-normal Azumaya S-algebra.

We mention without proof the following.

Theorem 15.2 *The association $d \mapsto (End_S(T), \sigma_d)$, as d ranges over derivations from Q to $Pic(T|S)$, yields a natural isomorphism*

$$\mathrm{H}^1(Q, Pic(T|S)) \longrightarrow \mathrm{XB}(S|S; \{e\}, Q) \cap \mathrm{XB}(T|S; G, Q)$$

of abelian groups in such a way that the resulting sequence

$$0 \longrightarrow \mathrm{H}^1(Q, Pic(T|S)) \longrightarrow \mathrm{XB}(T|S; G, Q) \longrightarrow \mathrm{H}^0(Q, \mathrm{B}(T|S)) \qquad (15.1)$$

is exact.

16 Relative theory and equivariant Brauer group

Given a morphism $(f, \varphi)\colon (S, Q, \kappa) \to (T, G, \lambda)$ in the change of actions category *Change* introduced in [14, Subsection 3.7], we denote by $\mathrm{EB}(T|S; G, Q)$ the kernel of the combined map

$$\mathrm{EB}(S, Q) \longrightarrow \mathrm{XB}(S, Q) \longrightarrow \mathrm{XB}(T, G);$$

this kernel $\mathrm{EB}(T|S; G, Q)$ is the subgroup of $\mathrm{EB}(S, Q)$ that consists of classes of Q-equivariant S-algebras (A, τ) so that $(A \otimes T, \tau_{(f,\varphi)})$ is an induced G-normal split algebra and hence, in view of [14, Corollary 7.7], an induced G-equivariant split algebra; see [14, Proposition 4.10(ii)] for the notation $\tau_{(f,\varphi)}$. Thus, in particular, $\mathrm{EB}(S|S; Q, Q)$ is the kernel of the canonical homomorphism

$$\mathrm{EB}(S, Q) \longrightarrow \mathrm{XB}(S, Q)$$

whereas $\mathrm{EB}(S|S; \{e\}, Q)$ is the kernel of the forgetful homomorphism from $\mathrm{EB}(S, Q)$ to $\mathrm{B}(S)$. It is obvious that the restriction homomorphism

$$\mathrm{res}\colon \mathrm{EB}(S, Q) \longrightarrow \mathrm{XB}(S, Q)$$

induces a homomorphism

$$\mathrm{res}\colon \mathrm{EB}(T|S; G, Q) \longrightarrow \mathrm{XB}(T|S; G, Q).$$

Consider a Q-normal Galois extension $T|S$ of commutative rings, with structure extension $\mathrm{e}_{(T|S)}\colon N \rightarrowtail G \overset{\pi_Q}{\twoheadrightarrow} Q$ and structure homomorphism $\kappa_G\colon G \to \mathrm{Aut}^S(T)$, cf. Section 12 above, and denote the injection of S into T by $i\colon S \to T$. Then the abelian groups $\mathrm{EB}(T|S; G, Q)$ and $\mathrm{XB}(T|S; G, Q)$ are defined relative to the morphism $(i, \pi_Q)\colon (S, Q, \kappa_Q) \longrightarrow (T, G, \kappa_G)$ in the change of actions category *Change* associated with the data, cf. (12.3) above.

Theorem 16.1 *Suppose that Q is a finite group. Then the sequence*

$$\mathrm{EB}(T|S; G, Q) \overset{res}{\to} \mathrm{XB}(T|S; G, Q) \overset{t}{\to} \mathrm{H}^3(Q, \mathrm{U}(S)) \overset{\inf}{\to} \mathrm{H}^3(G, \mathrm{U}(T)) \quad (16.1)$$

is exact and natural.

Proof The statement of the theorem is a consequence of [14, Theorems 6.1, 8.1 (ii), 9.1] and Theorems 14.9 and 15.1.

For if (A, σ) represents a member of $\mathrm{XB}(T|S; G, Q)$ with zero Teichmüller class, by [14, Theorem 6.1], we may assume (A, σ) to be equivariant, i.e., $\sigma = \sigma_\tau$ for some equivariant structure τ. Now the G-normal algebra $(A \otimes T, \sigma_{(i,\pi^G)})$ represents zero in $\mathrm{XB}(T, G)$ and hence is an induced G-normal split algebra, by [14, Theorem 8.1 (ii)]. By [14, Corollary 7.7], $(A \otimes T, \tau_{(i,\pi^G)})$ is an induced G-equivariant split algebra. □

Let $R = S^Q$, let $\mathrm{e}_G\colon \mathrm{U}(T) \overset{i_{\mathrm{e}_G}}{\rightarrowtail} \Gamma_G \overset{\pi_{\mathrm{e}_G}}{\twoheadrightarrow} G$ be a group extension, and denote the restriction to N of the group extension e_G by $\mathrm{e}_N\colon \mathrm{U}(T) \rightarrowtail \Gamma_N \overset{\pi_{\mathrm{e}_N}}{\twoheadrightarrow} N$. Then the crossed product S-algebra $A = (T, N, \mathrm{e}_N, \pi_{\mathrm{e}_N})$ and the crossed product R-algebra

$$(T, G, \mathrm{e}_G, \kappa_G \circ \pi_{\mathrm{e}_G})$$

are defined, the former being an Azumaya S-algebra, since $T|S$ is a Galois extension of commutative rings with Galois group N (cf. [14, Proposition 5.4(xi)]), and $(T, G, \mathrm{e}_G, \kappa_G \circ \pi_{\mathrm{e}_G})$ contains A as a subalgebra. Consider the resulting group extension $\mathrm{e}_Q\colon \Gamma_N \overset{j_{\mathrm{e}_Q}}{\rightarrowtail} \Gamma_G \overset{\pi_{\mathrm{e}_Q}}{\twoheadrightarrow} Q$, of the kind [14, (5.1)], and introduce the notation $i^{\Gamma_N}\colon \Gamma_N \to \mathrm{U}(A)$ for the obvious injection. Conjugation in Γ_G induces an action $\vartheta_{\mathrm{e}_G}\colon \Gamma_G \to \mathrm{Aut}(A)$ of Γ_G on A such that the pair $(i^{\Gamma_N}, \vartheta_{\mathrm{e}_G})$ is a morphism $(\Gamma_N, \Gamma_G, j_{\mathrm{e}_Q}) \to (\mathrm{U}(A), \mathrm{Aut}(A), \partial)$ of crossed modules of the kind [14, (5.2)], and this morphism, in turn, induces a Q-normal structure $\sigma_{\vartheta_{\mathrm{e}_G}}\colon Q \to \mathrm{Out}(A)$ on A; thus the crossed product R-algebra

$$(T, G, \mathrm{e}_G, \kappa_G \circ \pi_{\mathrm{e}_G})$$

can now be written as the crossed product R-algebra $(A, Q, \mathrm{e}_Q, \vartheta_{\mathrm{e}_G})$ relative to the group extension e_Q and the morphism $(i^{\Gamma_N}, \vartheta_{\mathrm{e}_G})$ of crossed modules, cf. [14, Section 5]. In particular, the left A-module M_{e_Q} that underlies the algebra

$$(T, G, \mathrm{e}_G, \kappa_G \circ \pi_{\mathrm{e}_G}) \cong (A, Q, \mathrm{e}_Q, \vartheta_{\mathrm{e}_G})$$

is free with basis in one-one correspondence with the elements of Q, and the Q-equivariant structure $\tau_{\mathrm{e}_Q}\colon Q \longrightarrow \mathrm{Aut}({}_A\mathrm{End}(M_{\mathrm{e}_Q}))$ given as [14, (5.5)] is defined. When the group Q is finite, the algebra ${}_A\mathrm{End}(M_{\mathrm{e}_Q})$ is an Azumaya S-algebra.

Proposition 16.2 *Suppose that the group Q is finite. Then the assignment to a group extension e_G of G by $\mathrm{U}(T)$ of the Q-equivariant algebra $({}_A End(M_{\mathrm{e}_Q})^{\mathrm{op}}, \tau_{\mathrm{e}_Q}^{\mathrm{op}})$ yields a homomorphism*

$$\mathrm{cpr}\colon \mathrm{H}^2(G, \mathrm{U}(T)) \longrightarrow \mathrm{EB}(T|S; G, Q) \tag{16.2}$$

of abelian groups that is natural on the change of actions category $\mathcal{C}hange$. In the special case where $T = S$ and N is the trivial group, the homomorphism (16.2) *comes essentially down to* [14, (9.2)], *viz.*

$$\mathrm{cpr}\colon \mathrm{H}^2(Q, \mathrm{U}(S)) \longrightarrow \mathrm{EB}(S|S; Q, Q). \tag{16.3}$$

□

17 The eight term exact sequence

Given a morphism $(f,\varphi)\colon (S,Q,\kappa)\to(T,G,\lambda)$ in the change of actions category Change introduced in [14, Subsection 3.7], the group Q being finite, the corresponding relative version of the exact sequence [14, (10.1)] takes the following form:

$$\begin{aligned}\cdots \xrightarrow{\omega_{\mathcal{P}ic_{S,Q}}} \mathrm{H}^2(Q,\mathrm{U}(S)) &\xrightarrow{\mathrm{cpr}} \mathrm{EB}(T|S;G,Q)\\ &\xrightarrow{\mathrm{res}} \mathrm{XB}(T|S;G,Q) \xrightarrow{t} \mathrm{H}^3(Q,\mathrm{U}(S))\end{aligned} \tag{17.1}$$

Remark 17.1 In the special case where $T=S$ and G is the trivial group, in view of the isomorphism [14, (8.3)] from $\mathrm{XB}(S|S;\{e\},Q)$ onto $\mathrm{H}^1(Q,\mathrm{Pic}(S))$, the sequence (17.1) has the form of the C(hase-)R(osenberg-)A(uslander-)B(rumer) sequence [3, Theorem 7.6 p. 62], [1]. Other versions of the CRAB-sequence were obtained by Childs [4, Theorem 2.2], Fröhlich and Wall [7, Theorem 1], [6], [9, Theorem 4.2] (upper and middle long sequence), Hattori [10], Kanzaki [16], Ulbrich [18], Yokogawa [20], and Villamayor-Zelinski [19].

