The third homology of SL_2 of local rings

Kevin Hutchinson¹

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Abstract We describe the third homology of SL_2 of local rings, over $\mathbb{Z}[\frac{1}{2}]$, in terms of a refined Bloch group. We use this result to elucidate the relationship of this homology group to the indecomposable part of K_3 of the ring, extending and generalizing recent results in the case of fields. In particular, we prove that if A is a local domain with sufficiently large (possibly finite) residue field then the natural map $H_3(SL_2(A), \mathbb{Z}[\frac{1}{2}]) \rightarrow K_3^{\text{ind}}(A)[\frac{1}{2}]$ induces an isomorphism $H_3(SL_2(A), \mathbb{Z}[\frac{1}{2}])_{A^\times} \cong K_3^{\text{ind}}(A)[\frac{1}{2}]$ on coinvariants for the natural action of units $A^\times$. We prove that the action of $A^\times$ on $H_3(SL_2(A), \mathbb{Z}[\frac{1}{2}])$ is trivial when A is a complete discrete valuation ring with finite residue field of odd characteristic, and we show by example that this action is non-trivial for certain complete discrete valuation rings with infinite residue field.

Keywords K -theory · Group homology · Special linear group

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1 Introduction

1.1 Overview and background

This paper is concerned with understanding and calculating the structure of the third homology of SL_2 of local integral domains over $\mathbb{Z}[\frac{1}{2}]$. We generalize the main theorem

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✉ Kevin Hutchinson
kevin.hutchinson@ucd.ie

¹ School of Mathematics and Statistics, University College Dublin, Dublin, Ireland

of [4] from fields to local domains. We then combine this with the main result of [6] to prove a localization exact sequence for the third homology of SL_2 of certain discrete valuation rings.

In their study of Hilbert's third problem for scissors congruence of 3-dimensional hyperbolic polyhedra, Dupont and Sah [1, Appendix] proved the *Bloch–Wigner* theorem: There is a short exact sequence

$$0 \rightarrow \mu_{\mathbb{C}} \rightarrow H_3(\mathrm{SL}_2(\mathbb{C}), \mathbb{Z}) \rightarrow \mathcal{B}(\mathbb{C}) \rightarrow 0$$

where $\mathcal{B}(F)$ denotes the Bloch group of the field F . By definition, $\mathcal{B}(F)$ is the kernel of the Dehn invariant $\lambda : \mathcal{P}(F) \rightarrow (F^\times \otimes F^\times) / \langle x \otimes y + y \otimes x \rangle := S_{\mathbb{Z}}^2(F^\times)$ where $\mathcal{P}(F)$ is the *scissors congruence group* of the field F and is given by an explicit presentation. For details, see Sect. 2 below.

In fact the natural map from $H_3(\mathrm{SL}_2(\mathbb{C}), \mathbb{Z})$ to $K_3^{\mathrm{ind}}(\mathbb{C})$, the indecomposable K_3 of $\mathbb{C}$, is an isomorphism. From this viewpoint, Suslin [12, Theorem] subsequently generalised the Bloch–Wigner theorem to all (infinite) fields F : There is a natural exact sequence

$$0 \rightarrow \mathrm{tor}(\widetilde{\mu_F}, \mu_F) \rightarrow K_3^{\mathrm{ind}}(F) \rightarrow \mathcal{B}(F) \rightarrow 0$$

where $\mathrm{tor}(\widetilde{\mu_F}, \mu_F)$ is the unique nontrivial extension of $\mathrm{tor}(\mu_F, \mu_F)$ by $\mathbb{Z}/2$ if the characteristic of F is not 2, and $\mathrm{tor}(\widetilde{\mu_F}, \mu_F) = \mathrm{tor}(\mu_F, \mu_F)$ in characteristic 2. Suslin's result has recently been generalized to *rings with many units* by Mirzaii and Mokari [11]. This class of rings includes, in particular, local rings with infinite residue field. Mirzaii has also recently extended the theorem to local rings with (sufficiently large) finite residue field [9].

However, for general fields, the natural map $H_3(\mathrm{SL}_2(F), \mathbb{Z}) \rightarrow K_3^{\mathrm{ind}}(F)$ is surjective but has a large kernel. (For surjectivity, see [7, Lemma 5.1] when F is infinite and [4, Corollary 3.9] when F is finite. See [3] for examples where the kernel is not finitely-generated). The integral homology of $\mathrm{SL}_2(F)$ is naturally a module for the group ring $R_F := \mathbb{Z}[F^\times / (F^\times)^2]$ of square classes. This action is generally nontrivial, while the corresponding natural action of R_F on $K_3^{\mathrm{ind}}(F)$ is the trivial action. The Bloch–Wigner sequence of Dupont–Sah, viewed as a statement about $H_3(\mathrm{SL}_2(F), \mathbb{Z})$ was generalised to arbitrary fields in [4]. For any field F there is a natural short exact sequence of R_F -modules

$$0 \rightarrow \mathrm{tor}(\mu_F, \mu_F) \left[\frac{1}{2} \right] \rightarrow H_3(\mathrm{SL}_2(F), \mathbb{Z} \left[\frac{1}{2} \right]) \rightarrow \mathcal{RB}(F) \left[\frac{1}{2} \right] \rightarrow 0. \quad (1)$$

(To avoid 2-torsion complications, we state the result for $\mathbb{Z} \left[\frac{1}{2} \right]$ -coefficients.) Here $\mathcal{RB}(F)$ is the *refined Bloch group* of F , defined in a similar way to $\mathcal{B}(F)$, but in the category of R_F -modules (see Sect. 2 below). In particular, applying $F^\times$ -coinvariants to this exact sequence we recover Suslin's Bloch–Wigner exact sequence for $K_3^{\mathrm{ind}}(F)$, at least over $\mathbb{Z} \left[\frac{1}{2} \right]$.

The first main result below (Theorem 3.22) is a generalization of exact sequence (1) from fields to local integral domains. The residue field may be finite, apart from

a few small exceptions arising from some limitations of our method of proof. As an application, and using the results of Mirzaii and Mokari referred to above, we prove (Corollary 3.23) that for any local domain A with sufficiently large residue field the natural map $H_3(SL_2(A), \mathbb{Z}) \rightarrow K_3^{\text{ind}}(A)$ induces an isomorphism

$$H_3(SL_2(A), \mathbb{Z}[\frac{1}{2}])_{A^\times} \cong K_3^{\text{ind}}(A)[\frac{1}{2}].$$

The second part of the paper is devoted to proving a localization theorem (Theorem 7.4) relating the third homology of SL_2 of certain discrete valuation rings to the third homology of SL_2 of the field of quotients: Let F be a field with discrete valuation v . Let $\mathcal{O}_v$ be the associated valuation ring and let k be the residue field. Let U_1 be the group of units in $\mathcal{O}_v$ which map to 1 in k . Suppose that $U_1 = U_1^2$. We assume also another technical condition, affecting only 3-torsion. Then there is a natural exact sequence

$$0 \rightarrow H_3(SL_2(\mathcal{O}_v), \mathbb{Z}[\frac{1}{2}]) \rightarrow H_3(SL_2(F), \mathbb{Z}[\frac{1}{2}]) \rightarrow \mathcal{RP}_1(k)[\frac{1}{2}] \rightarrow 0. \quad (2)$$

Here the module $\mathcal{RP}_1(k)$ is the *refined scissors congruence group*. It is a submodule of $\mathcal{RP}(k)$ which contains $\mathcal{RB}(k)$. It is defined in Sect. 2 below. It has already arisen elsewhere in connection with the calculation and properties of the third homology of SL_2 : in [5] it is shown that for any infinite field k there is a natural short exact

$$0 \rightarrow H_3(SL_2(k), \mathbb{Z}[\frac{1}{2}]) \rightarrow H_3(SL_2(k[t, t^{-1}]), \mathbb{Z}[\frac{1}{2}]) \rightarrow \mathcal{RP}_1(k)[\frac{1}{2}] \rightarrow 0.$$

The module $\mathcal{RP}_1(k)$ can be calculated explicitly in terms of classical scissors congruence groups (see [6] for many examples). Using such calculations, we can determine $H_3(SL_2(A), \mathbb{Z}[\frac{1}{2}])$ for certain local domains. For example

$$H_3(SL_2(\mathbb{Z}_p), \mathbb{Z}[\frac{1}{2}]) \cong K_3^{\text{ind}}(\mathbb{Q}_p)[\frac{1}{2}]$$

for any prime number $p \geq 11$ (see Example 7.8), while for any prime p

$$H_3(SL_2(\mathbb{Q}_p[[x]]), \mathbb{Z}[\frac{1}{2}]) \cong K_3^{\text{ind}}(\mathbb{Q}_p[[x]])[\frac{1}{2}] \oplus \mathcal{P}(\mathbb{F}_p)[\frac{1}{2}]$$

where the second factor on the right is a cyclic group whose order is the odd part of $p+1$ (see Example 7.10).

1.2 Some notation

For a commutative ring A , we let $A^\times$ denote the group of units of A . For $x \in A^\times$, we will let $\langle x \rangle \in A^\times / (A^\times)^2$ denote the corresponding square class. Let R_A denote the integral group ring $\mathbb{Z}[A^\times / (A^\times)^2]$ of the group $A^\times / (A^\times)^2$. We will use the notation $\langle\langle x \rangle\rangle$ for the basis elements, $\langle x \rangle - 1$, of the augmentation ideal $\mathcal{I}_A$ of R_A .

For any $a \in A^\times$, we will let p_a^+ and p_a^- denote respectively the elements $1 + \langle a \rangle$ and $1 - \langle a \rangle$ in R_A .

For any abelian group G we will let $G[\frac{1}{2}]$ denote $G \otimes \mathbb{Z}[\frac{1}{2}]$.

For an integer n , we will let n_{odd} denote the odd part of n . Thus if G is a finite abelian group of order n , then $G[\frac{1}{2}]$ is a finite abelian group of order n_{odd} .

We let e_a^+ and e_a^- denote respectively the mutually orthogonal idempotents

$$e_a^+ := \frac{p_a^+}{2} = \frac{1 + \langle a \rangle}{2}, \quad e_a^- := \frac{p_a^-}{2} = \frac{1 - \langle a \rangle}{2} \in \mathbb{R}_A[\frac{1}{2}].$$

(Of course, these operators depend only on the class of a in $A^\times / (A^\times)^2$.)

For a prime power q , the finite field with q elements is denoted by $\mathbb{F}_q$.

Given an abelian group G we let $S_{\mathbb{Z}}^2(G)$ denote the group

$$\frac{G \otimes_{\mathbb{Z}} G}{\langle x \otimes y + y \otimes x \mid x, y \in G \rangle}$$

and, for $x, y \in G$, we denote by $x \circ y$ the image of $x \otimes y$ in $S_{\mathbb{Z}}^2(G)$.

2 Scissors congruence groups and Bloch groups

In this section we review some known results about the third homology of SL_2 and its relation to K -theory and scissors congruence groups.

2.1 Indecomposable K_3

Let A be either a local ring or a field. Let $K_\bullet(A)$ denote the Quillen K -theory of A and let $K_\bullet^M(A)$ be the Milnor K -theory. There is a natural homomorphism of graded rings $K_\bullet^M(A) \rightarrow K_\bullet(A)$. Indecomposable K_3 of A is the group $K_3^{\text{ind}}(A) := \text{Coker}(K_3^M(A) \rightarrow K_3(A))$. We will require the following theorem from K -theory:

Theorem 2.1 *Let A be discrete valuation ring with field of fractions K and residue field F . Suppose that either $\text{char}(K) = \text{char}(F)$ or that F is algebraic over $\mathbb{F}_p$. Then the inclusion $A \rightarrow K$ induces an isomorphism $K_3^{\text{ind}}(A) \cong K_3^{\text{ind}}(K)$.*

Proof Let π be a uniformizer. There is a commutative diagram with exact columns

$$\begin{array}{ccccccc} K_3^M(A) & \longrightarrow & K_3^M(K) & \xrightarrow{\delta_\pi} & K_2^M(F) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow = & & \\ 0 \longrightarrow & K_3(A) & \longrightarrow & K_3(K) & \xrightarrow{\delta_\pi} & K_2(F) & \longrightarrow 0 \\ \downarrow & & \downarrow & & & & \\ K_3^{\text{ind}}(A) & \longrightarrow & K_3^{\text{ind}}(K) & & & & \\ \downarrow & & \downarrow & & & & \\ 0 & & 0 & & & & \end{array}$$

The exactness of the top row is well-known (see [14, Chapter V, Corollary 6.6.2] for example). The second row is exact by Gersten's conjecture, which is known for the case of equicharacteristic discrete valuation rings or discrete valuation rings with finite residue fields (see, for example, [14, Chapter V, p. 454]). The result follows by the snake lemma. $\square$

2.2 Scissors congruence groups and Bloch groups

For a commutative ring A , let $\mathcal{W}_A$ denote the set $\{u \in A^\times : 1 - u \in A^\times\}$. The *scissors congruence group* or *pre-Bloch group*, $\mathcal{P}(A)$, is the group generated by the elements $[x]$, $x \in \mathcal{W}_A$, subject to the relations

$$R_{x,y} : [x] - [y] + \left[\frac{y}{x}\right] - \left[\frac{1-x^{-1}}{1-y^{-1}}\right] + \left[\frac{1-x}{1-y}\right] \quad \text{for all } x, y, x/y \in \mathcal{W}_A.$$

For a commutative ring A , the map

$$\lambda : \mathcal{P}(A) \rightarrow S_{\mathbb{Z}}^2(A^\times), \quad [x] \mapsto (1-x) \circ x$$

is well-defined, and the *Bloch group* of A , $\mathcal{B}(A) \subset \mathcal{P}(A)$, is defined to be the kernel of λ .

For a field F , the Bloch group is known to be closely related to the indecomposable K_3 : There is a natural exact sequence

$$0 \rightarrow \widetilde{\text{tor}(\mu_F, \mu_F)} \rightarrow K_3^{\text{ind}}(F) \rightarrow \mathcal{B}(F) \rightarrow 0$$

where $\widetilde{\text{tor}(\mu_F, \mu_F)}$ is the unique nontrivial extension of $\text{tor}(\mu_F, \mu_F)$ by $\mathbb{Z}/2$. (See Suslin [12] for infinite fields and [4] for finite fields.)

More recently, this has been generalized to a much larger class of rings by Mirzaii and Mokari:

Theorem 2.2 (Mirzaii–Mokari [11], Mirzaii [9]) *Let A be an integral domain with many units or a local domain whose residue field has order at least 11 and is not 16 or 32. Then there is a natural exact sequence*

$$0 \rightarrow \widetilde{\text{tor}(\mu_A, \mu_A)} \rightarrow K_3^{\text{ind}}(A) \rightarrow \mathcal{B}(A) \rightarrow 0.$$

Remark 2.3 We will not define the term *ring with many units*; for the present purposes it is enough to know that a local ring with infinite residue field is a ring with many units.

2.3 The refined Bloch group and $H_3(SL_2(A), \mathbb{Z})$

For any local commutative ring A there is a natural homomorphism

$$H_3(SL_2(A), \mathbb{Z}) \rightarrow K_3^{\text{ind}}(A) \quad (3)$$

which is surjective if A is a field. We let $H_3(SL_2(A), \mathbb{Z})_0$ denote the kernel of this map.

For any commutative ring A , the group extension

$$1 \rightarrow SL_2(A) \rightarrow GL_2(A) \rightarrow A^\times \rightarrow 1$$

induces an action—by conjugation—of $A^\times$ on $H_\bullet(SL_2(A), \mathbb{Z})$ which factors through $A^\times/(A^\times)^2$. Thus $H_3(SL_2(A), \mathbb{Z})$ is naturally an R_A -module, and for general fields or rings, in order to give a Bloch-type description of it, we must incorporate the R_A -module structure at each stage of the process.

The following is [10, Theorem 3.7]:

Proposition 2.4 *For a ring with many units A the map (3) induces an isomorphism*

$$H_3(SL_2(A), \mathbb{Z}[\tfrac{1}{2}])_{A^\times} \cong K_3^{\text{ind}}(A)[\tfrac{1}{2}].$$

Remark 2.5 We extend this result to local domains with finite residue field below (Corollary 3.23).

Corollary 2.6 *Let A be a local ring with infinite residue field. Let $H_3(SL_2(A), \mathbb{Z})_0$ denote the kernel of the surjective homomorphism $H_3(SL_2(A), \mathbb{Z}) \rightarrow K_3^{\text{ind}}(A)$. Then*

$$H_3(SL_2(A), \mathbb{Z}[\tfrac{1}{2}])_0 = \mathcal{I}_A H_3(SL_2(A), \mathbb{Z}[\tfrac{1}{2}]).$$

However, as our calculations—in [3] and below—show, the action of $A^\times/(A^\times)^2$ on $H_3(SL_2(A), \mathbb{Z})$ is in general non-trivial.

