J. Homotopy Relat. Struct. (2017) 12:727–739
DOI 10.1007/s40062-016-0163-x

The mod 2 dual Steenrod algebra as a subalgebra of the mod 2 dual Leibniz-Hopf algebra

Neşet Deniz Turgay[1] · **Shizuo Kaji**[2]

Received: 18 November 2015 / Accepted: 23 October 2016 / Published online: 15 November 2016

Abstract The mod 2 Steenrod algebra $\mathcal{A}_2$ can be defined as the quotient of the mod 2 Leibniz–Hopf algebra $\mathcal{F}_2$ by the Adem relations. Dually, the mod 2 dual Steenrod algebra $\mathcal{A}_2^*$ can be thought of as a sub-Hopf algebra of the mod 2 dual Leibniz–Hopf algebra $\mathcal{F}_2^*$. We study $\mathcal{A}_2^*$ and $\mathcal{F}_2^*$ from this viewpoint and give generalisations of some classical results in the literature.

Keywords Leibniz–Hopf algebra · Steenrod algebra · Adem relation · Hopf algebra · Conjugation · Antipode

Mathematics Subject Classification 55S10 · 16T05 · 57T05

Communicated by Stewart Priddy.

The second named author was partially supported by KAKENHI, Grant-in-Aid for Young Scientists (B) 26800043 and JSPS Postdoctoral Fellowships for Research Abroad.

✉ Shizuo Kaji
skaji@yamaguchi-u.ac.jp

Neşet Deniz Turgay
Deniz_Turgay@Yahoo.com

1 Department of Mathematics, Faculty of Arts and Sciences, Eastern Mediterranean University, Gazimagusa, TRNC, Mersin 10, Turkey

2 Department of Mathematical Sciences, Faculty of Science, Yamaguchi University, 1677-1, Yoshida, Yamaguchi 753-8512, Japan

1 The mod 2 Leibniz–Hopf algebra and its dual

Let $\mathcal{F}_2$ be the free associative algebra over $\mathbb{F}_2$ generated by the indeterminates $S^1, S^2, S^3, \ldots$ of degree $|S^i| = i$. We often denote the unit 1 by S^0. This algebra is equipped with a co-commutative co-product given by

$$\Delta(S^n) = \sum_{i=0}^{n} S^i \otimes S^{n-i}, \tag{1}$$

which makes it a graded connected Hopf algebra. This algebra $\mathcal{F}_2$ is often called the mod 2 Leibniz–Hopf algebra. As an $\mathbb{F}_2$-module, $\mathcal{F}_2$ has the following canonical basis:

$$\{S^I := S^{i_1} S^{i_2} \cdots S^{i_n} \mid I = (i_1, i_2, \ldots, i_n) \in \mathbb{N}^n, 0 \leq n < \infty\},$$

where we regard $S^I = 1$ when $n = 0$.

Note that the integral counterpart of $\mathcal{F}_2$ is called the Leibniz–Hopf algebra and is isomorphic to the *ring of non-commutative symmetric functions* [7] and the *Solomon Descent algebra* [17]. Its graded dual is the ring of quasi-symmetric functions with the outer co-product, which has been studied by Hazewinkel, Malvenuto, and Reutenauer in [8–12].

The mod 2 Steenrod algebra $\mathcal{A}_2$ is defined to be the quotient Hopf algebra of $\mathcal{F}_2$ by the ideal generated by the Adem relations:

$$S^i S^j - \sum_{k=0}^{\lfloor i/2 \rfloor} \binom{j-k-1}{i-2k} S^{i+j-k} S^k. \tag{2}$$

Denote the quotient map by $\pi : \mathcal{F}_2 \to \mathcal{A}_2$ and $Sq^i = \pi(S^i)$. It is well-known (see, for example, [18]) that the *admissible monomials*

$$\{Sq^J := Sq^{j_1} Sq^{j_2} \cdots Sq^{j_n} \mid J = (j_1, j_2, \ldots, j_n) \in \mathbb{N}^n_{>0}, 0 \leq n < \infty, j_{k-1} \geq 2 j_k \forall k\}$$

form a module basis for $\mathcal{A}_2$. We will adhere to this purely algebraic definition and will not use any other known facts about $\mathcal{A}_2$.