Consider a Q-normal Galois extension $T|S$ of commutative rings, with structure extension $\mathrm{e}_{(T|S)}\colon N \rightarrowtail G \overset{\pi_Q}{\twoheadrightarrow} Q$ and structure homomorphism $\kappa_G\colon G\to \mathrm{Aut}^S(T)$, cf. Section 12 above, and denote the injection of S into T by $i\colon S\to T$. Then the abelian groups $\mathrm{EB}(T|S;G,Q)$ and $\mathrm{XB}(T|S;G,Q)$ are defined relative to the morphism $(i,\pi_Q)\colon (S,Q,\kappa_Q)\longrightarrow (T,G,\kappa_G)$ in the change of actions category Change associated with the data, cf. (12.3) above.

Theorem 17.2 *The group Q being finite, the extension*

$$\begin{aligned}0 \to \mathrm{H}^1(Q,\mathrm{U}(S)) &\overset{j_{\mathcal{P}ic_{S,Q}}}{\to} \mathrm{EPic}(S,Q) \overset{\mu_{\mathcal{P}ic_{S,Q}}}{\to} (Pic(S))^Q \overset{\omega_{\mathcal{P}ic_{S,Q}}}{\to} \mathrm{H}^2(Q,\mathrm{U}(S))\\ \overset{\mathrm{cpr}}{\to} \mathrm{EB}(T|S;G,Q) &\overset{res}{\to} \mathrm{XB}(T|S;G,Q) \overset{t}{\to} \mathrm{H}^3(Q,\mathrm{U}(S)) \overset{\inf}{\to} \mathrm{H}^3(G,\mathrm{U}(T))\end{aligned} \tag{17.2}$$

of the exact sequence [14, (3.15)] *is defined and yields an eight term exact sequence that is natural in terms of the data. If, furthermore, $S|R$ and $T|R$ are Galois extensions of commutative rings over $R=S^Q=T^G$, with Galois groups Q and G, respectively, then, with $Pic(S|R)$, $Pic(R)$ and $\mathrm{B}(T|R)$ substituted for, respectively $\mathrm{H}^1(Q,\mathrm{U}(S))$, $\mathrm{EPic}(S,Q)$ and $\mathrm{EB}(S,Q)$, where $R=S^Q$, the homomorphisms* cpr *and* res *being modified accordingly, the sequence is exact as well.*

Proof This is an immediate consequence of Theorem 16.1 and [14, Theorem 10.1]. □

Remark 17.3 In terms of the notation $B_0(R;\Gamma)$ for the group that corresponds to our $\mathrm{EB}(S|S;Q,Q)$ (where our notation Q and S corresponds to Γ and R, respectively), a homomorphism of the kind (16.3) above is given in [9, Theorem 4.2]. After the statement of Theorem 4.2, the authors of [9] remark that there is no direct construction for the map from $\mathrm{H}^2(\Gamma;\mathrm{U}(R))$ to $B_0(R;\Gamma)$. Our construction of (16.3) is direct, however.

Remark 17.4 In the special case where $T|S|R$ are ordinary Galois extensions of fields, the exact sequence boils down to the classical low degree five term exact sequence

$$\begin{aligned} 0 \to \mathrm{H}^2(Q, \mathrm{U}(S)) \to \mathrm{H}^2(G, \mathrm{U}(T)) \to \mathrm{H}^2(N, \mathrm{U}(T))^Q \\ \to \mathrm{H}^3(Q, \mathrm{U}(S)) \to \mathrm{H}^3(G, \mathrm{U}(T)), \end{aligned} \tag{17.3}$$

see [11, p. 130].

18 Relationship with the eight term exact sequence in the cohomology of a group extension

Let $T|S$ be a Q-normal Galois extension of commutative rings, with structure extension $\mathrm{e}_{(T|S)}\colon N \rightarrowtail G \overset{\pi_Q}{\twoheadrightarrow} Q$ and structure homomorphism $\kappa_G\colon G \to \mathrm{Aut}^S(T)$, cf. Section 12 above; in particular, N is a finite group. Since $\mathrm{U}(T)^N$ coincides with $\mathrm{U}(S)$, the eight term exact sequence in [13] associated with the group extension $\mathrm{e}_{(T|S)}$ and the G-module $\mathrm{U}(T)$, reproduced as (13.4) above, has the following form:

$$\begin{aligned} 0 \longrightarrow & \mathrm{H}^1(Q, \mathrm{U}(S)) \xrightarrow{\mathrm{inf}} \mathrm{H}^1(G, \mathrm{U}(T)) \xrightarrow{\mathrm{res}} \mathrm{H}^1(N, \mathrm{U}(T))^Q \\ \xrightarrow{\Delta} & \mathrm{H}^2(Q, \mathrm{U}(S)) \xrightarrow{\mathrm{inf}} \mathrm{H}^2(G, \mathrm{U}(T)) \xrightarrow{j} \mathrm{Xpext}(G, N; \mathrm{U}(T)) \\ \xrightarrow{\Delta} & \mathrm{H}^3(Q, \mathrm{U}(S)) \xrightarrow{\mathrm{inf}} \mathrm{H}^3(G, \mathrm{U}(T)). \end{aligned} \tag{18.1}$$

18.1 Relationship between the two long exact sequences

Consider the morphism $(i, \pi_Q)\colon (S, Q, \kappa_Q) \longrightarrow (T, G, \kappa_G)$ associated to the given Q-normal Galois extension, cf. 12.3, in the change of actions category $\mathcal{C}hange$ introduced in [14, Subsection 3.7]. The abelian groups $\mathrm{EB}(T|S;\, G, Q)$ and $\mathrm{XB}(T|S;\, G, Q)$ are now defined relative to this morphism.

The assignment to a crossed pair $(\mathrm{e}\colon \mathrm{U}(T) \rightarrowtail \Gamma \twoheadrightarrow N,\ \psi\colon Q \to \mathrm{Out}_G(\mathrm{e}))$ with respect to $\mathrm{e}_{(T|S)}$ and $\mathrm{U}(T)$ of its associated crossed pair algebra $(A_{\mathrm{e}}, \sigma_\psi)$, cf. Section 13 above, yields a homomorphism

$$\mathrm{cpa}\colon \mathrm{Xpext}(G, N; \mathrm{U}(T)) \longrightarrow \mathrm{XB}(T|S;\, G, Q). \tag{18.2}$$

Let $\mathrm{EPic}(T|S, Q)$ denote the kernel of the induced homomorphisms

$$\mathrm{EPic}(S, Q) \xrightarrow{\mu_{\mathcal{P}ic_{S,Q}}} \mathrm{Pic}(S) \xrightarrow{i_*} \mathrm{Pic}(T)$$

and $\mathrm{Pic}(T|S)$ that of the induced homomorphism $i_*\colon \mathrm{Pic}(S) \to \mathrm{Pic}(T)$. With T and G substituted for S and Q, respectively, the isomorphism [14, (3.17)] takes the form

$$j_{\mathcal{P}ic_{T,G}}\colon \mathrm{H}^1(G, \mathrm{U}(T)) \longrightarrow \mathrm{EPic}(T|T, G), \tag{18.3}$$

and Galois descent, cf. [14, Subsection 2.2 (ii)], yields an isomorphism

$$\mathrm{EPic}(T|S, Q) \longrightarrow \mathrm{EPic}(T|T, G)$$

whence (18.3) induces a homomorphism

$$\mathrm{H}^1(G, \mathrm{U}(T)) \longrightarrow \mathrm{EPic}(T|S, Q) \tag{18.4}$$

of abelian groups. The homomorphism (18.4) admits, of course, a straightforward direct description. Likewise, with T and N substituted for Q and S, respectively, the isomorphism [14, (3.17)] takes the form

$$j_{\mathcal{P}ic_{T,N}} : \mathrm{H}^1(N, \mathrm{U}(T)) \longrightarrow \mathrm{EPic}(T|T, N), \tag{18.5}$$

and Galois descent, cf. [14, Subsection 2.2 (ii)], yields an isomorphism

$$\mathrm{Pic}(T|S) \longrightarrow \mathrm{EPic}(T|T, N)$$

whence (18.5) induces an isomorphism

$$\mathrm{H}^1(N, \mathrm{U}(T)) \longrightarrow \mathrm{Pic}(T|S) \tag{18.6}$$

of abelian groups, necessarily compatible with the Q-module structures; the isomorphism (18.6) is entirely classical. Below we do not distinguish in notation between (18.4) and its composite

$$\mathrm{H}^1(G, \mathrm{U}(T)) \longrightarrow \mathrm{EPic}(T, Q)$$

with the canonical injection of $\mathrm{EPic}(T|S, Q)$ into $\mathrm{EPic}(T, Q)$, nor between (18.6) and its composite $\mathrm{H}^1(N, \mathrm{U}(T)) \to \mathrm{Pic}(S)$ with the canonical injection $\mathrm{Pic}(T|S) \to \mathrm{Pic}(S)$. Direct inspection establishes the following.

Theorem 18.1 *The group Q being finite, the homomorphisms* (18.4), (18.6), (16.2), *and* (18.2) *of abelian groups are natural on the category $\mathcal{C}hange$ and induce a morphism of exact sequences from* (18.1) *to* (17.2). □

Remark 18.2 Consider the classical case where R, S, and T are fields. Now the group $\mathrm{Xpext}(G, N; \mathrm{U}(T))$ comes down to $\mathrm{H}^2(N, \mathrm{U}(T))^Q$ and, likewise, the group $\mathrm{XB}(T|S; G, Q)$ to $\mathrm{B}(T|S)^Q$, and (18.2) boils down to the classical isomorphism

$$\mathrm{H}^2(N, \mathrm{U}(T))^Q \to \mathrm{B}(T|S)^Q.$$

Furthermore, the groups $\mathrm{H}^1(N, \mathrm{U}(T))$, $\mathrm{H}^1(G, \mathrm{U}(T))$, $\mathrm{EPic}(T|S, Q)$, and $\mathrm{Pic}(T|S)$ are zero, and (16.2) is an isomorphism. Thus the morphism (18.1) → (17.2) of exact sequences in Theorem 18.1 above is then an isomorphism of exact sequences.