We define $\mathcal{RP}(A)$ to be the R_A -module with generators $[x]$, $x \in \mathcal{W}_A$ subject to the relations

$$\begin{aligned} S_{x,y} : [x] - [y] + \langle x \rangle \left[\frac{y}{x} \right] - \langle x^{-1} - 1 \rangle \left[\frac{1 - x^{-1}}{1 - y^{-1}} \right] \\ + \langle 1 - x \rangle \left[\frac{1 - x}{1 - y} \right], \quad \text{for all } x, y, y/x \in \mathcal{W}_A. \end{aligned}$$

Of course, from the definition it follows immediately that $\mathcal{P}(A) = (\mathcal{RP}(A))_{A^\times} = H_0(A^\times, \mathcal{RP}(A))$.

Let $\Lambda = (\lambda_1, \lambda_2)$ be the R_A -module homomorphism

$$\mathcal{RP}(A) \rightarrow \mathcal{I}_A^2 \oplus S_{\mathbb{Z}}^2(A^\times)$$

where $\lambda_1 : \mathcal{RP}(A) \rightarrow \mathcal{I}_A^2$ is the map $[x] \mapsto \langle\langle 1 - x \rangle\rangle \langle\langle x \rangle\rangle$, and λ_2 is the composite

$$\mathcal{RP}(A) \twoheadrightarrow \mathcal{P}(A) \xrightarrow{\lambda} S_{\mathbb{Z}}^2(A^\times).$$

(It is straightforward to verify directly that Λ is well-defined module homomorphism.)

The *refined Bloch group* of the commutative ring A (with at least 4 elements) is the R_A -module

$$\mathcal{RB}(A) := \text{Ker}(\Lambda : \mathcal{RP}(A) \rightarrow \mathcal{I}_A^2 \oplus S_{\mathbb{Z}}^2(A^\times)).$$

For a field F , the refined Bloch group bears the same relation to $H_3(SL_2(F), \mathbb{Z}[\frac{1}{2}])$ as the classical Bloch group does to $K_3^{\text{ind}}(F)$:

Theorem 2.7 ([4]) *Let F be a field with at least 29 elements. Then there is a natural short exact sequence*

$$0 \rightarrow \text{tor}(\mu_F, \mu_F) \left[\frac{1}{2} \right] \rightarrow H_3(SL_2(F), \mathbb{Z} \left[\frac{1}{2} \right]) \rightarrow \mathcal{RB}(F) \left[\frac{1}{2} \right] \rightarrow 0.$$

Remark 2.8 If we take coinvariants for the action of $F^\times$ in this sequence, the resulting sequence is exact, and it is precisely the sequence of Theorem 2.2 tensored with $\mathbb{Z}[\frac{1}{2}]$.

2.4 The refined scissors congruence group $\mathcal{RP}_1(A)$

The *refined scissors congruence group* of A is defined to be the R_A -module

$$\mathcal{RP}_1(A) := \text{Ker}(\lambda_1 : \mathcal{RP}(A) \rightarrow \mathcal{I}_A^2).$$

We will require the following facts below:

Proposition 2.9 *Let A be a commutative ring. Then*

(1) *The natural map $\mathcal{RP}_1(A) \rightarrow \mathcal{P}(A)$ induces an isomorphism*

$$\mathcal{RP}_1(A) \left[\frac{1}{2} \right]_{A^\times} \cong \mathcal{P}(A) \left[\frac{1}{2} \right].$$

(2) *The natural map $\mathcal{RB}(A) \rightarrow \mathcal{B}(A)$ induces an isomorphism*

$$\mathcal{RB}(A) \left[\frac{1}{2} \right]_{A^\times} \cong \mathcal{B}(A) \left[\frac{1}{2} \right].$$

(3) *The inclusion map $\mathcal{RB}(A) \rightarrow \mathcal{RP}_1(A)$ induces an isomorphism*

$$\mathcal{I}_A \mathcal{RB}(A) \left[\frac{1}{2} \right] \cong \mathcal{I}_A \mathcal{RP}_1(A) \left[\frac{1}{2} \right].$$

Proof Since $A^\times/(A^\times)^2$ is annihilated by 2 it follows that taking $A^\times/(A^\times)^2$ -coinvariants is an exact functor on the category of $R_A[\frac{1}{2}]$ -modules.

- (1) Taking $A^\times / (A^\times)^2$ -coinvariants of the exact sequence $0 \rightarrow \mathcal{RP}_1(A) \left[\frac{1}{2} \right] \rightarrow \mathcal{RP}(A) \left[\frac{1}{2} \right]$ gives an injective map

$$0 \rightarrow \mathcal{RP}_1(A) \left[\frac{1}{2} \right]_{A^\times} \rightarrow \mathcal{RP}(A) \left[\frac{1}{2} \right]_{A^\times} = \mathcal{P}(A) \left[\frac{1}{2} \right].$$

On the other hand, the map $\mathcal{RP}_1(A) \left[\frac{1}{2} \right] \rightarrow \mathcal{P}(A) \left[\frac{1}{2} \right]$ is surjective: Let $a \in \mathcal{W}_A$. Direct calculation—or see Lemma 4.5 below—shows that

$$g(a) := p_{-1}^+ [a] + \langle 1 - a \rangle \left([a] + \langle -1 \rangle [a^{-1}] \right) \in \text{Ker}(\lambda_1) = \mathcal{RP}_1(A)$$

and $g(a)$ maps to $2[a] \in \mathcal{P}(A)$.

- (2) Taking $A^\times / (A^\times)^2$ -coinvariants of the exact sequence of $R_A \left[\frac{1}{2} \right]$ -modules

$$0 \longrightarrow \mathcal{RB}(A) \left[\frac{1}{2} \right] \longrightarrow \mathcal{RP}_1(A) \left[\frac{1}{2} \right] \xrightarrow{\lambda_2} S_{\mathbb{Z}}^2(A^\times) \left[\frac{1}{2} \right]$$

gives the exact sequence

$$0 \longrightarrow \mathcal{RB}(A) \left[\frac{1}{2} \right]_{A^\times} \longrightarrow \mathcal{RP}_1(A) \left[\frac{1}{2} \right]_{A^\times} = \mathcal{P}(A) \left[\frac{1}{2} \right] \xrightarrow{\lambda} S_{\mathbb{Z}}^2(A^\times) \left[\frac{1}{2} \right]$$

and hence

$$\mathcal{RB}(A) \left[\frac{1}{2} \right]_{A^\times} = \text{Ker}(\lambda : \mathcal{P}(A) \left[\frac{1}{2} \right] \rightarrow S_{\mathbb{Z}}^2(A^\times) \left[\frac{1}{2} \right]) = \mathcal{B}(A) \left[\frac{1}{2} \right].$$

- (3) Let $\Sigma(A) := \text{Im}(\lambda : \mathcal{P}(A) \rightarrow S_{\mathbb{Z}}^2(A^\times))$. Then by (1) and (2) there is a commutative diagram of $R_A \left[\frac{1}{2} \right]$ -modules with exact rows and columns:

$$\begin{array}{ccccccc}
 & 0 & & 0 & & & \\
 & \downarrow & & \downarrow & & & \\
 & \mathcal{I}_A \mathcal{RB}(A) \left[\frac{1}{2} \right] & \longrightarrow & \mathcal{I}_A \mathcal{RP}_1(A) \left[\frac{1}{2} \right] & & & \\
 & \downarrow & & \downarrow & & & \\
 0 & \longrightarrow & \mathcal{RB}(A) \left[\frac{1}{2} \right] & \longrightarrow & \mathcal{RP}_1(A) \left[\frac{1}{2} \right] & \xrightarrow{\lambda_2} & \Sigma(A) \left[\frac{1}{2} \right] \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow = \\
 0 & \longrightarrow & \mathcal{B}(A) \left[\frac{1}{2} \right] & \longrightarrow & \mathcal{P}(A) \left[\frac{1}{2} \right] & \xrightarrow{\lambda} & \Sigma(A) \left[\frac{1}{2} \right] \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & 0 & & 0 & & &
 \end{array}$$

Now apply the snake lemma. □

3 The third homology of SL_2 of local rings

In this section we generalize Theorem 2.7 to local domains with sufficiently large residue field (Theorem 3.22 below). The proof follows closely the proof of the corresponding result for fields given in [3]. The first step is to give a homological description of the modules $\mathcal{RP}(A)$ and $\mathcal{RP}_1(A)$.

3.1 Alternative description of $\mathcal{RP}(A)$

Let A be a commutative ring. A row vector $\mathbf{u} = (u_1, u_2) \in A^2$ is said to be *unimodular* if $Au_1 + Au_2 = A$. Equivalently, $\mathbf{u}$ is unimodular if there exists $\mathbf{v} \in A^2$ such that

$$\begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} \in GL_2(A).$$

We let $U_2 = U_2(A)$ denote the set of 2-dimensional unimodular row vectors of A . U_2 is a right $GL_2(A)$ -set. In particular this induces an action of $A^\times = Z(GL_2(A))$ acting as multiplication by scalars.

Let

$$\mathcal{U}_n^{\text{gen}} = \mathcal{U}_n^{\text{gen}}(A) := \left\{ (\mathbf{u}_1, \dots, \mathbf{u}_n) \in U_2^n : \begin{bmatrix} \mathbf{u}_i \\ \mathbf{u}_j \end{bmatrix} \in GL_2(A) \text{ for all } i \neq j \right\}.$$

$(A^\times)^n$ acts entry-wise on $\mathcal{U}_n^{\text{gen}}$ and we let $X_n = X_n(A) = \mathcal{U}_n^{\text{gen}} / (A^\times)^n$. Observe that $\mathcal{U}_n^{\text{gen}}$ and X_n are right $GL_2(A)$ -sets (with the natural diagonal action).

In particular, $X_1 = U_2 / A^\times$ and $X_n \subset X_1^n$. If $\mathbf{u} = (u_1, u_2) \in U_2$ we will denote the corresponding class in X_1 by $\bar{\mathbf{u}}$ or $[u_1, u_2]$.

We have two natural injective maps from A to X_1 :

$$\iota_+ : A \rightarrow X_1, a \mapsto a_+ := [a, 1] \text{ and } \iota_- : A \rightarrow X_1, a \mapsto a_- := [1, a]$$

Clearly, $a_+ = b_-$ in X_1 if and only if $a, b \in A^\times$ and $b = a^{-1}$. We will identify $A^\times$ with its image in X_1 under the map ι_+ .

If $(\mathbf{u}, \mathbf{v}) \in \mathcal{U}_2^{\text{gen}}$, we set

$$d(\mathbf{u}, \mathbf{v}) := \det \left(\begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} \right) \in A^\times$$

and

$$T_{\mathbf{u}, \mathbf{v}} := \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 & -1 \\ d(\mathbf{u}, \mathbf{v}) & 0 \end{bmatrix} \in SL_2(A).$$

Thus, for $(\mathbf{u}, \mathbf{v}) \in \mathcal{U}_2^{\text{gen}}$ with $d = d(\mathbf{u}, \mathbf{v})$ we have

$$\mathbf{u} \cdot T_{\mathbf{u}, \mathbf{v}} = (0, -1) \quad \text{and} \quad \mathbf{v} \cdot T_{\mathbf{u}, \mathbf{v}} = (d, 0) \text{ in } U_2$$

and hence

$$\bar{\mathbf{u}} \cdot T_{\mathbf{u}, \mathbf{v}} = 0_+, \quad \bar{\mathbf{v}} \cdot T_{\mathbf{u}, \mathbf{v}} = 0_- \text{ in } X_1.$$

Let $\phi : \mathcal{U}_3^{\text{gen}} \rightarrow X_1$ be the map defined by

$$\phi(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \bar{\mathbf{w}} \cdot T_{\mathbf{u}, \mathbf{v}} \in X_1.$$

Lemma 3.1 *Let A be a commutative ring and let $(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathcal{U}_3^{\text{gen}}$. Then*

$$\phi(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \left(\frac{d(\mathbf{u}, \mathbf{w}) \cdot d(\mathbf{u}, \mathbf{v})}{d(\mathbf{v}, \mathbf{w})} \right)_+ \in X_1.$$

In particular, $\phi(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in A^\times \subset X_1$.

Proof A straightforward direct calculation gives

$$\mathbf{w} \cdot T_{\mathbf{u}, \mathbf{v}} = \left(d(\mathbf{u}, \mathbf{w}), \frac{d(\mathbf{v}, \mathbf{w})}{d(\mathbf{u}, \mathbf{v})} \right) \in U_2.$$

□

Corollary 3.2 *Let A be a commutative ring and let $(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathcal{U}_3^{\text{gen}}$.*

(1) *For all $a, b, c \in A^\times$ we have*

$$\phi(\mathbf{u} \cdot a, \mathbf{v} \cdot b, \mathbf{w} \cdot c) = \phi(\mathbf{u}, \mathbf{v}, \mathbf{w}) \cdot a^2.$$

(2) *For all $X \in \text{GL}_2(A)$ we have*

$$\phi((\mathbf{u}, \mathbf{v}, \mathbf{w}) \cdot X) = \phi(\mathbf{u}, \mathbf{v}, \mathbf{w}) \cdot \det(X).$$

Now, for $n \geq 1$, let

$$Y_n = Y_n(A) := \{(y_1, \dots, y_n) \in (A^\times)^n : y_i - y_j \in A^\times \text{ for all } i \neq j\}.$$

We will consider Y_n as a right $A^\times$ -set via $(y_1, \dots, y_n) \cdot a := (y_1 a, \dots, y_n a)$.

For $n \geq 3$, let $\Phi_n : \mathcal{U}_n^{\text{gen}} \rightarrow (A^\times)^{n-2}$ be the map

$$\Phi_n(\mathbf{u}_1, \dots, \mathbf{u}_n) := (\phi(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3), \dots, \phi(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_i), \dots, \phi(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_n)).$$

Lemma 3.3 *For all $(\mathbf{u}_1, \dots, \mathbf{u}_n) \in \mathcal{U}_n^{\text{gen}}$, we have $\Phi_n(\mathbf{u}_1, \dots, \mathbf{u}_n) \in Y_{n-2}$.*

Proof For $3 \leq i \leq n$, let $y_i := \phi(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_i)$. Let $3 \leq i < j \leq n$. Then

$$\begin{bmatrix} \mathbf{u}_i \\ \mathbf{u}_j \end{bmatrix} \in \text{GL}_2(A) \implies \begin{bmatrix} \mathbf{u}_i \\ \mathbf{u}_j \end{bmatrix} \cdot T_{\mathbf{u}_1, \mathbf{u}_2} \in \text{GL}_2(A).$$

But

$$\begin{bmatrix} \mathbf{u}_i \\ \mathbf{u}_j \end{bmatrix} \cdot T_{\mathbf{u}_1, \mathbf{u}_2} = \begin{bmatrix} y_i & 1 \\ y_j & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

for some $a, b \in A^\times$. On taking the determinant, it follows that $y_i - y_j \in A^\times$. $\square$

From Corollary 3.2 we immediately deduce:

Lemma 3.4 *Let $(\mathbf{u}_1, \dots, \mathbf{u}_n) \in \mathcal{U}_n^{\text{gen}}$.*

(1) *For all $a_1, \dots, a_n \in A^\times$ we have*

$$\Phi_n(\mathbf{u}_1 \cdot a_1, \dots, \mathbf{u}_n \cdot a_n) = \Phi_n(\mathbf{u}_1, \dots, \mathbf{u}_n) \cdot a_1^2.$$

(2) *For all $X \in GL_2(A)$ we have*

$$\Phi_n((\mathbf{u}_1, \dots, \mathbf{u}_n) \cdot X) = \Phi_n(\mathbf{u}_1, \dots, \mathbf{u}_n) \cdot \det(X).$$

It follows that Φ_n induces a well-defined map of orbit sets (which we will continue to denote Φ_n)

$$X_n/SL_2(A) \rightarrow Y_{n-2}/(A^\times)^2.$$

Furthermore, this is a map of right $A^\times/(A^\times)^2$ -sets (noting that the matrix $X \in GL_2(A)$ acts via the square class of $\det(X)$ on the left).

Proposition 3.5 *For all $n \geq 3$, Φ_n induces a bijection of $A^\times/(A^\times)^2$ -sets*

$$X_n/SL_2(A) \leftrightarrow Y_{n-2}/(A^\times)^2.$$

Proof Let $\Psi_n : Y_{n-2} \rightarrow \mathcal{U}_n^{\text{gen}}$ be the map

$$(y_3, \dots, y_n) \mapsto ((0, -1), (1, 0), (y_3, 1), \dots, (y_n, 1)).$$

Then $\Phi_n \circ \Psi_n = \text{Id}_{Y_{n-2}}$ since $T_{(0,-1),(1,0)}$ is the identity matrix.

Now let $\bar{\Psi}_n$ be the induced map from Y_{n-2} to X_n , given by the formula

$$(y_3, \dots, y_n) \mapsto (0_+, 0_-, (y_3)_+, \dots, (y_n)_+).$$

For any $y \in A$ and $a \in A^\times$ we have

$$[y, 1] \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} = [ya, a^{-1}] = [ya^2, 1] = (ya^2)_+ \text{ in } X_1.$$

It follows that $\bar{\Psi}_n$ induces a well-defined map $Y_{n-2}/(A^\times)^2 \rightarrow X_n/SL_2(A)$ satisfying $\Phi_n \circ \bar{\Psi}_n = \text{Id}_{Y_{n-2}/(A^\times)^2}$.