By taking the graded dual of π, we obtain the following inclusion of Hopf algebras

$$\pi^* : \mathcal{A}_2^* \to \mathcal{F}_2^*.$$

$\mathcal{F}_2^*$ is given a module basis S_I dual to S^I, that is,

$$\langle S^{I'}, S_I \rangle = \begin{cases} 1 & (I = I') \\ 0 & (I \neq I') \end{cases}.$$

Similarly, we have the dual basis $\{Sq_J \mid J \text{ admissible}\}$ for $\mathcal{A}_2^*$ determined by

$$\langle Sq^{J'}, Sq_J\rangle = \begin{cases} 1 & (J = J') \\ 0 & (J \neq J') \end{cases}.$$

The commutative product among the basis elements in $\mathcal{F}_2^*$ is given by the *overlapping shuffle product* (see §2) and the co-product is given by

$$\Delta(S_{a_1,\ldots,a_n}) = S_{a_1,\ldots,a_n} \otimes 1 + 1 \otimes S_{a_1,\ldots,a_n} + \sum_{i=1}^{n-1} S_{a_1,\ldots,a_i} \otimes S_{a_{i+1},\ldots,a_n}. \quad (3)$$

The purpose of this paper is to deduce some of the classical results on $\mathcal{A}_2^*$ and its generalisations by considering it as a subalgebra of $\mathcal{F}_2^*$. We are particularly interested in the following problems.

Problem 1 (i) Determine the coefficients in

$$\pi^*(Sq_J) = \sum_I C_J^I S_I \quad (4)$$

for all admissible sequences J. This is important since in the dual it is equivalent to computing the coefficients of the Adem relations

$$Sq^I = \sum_{J:\text{admissible}} C_J^I Sq^J \quad (5)$$

for all sequences I.

(ii) Give an expansion of the dual Milnor bases in terms of the dual admissible monomial bases, i.e., determine the coefficient B_J^L in

$$\xi^L = \sum_{J:\text{admissible}} B_J^L Sq_J,$$

where $\xi_n = Sq_{2^{n-1},2^{n-2}\ldots 2^1,2^0}$ and $\xi^L = \xi_1^{l_1}\xi_2^{l_2}\cdots\xi_n^{l_n}$ for $L = (l_1, l_2, \ldots, l_n)$.

(iii) Generalise Milnor's conjugation formula [13] in $\mathcal{A}_2^*$ to $\mathcal{F}_2^*$. The formula for $\mathcal{A}_2^*$ is:

$$\chi(\xi_n) = \sum_\alpha \prod_{i=1}^{l(\alpha)} \xi_{\alpha(i)}^{2^{\sigma(i)}},$$

where $\alpha = (\alpha(1)|\alpha(2)|\ldots|\alpha(l(\alpha))$ runs through all the compositions of the integer n and $\sigma(i) = \sum_{j=1}^{i-1} \alpha(j)$.

Several different methods are known for resolving (i) and (ii) (see for example, [15,19]), but our argument (Sect. 4) is new in that it is purely combinatorial using the overlapping shuffle product on $\mathcal{F}_2^*$. We implemented our algorithm into a Maple code [16]. In Sect. 5 we discuss the conjugation (or antipode) in $\mathcal{F}_2^*$ and give an answer to (iii). Finally, we give an explicit duality between the conjugation invariants in $\mathcal{F}_2$ and $\mathcal{F}_2^*$ in Sect. 6.

2 Overlapping Shuffle product

We recall the definition of the overlapping shuffle product ([2, Section 2],[8]). Let $\mathcal{W}$ be the set of finite sequences of natural numbers:

$$\mathcal{W} = \{(i_1, i_2, \dots, i_n) \mid 0 \le n < \infty\}.$$

Note that we allow the length 0 sequence. Consider the $\mathbb{F}_2$-module $\mathbb{F}_2\langle\mathcal{W}\rangle$ freely generated by $\mathcal{W}$. For a sequence $I = (i_1, i_2, \dots, i_n)$, denote its tail partial sequence $(i_k, i_{k+1}, \dots, i_n)$ by I_k. When $n < k$, we regard I_k as the length 0 sequence. We use the convention

$$\begin{aligned}&(a_1, a_2, \dots, a_k, (b_1, \dots, b_i) + (c_1, \dots, c_j))\\ &\quad := (a_1, a_2, \dots, a_k, b_1, \dots, b_i) + (a_1, a_2, \dots, a_k, c_1, \dots, c_j).\end{aligned}$$