18.2 An application

Let $T|S$ be a Galois extension of commutative rings, with Galois group N, suppose that T carries a Q-action that extends the given Q-action on S, and define the group $\mathrm{EB}(T|S, Q)$ to be the kernel of the induced homomorphism

$$\mathrm{EB}(S, Q) \longrightarrow \mathrm{EB}(T, Q).$$

Relative to the induced Q-action on N, the semi-direct product group $N \rtimes Q$ is defined, and $T|S$ is a Q-normal Galois extension of rings, having as structure extension the split extension $\mathrm{e}_{(T|S)} \colon N \rightarrowtail N \rtimes Q \twoheadrightarrow Q$. Consider the commutative diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{EB}(\widetilde{T|S}, Q) & \longrightarrow & \mathrm{EB}(T|S; N \rtimes Q, Q) & \longrightarrow & \mathrm{EB}(T|T; Q, Q) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{EB}(T|S, Q) & \longrightarrow & \mathrm{EB}(S, Q) & \longrightarrow & \mathrm{EB}(T, Q) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{XB}(T; Q, N \rtimes Q) & \longrightarrow & \mathrm{XB}(T, N \rtimes Q) & \longrightarrow & \mathrm{XB}(T, Q) \end{array} \tag{18.7}$$

of abelian groups with exact rows and columns, the abelian group $\mathrm{EB}(\widetilde{T|S}, Q)$, necessarily (isomorphic to) a subgroup of $\mathrm{EB}(T|S, Q)$, being defined by the requirement that the upper row be exact.

The group N being finite, suppose now that Q is a finite group as well. The corresponding homomorphism (16.2), viz.

$$\mathrm{cpr} \colon \mathrm{H}^2(N \rtimes Q, \mathrm{U}(T)) \longrightarrow \mathrm{EB}(T|S; N \rtimes Q, Q),$$

and the homomorphism (16.3), with T substituted for S, viz.

$$\mathrm{cpr} \colon \mathrm{H}^2(Q, \mathrm{U}(T)) \longrightarrow \mathrm{EB}(T|T; Q, Q),$$

yield the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker(\mathrm{res}) & \longrightarrow & \mathrm{H}^2(N \rtimes Q, \mathrm{U}(T)) & \xrightarrow{\mathrm{res}} & \mathrm{H}^2(Q, \mathrm{U}(T)) \\ & & \downarrow & & \big\downarrow{\scriptstyle \mathrm{cpr}} & & \big\downarrow{\scriptstyle \mathrm{cpr}} \\ 0 & \longrightarrow & \mathrm{EB}(\widetilde{T|S}, Q) & \longrightarrow & \mathrm{EB}(T|S; N \rtimes Q, Q) & \longrightarrow & \mathrm{EB}(T|T; Q, Q) \end{array} \tag{18.8}$$

with exact rows and hence a homomorphism

$$\ker(\mathrm{res}) \longrightarrow \mathrm{EB}(T|S, Q)$$

of abelian groups. Suppose, furthermore, that S and T are fields. Then the homomorphism

$$\mathrm{XB}(T, N \rtimes Q) \longrightarrow \mathrm{XB}(T, Q)$$

in the lower row of the diagram (18.7) comes down to the obvious injection

$$\mathrm{B}(T)^{N \rtimes Q} \longrightarrow \mathrm{B}(T)^{Q}$$

whence the group $\mathrm{XB}(T; Q, N \rtimes Q)$ is now trivial and the inclusion

$$\mathrm{EB}\widetilde{(T|S, Q)} \subseteq \mathrm{EB}(T|S, Q)$$

is the identity. Moreover, the right-hand and the middle vertical arrow in (18.8) are isomorphisms whence the induced homomorphism $\ker(\mathrm{res}) \to \mathrm{EB}(T|S, Q)$ is an isomorphism. This observation recovers and casts new light on the main result of [2], obtained there via relative group cohomology. Our argument is elementary and does not invoke relative group cohomology. Indeed, the main point of our reasoning is the identification of the group cohomology group $\mathrm{H}^2(N \rtimes Q, \mathrm{U}(T))$ with the group $\mathrm{EB}(T|S; N \rtimes Q, Q)$; under the present circumstances, this group is the subgroup of the Q-equivariant Brauer group $\mathrm{EB}(S, Q)$ of S that consists of classes of Q-equivariant central simple S-algebras A such that $A \otimes T$ is a matrix algebra over T. Likewise, the group $\mathrm{EB}(T|T; Q, Q)$ is the subgroup of the Q-equivariant Brauer group $\mathrm{EB}(T, Q)$ of T that consists of classes of Q-equivariant matrix algebras over T. The group $\mathrm{EB}(T|S, Q)$ then appears as the kernel of the canonical homomorphism $\mathrm{EB}(T|S; N \rtimes Q, Q) \to \mathrm{EB}(T|T; Q, Q)$ and, in view of the identifications of $\mathrm{H}^2(N \rtimes Q, \mathrm{U}(T))$ with $\mathrm{EB}(T|S; N \rtimes Q, Q)$ and of $\mathrm{H}^2(Q, \mathrm{U}(T))$ with $\mathrm{EB}(T|T; Q, Q)$, the identification of $\ker(\mathrm{res}\colon \mathrm{H}^2(N \rtimes Q, \mathrm{U}(T)) \to \mathrm{H}^2(Q, \mathrm{U}(T)))$ with $\mathrm{EB}(T|S, Q)$ is immediate. In particular, when the group Q is trivial, that result comes down to the classical Brauer–Hasse–Noether isomorphism between the corresponding second group cohomology group and the corresponding subgroup of the ordinary Brauer group.

18.3 A variant of the relative theory

In the situation of the relative versions (17.1) and (17.2) of the long exact sequence [14, (10.1)], in general, there is no obvious reason for a homomorphism ω from $\mathrm{H}^0(Q, \mathrm{B}(T|S))$ to $\mathrm{H}^2(Q, \mathrm{Pic}(T|S))$ to exist that would complete

$$\begin{array}{ccc} \mathrm{H}^0(Q, \mathrm{B}(T|S)) & & \mathrm{H}^2(Q, \mathrm{Pic}(T|S)) \\ \downarrow & & \downarrow \\ \mathrm{H}^0(Q, \mathrm{B}(S)) & \xrightarrow{\ \omega\ } & \mathrm{H}^2(Q, \mathrm{Pic}(S)) \end{array}$$

to a commutative square and hence would complete the exact sequence (15.1) to a corresponding relative version of an exact sequence of the kind [14, (3.14)]. We now show that a variant of the relative theory includes such a homomorphism.

The object (S, Q, κ_Q) of the category $\mathcal{Change}$ being given, let (T, G, κ_G) be another object of $\mathcal{Change}$, and let $(f, \varphi)\colon (S, Q, \kappa_Q) \to (T, G, \kappa_G)$ be a morphism in $\mathcal{Change}$ having $\varphi\colon G \to Q$ surjective, cf. [14, Subsection 3.7].

18.3.1 The standard approach

We say that two Q-normal Azumaya S-algebras (A_1, σ_1) and (A_2, σ_2) such that $T \otimes A_1$ and $T \otimes A_2$ are matrix algebras over T are *relatively Brauer equivalent* if there are faithful finitely generated projective S-modules modules M_1 and M_2 having the property that $T \otimes M_1$ and $T \otimes M_2$ are free as T-modules, together with induced Q-normal structures

$$\rho_1\colon Q \to \mathrm{Out}(B_1),\ B_1 = \mathrm{End}_S(M_1),\ \rho_2\colon Q \to \mathrm{Out}(B_2),\ B_2 = \mathrm{End}_S(M_2),$$

such that $(A_1 \otimes B_1, \sigma_1 \otimes \rho_1)$ and $(A_2 \otimes B_2, \sigma_2 \otimes \rho_2)$ are isomorphic Q-normal S-algebras. Just as for $\mathrm{XB}(S, Q)$, under the operations of tensor product and that of taking opposite algebras, the equivalence classes constitute an abelian group, the identity element being represented by (S, κ_Q). We refer to this group as the *T-relative Q-crossed Brauer group of S with respect to the morphism (f, φ)* in $\mathcal{Change}$, denote this group by $\mathrm{XB}_{\mathrm{fr}}(T|S; G, Q)$, and we refer to the construction just given as the *standard construction*. The *T-relative Q-equivariant Brauer group* $\mathrm{EB}_{\mathrm{fr}}(T|S; G, Q)$ *with respect to the morphism* (f, φ) in $\mathcal{Change}$ arises in the same way as the relative Q-crossed Brauer group, save that, in the definition, ‘equivariant’ is substituted for ‘crossed’, and we likewise say that this construction is the *standard construction*. In particular, when we forget the actions, that is, we take the groups G and Q to be trivial, this construction yields an abelian group $\mathrm{B}_{\mathrm{fr}}(T|S)$ which we refer to as the *T-relative Brauer group of S*, obtained by the *standard construction*.

The group $\mathrm{B}_{\mathrm{fr}}(T|S)$ acquires a Q-module structure. Indeed, let $R = S^Q$. Given an S-module M and $x \in Q$, let ${}^xM$ denote the S-module whose underlying R-module is just M, and whose S-module structure is given by

$$S \otimes M \longrightarrow M,\ (s \otimes q) \longmapsto {}^xs\, q,\ s \in S,\ q \in M.$$

Consider a faithful finitely generated projective S-module M such that $T \otimes M$ is a free T-module, let $x \in Q$, and pick a pre-image $y \in G$ of $x \in Q$. Then the association

$$T \otimes {}^xM \longrightarrow {}^y(T \otimes M),\ t \otimes q \longmapsto {}^yt \otimes q, \tag{18.9}$$

yields an isomorphism of T-modules, and since $T \otimes M$ is a free T-module, so is ${}^y(T \otimes M)$; further,

$$T \otimes {}^x\mathrm{End}_S(M) \cong {}^y(T \otimes \mathrm{End}_S(M)) \cong {}^y(T \otimes \mathrm{End}_S(M)) \cong \mathrm{End}_S({}^y(T \otimes M))$$

is a matrix algebra over T. Likewise, given an Azumaya S-algebra A such that $T \otimes A$ is a matrix algebra over T and $x \in Q$, to show that $T \otimes^x A$ is a matrix algebra over T, pick a pre-image $y \in G$ of $x \in Q$ and note that the corresponding association (18.9) yields an isomorphism of T-algebras. Since $T \otimes A$ is a matrix algebra over T, so is ${}^y(T \otimes A)$.