It remains to show that $\tilde{\Psi}_n : Y_{n-2}/(A^\times)^2 \rightarrow X_n/\mathrm{SL}_2(A)$ is surjective: If $(\mathbf{u}_1, \dots, \mathbf{u}_n) \in \mathcal{U}_n^{\mathrm{gen}}$ then in $X_n/\mathrm{SL}_2(A)$ we have

$$(\bar{\mathbf{u}}_1, \dots, \bar{\mathbf{u}}_n) = (\bar{\mathbf{u}}_1, \dots, \bar{\mathbf{u}}_n) \cdot T_{\mathbf{u}_1, \mathbf{u}_2} = (0_+, 0_-, (y_3)_+, \dots, (y_n)_+) = \tilde{\Psi}_n(y_3, \dots, y_n)$$

where $y_i = \phi(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_i)$ for $i \geq 3$. $\square$

Taking the quotient set for the action of $A^\times/(A^\times)^2$ on both sides, we deduce:

Corollary 3.6 *For $n \geq 3$, Φ_n induces a natural bijection*

$$X_n/\mathrm{GL}_2(A) \leftrightarrow Y_{n-2}/A^\times.$$

For $n \geq 1$, we let

$$Z_n = Z_n(A) := \{(z_1, \dots, z_n) \in \mathcal{W}_A^n : z_i/z_j \in \mathcal{W}_A \text{ for all } i \neq j\}$$

and we let $Z_0 = Z_0(A) := \{1\}$.

We observe that for $n \geq 1$ there is a natural bijection of $A^\times$ -sets

$$Y_n \leftrightarrow A^\times \times Z_{n-1}, \quad (y_1, \dots, y_n) \leftrightarrow \left(y_1, \left(\frac{y_2}{y_1}, \dots, \frac{y_n}{y_1} \right) \right)$$

(where $A^\times$ acts on the first factor of the right-hand side). Thus, taking quotient sets for the action of $(A^\times)^2$ and $A^\times$, we obtain:

Corollary 3.7 *For all $n \geq 3$ we have natural bijections*

$$\begin{aligned} X_n/\mathrm{SL}_2(A) &\leftrightarrow A^\times/(A^\times)^2 \times Z_{n-3} \\ X_n/\mathrm{GL}_2(A) &\leftrightarrow Z_{n-3} \end{aligned}$$

Remark 3.8 Retracing our steps above, an explicit formula for the first bijection is

$$\begin{aligned} (\bar{\mathbf{u}}_1, \dots, \bar{\mathbf{u}}_n) &\mapsto \left(\langle \phi(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) \rangle, \left(\frac{\phi(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_4)}{\phi(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)}, \dots, \frac{\phi(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_n)}{\phi(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)} \right) \right) \\ &= \left(\left\langle \frac{d(\mathbf{u}_1, \mathbf{u}_3)d(\mathbf{u}_1, \mathbf{u}_2)}{d(\mathbf{u}_2, \mathbf{u}_3)} \right\rangle, \left(\frac{d(\mathbf{u}_1, \mathbf{u}_4)d(\mathbf{u}_2, \mathbf{u}_3)}{d(\mathbf{u}_2, \mathbf{u}_4)d(\mathbf{u}_1, \mathbf{u}_3)}, \dots, \frac{d(\mathbf{u}_1, \mathbf{u}_n)d(\mathbf{u}_2, \mathbf{u}_3)}{d(\mathbf{u}_2, \mathbf{u}_n)d(\mathbf{u}_1, \mathbf{u}_3)} \right) \right) \end{aligned}$$

and hence the formula for the second is

$$(\bar{\mathbf{u}}_1, \dots, \bar{\mathbf{u}}_n) \mapsto \left(\frac{d(\mathbf{u}_1, \mathbf{u}_4)d(\mathbf{u}_2, \mathbf{u}_3)}{d(\mathbf{u}_2, \mathbf{u}_4)d(\mathbf{u}_1, \mathbf{u}_3)}, \dots, \frac{d(\mathbf{u}_1, \mathbf{u}_n)d(\mathbf{u}_2, \mathbf{u}_3)}{d(\mathbf{u}_2, \mathbf{u}_n)d(\mathbf{u}_1, \mathbf{u}_3)} \right).$$

Remark 3.9 If $A = F$ is a field, then clearly $X_1 = (F^2 \setminus \{0\})/F^\times = \mathbb{P}^1(F)$ and more generally X_n is naturally the set of n -tuples of *distinct* points of $\mathbb{P}^1(F)$. On the other

hand, $\mathcal{W}_F = F^\times \setminus \{1\} = \mathbb{P}^1(F) \setminus \{\infty, 0, 1\}$ and Z_n consists of n -tuples of distinct points of $F^\times \setminus \{1\}$.

The point $x \in F$ is identified with the point of $\mathbb{P}^1(F)$ represented by $(x, 1)$. Since $d((x, 1), (y, 1)) = x - y$, the bijection $X_n/GL_2(F) \leftrightarrow Z_{n-3}$ is thus given by the formula

$$(x_1, \dots, x_n) \mapsto (\{x_1 : x_2 : x_3 : x_4\}, \dots, \{x_1 : x_2 : x_3 : x_n\})$$

where

$$\{x_1 : x_2 : x_3 : x_4\} = \frac{(x_1 - x_4)(x_2 - x_3)}{(x_1 - x_2)(x_3 - x_4)}$$

is the classic cross ratio.

Corollary 3.10 *For all $n \geq 3$ there are natural isomorphisms of R_A -modules*

$$\mathbb{Z}[X_n]_{SL_2(A)} \cong R_A[Z_{n-3}]$$

and natural isomorphisms of $\mathbb{Z}$ -modules

$$\mathbb{Z}[X_n]_{GL_2(A)} \cong \mathbb{Z}[Z_{n-3}].$$

Proof If G is a group and if X is a right G -set, then for any ring R there is a natural isomorphism

$$R[X]_G \cong R[X/G], \quad \bar{x} \mapsto \bar{x}.$$

Thus, for $n \geq 3$,

$$\begin{aligned} \mathbb{Z}[X_n]_{SL_2(A)} &\cong \mathbb{Z}[X_n/SL_2(A)] \cong \mathbb{Z}[A^\times/(A^\times)^2 \times Z_{n-3}] \\ &\cong \mathbb{Z}[A^\times/(A^\times)^2][Z_{n-3}] = R_A[Z_{n-3}]. \end{aligned}$$

□

For $n \geq 1$, let $\delta_n : \mathbb{Z}[X_{n+1}] \rightarrow \mathbb{Z}[X_n]$ be the simplicial boundary map

$$(\bar{\mathbf{u}}_1, \dots, \bar{\mathbf{u}}_{n+1}) \mapsto \sum_{i=1}^{n+1} (-1)^{i+1} (\bar{\mathbf{u}}_1, \dots, \widehat{\bar{\mathbf{u}}_i}, \dots, \bar{\mathbf{u}}_{n+1})$$

and let $\mathcal{A}(A) := \text{Coker}(\delta_4)$. Note that $(\mathbb{Z}[X_n], \delta_n)$ is a complex of $GL_2(A)$ -modules and that $\mathcal{A}(A)$ is thus also a $GL_2(A)$ -module.

Proposition 3.11 *For any commutative ring A , $\mathcal{RP}(A) \cong \mathcal{A}(A)_{SL_2(A)}$ as R_A -modules, and $\mathcal{P}(A) \cong \mathcal{A}(A)_{GL_2(A)}$ as $\mathbb{Z}$ -modules.*

Proof By right exactness of coinvariants, $\mathcal{A}(A)_{\mathrm{SL}_2(A)}$ is naturally identified with the cokernel of the map $\bar{\delta}_4 : \mathbb{Z}[X_5]_{\mathrm{SL}_2(A)} \rightarrow \mathbb{Z}[X_4]_{\mathrm{SL}_2(A)}$ of R_A -modules induced by δ_4 . Now

$$\mathbb{Z}[X_5]_{\mathrm{SL}_2(A)} \cong R_A[Z_2] \quad \text{and} \quad \mathbb{Z}[X_4]_{\mathrm{SL}_2(A)} \cong R_A[Z_1],$$

and, under these identifications, the map $\bar{\delta}_4$ is described as follows: $(z_1, z_2) \in R_A[Z_2]$ corresponds to $(1, z_1, z_2) \in Y_3/(A^\times)^2$ and this in turn corresponds to the element $(0_+, 0_-, 1_+, (z_1)_+, (z_2)_+) \in \mathbb{Z}[X_5]_{\mathrm{SL}_2(A)}$. The image of this under $\bar{\delta}_4$ is

$$\begin{aligned} & (0_-, 1_+, (z_1)_+, (z_2)_+) - (0_+, 1_+, (z_1)_+, (z_2)_+) \\ & + (0_+, 0_-, (z_1)_+, (z_2)_+) - (0_+, 0_-, 1_+, (z_2)_+) + (0_+, 0_-, 1_+, (z_1)_+, (z_2)_+) \end{aligned}$$

in $\mathbb{Z}[X_4]_{\mathrm{SL}_2(A)}$. Recalling that $(\bar{\mathbf{u}}_1, \dots, \bar{\mathbf{u}}_4) \in \mathbb{Z}[X_4]_{\mathrm{SL}_2(A)}$ corresponds to

$$\left\langle \frac{d(\mathbf{u}_1, \mathbf{u}_3)d(\mathbf{u}_1, \mathbf{u}_2)}{d(\mathbf{u}_2, \mathbf{u}_3)} \right\rangle \left(\frac{d(\mathbf{u}_1, \mathbf{u}_4)d(\mathbf{u}_2, \mathbf{u}_3)}{d(\mathbf{u}_2, \mathbf{u}_4)d(\mathbf{u}_1, \mathbf{u}_3)} \right) \in R_A[Z_1]$$

and observing that $d(a_+, b_+) = a - b$ and $d(0_-, a_+) = 1$ for all $a \neq b \in A$, we see that

$$\begin{aligned} \bar{\delta}_4(z_1, z_2) &= \langle 1 - z_1 \rangle \left(\frac{1 - z_1}{1 - z_2} \right) - \langle z_1^{-1} - 1 \rangle \left(\frac{1 - z_1^{-1}}{1 - z_2^{-1}} \right) \\ &+ \langle z_1 \rangle \left(\frac{z_2}{z_1} \right) - (z_2) + (z_1) \in R_A[Z_1]. \end{aligned}$$

Thus the map $R_A[Z_1] \rightarrow \mathcal{RP}(A)$, $(z) \mapsto [z]$ induces an isomorphism

$$\mathrm{Coker}(\bar{\delta}_4) \cong \mathcal{RP}(A).$$

□

Remark 3.12 We will call the (composite) map

$$\begin{aligned} \mathbb{Z}[X_4] &\rightarrow R_A[Z_1] \rightarrow \mathcal{RP}(A), \quad (\bar{\mathbf{u}}_1, \dots, \bar{\mathbf{u}}_4) \\ &\mapsto \left\langle \frac{d(\mathbf{u}_1, \mathbf{u}_3)d(\mathbf{u}_1, \mathbf{u}_2)}{d(\mathbf{u}_2, \mathbf{u}_3)} \right\rangle \left(\frac{d(\mathbf{u}_1, \mathbf{u}_4)d(\mathbf{u}_2, \mathbf{u}_3)}{d(\mathbf{u}_2, \mathbf{u}_4)d(\mathbf{u}_1, \mathbf{u}_3)} \right) \end{aligned}$$

the *refined cross ratio map*, and will denote it by cr . In the special case where $\mathbf{u}_i = \iota_+(x_i)$ for $x_i \in A$, it takes the form

$$(x_1, x_2, x_3, x_4) \mapsto \left\langle \frac{(x_1 - x_3)(x_1 - x_2)}{x_2 - x_3} \right\rangle \left[\frac{(x_1 - x_4)(x_2 - x_3)}{(x_1 - x_2)(x_3 - x_4)} \right].$$

3.2 The isomorphism $H_n(T, \mathbb{Z}) \cong H_n(B, \mathbb{Z})$

In order to prove Proposition 3.19 below, we follow the strategy of Suslin's proof of Theorem 1.8 in [13].

Lemma 3.13 ([13, Lemma 1.1]) *Suppose that $\phi_1, \dots, \phi_m : k \rightarrow F$ are field embeddings such that for any $x \in k^\times$ we have $\prod_{i=1}^m \phi_i(x) = 1$. Then k is a finite field of order p^f with $m \geq (p-1) \cdot f$.*

Remark 3.14 This simple but useful result can be extended in many directions. For example: (See [8, Lemma 2.2.4]) Let A be a ring with many units. Let B be any ring. For any $m \geq 1$, there do not exist ring homomorphisms $\phi_1, \dots, \phi_m : A \rightarrow B$ satisfying $\prod_{i=1}^m \phi_i(x) = 1$ for all $x \in A^\times$.

Local rings with infinite residue fields are rings with many units, but we will want to include the case of local rings with finite residue field below.

Corollary 3.15 *Suppose that $\phi_1, \dots, \phi_m : k \rightarrow F$ are field embeddings such that for any $x \in k^\times$ we have $\prod_{i=1}^m \phi_i(x^r) = 1$. Then k is a finite field of order p^f and $mr = (p-1) \cdot t$ for some $t \geq f$.*

Proof We have $1 = \prod_{i=1}^m \phi_i(x^r) = \prod_{i=1}^m \phi_i(x)^r := \prod_{i=1}^{mr} \psi_i(x)$, and thus k is finite of characteristic $p > 0$ and $mr \geq (p-1)f$ by Lemma 3.13.

On the other hand, if $a \in \mathbb{F}_p \subset k$ is a primitive root modulo p , then $1 = \prod_i \phi_i(a^r) = a^{mr}$ and thus $p-1 \mid mr$. $\square$

Corollary 3.16 *Let A be a local ring with maximal ideal $\mathcal{M}$ and residue field k . Suppose that $r \geq 1$ and $\phi_1, \dots, \phi_m : A \rightarrow F$ are homomorphisms from A to the field F satisfying $\prod_{i=1}^m \phi_i(u^r) = 1$ for all $u \in A^\times$. Then k is a finite field with p^f elements and $mr = (p-1)t$ where $t \geq f$. In particular, $(p-1)f \leq mr$.*

Proof F must have positive characteristic, for otherwise we can choose $1 < n \in A^\times \cap \mathbb{Z}$, and the hypothesis gives $n^{mr} = 1$ in F .

Let $\text{char}(F) = p > 0$. Replacing A by A/pA if necessary, we can assume that A is an $\mathbb{F}_p$ -algebra. We complete the proof by showing that $\mathcal{M} \subset \text{Ker}(\phi_i)$ for all i (and hence that the ϕ_i factor through k):

Let $x \in \mathcal{M}$. For $i = 1, \dots, m$, let $x_i = \phi_i(x) \in F$. If $f(T) \in \mathbb{F}_p[T]$ satisfies $f(0) \neq 0$, then $f(x) \in A^\times$. In this case we have

$$1 = \prod_{i=1}^m \phi_i(f(x))^r = \prod_{i=1}^m f(x_i)^r.$$

Thus, let I be the ideal of $\mathbb{F}_p[T_1, \dots, T_m]$ generated by the set

$$\left\{ \left(\prod_{i=1}^m f(T_i)^r \right) - 1 \mid f(T) \in \mathbb{F}_p[T] \text{ with } f(0) \neq 0 \right\}.$$

Let V be the corresponding variety. Then $(x_1, \dots, x_m) \in V(F)$.

We observe that $(0, \dots, 0) \in V$ if and only if $p - 1 | mr$.

On the other hand, suppose that $(a_1, \dots, a_m) \in \bar{\mathbb{F}}_p^m$ is algebraic and that $a_j \neq 0$ for some j . Then there exists $f(T) \in \mathbb{F}_p[T]$ with $f(0) \neq 0$ and $f(a_j) = 0$. It follows that $\prod_i f(a_i)^r = 0$ and hence $(a_1, \dots, a_m) \notin V(\bar{\mathbb{F}}_p)$. Thus

$$V(\bar{\mathbb{F}}_p) = \begin{cases} \{0\}, & p - 1 | mr \\ \emptyset, & \text{otherwise.} \end{cases}$$

It follows from the Nullstellensatz that the ideal, J , of V in $\bar{\mathbb{F}}_p[T_1, \dots, T_m]$ is given by

$$J = \begin{cases} \langle T_1, \dots, T_m \rangle, & p - 1 | mr \\ \bar{\mathbb{F}}_p[T_1, \dots, T_m], & \text{otherwise.} \end{cases}$$

and hence, for *any* field K we have

$$V(K) = \begin{cases} \{0\}, & p - 1 | mr \\ \emptyset, & \text{otherwise.} \end{cases}$$

Since $(x_1, \dots, x_m) \in V(F)$, it follows that $p - 1 | mr$ and $x_i = \phi_i(x) = 0$ for all i . $\square$

For $r \geq 1$, we denote by $A(r)$ the $\mathbb{Z}[A^\times]$ -module obtained by making $u \in A^\times$ act on A as multiplication by u^r .