The *overlapping shuffle product* on $\mathbb{F}_2\langle\mathcal{W}\rangle$ is defined as follows:

Definition 1 For $A = (a_1, a_2, \dots, a_n)$ and $B = (b_1, b_2, \dots, b_m)$, define their product inductively by

$$A \cdot B := \begin{cases} A & (m = 0)\\ B & (n = 0)\\ \sum_{0\le i\le n}(a_1, \dots, a_i, b_1, A_{i+1} \cdot B_2) & \\ \quad + \sum_{1\le i\le n}(a_1, \dots, a_i + b_1, A_{i+1} \cdot B_2) & (\text{otherwise}).\end{cases}$$

The product on $\mathbb{F}_2\langle\mathcal{W}\rangle$ is defined by the linear extension of the above.

We say a term in $A \cdot B$ is *a-first* if there exists k such that a_k goes[1] to an entry to the left of b_k and a_i goes to the same entry as b_i (that is, the entry makes $a_i + b_i$) for all $i < k$. For example, $(a_1 + b_1, a_2, b_2, b_3, a_3)$ is a-first while $(a_1 + b_1, b_2, a_2, a_3, b_3)$ is not. Observe that

Lemma 1 *For equal length sequences, we have*

$$(a_1, \dots, a_n) \cdot (b_1, \dots, b_n) = (a_1 + b_1, \dots, a_n + b_n) + Z + \tau(Z),$$

where Z is a sum of a-first terms and τ flips the occurrence of a_i and b_i for all i. In particular, the product is commutative.

[1] When calculated symbolically.

Example 1

$$\begin{aligned}(a_1, a_2)\cdot(b_1, b_2) = (a_1 + b_1, a_2 + b_2)\\ &+ (a_1 + b_1, a_2, b_2) + (a_1, b_1, a_2 + b_2) + (a_1, b_1 + a_2, b_2)\\ &+ (a_1, a_2, b_1, b_2) + (a_1, b_1, a_2, b_2) + (a_1, b_1, b_2, a_2)\\ &+ (b_1 + a_1, b_2, a_2) + (b_1, a_1, b_2 + a_2) + (b_1, a_1 + b_2, a_2)\\ &+ (b_1, b_2, a_1, a_2) + (b_1, a_1, b_2, a_2) + (b_1, a_1, a_2, b_2),\end{aligned}$$

where the second line consists of a-first terms and the third line is the τ-image of the second line.

Corollary 1 *For* $A = (a_1, \ldots, a_n)$,

$$A \cdot A = (2a_1, \ldots, 2a_n), \quad A^{2^m} = (2^m a_1, \ldots, 2^m a_n).$$

Proof In this case, the flip map τ in Lemma 1 is the identity. □

It is easy to see from the duality relation $\langle S_I S_J, S^K\rangle = \langle S_I \otimes S_J, \Delta(S^K)\rangle$ that the product on $\mathcal{F}_2^*$ dual to (1) is given by $S_I S_J = \sum_{K\in I\cdot J} S_K$.

3 Dual Steenrod algebra as a sub-Hopf algebra of $\mathcal{F}_2^*$

To identify the image of the inclusion $\pi^* : \mathcal{A}_2^* \to \mathcal{F}_2^*$, we prove some lemmas in this section. Let $\xi_n = Sq_{2^{n-1},2^{n-2},\ldots,2^0}$.