By construction, the canonical homomorphism

$$\mathrm{B}_{\mathrm{fr}}(T|S) \longrightarrow \mathrm{B}(T|S)$$

is a morphism of Q-modules but in general there is no reason for this homomorphism to be injective nor to be surjective. The assignment to a Q-equivariant Azumaya S-algebra representing a member of $\mathrm{EB}_{\mathrm{fr}}(T|S; G, Q)$ of the associated Q-normal Azumaya S-algebra yields a homomorphism $\mathrm{res}_{\mathrm{fr}} \colon \mathrm{EB}_{\mathrm{fr}}(T|S; G, Q) \to \mathrm{XB}_{\mathrm{fr}}(T|S; G, Q)$ of abelian groups, the assignment to a Q-normal Azumaya S-algebra (A, σ) representing a member of $\mathrm{XB}_{\mathrm{fr}}(T|S; G, Q)$ of its Teichmüller complex $\mathrm{e}_{(A,\sigma)}$ yields a homomorphism $t_{\mathrm{fr}} \colon \mathrm{XB}_{\mathrm{fr}}(T|S; G, Q) \longrightarrow \mathrm{H}^3(Q, \mathrm{U}(S))$ of abelian groups and, when the group Q is finite, the construction of the homomorphism

$$\mathrm{cpr} \colon \mathrm{H}^2(Q, \mathrm{U}(S)) \to \mathrm{EB}(T|S; G, Q),$$

cf. (16.3) above, lifts to a homomorphism

$$\mathrm{cpr}_{\mathrm{fr}} \colon \mathrm{H}^2(Q, \mathrm{U}(S)) \longrightarrow \mathrm{EB}_{\mathrm{fr}}(T|S; G, Q).$$

Remark 18.3 The abelian groups $\mathrm{EB}(T|S; G, Q)$ and $\mathrm{XB}(T|S; G, Q)$ being defined relative to the given morphism (f, φ) in *Change*, the obvious maps yield homomorphisms

$$\mathrm{EB}_{\mathrm{fr}}(T|S; G, Q) \longrightarrow \mathrm{EB}(T|S; G, Q) \tag{18.10}$$
$$\mathrm{XB}_{\mathrm{fr}}(T|S; G, Q) \longrightarrow \mathrm{XB}(T|S; G, Q) \tag{18.11}$$

of abelian groups that make the diagram

$$\begin{array}{ccccc}
\mathrm{EB}_{\mathrm{fr}}(T|S; G, Q) & \xrightarrow{\mathrm{res}_{\mathrm{fr}}} & \mathrm{XB}_{\mathrm{fr}}(T|S; G, Q) & \xrightarrow{t_{\mathrm{fr}}} & \mathrm{H}^3(Q, \mathrm{U}(S)) \\
\downarrow & & \downarrow & & \| \\
\mathrm{EB}(T|S; G, Q) & \xrightarrow{\mathrm{res}} & \mathrm{XB}(T|S; G, Q) & \xrightarrow{t} & \mathrm{H}^3(Q, \mathrm{U}(S))
\end{array}$$

commutative and, when the group Q is finite, the homomorphisms

$$\mathrm{cpr}_{\mathrm{fr}} \colon \mathrm{H}^2(Q, \mathrm{U}(S)) \to \mathrm{EB}_{\mathrm{fr}}(T|S; G, Q) \quad \text{and}$$
$$\mathrm{cpr} \colon \mathrm{H}^2(Q, \mathrm{U}(S)) \to \mathrm{EB}(T|S; G, Q)$$

extend the diagram to a larger commutative diagram having four terms in each row. However, there is no reason for the homomorphisms (18.10) or (18.11) to be injective nor to be surjective, nor is there a reason, when Q is a finite group, for

$$\mathrm{cpr}_{\mathrm{fr}} \colon \mathrm{H}^2(Q, \mathrm{U}(S)) \longrightarrow \mathrm{EB}_{\mathrm{fr}}(T|S; G, Q)$$

to be injective or surjective. In the classical situation where R, S, T are fields etc., these homomorphisms are, of course, isomorphisms.

Let $\mathrm{Pic}(T|S)$ denote the kernel of the homomorphism $\mathrm{Pic}(S) \to \mathrm{Pic}(T)$ induced by the ring homomorphism $f \colon S \to T$, necessarily a morphism of G-modules when G acts on S through $\varphi \colon G \to Q$ whence, in particular, the abelian subgroup $\mathrm{Pic}(T|S)^Q$ of Q-invariants is defined, and let $\mathrm{EPic}(T|S, Q)$ denote the kernel of the homomorphism $\mathrm{EPic}(S, Q) \to \mathrm{EPic}(T, G)$ induced by the morphism (f, φ) in $\mathcal{Change}$. It is immediate that the low degree exact sequence [14, (3.14)] restricts to the exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{H}^1(Q, \mathrm{U}(S)) &\stackrel{j_{\mathcal{Pic}_{S,Q}}}{\longrightarrow} \mathrm{EPic}(T|S, Q) \\ &\stackrel{\mu_{\mathcal{Pic}_{S,Q}}|}{\longrightarrow} \mathrm{Pic}(T|S)^Q \stackrel{\omega_{\mathcal{Pic}_{S,Q}}|}{\longrightarrow} \mathrm{H}^2(Q, \mathrm{U}(S)) \end{aligned} \tag{18.12}$$

of abelian groups. In the Appendix (cf. Subsection 19.2 below), we shall show that, with a suitably defined Picard category $\mathcal{Pic}_{T|S;G,Q}$ substituted for $\mathcal{C}_Q$, the sequence (18.12) is as well a special case of the exact sequence [14, (3.10)].

Theorem 18.4 *Suppose that the group Q is finite. Then the extension*

$$\begin{aligned} \cdots \stackrel{\omega_{\mathcal{Pic}_{S,Q}}}{\longrightarrow} \mathrm{H}^2(Q, \mathrm{U}(S)) &\stackrel{\mathrm{cpr}_{\mathrm{fr}}}{\longrightarrow} \mathrm{EB}_{\mathrm{fr}}(T|S; G, Q) \\ &\stackrel{res_{\mathrm{fr}}}{\longrightarrow} \mathrm{XB}_{\mathrm{fr}}(T|S; G, Q) \stackrel{t_{\mathrm{fr}}}{\longrightarrow} \mathrm{H}^3(Q, \mathrm{U}(S)) \end{aligned} \tag{18.13}$$

of the exact sequence (18.12) *is defined and yields a seven term exact sequence that is natural in terms of the data.*

Proof Essentially the same reasoning as that for [14, Theorem 10.1] establishes this theorem as well. We explain only the requisite salient modifications.
Exactness at $\mathrm{XB}_{\mathrm{fr}}(T|S; G, Q)$: This follows again from [14, Theorem 6.1] or [14, Theorem 9.1].
Exactness at $\mathrm{H}^2(Q, \mathrm{U}(S))$: Let J represent a class in $(\mathrm{Pic}(T|S))^Q$, and proceed as in the proof of the exactness at $\mathrm{H}^2(Q, \mathrm{U}(S))$ in [14, Theorem 10.1]. Now $T \otimes J$ is free as a T-module and, with reference to the associated group extension e_J, cf. [14, (10.2)], by construction, M_{e_J} is free as an S-module whence $T \otimes M_{\mathrm{e}_J}$ is free as a T-module. Hence

$$T \otimes \mathrm{Hom}_S(J, M_{\mathrm{e}_J}) \cong \mathrm{Hom}_T(T \otimes J, T \otimes M_{\mathrm{e}_J})$$

is free as a T-module. Consequently $(\mathrm{End}_S(M_{\mathrm{e}_J}), \tau_{\mathrm{e}_J})$ represents zero in the group $\mathrm{EB}_{\mathrm{fr}}(T|S; G, Q)$.

Conversely, let $\mathrm{e} \colon \mathrm{U}(S) \rightarrowtail \Gamma \twoheadrightarrow Q$ be a group extension, and proceed as in the proof of the exactness at $\mathrm{H}^2(Q, \mathrm{U}(S))$ in [14, Theorem 10.1]. Thus suppose that $(\mathrm{End}_S(M_{\mathrm{e}}), \tau_{\mathrm{e}})$ represents zero in $\mathrm{EB}_{\mathrm{fr}}(T|S; G, Q)$. Then there are $S^t Q$-modules M_1 and M_2 whose underlying S-modules are faithful and finitely generated projective

such that the following hold, where we denote by $\tau_1\colon Q \to \mathrm{Aut}(\mathrm{End}_S(M_1))$ and $\tau_2\colon Q \to \mathrm{Aut}(\mathrm{End}_S(M_2))$ the associated trivially induced Q-equivariant structures: The algebras $(\mathrm{End}_S(M_{\mathrm{e}}), \tau_{\mathrm{e}}) \otimes (\mathrm{End}_S(M_1), \tau_1)$ and $(\mathrm{End}_S(M_2), \tau_2)$ are isomorphic as Q-equivariant S-algebras and, furthermore, the T-modules $T \otimes M_1$ and $T \otimes M_2$ are free as T-modules. Consequently the T-module $T \otimes S$ arising from the finitely generated and projective rank one S-module $J = \mathrm{Hom}_{\mathrm{End}_S(M_{\mathrm{e}} \otimes M_1)}(M_{\mathrm{e}} \otimes M_1, M_2)$ is free of rank one whence $[J] \in \mathrm{Pic}(T|S)$. The group extension e_J, cf. [14, (10.2)], is now defined relative to J, whence $[J] \in (\mathrm{Pic}(T|S))^Q$, and the Γ-action on J induces a homomorphism $\Gamma \to \mathrm{Aut}(J, Q)$ which yields a congruence $(1, \cdot, 1)\colon \mathrm{e} \to \mathrm{e}_J$ of group extensions, and this congruence entails that $\omega_{\mathcal{Pic}_{S,Q}}[J] = [\mathrm{e}] \in \mathrm{H}^2(Q, \mathrm{U}(S))$.
Exactness at $\mathrm{EB}_{\mathrm{fr}}(T|S; G, Q)$: The argument in the proof of [14, Theorem 10.1] which shows that the composite $\mathrm{res} \circ \mathrm{cpr}$ is zero shows as well that the composite $\mathrm{res}_{\mathrm{fr}} \circ \mathrm{cpr}_{\mathrm{fr}}$ is zero.