Lemma 3.17 *Let $m, r \geq 1$. Let $n_1, \dots, n_k$ satisfy $n_1 + \dots + n_k = m$ and $n_i \geq 1$. Let A be a local ring with residue field k . If k is finite of order p^f we suppose that $mr < (p - 1)f$.*

Let $T^n(A(r))$ denote either $\wedge_{\mathbb{Z}}^n(A(r))$ or $\text{Sym}_{\mathbb{Z}}^n(A(r))$, considered as $A^\times$ modules with the diagonal action.

Then

$$H_i(A^\times, T^{n_1}(A(r)) \otimes \dots \otimes T^{n_k}(A(r))) = 0$$

for all $i \geq 0$.

Proof This follows from Corollary 3.16 by the same argument verbatim as that by which Suslin proves Corollary 1.6 from Lemma 1.1 in [13]. $\square$

Lemma 3.18 *Let $m, r \geq 1$. Let A be a local integral domain with residue field k . If k is finite of order p^f we suppose that $mr < (p - 1)f$.*

For all $i \geq 0$ we have

$$H_i(A^\times, H_m(A(r), \mathbb{Z})) = 0.$$

Proof If $\text{char}(A) = 0$, then $H_m(A(r), \mathbb{Z}) = \wedge_{\mathbb{Z}}^m(A(r))$ and the statement follows at once from Lemma 4.5.

Otherwise A is an $\mathbb{F}_p$ -algebra for some $p > 0$. Then $H_m(A(r), \mathbb{F}_p)$ is a direct sum of modules of the form $\wedge_{\mathbb{Z}}^s(A(r)) \otimes \text{Sym}_{\mathbb{Z}}^t(A(r))$ with $s + t \leq m$. It follows from Lemma 3.17 that $H_i(A^\times, H_m(A(r), \mathbb{F}_p)) = 0$.

On the other hand, the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{p} \mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p \longrightarrow 0$$

induces a long exact homology sequence for $H_\bullet(A(r), \cdot)$, which decomposes into short exact sequences

$$0 \rightarrow H_k(A(r), \mathbb{Z}) \rightarrow H_k(A(r), \mathbb{F}_p) \rightarrow H_{k-1}(A(r), \mathbb{Z}) \rightarrow 0 \quad (k \geq 2)$$

and an isomorphism

$$H_1(A(r), \mathbb{Z}) \cong A(r) \cong H_1(A(r), \mathbb{F}_p).$$

The vanishing of $H_i(A^\times, H_m(A(r), \mathbb{Z}))$ for all $i \geq 0$ then follows from a straightforward induction on m . $\square$

We let $T = T(A)$ denote the subgroup of $SL_2(A)$ consisting of diagonal matrices:

$$T(A) := \left\{ \begin{bmatrix} u & 0 \\ 0 & u^{-1} \end{bmatrix} \mid u \in A^\times \right\}.$$

Thus $T(A) \cong A^\times$. We let $B = B(A)$ denote the subgroup consisting of lower triangular matrices:

$$B(A) := \left\{ \begin{bmatrix} u & a \\ 0 & u^{-1} \end{bmatrix} \mid u \in A^\times, a \in A \right\}.$$

Thus there is natural (split) group extension

$$1 \rightarrow V \rightarrow B \rightarrow T \rightarrow 1 \quad (4)$$

where

$$V = T(A) := \left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \mid a \in A \right\} \cong A.$$

Here $T \cong A^\times$ acts on $V \cong A$ by conjugation. With the given identifications, $u \in A^\times$ acts on $a \in A$ as multiplication by u^2 . Thus $V \cong A(2)$ as a $\mathbb{Z}[A^\times]$ -module.

Proposition 3.19 *Let $n \geq 1$. Let A be a local integral domain with residue field k . If k is finite of order p^f we suppose that $(p-1)f > 2n$.*

The natural maps $B \rightarrow T$ and $T \rightarrow B$ induce isomorphisms on homology

$$H_n(T, \mathbb{Z}) \cong H_n(B, \mathbb{Z})$$

Proof The Hochschild–Serre spectral sequence associated to the extension (4) takes the form

$$E_{i,j}^2 = H_i(T, H_j(V, \mathbb{Z})) = H_i(A^\times, H_j(A(2), \mathbb{Z})) \Rightarrow H_{i+j}(B, \mathbb{Z}).$$

By Lemma 3.18, it follows that $E_{i,j}^2 = 0$ if $0 < j \leq n$, and $E_{i,0}^2 = H_i(T, \mathbb{Z})$ for all i .

Hence $H_n(B, \mathbb{Z}) = H_n(T, \mathbb{Z})$. $\square$

Remark 3.20 In particular, $H_n(T, \mathbb{Z}) \cong H_n(B, \mathbb{Z})$ for all $n \leq 3$ provided

$$|k| \notin \{p^f \mid p \text{ prime and } (p-1)f \leq 6\} = \{2, 3, 4, 5, 7, 8, 9, 16, 27, 32, 64\}.$$

In the remainder of the paper we will say that a field k is *sufficiently large* if either k is infinite or if $|k| = p^f$ where $(p-1)f > 6$.

3.3 The complex $L_\bullet$

For a commutative ring A , we let $L_n = L_n(A) := \mathbb{Z}[X_{n+1}]$. Equipped with the boundary $\delta_n : L_n \rightarrow L_{n-1}$ this yields a complex, $L_\bullet$, of right $\mathrm{GL}_2(A)$ -modules. Restricting the group action, this is also a complex of $\mathrm{SL}_2(A)$ -modules.

We now restrict attention to the case where A is a commutative local ring with residue field k . We let $\pi : A \rightarrow k$ denote the canonical surjective quotient map. So $A^\times = \pi^{-1}(k^\times)$. More generally, if $X \in M_n(A)$ is an $n \times n$ matrix with coefficients in A , we let $\pi(X) \in M_n(k)$ denote the matrix obtained by applying π to each entry of X . Since A is a local ring $X \in \mathrm{GL}_n(A)$ if and only if $\pi(X) \in \mathrm{GL}_n(k)$.

Similarly, $\mathbf{u} = (u_1, u_2) \in U_2(A)$ if and only if $\pi(\mathbf{u}) \in U_2(k) = k^2 \setminus \{0\}$ and $\bar{\mathbf{u}} \in X_1(A)$ if and only if $\pi(\bar{\mathbf{u}}) \in X_1(k) = \mathbb{P}^1(k)$. Furthermore,

$$(\mathbf{u}_1, \dots, \mathbf{u}_n) \in \mathcal{U}_n^{\mathrm{gen}}(A) \iff (\pi(\mathbf{u}_1), \dots, \pi(\mathbf{u}_n)) \in \mathcal{U}_n^{\mathrm{gen}}(k)$$

and hence

$$(\bar{\mathbf{u}}_1, \dots, \bar{\mathbf{u}}_n) \in X_n(A) \iff (\overline{\pi(\mathbf{u}_1)}, \dots, \overline{\pi(\mathbf{u}_n)}) \in X_n(k).$$

Lemma 3.21 $H_n(L_\bullet) = 0$ for $1 \leq n < |k|$.

Proof When $A = k$ is a field, the argument is given in [4, Lemma 4.4]. This argument is easily adapted to the current situation as follows:

For any subset S of $\mathbb{P}^1(k)$, let $D_n(S)$ denote the subgroup of $L_n(A)$ generated by those $(n+1)$ -tuples $(\bar{\mathbf{u}}_1, \dots, \bar{\mathbf{u}}_{n+1}) \in X_{n+1}(A)$ which satisfy $S \subset \{\overline{\pi(\mathbf{u}_1)}, \dots, \overline{\pi(\mathbf{u}_{n+1})}\}$. Thus $D_n(S) = 0$ if $|S| > n+1$. Furthermore, $D_n(S_1 \cup S_2) = D_n(S_1) \cap D_n(S_2)$ for any $S_1, S_2 \subset \mathbb{P}^1(k)$.

Now for each $x \in \mathbb{P}^1(k)$, choose $\mathbf{u}_x \in U_2(A)$ satisfying $\overline{\pi(\mathbf{u}_x)} = x$ and for $n \geq 0$ define a homomorphism $S_x : L_n \rightarrow L_{n+1}$ by

$$S_x(\bar{\mathbf{u}}_1, \dots, \bar{\mathbf{u}}_{n+1}) = \begin{cases} (\bar{\mathbf{u}}_x, \bar{\mathbf{u}}_1, \dots, \bar{\mathbf{u}}_{n+1}), & x \notin \{\overline{\pi(\mathbf{u}_1)}, \dots, \overline{\pi(\mathbf{u}_{n+1})}\} \\ 0, & \text{otherwise} \end{cases}$$

Thus if $(\bar{\mathbf{u}}_1, \dots, \bar{\mathbf{u}}_{n+1}) \in X_{n+1}(A)$ and if $x \notin \{\overline{\pi(\mathbf{u}_1)}, \dots, \overline{\pi(\mathbf{u}_{n+1})}\}$ then

$$\delta S_x(\bar{\mathbf{u}}_1, \dots, \bar{\mathbf{u}}_{n+1}) = (\bar{\mathbf{u}}_1, \dots, \bar{\mathbf{u}}_{n+1}) - S_x \delta(\bar{\mathbf{u}}_1, \dots, \bar{\mathbf{u}}_{n+1}).$$

On the other hand, if $x = \overline{\pi(\mathbf{u}_j)}$ for some j then

$$S_x \delta(\bar{\mathbf{u}}_1, \dots, \bar{\mathbf{u}}_{n+1}) = (-1)^{j+1}(\bar{\mathbf{u}}_x, \bar{\mathbf{u}}_1, \dots, \widehat{\bar{\mathbf{u}}_j}, \dots, \bar{\mathbf{u}}_{n+1}).$$

and hence

$$\begin{aligned} 0 &= \delta S_x(\bar{\mathbf{u}}_1, \dots, \bar{\mathbf{u}}_{n+1}) \\ &= (\bar{\mathbf{u}}_1, \dots, \bar{\mathbf{u}}_{n+1}) - S_x \delta(\bar{\mathbf{u}}_1, \dots, \bar{\mathbf{u}}_{n+1}) \\ &\quad - \left\{ S_x \delta(\bar{\mathbf{u}}_1, \dots, \bar{\mathbf{u}}_{n+1}) - (-1)^j(\bar{\mathbf{u}}_x, \bar{\mathbf{u}}_1, \dots, \widehat{\bar{\mathbf{u}}_j}, \dots, \bar{\mathbf{u}}_{n+1}) \right\}. \end{aligned}$$

In either case we have

$$\delta S_x(\bar{\mathbf{u}}_1, \dots, \bar{\mathbf{u}}_{n+1}) = (\bar{\mathbf{u}}_1, \dots, \bar{\mathbf{u}}_{n+1}) - S_x \delta(\bar{\mathbf{u}}_1, \dots, \bar{\mathbf{u}}_{n+1}) + w$$

where $w \in D_n(\{x\})$. Furthermore, if $(\bar{\mathbf{u}}_1, \dots, \bar{\mathbf{u}}_{n+1}) \in D_n(S)$ for some subset S of $\mathbb{P}^1(k)$ then $w \in D_n(S \cup \{x\})$.

Suppose now that $1 \leq n < |k|$ and that $x_1, \dots, x_{n+2}$ are $n+2$ distinct points of $\mathbb{P}^1(k)$. Let $z \in L_n(A)$ be a cycle. Then

$$(\delta S_{x_1} - \text{Id})z = S_{x_1} \delta(z) + z_1 = z_1$$

where $z_1 \in D_n(\{x_1\})$ and z_1 is again a cycle.

Thus $(\delta S_{x_2} - \text{Id})z_1 = z_2$ where z_2 is a cycle belonging to $D_n(\{x_1, x_2\})$. Repeating the process we get

$$(\delta S_{x_{n+2}} - \text{Id})(\delta S_{x_{n+1}} - \text{Id}) \cdots (\delta S_{x_1} - \text{Id})z \in D_n(\{x_1, \dots, x_{n+2}\}) = 0.$$

This equation has the form $\delta(y) + (-1)^{n+2}z = 0$ and hence $z = \delta((-1)^{n+1}y)$ is a boundary as required. $\square$

3.4 The third homology of SL_2 of local domains

Recall that we say a field k is *sufficiently large* if either k is infinite or $|k| = p^f$ with $(p-1)f > 6$; see Remark 3.20.

Theorem 3.22 *Let A be a local integral domain with sufficiently large residue field k . Then there is a natural short exact sequence of $R_A[\frac{1}{2}]$ -modules*

$$0 \rightarrow \operatorname{tor}(\mu_A, \mu_A) \left[\frac{1}{2} \right] \rightarrow H_3 \left(\operatorname{SL}_2(A), \mathbb{Z} \left[\frac{1}{2} \right] \right) \rightarrow \mathcal{RB}(A) \left[\frac{1}{2} \right] \rightarrow 0.$$

Proof In the case where A is a field, the proof can be found in [4, section 4]. We indicate here the adaptations needed to extend that proof to the current context:

Associated to the complex $L_\bullet = L_\bullet(A)$ there is hyperhomology spectral sequence of the form

$$E_{p,q}^1 = H_p \left(\operatorname{SL}_2(A), L_q \left[\frac{1}{2} \right] \right) \Rightarrow H_{p+q} \left(\operatorname{SL}_2(A), L_\bullet \left[\frac{1}{2} \right] \right)$$

and furthermore the augmentation $L_0 \rightarrow \mathbb{Z}$ induces an isomorphism

$$H_n \left(\operatorname{SL}_2(A), L_\bullet \left[\frac{1}{2} \right] \right) \cong H_n \left(\operatorname{SL}_2(A), \mathbb{Z} \left[\frac{1}{2} \right] \right)$$

for $n \leq 3$ by Lemma 3.21.

The $\operatorname{SL}_2(A)$ -modules L_n are permutation modules, so the E^1 -terms are calculated using Shapiro's Lemma:

$\operatorname{SL}_2(A)$ acts transitively on X_1 and the stabilizer of $0_+ \in X_1$ is $B = B(A)$. Thus

$$L_0 = \mathbb{Z}[X_1] \cong \mathbb{Z}[B \backslash \operatorname{SL}_2(A)] \cong \operatorname{Ind}_{\mathbb{Z}[B]}^{\mathbb{Z}[\operatorname{SL}_2(A)]} \mathbb{Z}$$

and hence

$$E_{p,0}^1 = H_p \left(\operatorname{SL}_2(A), L_0 \left[\frac{1}{2} \right] \right) \cong H_p \left(B, \mathbb{Z} \left[\frac{1}{2} \right] \right).$$

Similarly, $\operatorname{SL}_2(A)$ acts transitively on X_2 and the stabilizer of $(0_+, 0_-)$ is $T = T_A$, so that

$$E_{p,1}^1 = H_p \left(\operatorname{SL}_2(A), L_1 \left[\frac{1}{2} \right] \right) \cong H_p \left(T, \mathbb{Z} \left[\frac{1}{2} \right] \right).$$

For $n \geq 3$, the stabilizer in $\operatorname{SL}_2(A)$ of $(\bar{\mathbf{u}}_1, \dots, \bar{\mathbf{u}}_n) \in X_n$ is $Z(\operatorname{SL}_2(A)) \cong \mu_2(A)$. By Corollary 3.10 it follows that for $q \geq 2$ we have

$$E_{p,q}^1 = R_A \left[\frac{1}{2} \right] [Z_{q-2}] \otimes H_p(\mu_2(A), \mathbb{Z}) = \begin{cases} R_A \left[\frac{1}{2} \right] [Z_{q-2}], & p = 0 \\ 0, & p > 0 \end{cases}$$

where $Z_n = Z_n(A)$ as above.

Now $T \cong A^\times$. Thus $E_{p,q}^1 \cong H_p(A^\times, \mathbb{Z}[\frac{1}{2}])$ for $p \leq 3$ and $q \in \{0, 1\}$.

Now let

$$w := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \operatorname{SL}_2(A).$$

Then $w(0_+) = 0_-$ and $w(0_-) = 0_+$. It follows easily that the differential

$$d^1 : E_{p,1}^1 = H_p(T, \mathbb{Z}[\frac{1}{2}]) \rightarrow H_p(T, \mathbb{Z}[\frac{1}{2}]) = E_{p,0}^1$$

is the map

$$H_p(T, \mathbb{Z}[\frac{1}{2}]) \xrightarrow{w_p^{-1}} H_p(T, \mathbb{Z}[\frac{1}{2}])$$

where $w_p : H_p(T, \mathbb{Z}[\frac{1}{2}]) \rightarrow H_p(T, \mathbb{Z}[\frac{1}{2}])$ is the map induced by conjugation by w . However, conjugating by w is just the inversion map on $A^\times \cong T$. Thus $d^1 : \mathbb{Z}[\frac{1}{2}] = E_{0,1}^1 \rightarrow E_{0,0}^1 = \mathbb{Z}[\frac{1}{2}]$ is the zero map. $d^1 : A^\times[\frac{1}{2}] = E_{1,1}^1 \rightarrow E_{1,0}^1 = A^\times[\frac{1}{2}]$ is the map $u \mapsto u^{-2}$ and hence is an isomorphism. $d^1 : \wedge_{\mathbb{Z}}^2(A^\times[\frac{1}{2}]) = E_{2,1}^1 \rightarrow E_{2,0}^1 = \wedge_{\mathbb{Z}}^2(A^\times[\frac{1}{2}])$ is the zero map.