Lemma 2 (cf. [2,19]) *We have*

$$\begin{aligned}\pi^*(Sq_{2^n}) &= S_{2^n},\\ \pi^*(\xi_n) &= S_{2^{n-1},2^{n-2},\ldots,2^0}.\end{aligned}$$

Proof For the first equation, we have to show that for any non-admissible sequence I, the right-hand side of

$$Sq^I = \sum_{J:\text{admissible}} C^I_J Sq^J$$

does not contain Sq^{2^n}. If there exists such an I, we can assume it has length two, that is, $I = (i, j)$. (Because the right-hand side is obtained by successively applying the length two relations.) By the Adem relations in Eq. (2), we have $i + j = 2^n$ and

$$1 \equiv \binom{j-1}{i} \equiv \binom{2^n - 1 - i}{i} \mod 2.$$

However, the binary expressions of $2^n - 1 - i$ and i are complementary and the binary expression of $2^n - 1 - i$ contains at least one digit with 0. Hence, by Lucas' Theorem, we have $\binom{2^n-1-i}{i} \equiv 0 \mod 2$; we arrive at a contradiction.

For the second equation, suppose that there exists an $I = (i, j)$ such that $i < 2j$ and

$$Sq^{i,j} = \sum_{k=0}^{\lfloor i/2 \rfloor} \binom{j-k-1}{i-2k} Sq^{i+j-k} Sq^k$$

contains $Sq^{2^{n-k}}$ or $Sq^{2^{n-k}} Sq^{2^{n-k-1}}$ as a summand. The former case is already ruled out by the first equation. For the latter case to happen, we should have

$$i + j = 2^{n-k} + 2^{n-k-1}, \quad \lfloor i/2 \rfloor \geq 2^{n-k-1}.$$

But this implies $j \leq 2^{n-k-1}$ so $i \geq 2j$; we arrive at a contradiction. □

Put $\bar{\xi}_n = \pi^*(\xi_n) = S_{2^{n-1},2^{n-2},\ldots,2^0}$. We denote by $\widetilde{\mathcal{A}}_2^*$ the subalgebra of $\mathcal{F}_2^*$ generated by $\{\bar{\xi}_n \mid 0 < n\}$. For a sequence $L = (l_1, l_2, \ldots, l_n)$ of non-negative integers, we denote $\bar{\xi}_1^{l_1} \bar{\xi}_2^{l_2} \cdots \bar{\xi}_n^{l_n}$ by $\bar{\xi}^L$. Then, the monomials $\bar{\xi}^L$ span $\widetilde{\mathcal{A}}_2^*$. Now, we identify $\widetilde{\mathcal{A}}_2^*$ with $\mathrm{Im}(\pi^*)$.

Recall the definition of the *excess vector* of an admissible sequence $J = (j_1, j_2, \ldots, j_n)$:

$$\gamma(j_1, j_2, \ldots, j_n) = (j_1 - 2j_2, j_2 - 2j_3, \ldots, j_{n-1} - 2j_n, j_n).$$

This gives a bijection between admissible sequences and sequences of non-negative integers. The inverse is given by

$$\gamma^{-1}(l_1, l_2, \ldots, l_n) = (l_1 + 2l_2 + 2^2 l_3 + \cdots + 2^{n-1} l_n, \ldots, l_{n-1} + 2l_n, l_n).$$

We put the right lexicographic order on $\mathcal{W}$, i.e.,

$$(a_1, a_2, \ldots, a_n) > (b_1, b_2, \ldots, b_m) \Leftrightarrow (n > m) \text{ or } (\exists k, a_k > b_k \text{ and } a_i = b_i \forall i > k).$$

This induces an ordering on the basis elements S_I which is compatible with the overlapping shuffle product. Observe that the lowest term in the product $S_I \cdot S_{I'}$ for $I = (i_1, i_2, \ldots)$ and $I' = (i'_1, i'_2, \ldots)$ is $S_{(i_1+i'_1, i_2+i'_2, \ldots)}$.

Lemma 3 *For an admissible sequence J,*

$$\langle \bar{\xi}^{\gamma(J)}, S^I \rangle = \begin{cases} 1 & (I = J) \\ 0 & (I < J). \end{cases}$$

Proof We proceed by induction on $J = (j_1, \ldots, j_n)$. Put $J' = (j_1 - 2^{n-1}, j_2 - 2^{n-2}, \ldots, j_n - 2^0)$. Then by induction hypothesis,

$$\bar{\xi}^{\gamma(J')} = S_{J'} + (\text{terms higher than } S_{J'}).$$

It follows that

$$\begin{aligned}\bar{\xi}^{\gamma(J)} &= \bar{\xi}^{\gamma(J')} \cdot \bar{\xi}_n \\ &= (S_{J'} + (\text{terms higher than } S_{J'})) \cdot S_{2^{n-1},2^{n-2},\dots,2^0} \\ &= S_J + (\text{terms higher than } S_J).\end{aligned}$$