To show that $\ker(\mathrm{res}_{\mathrm{fr}}) \subseteq \mathrm{im}(\mathrm{cpr}_{\mathrm{fr}})$, let (A, τ) be a Q-equivariant Azumaya S-algebra representing a member of $\mathrm{EB}_{\mathrm{fr}}(T|S; G, Q)$, and suppose that the class of its associated Q-normal algebra (A, σ_τ) goes to zero in $\mathrm{XB}_{\mathrm{fr}}(T|S; G, Q)$. As in the proof of the exactness at $\mathrm{EB}(S, Q)$ in [14, Theorem 10.1], there are two induced Q-equivariant split algebras $(\mathrm{End}_S(M_1), \tau_1)$ and $(\mathrm{End}_S(M_2), \tau_2)$ over faithful finitely generated projective S-modules M_1 and M_2, respectively, such that $(A, \tau) \otimes (\mathrm{End}_S(M_1), \tau_1)$ and $(\mathrm{End}_S(M_2), \tau_2)$ are isomorphic as Q-equivariant central S-algebras but now we may furthermore take M_1 and M_2 to have the property that the T-modules $T \otimes M_1$ and $T \otimes M_2$ are free of finite rank. Essentially the same reasoning as that in the proof of the exactness at $\mathrm{EB}(S, Q)$ in [14, Theorem 10.1] yields a group extension

$$\mathrm{e}\colon \mathrm{U}(S) \rightarrowtail \Gamma \twoheadrightarrow Q$$

such that

$$\mathrm{cpr}_{\mathrm{fr}}([\mathrm{e}]) = [(\mathrm{End}_S(M_{\mathrm{e}}), \tau_{\mathrm{e}})] = [(A, \tau)] \in \mathrm{EB}_{\mathrm{fr}}(T|S; G, Q).$$

□

Consider a Q-normal Galois extension $T|S$ of commutative rings, with structure extension $\mathrm{e}_{(T|S)}\colon N \rightarrowtail G \overset{\pi_Q}{\twoheadrightarrow} Q$ and structure homomorphism

$$\kappa_G\colon G \longrightarrow \mathrm{Aut}^S(T),$$

cf. Section 12 above, and take the morphism (f, φ) to be the morphism

$$(i, \pi_Q)\colon (S, Q, \kappa_Q) \longrightarrow (T, G, \kappa_G)$$

in $\mathcal{Change}$ associated to that Q-normal Galois extension, cf. (12.3).

Theorem 18.5 *Suppose that the group Q is finite. Then the extension*

$$\begin{aligned} 0 &\longrightarrow \mathrm{H}^1(Q, \mathrm{U}(S)) \xrightarrow{j_{\mathcal{P}ic_{S,Q}}|} \mathrm{EPic}(T|S, Q) \xrightarrow{\mu_{\mathcal{P}ic_{S,Q}}|} (Pic(T|S))^Q \\ &\xrightarrow{\omega_{\mathcal{P}ic_{S,Q}}|} \mathrm{H}^2(Q, \mathrm{U}(S)) \xrightarrow{\mathrm{cpr}_{\mathrm{fr}}} \mathrm{EB}_{\mathrm{fr}}(T|S; G, Q) \xrightarrow{res_{\mathrm{fr}}} \mathrm{XB}_{\mathrm{fr}}(T|S; G, Q) \\ &\xrightarrow{t_{\mathrm{fr}}} \mathrm{H}^3(Q, \mathrm{U}(S)) \xrightarrow{\mathrm{inf}} \mathrm{H}^3(G, \mathrm{U}(T)) \end{aligned} \tag{18.14}$$

of the exact sequence (18.12) *is defined and yields an eight term exact sequence that is natural in terms of the data.*

Proof Essentially the same reasoning as that for Theorem 17.2 establishes this theorem as well. We leave the details to the reader. □

The homomorphism (18.2) now lifts to a homomorphism

$$\mathrm{Xpext}(G, N; \mathrm{U}(T)) \longrightarrow \mathrm{XB}_{\mathrm{fr}}(T|S; G, Q) \tag{18.15}$$

such that (18.2) may be written as the composite

$$\mathrm{Xpext}(G, N; \mathrm{U}(T)) \longrightarrow \mathrm{XB}_{\mathrm{fr}}(T|S; G, Q) \longrightarrow \mathrm{XB}(T|S; G, Q) \tag{18.16}$$

and, when Q and hence G is a finite group, the homomorphism (16.2) lifts to a homomorphism

$$\mathrm{H}^2(G, \mathrm{U}(T)) \longrightarrow \mathrm{EB}_{\mathrm{fr}}(T|S; G, Q) \tag{18.17}$$

such that (16.2) may be written as the composite

$$\mathrm{H}^2(G, \mathrm{U}(T)) \longrightarrow \mathrm{EB}_{\mathrm{fr}}(T|S; G, Q) \longrightarrow \mathrm{EB}(T|S; G, Q).$$

Theorem 18.1, adjusted to the present circumstances, takes the following form which, again, we spell out without proof.

Theorem 18.6 *The group Q being finite, the maps* (18.15), (18.4), (18.6), *and* (18.17) *are natural homomorphisms of abelian groups and induce a morphism*

$$(18.1) \longrightarrow (18.14)$$

of exact sequences.

18.3.2 The Morita equivalence approach

We define the *Q-graded relative Brauer precategory associated with the morphism* (f, φ) in $\mathcal{C}hange$ to be the precategory $\mathcal{P}re\mathcal{B}_{T|S;G,Q}$ that has as its *objects* the Azumaya S-algebras A such that $T \otimes A$ is a matrix algebra over T, a *morphism*

$$([M], x) \colon A \longrightarrow B$$

in $\mathcal{P}re\mathcal{B}_{T|S;G,Q}$ *of grade* $x \in Q$ between two Azumaya algebras A and B in $\mathcal{B}_{T|S;G,Q}$, necessarily an isomorphism in $\mathcal{P}re\mathcal{B}_{T|S;G,Q}$, being a morphism in $\mathcal{B}_{S,Q}$, that is, a pair $([M], x)$ where $[M]$ is an isomorphism class of an invertible (B, A)-bimodule M of grade $x \in Q$, such that, furthermore, $T \otimes M$ is free as a T-module. There is no reason for composition in the ambient category $\mathcal{B}_{S,Q}$ to induce an operation of composition in $\mathcal{P}re\mathcal{B}_{T|S;G,Q}$ since, given three Azumaya algebras A, B, C in $\mathcal{P}re\mathcal{B}_{T|S;G,Q}$ and morphisms $([{}_BM_A], x): A \to B$ and $([{}_AM_C], x): C \to A$ of grade $x \in Q$ in $\mathcal{P}re\mathcal{B}_{T|S;G,Q}$, while the composite $([{}_BM_A \otimes_A {}_AM_C], x): C \to B$ of grade $x \in Q$ in $\mathcal{B}_{S,Q}$ is defined, there is no reason for the $(T \otimes B, T \otimes C)$-bimodule

$$T \otimes ({}_BM_A \otimes_A {}_AM_C) \cong {}_{T\otimes B}(T \otimes M)_{T\otimes A} \otimes_{(T\otimes A)} {}_{T\otimes A}(T \otimes M)_{T\otimes C}$$

to be free as a T-module. To overcome this difficulty, we take the *Q-graded relative Brauer category associated with the morphism* (f, φ) in $\mathcal{C}hange$ to be the subcategory $\mathcal{B}_{T|S;G,Q}$ of $\mathcal{B}_{S,Q}$ generated by $\mathcal{P}re\mathcal{B}_{T|S;G,Q}$. Thus a morphism in $\mathcal{B}_{T|S;G,Q}$ of grade $x \in Q$ between two objects A and B of $\mathcal{B}_{T|S;G,Q}$ is a morphism $([{}_BM_A], x): A \to B$ in $\mathcal{B}_{S,Q}$ of grade $x \in Q$ such that there are objects $A_1,\ldots, A_n$ of $\mathcal{B}_{T|S;G,Q}$ and morphisms $([{}_{A_{j+1}}M_{A_j}], x): A_j \to A_{j+1}$ in $\mathcal{P}re\mathcal{B}_{T|S;G,Q}$ such that, when we write A as A_0 and B as A_n,

$$ {}_BM_A \cong {}_{A_n}M_{A_{n-1}} \otimes_{A_{n-1}} \cdots \otimes_{A_2} {}_{A_2}M_{A_1} \otimes_{A_1} {}_{A_1}M_{A_0}. \tag{18.18}$$

We then define composition, monoidal structure, the operation of inverse, and the unit object as in $\mathcal{B}_{S,Q}$. The resulting category $\mathcal{B}_{T|S;G,Q}$ is a group-like stably Q-graded symmetric monoidal category. Hence the category $\mathcal{R}ep(Q, \mathcal{B}_{T|S;G,Q})$ is group-like and thence $k\mathcal{R}ep(Q, \mathcal{B}_{T|S;G,Q})$ is an abelian group. When the groups G and Q are trivial, that is, we consider merely the homomorphism $f: S \to T$ of commutative rings, the same construction yields a precategory $\mathcal{P}re\mathcal{B}_{T|S}$ and, accordingly, the corresponding group-like symmetric monoidal category $\mathcal{B}_{T|S}$ which we refer to as the *relative Brauer category associated with the homomorphism* $f: S \to T$ of commutative rings. The category $\mathcal{B}_{T|S}$ has $\mathrm{U}(\mathcal{B}_{T|S}) = \mathrm{Pic}(T|S)$ as its unit group, is group-like, and $k\mathcal{B}_{T|S}$ is therefore an abelian group. The ring homomorphism $f: S \to T$ being a constituent of the morphism (f, φ) in $\mathcal{C}hange$ having φ surjective, the category $\mathcal{B}_{T|S;G,Q}$ has $\mathcal{K}er(\mathcal{B}_{T|S;G,Q}) = \mathcal{B}_{T|S}$ and

$$\mathrm{U}(\mathcal{B}_{T|S;G,Q}) = \mathrm{U}(\mathcal{B}_{T|S}) = \mathrm{Pic}(T|S)$$

as its unit group.

Given two objects A and B of $\mathcal{B}_{T|S;G,Q}$ we *define, with respect to the morphism* (f, φ) *in* $\mathcal{C}hange$, a *relative Morita equivalence of grade* $x \in Q$ *between* A *and* B to be a string of isomorphisms in $\mathcal{P}re\mathcal{B}_{T|S;G,Q}$ of the kind (18.18) above. It is immediate that, as in the classical situation, given two objects A_1 and A_2 of $\mathcal{B}_{T|S}$, a relative Brauer equivalence

$$A_1 \otimes \mathrm{End}_S(M_1) \cong A_2 \otimes \mathrm{End}_S(M_2)$$

between A_1 and A_2 induces a string

$$A_1 \simeq A_1 \otimes \mathrm{End}_S(M_1) \cong A_2 \otimes \mathrm{End}_S(M_2) \simeq A_2$$

of isomorphisms in $\mathcal{PreB}_{T|S}$ and hence a relative Morita equivalence between A_1 and A_2 (of grade $e \in Q$) whence the obvious association induces a homomorphism

$$\mathrm{B}_{\mathrm{fr}}(T|S) \longrightarrow k\mathcal{B}_{T|S} \tag{18.19}$$

of abelian groups, necessarily surjective. Moreover, since $\mathcal{Ker}(\mathcal{B}_{T|S;G,Q})$ is stably graded, $k\mathcal{B}_{T|S} = k\mathcal{Ker}(\mathcal{B}_{T|S;G,Q})$ acquires a Q-module structure, and the homomorphism (18.19) is a morphism of Q-modules.