Finally, $E_{3,1}^1 = E_{3,0}^1 = H_3(A^\times, \mathbb{Z}[\frac{1}{2}]) \cong \wedge_{\mathbb{Z}}^3(A^\times[\frac{1}{2}]) \oplus \text{tor}(\mu_A, \mu_A)[\frac{1}{2}]$. The map $d^1 : E_{3,1}^1 \rightarrow E_{3,0}^1$ is an isomorphism of the first factor and the zero map on the second factor.

The differential

$$d^1 : R_A[\frac{1}{2}] \cong H_0(SL_2(A), L_2) = E_{0,2}^1 \rightarrow E_{0,1}^1 = H_0(SL_2(A), L_1) \cong \mathbb{Z}[\frac{1}{2}]$$

is the natural augmentation sending $\langle u \rangle$ to 1 for any $u \in A^\times$.

As in the proof of [4, Theorem 4.3], the differential

$$d^1 : R_A[\frac{1}{2}][Z_1] \cong H_0(SL_2(A), L_3) = E_{0,3}^1 \rightarrow E_{0,2}^1 = H_0(SL_2(A), L_2) \cong R_A[\frac{1}{2}]$$

is the R_A -homomorphism sending (z) to $\langle\langle z \rangle\rangle \langle\langle 1 - z \rangle\rangle \in \mathcal{I}_A^2$ for any $z \in \mathcal{W}_A$.

By the proof of Proposition 3.11 above, the differential

$$d^1 : R_A[\frac{1}{2}][Z_2] \cong H_0(SL_2(A), L_4) = E_{0,4}^1 \rightarrow E_{0,3}^1 = H_0(SL_2(A), L_3) \cong R_A[\frac{1}{2}][Z_1]$$

is the map

$$(z_1, z_2) \mapsto \langle 1 - z_1 \rangle \left(\frac{1 - z_1}{1 - z_2} \right) - \langle z_1^{-1} - 1 \rangle \left(\frac{1 - z_1^{-1}}{1 - z_2^{-1}} \right) + \langle z_1 \rangle \left(\frac{z_2}{z_1} \right) - (z_2) + (z_1)$$

Thus $E_{0,3}^2 = \mathcal{RP}_1(A)[\frac{1}{2}]$. Clearly there are no nonzero d^2 -differentials (since either the domain or target of d^2 will be 0).

The argument now concludes exactly as in [4, section 4]: The differential $d^3 : E_{0,4}^3 \rightarrow \wedge_{\mathbb{Z}}^2(A^\times[\frac{1}{2}])$ is surjective (over $\mathbb{Z}[\frac{1}{2}]$) and hence $E_{2,1}^4 = E_{2,1}^\infty = 0$. There is a commutative diagram

$$\begin{array}{ccc}
 \mathcal{RP}_1(A) \left[\frac{1}{2} \right] & \xrightarrow{d^3} & \wedge_{\mathbb{Z}}^2(A^\times \left[\frac{1}{2} \right]) \\
 \downarrow & \searrow \lambda_2 & \downarrow \cong \\
 \mathcal{P}(A) \left[\frac{1}{2} \right] & \xrightarrow{\lambda} & S_{\mathbb{Z}}^2(A^\times \left[\frac{1}{2} \right])
 \end{array}$$

It follows that $E_{0,3}^4 = E_{0,3}^\infty = \text{Ker}(\lambda_2 : \mathcal{RP}_1(A) \left[\frac{1}{2} \right] \rightarrow S_{\mathbb{Z}}^2(A^\times \left[\frac{1}{2} \right]) = \mathcal{RB}(A) \left[\frac{1}{2} \right])$. This completes the proof of the theorem. $\square$

3.5 Local domains with finite residue fields

Corollary 3.23 *Let A be a local integral domain with sufficiently large residue field. Then the natural map $H_3(SL_2(A), \mathbb{Z}) \rightarrow K_3^{\text{ind}}(A)$ induces an isomorphism*

$$H_3(SL_2(A), \mathbb{Z} \left[\frac{1}{2} \right])_{A^\times} \cong K_3^{\text{ind}}(A) \left[\frac{1}{2} \right].$$

Proof Theorem 3.22 implies the exactness of the top row in the diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{tor}(\mu_A, \mu_A) \left[\frac{1}{2} \right] & \longrightarrow & H_3(SL_2(A), \mathbb{Z} \left[\frac{1}{2} \right]) & \longrightarrow & \mathcal{RB}(A) \left[\frac{1}{2} \right] \longrightarrow 0 \\
 & & \downarrow = & & \downarrow & & \downarrow = \\
 0 & \longrightarrow & \text{tor}(\mu_A, \mu_A) \left[\frac{1}{2} \right] & \longrightarrow & K_3^{\text{ind}}(A) \left[\frac{1}{2} \right] & \longrightarrow & \mathcal{B}(A) \left[\frac{1}{2} \right] \longrightarrow 0
 \end{array} \quad (5)$$

The bottom row is exact by [11, Theorem 5.1] (infinite residue field) and [9, Theorem 6.1] (finite residue field).

The corollary now follows by taking $A^\times$ -coinvariants on the top row, since $\text{tor}(\mu_A, \mu_A)_{A^\times} = \text{tor}(\mu_A, \mu_A)$ and $\mathcal{RB}(A) \left[\frac{1}{2} \right]_{A^\times} = \mathcal{B}(A) \left[\frac{1}{2} \right]$ by Proposition 2.9 (2). $\square$

Corollary 3.24 *Let A be a local integral domain with sufficiently large residue field. Then*

$$\mathcal{I}_A \mathcal{RP}_1(A) \left[\frac{1}{2} \right] \cong \mathcal{I}_A H_3(SL_2(R), \mathbb{Z} \left[\frac{1}{2} \right]) = H_3(SL_2(A), \mathbb{Z} \left[\frac{1}{2} \right])_0.$$

Proof Apply the snake lemma to diagram and use Proposition 2.9 (3). $\square$

4 Submodules and special elements in $\mathcal{RP}(A)$

In this section, A will denote a commutative local ring, with maximal ideal $\mathcal{M}$ and residue field $k = A/\mathcal{M}$. Furthermore, we will suppose that k has at least four elements.

In this case, we have $A^\times = A \setminus \mathcal{M}$ and $\mathcal{W}_A = A^\times \setminus U_1$, where $U_1 = U_{1,A} = 1 + \mathcal{M}$. In particular, if $x \in A^\times$, then $x \in \mathcal{W}_A \iff x^{-1} \in \mathcal{W}_A$.

4.1 The modules $\mathcal{K}_A^{(i)}$

As in [3], we define two families of elements of $\mathcal{RP}(A)$.

Given $x \in \mathcal{W}_A$ we define

$$\psi_1(x) := [x] + \langle -1 \rangle [x^{-1}]$$

and

$$\psi_2(x) := \langle 1 - x \rangle \left(\langle x \rangle [x] + [x^{-1}] \right) = \langle x^{-1} - 1 \rangle [x] + \langle 1 - x \rangle [x^{-1}].$$

Observe, from the definitions, that $\psi_i(x^{-1}) = \langle -1 \rangle \psi_i(x)$ for all $x \in \mathcal{W}_A$.

Lemma 4.1 *For $i = 1, 2$ we have*

- (1) $\psi_i(xy) = \langle x \rangle \psi_i(y) + \psi_i(x)$ whenever $x, y, xy \in \mathcal{W}_A$.
- (2) $\langle x \rangle \psi_i(x^{-1}) + \psi_i(x) = 0$ for all $x \in \mathcal{W}_A$.

Proof (1) The proof of Lemma 3.1 in [3] adapts without alteration.

- (2) Let $x \in \mathcal{W}_A$. Choose $y \in \mathcal{W}_A$ such that $xy \in \mathcal{W}_A$ also. (Note that this is possible because of the hypothesis that $|k| \geq 4$.)

Then by part (4.1) we have

$$\psi_i(xy) = \langle y \rangle \psi_i(x) + \psi_i(y) \implies \langle y \rangle \psi_i(x) = \psi_i(xy) - \psi_i(y)$$

and

$$\begin{aligned} \psi_i(y) &= \psi_i(xy \cdot x^{-1}) = \langle xy \rangle \psi_i(x^{-1}) + \psi_i(xy) \implies \langle xy \rangle \psi_i(x^{-1}) \\ &= \psi_i(y) - \psi_i(xy). \end{aligned}$$

From these we deduce that

$$\langle y \rangle \psi_i(x) = -\langle xy \rangle \psi_i(x^{-1}).$$

Multiplying both sides of this equation by $\langle y \rangle$ gives the result. □

Lemma 4.2 *Let $u \in U_1$. For any $w_1, w_2 \in \mathcal{W}_A$, we have*

$$\psi_i(w_1u) - \langle u \rangle \psi_i(w_1) = \psi_i(w_2u) - \langle u \rangle \psi_i(w_2)$$

for $i = 1, 2$.

Proof First suppose that $w_1 \not\equiv w_2 \pmod{U_1}$. Then $w_1 w_2^{-1} \in \mathcal{W}_A$ and hence

$$\begin{aligned}\psi_i(w_1 u) &= \psi_i\left((w_2^{-1} w_1) \cdot (w_2 u)\right) \\ &= \langle w_2 u \rangle \psi_i\left(w_1 w_2^{-1}\right) + \psi_i(w_2 u) \\ &= \langle u \rangle \left(\langle w_2 \rangle \psi_i\left(w_1 w_2^{-1}\right)\right) + \psi_i(w_2 u) \\ &= \langle u \rangle (\psi_i(w_1) - \psi_i(w_2)) + \psi_i(w_2 u)\end{aligned}$$

giving the result in this case.

On the other hand, if $w_1 \equiv w_2 \pmod{U_1}$ choose $w_3 \in \mathcal{W}_A$ with $w_3 \not\equiv w_1 \pmod{U_1}$. Then

$$\psi_i(w_1 u) - \langle u \rangle \psi_i(w_1) = \psi_i(w_3 u) - \langle u \rangle \psi_i(w_3) = \psi_i(w_2 u) - \langle u \rangle \psi_i(w_2).$$

□

We now extend the definition of $\psi_i(x)$ to allow $x \in U_1$. For $u \in U_1$, we define

$$\psi_i(u) := \psi_i(uw) - \langle u \rangle \psi_i(w)$$

for any $w \in \mathcal{W}_A$.

Proposition 4.3 *For $i = 1, 2$ the maps $A^\times \rightarrow \mathcal{RP}(A)$, $x \mapsto \psi_i(x)$ define 1-cocycles; i.e. we have*

$$\psi_i(xy) = \langle x \rangle \psi_i(y) + \psi_i(x)$$

for all $x, y \in A^\times$.

Proof If $x, y, xy \in \mathcal{W}_A$, this is part (1) of Lemma 4.1.

If $x, y \in \mathcal{W}_A$, but $xy \in U_1$, then

$$\begin{aligned}\psi_i(xy) &= \psi_i(xy \cdot y^{-1}) - \langle xy \rangle \psi_i(y^{-1}) \\ &= \psi_i(x) - \langle x \rangle \left(\langle y \rangle \psi_i(y^{-1})\right) \\ &= \psi_i(x) + \langle x \rangle \psi_i(y)\end{aligned}$$

using Lemma 4.1 (2) in the last step.

If $x \in U_1$ and $y \in \mathcal{W}_A$, the identity is just the definition of $\psi_i(x)$.

On the other hand, if $x \in \mathcal{W}_A$ and $y \in U_1$, then choose $w \in \mathcal{W}_A$ such that $xw \in \mathcal{W}_A$. We have

$$\begin{aligned}\langle x \rangle \psi_i(y) + \psi_i(x) &= \langle x \rangle (\psi_i(wy) - \langle y \rangle \psi_i(w)) + \psi_i(x) \\ &= \langle x \rangle \psi_i(wy) + \psi_i(x) - \langle xy \rangle \psi_i(w) \\ &= \psi_i(xyw) - \langle xy \rangle \psi_i(w) \\ &= \psi_i(xy).\end{aligned}$$

Finally, suppose that $x, y \in U_1$. Let $w \in \mathcal{W}_A$. Then

$$\begin{aligned}\psi_i(xy) &= \psi_i(xyw) - \langle xy \rangle \psi_i(w) \\ &= \langle x \rangle \psi_i(yw) + \psi_i(x) - \langle xy \rangle \psi_i(w) \\ &= \langle x \rangle (\psi_i(yw) - \langle y \rangle \psi_i(w)) + \psi_i(x) \\ &= \langle x \rangle \psi_i(y) + \psi_i(x)\end{aligned}$$

as required. $\square$

We recall here, from [3] some of the basic algebraic properties of the $\psi_i(x)$. (The proofs given in [3] for the case of fields adapt without change to the case of local rings.)

Proposition 4.4 *For $i \in \{1, 2\}$ we have:*

- (1) $\langle\langle x \rangle\rangle \psi_i(y) = \langle\langle y \rangle\rangle \psi_i(x)$ for all x, y
- (2) $\psi_i(xy^2) = \psi_i(x) + \psi_i(y^2)$ for all x, y
- (3) $\langle\langle x \rangle\rangle \psi_i(y^2) = 0$ for all x, y
- (4) $2 \cdot \psi_i(-1) = 0$ for all i
- (5) $\psi_i(x^2) = -\langle\langle x \rangle\rangle \psi_i(-1)$ for all x
- (6) $2 \cdot \psi_i(x^2) = 0$ for all x and if -1 is a square in $A^\times$ then $\psi_i(x^2) = 0$ for all x .
- (7) $\langle\langle x \rangle\rangle \langle\langle y \rangle\rangle \psi_i(-1) = 0$ for all x, y
- (8) $\langle -1 \rangle \langle\langle x \rangle\rangle \psi_i(y) = \langle\langle x \rangle\rangle \psi_i(y)$ for all x, y
- (9) *Let*

$$\epsilon(A) := \begin{cases} 1, & -1 \in (A^\times)^2 \\ 2, & -1 \notin (A^\times)^2 \end{cases}$$

The map $A^\times / (A^\times)^2 \rightarrow \mathcal{RP}(A), \langle x \rangle \mapsto \epsilon(A) \psi_i(x)$ is a well-defined 1-cocycle.

Lemma 4.5 *For $i = 1, 2$ and for all $x \in A^\times$*

- (1) $\lambda_1(\psi_i(x)) = -p_{-1}^+ \langle\langle x \rangle\rangle = \langle\langle -x \rangle\rangle \langle\langle x \rangle\rangle \in \mathcal{I}_A^2$
- (2) $\lambda_2(\psi_i(x)) = (-x) \circ x \in S_{\mathbb{Z}}^2(A^\times)$.

Proof (1) For $x \in \mathcal{W}_A$, this is a straightforward calculation given in [3, Lemma 3.3].

For $x \in U_1$ it follows from the identity $\langle\langle xw \rangle\rangle - \langle x \rangle \langle\langle w \rangle\rangle = \langle\langle x \rangle\rangle$ in $\mathcal{R}_A$.

- (2) Since $A^\times$ acts trivially on $S_{\mathbb{Z}}^2(A^\times)$, we have for any $x \in \mathcal{W}_A$

$$\begin{aligned}\lambda_2(\psi_1(x)) &= \lambda_2(\psi_2(x)) = \lambda_2([x]) + \lambda_2\left(\begin{bmatrix} x^{-1} \end{bmatrix}\right) \\ &= (1-x) \circ x + (1-x^{-1}) \circ x^{-1} \\ &= (1-x) \circ x - \left(\frac{1-x}{-x}\right) \circ x \\ &= (-x) \circ x.\end{aligned}$$

On the other hand, if $x \in U_1$, the result follows from the identity

$$(-xw) \circ (xw) - (-w) \circ w = (-x) \circ x$$

in $S_{\mathbb{Z}}^2(A^\times)$.

□

Let $\mathcal{K}_A^{(i)}$ denote the R_A -submodule of $\mathcal{RP}(A)$ generated by the set $\{\psi_i(x) \mid x \in A^\times\}$.

Lemma 4.6 *Then for $i \in \{1, 2\}$*

$$\lambda_1(\mathcal{K}_A^{(i)}) = p_{-1}^+(\mathcal{I}_A) \subset \mathcal{I}_A^2$$

and $\text{Ker}(\lambda_1|_{\mathcal{K}_A^{(i)}})$ is annihilated by 4.

Proof The first statement follows from Lemma 4.5

For the second, the proof of Lemma 3.3 in [3] applies without change. □

Let

$$\widetilde{\mathcal{RP}}(A) := \mathcal{RP}(A)/\mathcal{K}_A^{(1)}$$

By Lemma 4.5,

$$\Lambda(\mathcal{K}_A^{(1)}) = p_{-1}^+\mathcal{I}_A \oplus T(A) \subset \mathcal{I}_A^2 \oplus S_{\mathbb{Z}}^2(A^\times)$$

where $T(A)$ is the subgroup of $S_{\mathbb{Z}}^2(A^\times)$ generated by the elements $(-x) \circ x$. Note that $T(A)$ is annihilated by 2.

We denote $S_{\mathbb{Z}}^2(A^\times)/T(A)$ by $\tilde{S}_{\mathbb{Z}}^2(A^\times)$. Then the R_A -homomorphism

$$\tilde{\Lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2) : \widetilde{\mathcal{RP}}(A) \rightarrow \frac{\mathcal{I}_A^2}{p_{-1}^+\mathcal{I}_A} \oplus \tilde{S}_{\mathbb{Z}}^2(A^\times)$$

is well-defined and we set $\widetilde{\mathcal{RP}}_1(A) := \text{Ker}(\tilde{\lambda}_1)$ and $\widetilde{\mathcal{RB}}(A) := \text{Ker}(\tilde{\Lambda})$.

Corollary 4.7 *The natural maps $\mathcal{RP}_1(A) \rightarrow \widetilde{\mathcal{RP}}_1(A)$ and $\mathcal{RB}(A) \rightarrow \widetilde{\mathcal{RB}}(A)$ are surjective with kernel annihilated by 4. In particular,*

$$\mathcal{RP}_1(A) \left[\frac{1}{2} \right] = \widetilde{\mathcal{RP}}_1(A) \left[\frac{1}{2} \right] \quad \text{and} \quad \mathcal{RB}(A) \left[\frac{1}{2} \right] = \widetilde{\mathcal{RB}}(A) \left[\frac{1}{2} \right].$$

Proof From the definitions,

$$\widetilde{\mathcal{RP}}_1(A) \cong \frac{\mathcal{RP}_1(A)}{\mathcal{RP}_1(A) \cap \mathcal{K}_A^{(1)}} \quad \text{and} \quad \widetilde{\mathcal{RB}}(A) \cong \frac{\mathcal{RB}(A)}{\mathcal{RB}(A) \cap \mathcal{K}_A^{(1)}}$$

and the modules $\mathcal{RP}_1(A) \cap \mathcal{K}_A^{(1)}$ and $\mathcal{RB}(A) \cap \mathcal{K}_A^{(1)}$ are both contained in $\text{Ker}(\lambda_1|_{\mathcal{K}_A^{(1)}})$ which is annihilated by 4. $\square$

4.2 The constant D_A

In [12], Suslin shows that, for an infinite field F , the elements $[x] + [1 - x] \in \mathcal{B}(F) \subset \mathcal{P}(F)$ are independent of x and that the resulting constant, $\tilde{C}_F$, has order dividing 6. Furthermore Suslin shows that $\tilde{C}_{\mathbb{R}}$ has exact order 6.

In [3, Lemma 3.5] it is shown that the elements

$$C(x) = [x] + \langle -1 \rangle [1 - x] + \langle 1 - x \rangle \psi_1(x) \in \mathcal{RB}(F)$$

are constant (for a field with at least 4 elements) and have order dividing 6. In fact the proof given there extends without alteration to the case of local rings:

Lemma 4.8 *Let A be a local ring whose residue field has at least 4 elements. For all $a, b \in \mathcal{W}_A$*

$$C(a) := [a] + \langle -1 \rangle [1 - a] + \langle 1 - a \rangle \psi_1(a) \in \mathcal{RB}(A) \quad \text{and} \quad C(a) = C(b).$$

We denote this constant by C_A and we set $D_A := 2C_A$. Of course, C_A maps to $\tilde{C}_A$ under the natural map $\mathcal{RP}(A) \rightarrow \mathcal{P}(A)$. Similarly, we denote the image of D_A in $\mathcal{P}(A)$ by $\tilde{D}_A$. Thus, of course, $\tilde{D}_A = 2\tilde{C}_A$ in $\mathcal{P}(A)$. In fact, these elements lie in $\mathcal{B}(A)$ by Lemma 4.8.

Let $\Phi(X)$ denote the polynomial $X^2 - X + 1 \in A[X]$.

Lemma 4.9 *Let A be a local ring whose residue field has at least 4 elements.*

- (1) $3C_A = \psi_1(-1)$ and $6C_A = 0$.
- (2) If $\Phi(X)$ has a root in A then $D_A = 0$ and $C_A = \psi_1(-1)$.

Proof The proof for the case when A is a field is given in [3, Corollary 3.7, Lemma 3.10]. It adapts without significant change to the current situation (see [2] for details). $\square$

Proposition 4.10 ([6]) *For a field F we have $D_F = 0$ if and only if $\Phi(X)$ has a root in F .*

Corollary 4.11 *For any field F , $D_F = 0$ if and only if $\tilde{D}_F = 0$, and the map $\mathcal{RB}(F) \rightarrow \mathcal{B}(F)$ induces an isomorphism $\mathbb{Z} \cdot D_F \cong \mathbb{Z} \cdot \tilde{D}_F$.*

Corollary 4.12 *Let A be a local ring with residue field k . Suppose either that $U_{1,A} = U_{1,A}^2$ and that $\text{char}(k) \neq 2, 3$ or that $\text{char}(k) = 2$ and A is henselian. Then the functorial map $\mathcal{RB}(A) \rightarrow \mathcal{RB}(k)$ induces an isomorphism $\mathbb{Z} \cdot D_A \cong \mathbb{Z} \cdot D_k$.*

Proof Clearly $D_A = 0 \implies D_k = 0$.

Conversely, if $D_k = 0$ then $\Phi(X)$ has a root in k . The hypotheses then imply that $\Phi(X)$ has a root in A and thus $D_A = 0$ also. $\square$

Remark 4.13 When $\text{char}(k) = 3$, the result may fail. For example, $\Phi(X)$ has no root in $\mathbb{Q}_3$, so that $D_{\mathbb{Q}_3} \neq 0$, and hence $D_{\mathbb{Z}_3} \neq 0$ also. But $D_{\mathbb{F}_3} = 0$.

Theorem 4.14 *Let A be a local ring whose residue field has at least 11 elements. Then*

- (1) *For all $x \in A^\times$, $\langle\langle x \rangle\rangle D_A = \psi_1(x) - \psi_2(x)$.*
- (2) *$\langle\langle x \rangle\rangle D_A = 0$ if $x \in A^\times$ is of the form $\pm\Phi(a)u^2$ for some $a, u \in A^\times$.*

Proof The proof when A is a field is [3, Theorem 3.12]. The details of the extension of this proof to local rings is given in the appendix to the Arxiv version of this article [2]. $\square$

4.3 The module $\mathcal{D}_A$

We let $\mathcal{D}_A$ denote the cyclic R_A -submodule of $\mathcal{RB}(A)$ generated by D_A . Of course, since $3D_A = 0$ always, in fact $\mathcal{D}_F$ is a module over the group ring $\mathbb{F}_3[A^\times/(A^\times)^2]$.

For a local ring A with residue field k , let $\tilde{\mathcal{W}}_A := \mathcal{W}_A \setminus \{a \in A \mid \Phi(\bar{a}) = 0 \text{ in } k\}$. Let $\mathcal{N}_A$ be the subgroup of $A^\times$ generated by elements of the form $\pm\Phi(x)u^2$, $x \in \tilde{\mathcal{W}}_A$, $u \in A^\times$. By Theorem 4.14 (2), we have $\langle x \rangle D_A = D_A$ if $x \in \mathcal{N}_A$. Let $\mathcal{G}_A = A^\times/\mathcal{N}_A$. Thus the action of $A^\times/(A^\times)^2$ on $\mathcal{D}_A$ factors through the quotient $\mathcal{G}_A$, and hence $\mathcal{D}_A$ is a cyclic module over the ring $\widehat{R}_A := \mathbb{F}_3[\mathcal{G}_A]$.

Lemma 4.15 *Let A be a local ring with residue field k . Then the natural map $\mathcal{D}_A \rightarrow \mathcal{D}_k$ is surjective.*

Furthermore, if $U_{1,A} = U_{1,A}^2$ and if $\mathcal{D}_k$ is free (of rank 1) as a $\widehat{R}_k$ -module, then $\mathcal{D}_A \cong \mathcal{D}_k$.

Proof Since $A^\times/(A^\times)^2$ maps onto $k^\times/(k^\times)^2$ and hence R_A maps onto R_k the first statement is clear.

For the second statement, note that the conditions ensure that $A^\times/(A^\times)^2 \cong k^\times/(k^\times)^2$ and under this isomorphism, $\mathcal{N}_A$ corresponds to $\mathcal{N}_k$. Thus $\widehat{R}_A \cong \widehat{R}_k$ and the composite map

$$\widehat{R}_A \twoheadrightarrow \mathcal{D}_A \twoheadrightarrow \mathcal{D}_k$$

is an isomorphism. $\square$

From [6, Corollary 7.11, Proposition 7.12] we itemize some conditions under which $\mathcal{D}_k$ is a free $\widehat{R}_k$ -module.

Proposition 4.16 *Let k be a field in which $X^2 - X + 1$ has no roots. Then $\mathcal{D}_k$ is a free $\widehat{R}_k$ -module of rank 1 in any of the following cases*

- (1) *k is finite.*
- (2) *k is real closed.*
- (3) *There is a discrete valuation on k with residue field $\bar{k}$ satisfying one of the following conditions:*

- (a) $U_1 = U_1^2$ and $X^2 - X + 1$ has no root in $\bar{k}$ and $\mathcal{D}_{\bar{k}}$ is a free $\widehat{\mathcal{R}_{\bar{k}}}$ -module
- (b) k is complete with respect to v and $\bar{k}$ is finite of characteristic not equal to 3
- (c) k is complete with respect to v and $\bar{k}$ is finite of characteristic 3 and $[k : \mathbb{Q}_3]$ is odd.

Remark 4.17 (1) If k is complete with respect to the valuation v and if $\text{char}(\bar{k}) \neq 2$ then $U_1 = U_1^2$. However, $\mathbb{Q}_2$ is complete with respect to the associated discrete valuation but $U_1 \neq U_1^2$ in this case.

(2) $X^2 - X + 1$ has a (repeated) root in $\mathbb{F}_3$, but does not have a root in $\mathbb{Q}_3$. Thus $\mathcal{D}_{\mathbb{F}_3} = 0$ but $\mathcal{D}_{\mathbb{Q}_3} \neq 0$. However, $\mathcal{I}_{\mathbb{Q}_3} \mathcal{D}_{\mathbb{Q}_3} = 0$ by Proposition 4.16(3)(c), since $\widehat{\mathcal{R}_{\mathbb{Q}_3}} = \mathbb{F}_3$ by [3, Lemma 6.2].

4.4 Reduced scissors congruence groups

We introduce some quotient groups of scissors congruence groups which will be required in our computations below.

First recall that

$$\widetilde{\mathcal{RP}}(A) = \frac{\mathcal{RP}(A)}{\mathcal{K}_A^{(1)}}, \quad \widetilde{\mathcal{RP}}_1(A) = \text{Ker}(\tilde{\lambda}_1) \quad \text{and} \quad \widetilde{\mathcal{RB}}(A) = \text{Ker}(\tilde{\Lambda}) \subset \widetilde{\mathcal{RP}}_1(A).$$

Furthermore, by Corollary 4.7, the maps $\mathcal{RP}_1(A) \rightarrow \widetilde{\mathcal{RP}}_1(A)$ and $\mathcal{RB}(A) \rightarrow \widetilde{\mathcal{RB}}(A)$ are surjective with kernel annihilated by 4.

Since $\mathcal{D}_A$ is annihilated by 3, it follows that the composite

$$\mathcal{D}_A \rightarrow \mathcal{RB}(A) \rightarrow \widetilde{\mathcal{RB}}(A)$$

is injective and we will identify $\mathcal{D}_A$ with its image in $\widetilde{\mathcal{RB}}(A)$.

We further define

$$\overline{\mathcal{RP}}(A) := \frac{\widetilde{\mathcal{RP}}(A)}{\mathcal{D}_A} = \frac{\mathcal{RP}(A)}{\mathcal{K}_A^{(1)} + \mathcal{D}_A}, \quad \overline{\mathcal{RP}}_1(A) := \frac{\widetilde{\mathcal{RP}}_1(A)}{\mathcal{D}_A} \quad \text{and} \quad \overline{\mathcal{RB}}(A) := \frac{\widetilde{\mathcal{RB}}(A)}{\mathcal{D}_A}.$$

Corollary 4.7 implies:

Lemma 4.18 *For any field or local ring A there are short exact sequences*

$$0 \rightarrow \mathcal{D}_A \rightarrow \mathcal{RP}_1(A) \left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \right] \rightarrow \overline{\mathcal{RP}}_1(A) \left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \right] \rightarrow 0.$$

and

$$0 \rightarrow \mathcal{D}_A \rightarrow \mathcal{RB}(A) \left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \right] \rightarrow \overline{\mathcal{RB}}(A) \left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \right] \rightarrow 0.$$

Observe also that $\mathcal{K}_A^{(2)} \subset \mathcal{K}_A^{(1)} + \mathcal{D}_A$ by Theorem 4.14(2). It follows that for $i = 1, 2$ and all $x \in A^\times$ we have $\psi_i(x) = 0$ in $\overline{\mathcal{RP}}(A)$.

Lemma 4.19 *Let A be a local ring whose residue field has at least 10 elements.*

- (1) *For all $x \in \mathcal{W}_A$, $[x^{-1}] = -\langle -1 \rangle [x] = -\langle x \rangle [x]$ in $\overline{\mathcal{RP}}(A)$.*
- (2) *For all $x \in \mathcal{W}_A$, $\langle\langle y \rangle\rangle [x] = 0$ in $\overline{\mathcal{RP}}(A)$ whenever $y \equiv -x \pmod{(A^\times)^2}$.*
- (3) *For all $x \in \mathcal{W}_A$, $[1 - x] = \langle -1 \rangle [x] = [x^{-1}]$ in $\overline{\mathcal{RP}}(A)$.*

Proof (1) The first equality follows from $\psi_1(x) = 0$, the second from $\psi_2(x) = 0$.
 (2) From $\langle x \rangle [x] = \langle -1 \rangle [x]$ it follows that $\langle y \rangle [x] = \langle -x \rangle [x] = [x]$.
 (3) Since $2C_A = D_A$ and $3C_A = \psi_1(-1) \in \mathcal{K}_A^{(1)}$, it follows that $C_A = 0$ in $\overline{\mathcal{RP}}(A)$, and thus we have (from the definition of C_A) that $0 = [x] + \langle -1 \rangle [1 - x]$ in $\overline{\mathcal{RP}}(A)$. □

5 Characters and $\mathcal{R}_A$ -modules

In this section we review some results from [6].

5.1 The character-theoretic local-global principle

Let G be a multiplicative abelian group satisfying $g^2 = 1$ for all $g \in G$. Let $\mathcal{R}$ denote the group ring $\mathbb{Z}[G]$. Let $\widehat{G}$ be the group of characters of G , i.e. the group of homomorphisms $\chi : G \rightarrow \mu_2$.

For any $\mathcal{R}$ -module M and any $\chi \in \widehat{G}$ we let $M_{(\chi)}$ denote the quotient of M by the submodule generated by the elements $(\chi(g) - g)m$, $g \in G$, $m \in M$. Thus $g \cdot x = \chi(g)x$ for all $g \in G$ and $x \in M_{(\chi)}$. For any $m \in M$, we will let $m_{(\chi)}$ denote the image of m in $M_{(\chi)}$.

We let χ_0 denote the trivial character on G . Thus, for any $\mathcal{R}$ -module M , $M_{(\chi_0)}$ is equal to the module M_G of coinvariants.

If $f : M \rightarrow N$ is a homomorphism of $\mathcal{R}$ -modules we let $f_{(\chi)}$ denote the induced homomorphism $M_{(\chi)} \rightarrow N_{(\chi)}$.

The following observation is proved in [6, Corollary 3.7]:

Lemma 5.1 *For any $\chi \in \widehat{G}$, the functor $M \rightarrow M_{(\chi)}$ is an exact functor on the category of $\mathcal{R}[\frac{1}{2}]$ -modules.*

We recall the following character-theoretic local global principle (see [6, Proposition 3.10]):

Proposition 5.2 *Let $f : M \rightarrow N$ be a homomorphism of $\mathcal{R}[\frac{1}{2}]$ -modules. Then f is injective (resp. surjective) if and only if $f_{(\chi)}$ is injective (resp. surjective) for all $\chi \in \widehat{G}$.*

Of course we will apply this in the case $G = A^\times / (A^\times)^2$ and hence $\mathcal{R} = \mathcal{R}_A$ for a local ring A .

Lemma 5.3 *Let A be a local ring whose residue field has at least 10 elements. Let $\chi \in A^\times / (A^\times)^2$.*

If $a \in \mathcal{W}_A$ satisfies $\chi(-a) = -1$ then $2 \cdot [a]_{(\chi)} = 0$ in $\overline{\mathcal{RP}}(A)_{(\chi)}$.