Proposition 18.7 *The homomorphism* (18.19) *is an isomorphism, that is, relative Brauer equivalence is equivalent to relative Morita equivalence.*

Proof The classical argument, suitably rephrased, carries over: Let A and B be two Azumaya S-algebras A in $\mathcal{B}_{T|S}$ and consider a *morphism* $[M]\colon A \to B$ in $\mathcal{PreB}_{T|S}$. We must show that A and B are relatively Brauer equivalent. Now $B^{\mathrm{op}} \cong {}_A\mathrm{End}(M)$ (the algebra of left A-endomorphisms of M), and

$$\mathrm{End}_S(M) \cong A \otimes ({}_A\mathrm{End}(M)) \cong A \otimes B^{\mathrm{op}}$$

whence

$$\mathrm{End}_S(M) \otimes B \cong A \otimes B^{\mathrm{op}} \otimes B \cong A \otimes \mathrm{End}_S(B).$$

Since $T \otimes M$ and $T \otimes B$ are free as T-modules, A and B are relatively Brauer equivalent. □

With N, T, S substituted for, respectively, Q, S, R, the standard homomorphism [14, (5.6)] from $\mathrm{H}^2(N, U)$ to $\mathrm{B}(T|S)$, necessarily a morphism of Q-modules, lifts to a morphism

$$\mathrm{H}^2(N, U)) \longrightarrow \mathrm{B}_{\mathrm{fr}}(T|S) \tag{18.20}$$

of Q-modules. By construction, the assignment to an automorphism in $\mathcal{B}_{T|S;G,Q}$ of an Azumaya algebra A in $\mathcal{B}_{T|S;G,Q}$ of its grade in Q yields a homomorphism

$$\pi^{\mathrm{Aut}_{\mathcal{B}_{T|S;G,Q}}(A)}\colon \mathrm{Aut}_{\mathcal{B}_{T|S;G,Q}}(A) \longrightarrow Q$$

which is surjective if and only if the Brauer class $[A] \in \mathrm{B}_{\mathrm{fr}}(T|S)$ of A in $\mathrm{B}_{\mathrm{fr}}(T|S) \cong k\mathcal{B}_{T|S}$ is fixed under Q, and the group $\mathrm{Aut}_{\mathcal{B}_{T|S;G,Q}}(A)$ associated to an Azumaya S-algebra A in $\mathcal{B}_{T|S}$ whose Brauer class $[A] \in \mathrm{B}_{\mathrm{fr}}(T|S)$ is fixed under Q fits into a group extension of the kind [14, (3.6)], viz.

$$\mathrm{e}_A^{\mathrm{Pic}(T|S)}\colon 1 \longrightarrow \mathrm{Pic}(T|S) \longrightarrow \mathrm{Aut}_{\mathcal{B}_{T|S;G,Q}}(A) \stackrel{\pi^{\mathrm{Aut}_{\mathcal{B}_{T|S;G,Q}}(A)}}{\longrightarrow} Q \longrightarrow 1, \tag{18.21}$$

with abelian kernel in such a way that the assignment to A of $\mathrm{e}_A^{\mathrm{Pic}(T|S)}$ yields a homomorphism

$$\omega_{\mathcal{B}_{T|S;G,Q}}\colon \mathrm{H}^0(Q, \mathrm{B}_{\mathrm{fr}}(T|S)) \longrightarrow \mathrm{H}^2(Q, \mathrm{Pic}(T|S)). \tag{18.22}$$

The sequence [14, (3.10)] now takes the form

$$\begin{aligned} 0 \longrightarrow \mathrm{H}^1(Q, \mathrm{Pic}(T|S)) &\stackrel{j_{\mathcal{B}_{T|S;G,Q}}}{\longrightarrow} k\mathcal{R}ep(Q, \mathcal{B}_{T|S;G,Q}) \\ \stackrel{\mu_{\mathcal{B}_{T|S;G,Q}}}{\longrightarrow} \mathrm{B}_{\mathrm{fr}}(T|S)^Q &\stackrel{\omega_{\mathcal{B}_{T|S;G,Q}}}{\longrightarrow} \mathrm{H}^2(Q, \mathrm{Pic}(T|S)) \end{aligned} \tag{18.23}$$

and is an exact sequence of abelian groups since the category $\mathcal{B}_{T|S;G,Q}$ is group-like. Furthermore, the association that defines the homomorphism [14, (8.4)] yields an injective homomorphism

$$\theta_{\mathrm{fr}}\colon \mathrm{XB}_{\mathrm{fr}}(T|S;\, G,\, Q) \longrightarrow k\mathcal{R}ep(Q, \mathcal{B}_{T|S;G,Q}) \tag{18.24}$$

in such a way that the diagram

$$\begin{array}{ccc} \mathrm{XB}_{\mathrm{fr}}(T|S;\, G,\, Q) & \xrightarrow{\theta_{\mathrm{fr}}} & k\mathcal{R}ep(Q, \mathcal{B}_{T|S;G,Q}) \\ \downarrow & & \downarrow \\ \mathrm{XB}(S, Q) & \xrightarrow{\theta} & k\mathcal{R}ep(Q, \mathcal{B}_{S,Q}) \end{array}$$

is commutative, the unlabeled vertical arrows being the obvious maps, and the argument for [14, Theorem 8.10 (iii)], adjusted to the present situation, shows that if Q (and hence G) is a finite group, the homomorphism θ_{fr} is surjective and hence an isomorphism of abelian groups. Thus when the group Q is finite, the exact sequence (18.23) is available with $\mathrm{XB}_{\mathrm{fr}}(T|S;\, G,\, Q)$ substituted for $k\mathcal{R}ep(Q, \mathcal{B}_{T|S;G,Q})$.

Consider a Q-normal Galois extension $T|S$ of commutative rings, with structure extension $\mathrm{e}_{(T|S)}\colon N \rightarrowtail G \stackrel{\pi_Q}{\twoheadrightarrow} Q$ and structure homomorphism

$$\kappa_G\colon G \longrightarrow \mathrm{Aut}^S(T),$$

cf. Section 12 above, and take the morphism (f, φ) to be the morphism

$$(i, \pi_Q)\colon (S, Q, \kappa_Q) \longrightarrow (T, G, \kappa_G)$$

in *Change* associated with that Q-normal Galois extension, cf. 12.3. Comparison of the exact sequences [14, (3.14)] and (18.23) with [13, (1.9)] yields the following result, which we spell out without proof.

Theorem 18.8 *Write* $U = \mathrm{U}(T)$. *The various groups and homomorphisms fit into a commutative diagram*

$$\begin{array}{ccccccc}
\mathrm{H}^1(Q, \mathrm{H}^1(N, U)) & \rightarrowtail & \mathrm{Xpext}(G, N; U) & \longrightarrow & \mathrm{H}^0(Q, \mathrm{H}^2(N, U)) & \xrightarrow{d_2} & \mathrm{H}^2(Q, \mathrm{H}^1(N, U)) \\
\downarrow \cong & & \downarrow & & \downarrow & & \downarrow \cong \\
\mathrm{H}^1(Q, Pic(T|S)) & \overset{j}{\rightarrowtail} & k\mathcal{R}ep(Q, \mathcal{B}_{T|S;G,Q}) & \xrightarrow{\mu} & \mathrm{H}^0(Q, \mathrm{B}_{\mathrm{fr}}(T|S)) & \xrightarrow{\omega} & \mathrm{H}^2(Q, Pic(T|S)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{H}^1(Q, Pic(S)) & \overset{j}{\rightarrowtail} & k\mathcal{R}ep(Q, \mathcal{B}_{S,Q}) & \xrightarrow{\mu} & \mathrm{H}^0(Q, \mathrm{B}(S)) & \xrightarrow{\omega} & \mathrm{H}^2(Q, Pic(S))
\end{array}$$

with exact rows; here the top row is the exact sequence [13, (1.9)], *the middle row the sequence* (18.23), *the bottom row the exact sequence* [14, (3.14)], *the unlabeled arrow from* $\mathrm{H}^0(Q, \mathrm{H}^2(N, U))$ *to* $\mathrm{H}^0(Q, \mathrm{B}_{\mathrm{fr}}(T|S))$ *is induced by the homomorphism* (18.20), *and the other unlabeled arrows are either the obvious ones or have been introduced before. If, furthermore, the group* Q *is a finite group, the above diagram is available with* $\mathrm{XB}_{\mathrm{fr}}(T|S; G, Q)$ *substituted for* $k\mathcal{R}ep(Q, \mathcal{B}_{T|S;G,Q})$ *and* $\mathrm{XB}(S, Q)$ *for* $k\mathcal{R}ep(Q, \mathcal{B}_{S,Q})$.

Remark 18.9 The exact sequences (18.14) and (18.23) are presumably related with an equivariant Amitsur cohomology spectral sequence of the kind given in [4, Sections 1 and 2] and [3, Theorem 7.3 p. 61] in the same way as the exact sequences (13.4) and [13, (1.9)] are related with the spectral sequence associated with a group extension and a module over the extension group, cf. also [12].

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19 Appendix

As a service to the reader, we recollect some more material from the theory of stably graded symmetric monoidal categories [6,8,9] and use it to illustrate some of the constructions in the present paper.

Recall that an S-*progenerator* is a faithful finitely generated projective S-module. Given two Q-equivariant Azumaya S-algebras (A, τ_A) and (B, τ_B), a (B, A, Q)-*bimodule* (M, τ_M) is a (B, A)-*bimodule* M together with an $S^t Q$-module structure $\tau_M \colon Q \to \mathrm{Aut}(M)$ which is compatible with the Q-equivariant structures

$$\tau_A \colon Q \longrightarrow \mathrm{Aut}(A), \quad \tau_B \colon Q \longrightarrow \mathrm{Aut}(B)$$

in the sense that

$$^x(bya) = {}^xb\,{}^xy\,{}^xa,\ x \in Q,\ a \in A,\ b \in B. \tag{19.1}$$

The object (S, Q, κ_Q) of the category $\mathcal{C}hange$ being given, let (T, G, κ_G) be another object of $\mathcal{C}hange$, and let $(f, \varphi)\colon (S, Q, \kappa_Q) \to (T, G, \kappa_G)$ be a morphism in $\mathcal{C}hange$ having $\varphi\colon G \to Q$ surjective, cf. [14, Subsection 3.7].