Proof By definition, $(\langle -a \rangle + 1)x = 0$ for all $x \in \overline{\mathcal{RP}}(A)_{(\chi)}$. In particular, $(\langle -a \rangle + 1)[a]_{(\chi)} = 0$. However, $(\langle -a \rangle - 1)[a] = 0$ in $\overline{\mathcal{RP}}(A)$ by Lemma 4.19(2). $\square$

5.2 The action of $\langle -1 \rangle$ on $\mathcal{RP}_1(A)$

Proposition 5.4 *Let A be a local ring whose residue field has at least 10 elements. Then $\langle -1 \rangle \in R_A$ acts trivially on $\overline{\mathcal{RP}}_1(A) \left[\frac{1}{2} \right]$ and in fact $\mathcal{RP}_1(A) \left[\frac{1}{2} \right] = e_{-1}^+ \mathcal{RP}(A) \left[\frac{1}{2} \right]$.*

Proof The proof given in [6, Theorem 4.3, Corollary 4.4] in the case of a field F applies here, simply by replacing $F^\times \setminus \{1\}$ with $\mathcal{W}_A$. $\square$

Since $\langle -1 \rangle$ acts trivially on D_A and on $\text{tor}(\mu_A, \mu_A)$, we deduce as in [6]:

Corollary 5.5 *Let A be a local ring whose residue field has at least 10 elements. Then $\langle -1 \rangle \in R_A$ acts trivially on $\mathcal{RP}_1(A) \left[\frac{1}{2} \right]$ and on $H_3(SL_2(A), \mathbb{Z} \left[\frac{1}{2} \right])$.*

Remark 5.6 As in [6], it follows that $\mathcal{RP}_1(A) \left[\frac{1}{2} \right] = e_{-1}^+ \mathcal{RP}(A) \left[\frac{1}{2} \right]$ is a quotient of $\mathcal{RP}(A) \left[\frac{1}{2} \right]$ and hence admits a simple explicit presentation as a R_A -module.

6 The map from $\mathcal{RP}_1(A)$ to $\mathcal{RP}_1(k)$

In this section A will be a commutative local ring with residue field k .

6.1 The module $\mathcal{L}_A$

We let

$$\mathcal{L}_A := \langle [au] - [a] \in \overline{\mathcal{RP}}(A) \mid a \in \mathcal{W}_A, u \in U_1 \rangle_{R_A}.$$

Lemma 6.1 *There is a short exact sequence of R_A -modules*

$$0 \rightarrow \mathcal{L}_A \rightarrow \overline{\mathcal{RP}}(A) \rightarrow \overline{\mathcal{RP}}(k) \rightarrow 0.$$

Proof The functorial map $\overline{\mathcal{RP}}(A) \rightarrow \overline{\mathcal{RP}}(k)$ is clearly surjective and $\mathcal{L}_A$ is contained in its kernel.

Thus if we let $Q(A) := \overline{\mathcal{RP}}(A)/\mathcal{L}_A$, there is an induced surjective homomorphism of R_A -modules $Q(A) \rightarrow \overline{\mathcal{RP}}(k)$. We must show that this is an isomorphism.

I claim that $\langle u \rangle$ acts trivially on $Q(A)$ for any $u \in U_1$: Let $a \in \mathcal{W}_A$. Then $[a] = [au]$ in $Q(A)$ for any $u \in U_1$. For all $b \in \mathcal{W}_A$, $\langle -b \rangle [b] = [b]$ in $Q(A)$ by Lemma 4.19(2). Thus, for any $u \in U_1$ we have

$$\langle u \rangle [a] = \langle -au \rangle [a] = \langle -au \rangle [au] = [au] = [a]$$

in $Q(A)$, proving the claim.

It follows that the action of R_A on $Q(A)$ descends to an action of $\mathbb{Z}[A^\times/U_1 \cdot (A^\times)^2] = R_k$. This allows us to construct a well-defined inverse map

$$\overline{\mathcal{RP}}(k) \rightarrow Q(A) = \overline{\mathcal{RP}}(A)/\mathcal{L}_A, \quad [\bar{a}] \mapsto [a] + \mathcal{L}_A.$$

□

6.2 The map $\mathcal{RP}_1(A) \rightarrow \mathcal{RP}_1(k)$

Theorem 6.2 *Let A be a local ring with sufficiently large residue field k . Suppose that A satisfies the condition $U_1 = U_1^2$.*

Then the functorial map $\mathcal{RP}(A) \rightarrow \mathcal{RP}(k)$ induces an isomorphism

$$\mathcal{I}_A \overline{\mathcal{RP}}_1(A) \left[\frac{1}{2} \right] \cong \mathcal{I}_k \overline{\mathcal{RP}}_1(k) \left[\frac{1}{2} \right].$$

Proof By Proposition 5.4 and the character-theoretic local-global principle, it is enough to prove that $\overline{\mathcal{RP}}(A) \left[\frac{1}{2} \right]_{(\chi)} \cong \overline{\mathcal{RP}}(k) \left[\frac{1}{2} \right]_{(\chi)}$ for all characters χ satisfying $\chi(-1) = 1$ and $\chi \neq \chi_0$.

By Lemmas 5.1 and 6.1 we must prove that $\mathcal{L}_A \left[\frac{1}{2} \right]_{(\chi)} = 0$ for all such characters χ ; i.e. we must prove that $[a]_{(\chi)} = [au]_{(\chi)}$ for all $a \in \mathcal{W}_A$, $u \in U_1$ and χ satisfying $\chi(-1) = 1$, $\chi \neq \chi_0$.

Let $\chi \neq \chi_0$ be a character satisfying $\chi(-1) = 1$. Let $K = K(\chi) := \{a \in A^\times \mid \chi(a) = 1\}$.

By Lemma 5.3, if $a \in \mathcal{W}_A \setminus K$ then $[a]_{(\chi)} = 0$ in $\overline{\mathcal{RP}}(A) \left[\frac{1}{2} \right]_{(\chi)}$. Since $[1 - a] = \langle -1 \rangle [a]$ in $\overline{\mathcal{RP}}(A)$, it also follows that if $1 - a \notin K$ then $[a]_{(\chi)} = 0$.

Let $S_1 := \{a \in \mathcal{W}_A \mid a, 1 - a \in K\}$ and let $S_2 := \{a \in \mathcal{W}_A \mid a \in K, 1 - a \notin K\}$. Thus $[a]_{(\chi)} = 0$ if $a \notin S_1$.

Observe that if $a \in \mathcal{W}_A \cap K$ then

$$\chi(1 - a^{-1}) = \chi(-a^{-1}(1 - a)) = \chi(-1)\chi(a)\chi(1 - a) = \chi(1 - a).$$

Thus $a \in S_i$ if and only if $a^{-1} \in S_i$, for $i = 1, 2$.

Furthermore, $U_1 \subset K$ since $U_1 = U_1^2$ and for any $a \in \mathcal{W}_A$ we have $1 - a \equiv 1 - au \pmod{U_1} \implies 1 - a \equiv 1 - au \pmod{(A^\times)^2}$ and thus $\chi(1 - a) = \chi(1 - au)$.

It follows that for all $a \in \mathcal{W}_A$ and $u \in U_1$ we have $a \in S_i$ if and only if $au \in S_i$ for $i = 1, 2$.

Note also that $S_2 \neq \emptyset$: Let $a \notin K$. If $1 - a \in K$ then $1 - a \in S_2$. Otherwise $1 - a \notin K$ and hence $\chi(1 - a^{-1}) = \chi(-1)\chi(a)\chi(1 - a) = 1$ so that $1 - a^{-1} \in S_2$.

Now let $a \in S_1$, $b \in S_2$. Note that it follows that $ab \in \mathcal{W}_A$ since otherwise $u := ab \in U_1 \implies b = a^{-1}u \in S_1$.

We Claim that $[a]_{(\chi)} = [ab]_{(\chi)}$: We have

$$0 = (S_{a,ab})_{(\chi)} = [a]_{(\chi)} - [ab]_{(\chi)} + [b]_{(\chi)} - \left[b \cdot \frac{1-a}{1-ab} \right]_{(\chi)} + \left[\frac{1-a}{1-ab} \right]_{(\chi)}.$$

Note that $[b]_{(\chi)} = 0$ since $b \notin S_1$.

We will show that the last two terms also vanish. We consider two cases:

Case (i): $ab \in S_2$

In this case, $1 - ab \notin K \implies (1 - a)/(1 - ab) \notin K$ and $b \cdot (1 - a)/(1 - ab) \notin K$. Hence the last two terms are 0.

Case (ii): $ab \notin S_2$

Then $1 - ab \in K$. Thus $s := (1 - a)/(1 - ab) \in K$. However

$$1 - s = 1 - \frac{1 - a}{1 - ab} = a \cdot \frac{1 - b}{1 - ab} \notin K$$

since $a \in K$, $1 - b \notin K$, it follows that $[s]_{(\chi)} = 0$.

Similarly, replacing a, b by a^{-1}, b^{-1} it follows that $[t]_{(\chi)} = 0$ where

$$t := \frac{b(1 - a)}{1 - ab} = \frac{1 - a^{-1}}{1 - (ab)^{-1}}.$$

Thus the *Claim* is proven.

Now let $a \in \mathcal{W}_A, u \in U_1$. Fix $b \in S_2$.

If $a \notin S_1$ then $au \notin S_1$ and hence

$$[a]_{(\chi)} = 0 = [au]_{(\chi)}.$$

Otherwise, $a \in S_1$. Hence $au \in S_1$ and $u^{-1}b \in S_2$. By the *Claim* we have

$$[a]_{(\chi)} = [ab]_{(\chi)}$$

and also

$$[au]_{(\chi)} = \left[(au)(u^{-1}b) \right]_{(\chi)} = [ab]_{(\chi)}.$$

The theorem is thus proven. $\square$

Remark 6.3 The theorem does not hold for general local rings, however.

For example, if the residue field k is finite, then $\mathcal{I}_k \mathcal{R}\mathcal{P}_1(k) = 0$. Thus if the theorem holds for a local ring A with finite residue field then it implies that $\mathcal{I}_A \overline{\mathcal{R}\mathcal{P}_1}(A) \left[\frac{1}{2} \right] = 0$.

Let p be a prime number and consider $A = \mathbb{Z}_{(p)} = \{a/b \in \mathbb{Q} \mid p \nmid b\}$ with residue field $\mathbb{F}_p$. In [3] it is shown that there is a natural surjective homomorphism

$$\mathcal{I}_{\mathbb{Q}} \mathcal{R}\mathcal{P}_1(\mathbb{Q}) \left[\frac{1}{2} \right] \rightarrow \bigoplus_{q \text{ prime}} \mathcal{P}(\mathbb{F}_q) \left[\frac{1}{2} \right].$$

It is straightforward to verify that the resulting composite map

$$\mathcal{I}_A \mathcal{R}\mathcal{P}_1(A) \left[\frac{1}{2} \right] \rightarrow \mathcal{I}_{\mathbb{Q}} \mathcal{R}\mathcal{P}_1(\mathbb{Q}) \left[\frac{1}{2} \right] \rightarrow \bigoplus_{q \neq p} \mathcal{P}(\mathbb{F}_q) \left[\frac{1}{2} \right]$$

is surjective and thus $\mathcal{I}_A \mathcal{R}\mathcal{P}_1(A) \left[\frac{1}{2} \right] \neq 0$ and also $\mathcal{I}_A \overline{\mathcal{R}\mathcal{P}}_1(A) \left[\frac{1}{2} \right] \neq 0$.

This example suggests that some completeness or Henselian condition, such as the condition $U_1 = U_1^2$, is necessary for the theorem to hold.

6.3 Application to the third homology of SL_2 of local domains

For a local ring A with residue field k we let

$$\mathcal{D}(A/k) := \mathrm{Ker}(\mathcal{I}_A \mathcal{D}_A \rightarrow \mathcal{I}_k \mathcal{D}_k).$$

Thus $\mathcal{D}(A/k)$ is an R_A -submodule of $\mathcal{D}_A$. In particular, it is annihilated by 3.

Remark 6.4 Note that, by Lemma 4.15 and Proposition 4.16(1), if A is a local integral domain satisfying $U_1 = U_1^2$ and with finite residue field k then $\mathcal{D}(A/k) = 0$.

In fact, when k is a finite field then R_k acts trivially on $\mathcal{R}\mathcal{B}(k) = \mathcal{B}(k)$ and hence $\mathcal{I}_k \mathcal{D}_k = 0$. Thus, under the stated conditions, $\mathcal{I}_A \mathcal{D}_A = 0$ also.

Proposition 6.5 *Let A be a local integral domain with sufficiently large finite residue field k . Suppose that A satisfies the condition $U_1 = U_1^2$.*

Then $A^\times$ acts trivially on $H_3(\mathrm{SL}_2(A), \mathbb{Z} \left[\frac{1}{2} \right])$.

Equivalently, the natural map $H_3(\mathrm{SL}_2(A), \mathbb{Z} \left[\frac{1}{2} \right]) \rightarrow K_3^{\mathrm{ind}}(A) \left[\frac{1}{2} \right]$ is an isomorphism.

Proof By Corollaries 3.23 and 3.24 it is enough to prove that $\mathcal{I}_A \mathcal{R}\mathcal{P}_1(A) \left[\frac{1}{2} \right] = 0$.

There is a commutative diagram of R_A -modules with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{I}_A \mathcal{D}_A & \longrightarrow & \mathcal{I}_A \mathcal{R}\mathcal{P}_1(A) \left[\frac{1}{2} \right] & \longrightarrow & \mathcal{I}_A \overline{\mathcal{R}\mathcal{P}}_1(A) \left[\frac{1}{2} \right] \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \cong \\ 0 & \longrightarrow & \mathcal{I}_k \mathcal{D}_k & \longrightarrow & \mathcal{I}_k \mathcal{R}\mathcal{P}_1(k) \left[\frac{1}{2} \right] & \longrightarrow & \mathcal{I}_k \overline{\mathcal{R}\mathcal{P}}_1(k) \left[\frac{1}{2} \right] \longrightarrow 0. \end{array}$$

The right-hand vertical arrow is an isomorphism by Theorem 6.2. Thus we can extract the natural short exact sequence

$$0 \rightarrow \mathcal{D}(A/k) \rightarrow \mathcal{I}_A \mathcal{R}\mathcal{P}_1(A) \left[\frac{1}{2} \right] \rightarrow \mathcal{I}_k \mathcal{R}\mathcal{P}_1(k) \left[\frac{1}{2} \right] \rightarrow 0.$$

By remark 6.4, $\mathcal{D}(A/k) = 0$ and hence the middle vertical arrow is an isomorphism. Furthermore, since k is finite, $\mathcal{R}\mathcal{B}(k) = \mathcal{B}(k)$ and hence $\mathcal{I}_k \mathcal{R}\mathcal{B}(k) = 0$ [4, Lemma 7.1]. It follows that

$$\mathcal{I}_A \mathcal{R}\mathcal{B}(A) \left[\frac{1}{2} \right] = \mathcal{I}_A \mathcal{R}\mathcal{P}_1(A) \left[\frac{1}{2} \right] \cong \mathcal{I}_k \mathcal{R}\mathcal{P}_1(k) \left[\frac{1}{2} \right] = \mathcal{I}_k \mathcal{R}\mathcal{B}(k) \left[\frac{1}{2} \right] = 0.$$

□

Corollary 6.6 *Let A be a local integral domain with sufficiently large residue field k . Suppose that $U_1 = U_1^2$. There is a short exact sequence of R_A -modules*

$$0 \rightarrow \mathcal{D}(A/k) \rightarrow H_3(SL_2(A), \mathbb{Z}[\frac{1}{2}])_0 \rightarrow H_3(SL_2(k), \mathbb{Z}[\frac{1}{2}])_0 \rightarrow 0.$$

Proof As in the proof of Proposition 6.5, there is a short exact sequence

$$0 \rightarrow \mathcal{D}(A/k) \rightarrow \mathcal{I}_A \mathcal{R} \mathcal{P}_1(A) [\frac{1}{2}] \rightarrow \mathcal{I}_k \mathcal{R} \mathcal{P}_1(k) [\frac{1}{2}] \rightarrow 0.$$

We have

$$\mathcal{I}_A \mathcal{R} \mathcal{P}_1(A) [\frac{1}{2}] \cong H_3(SL_2(A), \mathbb{Z}[\frac{1}{2}])_0 \text{ and } \mathcal{I}_k \mathcal{R} \mathcal{P}_1(k) [\frac{1}{2}] \cong H_3(SL_2(k), \mathbb{Z}[\frac{1}{2}])_0$$

by Corollary 3.24. $\square$

7 The localization sequence for discrete valuation rings

7.1 Fields with discrete valuation

We recall some of the main results of [6]:

Let F be a field with discrete valuation v and residue field $k = k_v$. Let $\mathcal{O}_v$ denote the corresponding valuation ring. Let $U = U_v = \mathcal{O}_v^\times$.

Fix a uniformizer $\pi \in \mathcal{O}_v$. Then exists a surjective homomorphism of $\mathbb{Z}[U_v/U_v^2]$ -modules $\delta_\pi : H_3(SL_2(F), \mathbb{Z}[\frac{1}{2}]) \rightarrow \mathcal{R} \mathcal{P}_1(k) [\frac{1}{2}]$, which we proceed to describe. (We also denote by δ_π the restriction of this homomorphism to $H_3(SL_2(F), \mathbb{Z}[\frac{1}{2}])_0$.)