19.1 Examples of symmetric monoidal categories

- $\mathcal{M}od_S$: the category of S-modules, a symmetric monoidal category under the operation of tensor product, with S as unit object, and $\mathrm{U}(\mathcal{M}od_S) = \mathrm{U}(S)$;
- $\mathcal{G}en_S$ [6, §2 p. 17], [8, p. 229], [9, §2]: the symmetric monoidal subcategory of $\mathcal{M}od_S$, necessarily a groupoid, whose objects are the *S-progenerators*, with morphisms only the invertible ones, having S as its unit object and

  $$\mathrm{U}(\mathcal{G}en_S) = \mathrm{U}(S)$$

  as its unit group;
- $\mathcal{P}ic_S$: the symmetric monoidal subcategory of $\mathcal{G}en_S$, necessarily group-like, of invertible modules, written in [6, §2 p. 17], [9, §2] as $\mathcal{C}_R$, reproduced in [14, Subsection 3.6];
- $\mathcal{A}z_S$: the symmetric monoidal subcategory of $\mathcal{G}en_S$, necessarily a groupoid, having the Azumaya S-algebras as objects, invertible algebra morphisms between Azumaya S-algebras as morphisms, the ground ring S as its unit object, and unit group $\mathrm{U}(\mathcal{A}z_S)$ trivial [6, § 2 p. 18], [8, p. 229], [9, §2];
- $\mathcal{X}\mathcal{A}z_S$: the quotient category of $\mathcal{A}z_S$, necessarily a groupoid, having the same objects as $\mathcal{A}z_S$, and having as morphisms $A \to B$ between two objects A and B equivalence classes of morphisms $h\colon A \to B$ in $\mathcal{A}z_S$ under the equivalence relation

  $$h_1 \sim h_2\colon A \to B \text{ if } h_1 = h_2 \circ I_a \text{ for some } a \in \mathrm{U}(A)$$

  [6, §5 p. 43], [9, §2], where the notation I_a refers to the inner automorphism of A induced by $a \in \mathrm{U}(A)$; this category has S as its unit object, and its unit group $\mathrm{U}(\mathcal{X}\mathcal{A}z_S)$ is trivial;
- $\mathcal{B}_S$, the *Brauer category* of the commutative ring S, reproduced in [14, Subsection 3.2];
- with respect to the ring homomorphism $f\colon S \to T$, with the obvious interpretations, the relative categories $\mathcal{M}od_{T|S}$, $\mathcal{G}en_{T|S}$, $\mathcal{P}ic_{T|S}$, $\mathcal{A}z_{T|S}$, $\mathcal{X}\mathcal{A}z_{T|S}$, taken as full subcategories of, respectively, $\mathcal{M}od_S$, $\mathcal{G}en_S$, $\mathcal{P}ic_S$, $\mathcal{A}z_S$, $\mathcal{X}\mathcal{A}z_S$;
- $\mathcal{B}_{T|S}$, with respect to the ring homomorphism $f\colon S \to T$, the *relative Brauer category*, introduced in Subsection 18.3.2 above;
- $\mathcal{E}\mathcal{B}_{S,Q}$, the equivariant Brauer category $\mathcal{E}\mathcal{B}_{S,Q}$ of S relative to the given action of Q on S, written as $\mathcal{B}(R, \Gamma)$ in [6, §5 p. 41] and [9, §3]; its objects are the Q-equivariant Azumaya algebras (A, τ); given two Q-equivariant Azumaya algebras (A, τ_A) and (B, τ_B), a *morphism*

  $$[(M, \tau_M)]\colon (A, \tau_A) \longrightarrow (B, \tau_B)$$

in $\mathcal{EB}_{S,Q}$, necessarily an isomorphism in $\mathcal{EB}_{S,Q}$, is an isomorphism class $[(M, \tau_M)]$ of a (B, A, Q)-bimodule $(M, \tau_M \colon Q \to \mathrm{Aut}(M))$ whose underlying (B, A)-bimodule M is invertible; the operations of tensor product and that of assigning to a Q-equivariant Azumaya S-algebra its opposite algebra (as a Q-equivariant Azumaya S-algebra) turn $\mathcal{EB}_{S,Q}$ into a group-like symmetric monoidal category having (S, κ_Q) is its unit object and $\mathrm{U}(\mathcal{EB}_{S,Q}) = \mathrm{EPic}(S)$ as its unit group [9, §3], [9, Proposition 3.1].

- $\mathcal{EB}_{T|S;G,Q}$, the *relative equivariant Brauer category associated with the morphism* (f, φ) in $\mathcal{C}hange$; it has as its objects the Q-equivariant Azumaya algebras (A, τ) such that the G-equivariant Azumaya algebra $(T \otimes A, \tau^G)$ that arises by scalar extension has its underlying central T-algebra $T \otimes A$ isomorphic to a matrix algebra; given two Q-equivariant Azumaya algebras (A, τ_A) and (B, τ_B) in $\mathcal{EB}_{S,Q}$, a *morphism* $(A, \tau_A) \to (B, \tau_B)$ in the associated precategory $\mathcal{P}re\mathcal{EB}_{T|S;G,Q}$, necessarily an isomorphism in $\mathcal{EB}_{T|S;G,Q}$, is a morphism $[M, \tau_M] \colon (A, \tau_A) \longrightarrow (B, \tau_B)$ in $\mathcal{EB}_{S,Q}$, that is, an isomorphism class of a (B, A, Q)-bimodule $(M, \tau_M \colon Q \to \mathrm{Aut}(M))$ whose underlying (B, A)-bimodule M is invertible, such that, furthermore, the resulting T^tG-module $T \otimes M$ is free as a T-module. We then take $\mathcal{EB}_{T|S;G,Q}$ to be the resulting subcategory of $\mathcal{EB}_{S,Q}$ generated by $\mathcal{P}re\mathcal{EB}_{T|S;G,Q}$, that is, we define morphisms and composition of morphisms as finite strings in $\mathcal{EB}_{S,Q}$, of morphisms in $\mathcal{P}re\mathcal{EB}_{T|S;G,Q}$, and we define the monoidal structure, the operation of inverse, and the unit object as in $\mathcal{EB}_{S,Q}$. The resulting category $\mathcal{EB}_{T|S;G,Q}$ is a group-like symmetric monoidal category and has $\mathrm{U}(\mathcal{EB}_{T|S;G,Q}) = \mathrm{EPic}(T|S)$.

19.2 Examples of stably Q-graded symmetric monoidal categories

- $\mathcal{M}od_{S,Q}$, a stably Q-graded symmetric monoidal category that arises from $\mathcal{M}od_S$ as follows: Given two S-modules M and N, a *morphism* $M \to N$ *of* S*-modules of grade* $x \in Q$ is a pair (φ, x) having $\varphi \colon M \to N$ a morphism over $R = S^Q$ such that $\varphi(sy) = ({}^x s)y$ $(s \in S,\ y \in M)$ [8, p. 229], [9, §2].

Enhancing each of the categories $\mathcal{C} = \mathcal{G}en_S, \mathcal{P}ic_S, \mathcal{A}z_S, \mathcal{XA}z_S, \mathcal{B}_S$ in Subsection 19.1 above to a stably Q-graded symmetric monoidal category $\mathcal{C}_Q$ in the same was as enhancing the category $\mathcal{M}od_S$ of S-modules to the stably Q-graded symmetric monoidal category $\mathcal{M}od_{S,Q}$ just explained yields the following stably Q-graded symmetric monoidal categories:

- $\mathcal{G}en_{S,Q}$, written in [9] as $\mathcal{G}en_R$;
- $\mathcal{P}ic_{S,Q}$, written in [9, §3] as $\mathcal{C}_R$, reproduced in [14, Subsection 3.6];
- $\mathcal{A}z_{S,Q}$, written in [9, §2] as $\mathcal{A}z_R$;
- $\mathcal{XA}z_{S,Q}$, written in [6, §5 p. 43] as $Q - \widetilde{\mathcal{A}z_R}$ and in [9, §2] as $Q\mathcal{A}z_R$ (beware: the notation Q in [op. cit.] has nothing to do with our notation Q for a group, and the tilde-notation in [6, §5 p. 43] refers to the additional structure of a twisting and need not concern us here); morphisms are now enhanced via the Q-grading, that is to say, a morphism $([h], x) \colon A \to B$ in $\mathcal{A}z_{S,Q}$ of grade $x \in Q$ has $[h]$ an equivalence class of an isomorphism $h \colon A \to B$ of algebras over $R = S^Q$ such that (h, x) is, furthermore, a morphism in $\mathcal{M}od_{S,Q}$ of grade $x \in Q$;

– $\mathcal{B}_{S,Q}$, the stably Q-graded Brauer category associated with the commutative ring S and the Q-action $\kappa_Q \colon Q \to \mathrm{Aut}(S)$ on S, reproduced in [14, Subsection 3.5].

The morphism $(f, \varphi)\colon (S, Q, \kappa_Q) \to (T, G, \kappa_G)$ in $\mathcal{C}hange$ having φ surjective being given, similarly to the construction of the category $\mathcal{B}_{T|S;G,Q}$ in Subsection 18.3 above, for each of the stably Q-graded symmetric monoidal categories

$$\mathcal{C}_{S,Q} = \mathcal{M}od_{S,Q}, \mathcal{G}en_{S,Q}, \mathcal{P}ic_{S,Q},$$

the stably Q-graded symmetric monoidal category $\mathcal{C}_{T|S;G,Q}$ is the subcategory that arises from the ambient category $\mathcal{C}_{S,Q}$ in essentially the same way as $\mathcal{B}_{T|S;G,Q}$ arises from the ambient category $\mathcal{B}_{S,Q}$ save that there is no need to pass through a corresponding precategory: The objects of $\mathcal{C}_{T|S;G,Q}$ are those objects C of $\mathcal{C}_{S,Q}$ that have the property that $T \otimes C$ is free as a T-module, and $\mathcal{C}_{T|S;G,Q} = \mathcal{M}od_{T|S;G,Q}, \mathcal{G}en_{T|S;G,Q}, \mathcal{P}ic_{T|S;G,Q}$ is the respective full subcategory of $\mathcal{C}_{S,Q}$. Likewise, for the stably Q-graded symmetric monoidal categories $\mathcal{C}_{S,Q} = \mathcal{A}z_{S,Q}$ and $\mathcal{C}_{S,Q} = \mathcal{XA}z_{S,Q}$, the stably Q-graded symmetric monoidal category $\mathcal{C}_{T|S;G,Q}$ arises as the subcategory that has as its objects Azumaya S-algebras A such that $T \otimes A$ is a matrix algebra over T, and $\mathcal{A}z_{T|S;G,Q}$ is the corresponding full subcategory of $\mathcal{A}z_{S,Q}$ and $\mathcal{XA}z_{T|S;G,Q}$ that of $\mathcal{XA}z_{S,Q}$. Now, with $\mathcal{P}ic_{T|S;G,Q}$ substituted for $\mathcal{C}_{S,Q}$, the exact sequence [14, (3.10)] yields the exact sequence (18.12).