Given any R_k -module M , we define the induced R_F -module

$$\text{Ind}_k^F M := R_F \otimes_{\mathbb{Z}[U/U^2]} M.$$

Note that $F^\times/(F^\times)^2 = U/U^2 \times C_\pi$ where C_π denotes the multiplicative group of order 2 generated by the square class $\langle \pi \rangle$. Thus $R_F = \mathbb{Z}[U/U^2][C_\pi]$ and there is a decomposition of $\mathbb{Z}[U/U^2]$ -modules

$$\text{Ind}_k^F M \cong M \oplus \langle \pi \rangle \cdot M \cong M \oplus M.$$

We let $\rho_\pi : \text{Ind}_k^F M \rightarrow M$ denote the projection on the second factor.

There is a natural surjective specialization or residue homomorphism of R_F -modules

$$S := S_v : \mathcal{R} \mathcal{P}(F) \rightarrow \widetilde{\text{Ind}_k^F \mathcal{R} \mathcal{P}(k)}$$

$$[a] \mapsto \begin{cases} 1 \otimes [\bar{a}], & v(a) = 0 \\ 1 \otimes C_k, & v(a) > 0 \\ -(1 \otimes C_k), & v(a) < 0 \end{cases}$$

where $\widetilde{\mathcal{R} \mathcal{P}(k)}$ is a certain quotient of the R_k -module $\mathcal{R} \mathcal{P}(k)$.

S_v induces a well-defined homomorphism

$$S_v : \mathcal{RP}_1(F) \rightarrow \text{Ind}_k^F \widetilde{\mathcal{RP}}_1(k)$$

where $\widetilde{\mathcal{RP}}_1(k)$ is a quotient of $\mathcal{RP}_1(k)$ satisfying $\mathcal{RP}_1(k) \left[\frac{1}{2} \right] \cong \widetilde{\mathcal{RP}}_1(k) \left[\frac{1}{2} \right]$. (However, S_v does not restrict to a homomorphism $\mathcal{RB}(F) \rightarrow \text{Ind}_k^F \widetilde{\mathcal{RB}}(k)$.)

Then we define δ_π to be the composite

$$H_3(SL_2(F), \mathbb{Z} \left[\frac{1}{2} \right]) \longrightarrow \mathcal{RB}(F) \left[\frac{1}{2} \right] \longrightarrow \mathcal{RP}_1(F) \left[\frac{1}{2} \right] \xrightarrow{S_v} \text{Ind}_k^F \mathcal{RP}_1(k) \left[\frac{1}{2} \right] \xrightarrow{\rho_\pi} \mathcal{RP}_1(k) \left[\frac{1}{2} \right].$$

Lemma 7.1 *The composite*

$$H_3(SL_2(\mathcal{O}_v), \mathbb{Z} \left[\frac{1}{2} \right]) \longrightarrow H_3(SL_2(F), \mathbb{Z} \left[\frac{1}{2} \right]) \xrightarrow{\delta_\pi} \mathcal{RP}_1(k) \left[\frac{1}{2} \right]$$

is the zero map.

Proof By functoriality, the square of $\mathbb{Z}[U/U^2]$ -module homomorphisms

$$\begin{array}{ccc} H_3(SL_2(\mathcal{O}_v), \mathbb{Z}) & \longrightarrow & H_3(SL_2(F), \mathbb{Z}) \\ \downarrow & & \downarrow \\ \mathcal{RB}(\mathcal{O}_v) & \longrightarrow & \mathcal{RB}(F) \end{array}$$

commutes. Since δ_π factors through the right vertical map, it is sufficient to prove that the composite

$$\mathcal{RB}(\mathcal{O}_v) \left[\frac{1}{2} \right] \longrightarrow \mathcal{RB}(F) \left[\frac{1}{2} \right] \xrightarrow{S_v} \text{Ind}_k^F \widetilde{\mathcal{RP}}(k) \left[\frac{1}{2} \right] \xrightarrow{\rho_\pi} \widetilde{\mathcal{RP}}(k) \left[\frac{1}{2} \right]$$

is the zero map.

Now, $\mathcal{RP}(\mathcal{O}_v)$ is generated as a $R_{\mathcal{O}_v} = \mathbb{Z}[U/U^2]$ -module by elements $[a]$, $a \in \mathcal{W}_{\mathcal{O}_v} = U \setminus U_1$. By definition,

$$\rho_\pi(S_v([a])) = \rho_\pi(1 \otimes [\bar{a}]) = 0$$

for any such a . □

Let F be a field with discrete valuation $v : F^\times \rightarrow \mathbb{Z}$. We consider the module

$$\mathcal{D}_F(v) := \text{Ker}(S_v : \mathcal{I}_F \mathcal{D}_F \rightarrow \mathcal{I}_F(\text{Ind}_k^F \mathcal{D}_k)).$$

Observe that this module is annihilated by 3.

Proposition 7.2 *Let F be a field with discrete valuation v . Let π be a uniformizer. For $n \geq 1$, let $U_n = \{u \in U \mid v(1 - u) \geq n\}$. Suppose that there exists $n \geq 1$ such that $U_n \subset U_1^2$. Then there is a natural short exact sequence of $\mathbb{Z}[U_v/U_v^2]$ -modules*

$$0 \rightarrow \mathcal{D}_F(v) \rightarrow H_3(SL_2(F), \mathbb{Z}[\frac{1}{2}])_0 \rightarrow H_3(SL_2(k), \mathbb{Z}[\frac{1}{2}])_0 \oplus \mathcal{RP}_1(k)[\frac{1}{2}] \rightarrow 0$$

where δ_π is the composite of the map $H_3(SL_2(F), \mathbb{Z}[\frac{1}{2}])_0 \rightarrow H_3(SL_2(k), \mathbb{Z}[\frac{1}{2}])_0 \oplus \mathcal{RP}_1(k)[\frac{1}{2}]$ with projection on the second factor.

Proof This is [6, Proposition 7.1]. $\square$

Corollary 7.3 *Let F be a field with discrete valuation v and residue field k . Suppose that $U_1 = U_1^2$ and that $\mathcal{D}_F(v) = 0$. Let π be a uniformizer. Then there is a short exact sequence of R_k -modules*

$$0 \longrightarrow H_3(SL_2(k), \mathbb{Z}[\frac{1}{2}])_0 \longrightarrow H_3(SL_2(F), \mathbb{Z}[\frac{1}{2}])_0 \xrightarrow{\delta_\pi} \mathcal{RP}_1(k)[\frac{1}{2}] \longrightarrow 0.$$

7.2 The localization theorem

Theorem 7.4 *Let F be a field with discrete valuation v and residue field k . Suppose that either $\text{char}(F) = \text{char}(k)$ or that k is algebraic over $\mathbb{F}_p$ for some prime p . Suppose further that $U_1 = U_1^2$ and that $\mathcal{D}_F(v) = 0 = \mathcal{D}(\mathcal{O}_v/k)$. Let π be a uniformizer. Then there is a natural short exact sequence of $R_{\mathcal{O}_v}$ -modules*

$$0 \longrightarrow H_3(SL_2(\mathcal{O}_v), \mathbb{Z}[\frac{1}{2}]) \longrightarrow H_3(SL_2(F), \mathbb{Z}[\frac{1}{2}]) \xrightarrow{\delta_\pi} \mathcal{RP}_1(k)[\frac{1}{2}] \longrightarrow 0.$$

Proof We begin by observing that the triangle

$$\begin{array}{ccc} H_3(SL_2(\mathcal{O}_v), \mathbb{Z}[\frac{1}{2}])_0 & \xrightarrow{\cong} & H_3(SL_2(k), \mathbb{Z}[\frac{1}{2}])_0 \\ & \searrow & \swarrow \\ & H_3(SL_2(F), \mathbb{Z}[\frac{1}{2}])_0 & \end{array}$$

commutes (where the horizontal map is an isomorphism by Corollary 6.6). This follows from the commutativity of the square

$$\begin{array}{ccc} \mathcal{RP}(\mathcal{O}_v) & \longrightarrow & \widetilde{\mathcal{RP}}(k) \\ \downarrow & & \downarrow \\ \mathcal{RP}(F) & \xrightarrow{S_v} & \text{Ind}_k^F \widetilde{\mathcal{RP}}(k) \end{array} \quad , \quad \begin{array}{ccc} \langle u \rangle [v] & \longrightarrow & \langle \bar{u} \rangle [\bar{v}] \\ \downarrow & & \downarrow \\ \langle u \rangle [v] & \longrightarrow & 1 \otimes \langle \bar{u} \rangle [\bar{v}]. \end{array}$$

Thus by Corollary 7.3 the long vertical column in the diagram

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
 0 & \longrightarrow & H_3(\mathrm{SL}_2(\mathcal{O}_v), \mathbb{Z}[\frac{1}{2}])_0 & \longrightarrow & H_3(\mathrm{SL}_2(\mathcal{O}_v), \mathbb{Z}[\frac{1}{2}]) & \longrightarrow & K_3^{\mathrm{ind}}(\mathcal{O}_v)[\frac{1}{2}] \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \cong \\
 0 & \longrightarrow & H_3(\mathrm{SL}_2(F), \mathbb{Z}[\frac{1}{2}])_0 & \longrightarrow & H_3(\mathrm{SL}_2(F), \mathbb{Z}[\frac{1}{2}]) & \longrightarrow & K_3^{\mathrm{ind}}(F)[\frac{1}{2}] \longrightarrow 0 \\
 & & \downarrow \delta_\pi & & & & \\
 & & \mathcal{RP}_1(k)[\frac{1}{2}] & & & & \\
 & & \downarrow & & & & \\
 & & 0 & & & &
 \end{array}$$

is exact.

Since the right-hand vertical arrow is an isomorphism by Theorem 2.1, the result follows from the snake lemma. $\square$

7.3 The condition $\mathcal{D}_F(v) = 0 = \mathcal{D}(\mathcal{O}_v/k)$

Remark 7.5 If $X^2 - X + 1$ has a root in $\mathcal{O}_v$, then $D_{\mathcal{O}_v} = D_F = D_k = 0$ and therefore $\mathcal{D}_F(v) = 0 = \mathcal{D}(\mathcal{O}_v/k)$.

Proposition 7.6 Let F be a field with discrete valuation v and residue field k . Suppose that $U_1 = U_1^2$ and that $\mathcal{D}_k$ is free as a $\widehat{\mathcal{R}_k}$ -module. Suppose also that $X^2 - X + 1$ has no root in k . Then

$$\mathcal{D}_F(v) = 0 = \mathcal{D}(\mathcal{O}_v/k).$$

Proof By Lemma 4.15 the conditions ensure that the map $\mathcal{D}_{\mathcal{O}_v} \rightarrow \mathcal{D}_k$ is an isomorphism and hence that $\mathcal{D}(\mathcal{O}_v/k) = 0$.

On the other hand, $\mathcal{D}_F(v) = 0$ under these conditions by [6, Corollary 7.11]. $\square$

7.4 Some examples

Example 7.7 Let F be a field complete with respect to a discrete valuation v with sufficiently large finite residue field k of odd characteristic. Then $H_3(\mathrm{SL}_2(\mathcal{O}_v), \mathbb{Z}[\frac{1}{2}]) \cong K_3^{\mathrm{ind}}(\mathcal{O}_v)[\frac{1}{2}] \cong K_3^{\mathrm{ind}}(F)[\frac{1}{2}]$ by Proposition 6.5 and there is a (split) short exact sequence

$$0 \rightarrow H_3(\mathrm{SL}_2(\mathcal{O}_v), \mathbb{Z}[\frac{1}{2}]) \rightarrow H_3(\mathrm{SL}_2(F), \mathbb{Z}[\frac{1}{2}]) \rightarrow \mathcal{P}(k)[\frac{1}{2}] \rightarrow 0$$

(since $\mathcal{RP}_1(k) \left[\frac{1}{2} \right] = \mathcal{P}(k) \left[\frac{1}{2} \right]$ for a finite field). Recall that if $k = \mathbb{F}_q$ then $\mathcal{P}(k) \left[\frac{1}{2} \right]$ is cyclic of order $(q + 1)_{\text{odd}}$.

Example 7.8 In particular, if $p \geq 11$ is prime then $H_3(SL_2(\mathbb{Z}_p), \mathbb{Z} \left[\frac{1}{2} \right]) \cong K_3^{\text{ind}}(\mathbb{Q}_p) \left[\frac{1}{2} \right]$ and

$$H_3(SL_2(\mathbb{Q}_p), \mathbb{Z} \left[\frac{1}{2} \right]) \cong H_3(SL_2(\mathbb{Z}_p), \mathbb{Z} \left[\frac{1}{2} \right]) \oplus \mathbb{Z}/(p + 1)_{\text{odd}}.$$

Example 7.9 Let k be any (sufficiently large) field. By Theorem 7.4 there is a short exact sequence of R_k -modules

$$0 \longrightarrow H_3(SL_2(k[[x]]), \mathbb{Z} \left[\frac{1}{2} \right]) \longrightarrow H_3(SL_2(k((x))), \mathbb{Z} \left[\frac{1}{2} \right]) \xrightarrow{\delta_x} \mathcal{RP}_1(k) \left[\frac{1}{2} \right] \longrightarrow 0.$$

Note further that the surjective maps

$$H_3(SL_2(k[[x]]), \mathbb{Z} \left[\frac{1}{2} \right]) \rightarrow K_3^{\text{ind}}(k[[x]]) \left[\frac{1}{2} \right] \text{ and } \mathcal{RP}_1(k) \left[\frac{1}{2} \right] \rightarrow \mathcal{P}(k) \left[\frac{1}{2} \right]$$

both have kernels isomorphic to $\mathcal{I}_k \mathcal{RP}_1(k) \left[\frac{1}{2} \right]$. This module is in general quite large and has been computed for many fields in [6].

Example 7.10 Let p be any prime and take $k = \mathbb{Q}_p$ in the last example. Then $\mathcal{I}_k \mathcal{RP}_1(k) \left[\frac{1}{2} \right] \cong \mathcal{P}(\mathbb{F}_p) \left[\frac{1}{2} \right] \cong \mathbb{Z}/(p + 1)_{\text{odd}}$ (by [6, Proposition 7.1]). Thus there are (split) short exact sequences

$$0 \rightarrow \mathcal{P}(\mathbb{F}_p) \left[\frac{1}{2} \right] \rightarrow H_3(SL_2(\mathbb{Q}_p[[x]]), \mathbb{Z} \left[\frac{1}{2} \right]) \rightarrow K_3^{\text{ind}}(\mathbb{Q}_p[[x]]) \left[\frac{1}{2} \right] \rightarrow 0$$

and

$$0 \rightarrow \mathcal{P}(\mathbb{F}_p) \left[\frac{1}{2} \right] \rightarrow \mathcal{RP}_1(\mathbb{Q}_p) \left[\frac{1}{2} \right] \rightarrow \mathcal{P}(\mathbb{Q}_p) \left[\frac{1}{2} \right] \rightarrow 0.$$

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References

1. Dupont, J.L., Sah, C.H.: Scissors congruences. II. J. Pure Appl. Algebra **25**(2), 159–195 (1982)
2. Hutchinson, K.: The third homology of SL_2 of local rings. [arXiv:1309.5010](https://arxiv.org/abs/1309.5010)
3. Hutchinson, K.: A refined Bloch group and the third homology of SL_2 of a field. J. Pure Appl. Algebra **217**, 2003–2035 (2013)
4. Hutchinson, K.: A Bloch-Wigner complex for SL_2 . J. K-Theory **12**(1), 15–68 (2013)
5. Hutchinson, K.: On the low-dimensional homology of $SL_2(k[t, t^{-1}])$. J. Algebra **425**, 324–366 (2015)
6. Hutchinson, K.: The third homology of SL_2 of fields with discrete valuation. J. Pure Appl. Algebra **221**(5), 1076–1111 (2017)
7. Hutchinson, K., Tao, L.: The third homology of the special linear group of a field. J. Pure Appl. Algebra **213**, 1665–1680 (2009)
8. Knudson, K.P.: Homology of linear groups. Progress in Mathematics, vol. 193. Birkhäuser Verlag, Basel (2001)

9. Mirzaii, B.: A Bloch-Wigner exact sequence over local rings. [arXiv:1603.03649](https://arxiv.org/abs/1603.03649)
10. Mirzaii, B.: Bloch-Wigner theorem over rings with many units. *Math. Z.* **268**(1–2), 329–346 (2011)
11. Mirzaii, B., Mokari, F.Y.: A Bloch-Wigner theorem over rings with many units II. *J. Pure Appl. Algebra* **219**(11), 5078–5096 (2015)
12. Suslin, A.A.: K_3 of a field, and the Bloch group. *Trudy Mat. Inst. Steklov.* **183**, 180–199, 229 (1990). Translated in *Proc. Steklov Inst. Math.* **1991**(4), 217–239. Galois theory, rings, algebraic groups and their applications (Russian)
13. Suslin, A.A.: Homology of GL_n , characteristic classes and Milnor K -theory. In: *Algebraic K -theory, number theory, geometry and analysis* (Bielefeld, 1982), *Lecture Notes in Math.*, vol. 1046, pp. 357–375. Springer, Berlin (1984)
14. Weibel, C.A.: *The K -book*. In: *An introduction to algebraic K -theory*. *Graduate Studies in Mathematics*, vol. 145. American Mathematical Society, Providence (2013)