Remark 19.1 For an object of $\mathcal{G}en_{S,Q}$, that is, for a faithful finitely generated projective S-module M, the group $\mathrm{Aut}(M, Q)$ introduced in [14, Section 7] is canonically isomorphic to the group $\mathrm{Aut}_{\mathcal{G}en_{S,Q}}(M)$.

19.3 The standard constructions revisited

The endomorphism functor $\mathcal{E}nd\colon \mathcal{G}en_S \to \mathcal{A}z_S$ induces an exact sequence

$$0 \longrightarrow \mathrm{Pic}(S) \longrightarrow k\,\mathcal{G}en_S \stackrel{\mathrm{End}}{\longrightarrow} k\mathcal{A}z_S \longrightarrow \mathrm{B}(S) \longrightarrow 0 \qquad (19.2)$$

of abelian monoids [6, §5 p. 38], [8, Introduction], [9, §3]. This yields $\mathrm{Pic}(S)$ as the maximal subgroup of the abelian monoid $k\,\mathcal{G}en_S$ and recovers the *standard construction* of $\mathrm{B}(S)$, cf. [14, Subsection 4.2], as the cokernel of the homomorphism End of abelian monoids, the cokernel of a morphism of monoids being suitably interpreted (in terms of the associated equivalence relation and "cofinality", cf. [8, §12]). The obvious functor $\Omega\colon \mathcal{A}z_S \to \mathcal{B}_S$ induces the isomorphism $\mathrm{B}(S) \to k\mathcal{B}_S$ of abelian groups [6, §5 p. 38], [9, §3, Theorem 3.2 (i)] quoted in [14, Subsection 4.2].

Likewise, the endomorphism functor $\mathcal{E}nd\colon \mathcal{G}en_{S,Q} \to \mathcal{A}z_{S,Q}$ induces an exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathrm{EPic}(S, Q) \longrightarrow k\mathcal{R}ep(Q, \mathcal{G}en_{S,Q}) \stackrel{\mathrm{End}}{\longrightarrow} k\mathcal{R}ep(Q, \mathcal{A}z_{S,Q}) \\ &\longrightarrow \mathrm{EB}(S, Q) \longrightarrow 0 \end{aligned}$$

of abelian monoids [6, §5 p. 38], [8, Introduction], [9, §3]. This yields $\mathrm{EPic}(S, Q)$ as the maximal subgroup of the abelian monoid $k\mathcal{Rep}(Q, \mathcal{Gens}_{S,Q})$ and recovers the *standard construction* of $\mathrm{EB}(S, Q)$, cf. [14, Section 9], as the cokernel of the corresponding homomorphism End of abelian monoids. The obvious functor $\Omega\colon \mathcal{Az}_{S,Q} \to \mathcal{B}_{S,Q}$ induces an isomorphism $\mathrm{EB}(S, Q) \to k\mathcal{EB}_{S,Q}$ of abelian groups [6, §5 p. 38], [9, § 3, Theorem 3.2 (i)], that is, equivariant Brauer equivalence is equivalent to equivariant Morita equivalence. Moreover, that obvious functor Ω factors as

$$\mathcal{Az}_{S,Q} \stackrel{\Omega^{\mathcal{Az}}}{\longrightarrow} \mathcal{XAz}_{S,Q} \stackrel{\Omega^{\mathcal{XAz}}}{\longrightarrow} \mathcal{B}_{S,Q}, \tag{19.3}$$

and the functor $\Omega^{\mathcal{XAz}}\colon \mathcal{XAz}_{S,Q} \longrightarrow \mathcal{B}_{S,Q}$ induces the injection

$$\theta\colon \mathrm{XB}(S, Q) \longrightarrow k\mathcal{Rep}(Q, \mathcal{B}_{S,Q})$$

of abelian groups spelled out as [14, (8.4)].

Recall that, given a stably Q-graded category $\mathcal{C}_Q$, the notation $k_Q\mathcal{C}_Q$ refers to the monoid $k\mathcal{C} = k\mathcal{Ker}(\mathcal{C}_Q) = k\mathcal{C}_Q$, viewed as a Q-monoid, cf. [14, Subsection 3.4]. The functor $\mathcal{End}\colon \mathcal{Gens}_{S,Q} \to \mathcal{Az}_{S,Q}$ induces, furthermore, a homomorphism

$$\mathrm{H}^0(Q, k_Q\mathcal{Gens}_{S,Q}) \longrightarrow k\mathcal{Rep}(Q, \mathcal{XAz}_{S,Q}) \tag{19.4}$$

of monoids [9, §3]. Indeed, let M be an object of $\mathcal{Gens}_{S,Q}$. By construction, the grading homomorphism $\mathrm{Aut}_{\mathcal{Gens}_{S,Q}}(M) \to Q$ is surjective if and only if the isomorphism class of M in $k\mathcal{Gens}_{S,Q}$ is fixed under Q. Hence an object M of $\mathcal{Gens}_{S,Q}$ whose isomorphism class in $k\mathcal{Gens}_{S,Q}$ is fixed under Q determines the exact sequence

$$1 \longrightarrow \mathrm{Aut}_S(M) \longrightarrow \mathrm{Aut}_{\mathcal{Gens}_{S,Q}}(M) \longrightarrow Q \longrightarrow 1, \tag{19.5}$$

plainly congruent to the exact sequence [14, (7.4)]; in particular, the group $\mathrm{Aut}_{\mathcal{Gens}_{S,Q}}(M)$ is canonically isomorphic to the group $\mathrm{Aut}(M, Q)$, cf. [14, (7.3)]. Now, for any object M of $\mathcal{Gens}_S$, the groups $\mathrm{Aut}_{\mathcal{Gens}_S}(M)$, $\mathrm{Aut}_S(M)$, and $\mathrm{U}(\mathrm{End}_S(M))$ coincide, and the induced action of $\mathrm{Aut}_{\mathcal{Gens}_{S,Q}}(M)$ on $\mathrm{End}_S(M)$ yields a commutative diagram of the kind

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Aut}_S(M) & \longrightarrow & \mathrm{Aut}_{\mathcal{Gens}_{S,Q}}(M) & \longrightarrow & Q & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
 & & \mathrm{U}(\mathrm{End}_S(M)) & \longrightarrow & \mathrm{Aut}(\mathrm{End}_S(M), Q) & \longrightarrow & \mathrm{Out}(\mathrm{End}_S(M), Q) & \longrightarrow & 1
\end{array}$$

and hence an induced Q-normal structure $Q \to \mathrm{Out}(\mathrm{End}_S(M))$ on the split algebra $\mathrm{End}_S(M)$. Thus the endomorphism functor $\mathcal{End}\colon \mathcal{Gens}_{S,Q} \to \mathcal{Az}_{S,Q}$ induces an exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{H}^0(Q, \mathrm{Pic}(S)) \longrightarrow \mathrm{H}^0(Q, k_Q\mathcal{Gens}_{S,Q}) \stackrel{\mathrm{End}}{\longrightarrow} k\mathcal{Rep}(Q, \mathcal{XAz}_{S,Q}) \\ \longrightarrow \mathrm{XB}(S, Q) \longrightarrow 0 \end{aligned} \tag{19.6}$$

of abelian monoids [9, §3] which, in turn, recovers the *standard construction* of the crossed Brauer group $\mathrm{XB}(S, Q)$ of S relative to Q given in [14, Section 8.1]. The unit object of $\mathcal{XAz}_{S,Q}$ is represented by (S, κ_Q). This kind of construction is given in [6, Theorem 4 p. 43], [9, Section 3, a few lines before Theorem 3.2] (the cokernel of End being written as $QB(R, \Gamma)$). In general, for the "crossed" versions, the equivalence between Brauer and Morita equivalence persists only when the group Q is finite, that is the canonical homomorphism $\theta\colon \mathrm{XB}(S, Q) \to k\mathcal{Rep}(Q, \mathcal{B}_{S,Q})$ of abelian groups given as [14, (8.4)] is injective, see [14, Theorem 8.10 (i)], but to prove that θ is surjective we need the additional hypothesis that Q be a finite group, see [14, Theorem 8.10 (iii)].

The above constructions, applied, with respect to the morphism (f, φ) in $\mathcal{Change}$, to the functors $\mathcal{End}\colon \mathcal{Gen}_{T|S} \to \mathcal{Az}_{T|S}$ and $\mathcal{End}\colon \mathcal{Gen}_{T|S;G,Q} \to \mathcal{Az}_{T|S;G,Q}$, yield the exact sequences

$$\mathrm{Pic}(T|S) \rightarrowtail k\mathcal{Gen}_{T|S} \xrightarrow{\mathrm{End}} k\mathcal{Az}_{T|S} \twoheadrightarrow \mathrm{B}_{\mathrm{fr}}(T|S)$$

$$\mathrm{EPic}(T|S, Q) \rightarrowtail k\mathcal{Rep}(Q, \mathcal{Gen}_{T|S;G,Q}) \xrightarrow{\mathrm{End}} k\mathcal{Rep}(Q, \mathcal{Az}_{T|S;G,Q}) \twoheadrightarrow \mathrm{EB}_{\mathrm{fr}}(T|S; G, Q)$$

$$\mathrm{H}^0(Q, \mathrm{Pic}(T|S)) \rightarrowtail \mathrm{H}^0(Q, k_Q\,\mathcal{Gen}_{T|S;G,Q}) \xrightarrow{\mathrm{End}} k\mathcal{Rep}(Q, \mathcal{XAz}_{T|S;G,Q}) \twoheadrightarrow \mathrm{XB}_{\mathrm{fr}}(T|S; G, Q)$$

of abelian monoids. These recover the standard constructions of the abelian groups $\mathrm{B}_{\mathrm{fr}}(T|S)$, $\mathrm{EB}_{\mathrm{fr}}(T|S; G, Q)$, and $\mathrm{XB}_{\mathrm{fr}}(T|S; G, Q)$, cf. Subsection 18.3.1 above.

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