□

By this upper-triangularity, the monomials $\bar{\xi}^L$ are linearly independent and we have

Theorem 1

$$\mathrm{Im}(\pi^*) = \widetilde{\mathcal{A}}_2^* = \mathbb{F}_2[\bar{\xi}_1, \bar{\xi}_2, \dots,].$$

Proof By Lemma 3 in each degree $\widetilde{\mathcal{A}}_2^*$ has the same dimension as $\mathcal{A}_2^*$ (the number of admissible sequences). □

This is nothing but the well-known fact:

Corollary 2 [13]

$$\mathcal{A}_2^* = \mathbb{F}_2[\xi_1, \xi_2, \dots,],$$

where

$$\xi^{\gamma(J)} = Sq_J + (\textit{terms higher than } Sq_J).$$

4 Computation with π^*

Recall from [19, Section 4] the linear left inverse $r\colon \mathcal{F}_2^* \to \mathcal{A}_2^*$ of π^*:

$$r(S_I) = \begin{cases} Sq_I & (I : \text{admissible}) \\ 0 & (\text{otherwise}). \end{cases}$$

For (ii) of Problem 1, we can compute

$$\begin{aligned}\xi^{(l_1,l_2,\dots,l_n)} &= r\pi^*(\xi^{(l_1,l_2,\dots,l_n)}) \\ &= r(\bar{\xi}_1^{l_1}\bar{\xi}_2^{l_2}\cdots\bar{\xi}_n^{l_n}) \\ &= r((S_{2^0})^{l_1}(S_{2^1,2^0})^{l_2}\cdots(S_{2^{n-1},2^{n-2},\dots,2^0})^{l_n}) \qquad (6)\end{aligned}$$

and it reduces to computing admissible sequences occurring in the overlapping shuffle product.

For (i) of Problem 1, by Corollary 2 we have

$$\pi^*(\xi^{\gamma(J)}) = \pi^*(Sq_J + (\text{terms higher than } Sq_J))$$

and the left-hand side can be computed by the overlapping shuffle product. Thus, we can compute inductively the coefficients C^I_J in

$$\pi^*(Sq_J) = \sum_I C^I_J S_I.$$

We implemented the algorithm into a Maple code [16].

Example 2 We demonstrate the above algorithm in low degrees. First, compute π^*-image of monomials ξ^L:

$$\begin{aligned}
\pi^*(\xi_2^2) &= S_{2,1}S_{2,1} = S_{4,2} \\
\pi^*(\xi_1^3\xi_2) &= (S_3 + S_{1,2} + S_{2,1})S_{2,1} \\
&= S_{5,1} + S_{4,2} + S_{3,3} + S_{2,4} + S_{2,3,1} + S_{1,4,1} \\
&\quad + S_{3,1,2} + S_{2,2,2} + S_{1,2,3} + S_{2,1,2,1} + S_{1,2,1,2} \\
\pi^*(\xi_1^6) &= S_6 + S_{4,2} + S_{2,4}.
\end{aligned}$$

Taking r on the both sides of equations, we obtain

$$\xi_2^2 = Sq_{4,2}, \quad \xi_1^3\xi_2 = Sq_{5,1} + Sq_{4,2}, \quad \xi_1^6 = Sq_6 + Sq_{4,2}.$$

Again taking π^* on the both sides of the equations, we obtain

$$\begin{aligned}
\pi^*(Sq_{4,2}) &= S_{4,2} \\
\pi^*(Sq_{5,1} + Sq_{4,2}) &= S_{5,1} + S_{4,2} + S_{3,3} + S_{2,4} + S_{2,3,1} + S_{1,4,1} \\
&\quad + S_{3,1,2} + S_{2,2,2} + S_{1,2,3} + S_{2,1,2,1} + S_{1,2,1,2} \\
\pi^*(Sq_6 + Sq_{4,2}) &= S_6 + S_{4,2} + S_{2,4}.
\end{aligned}$$

Finally, by using the upper-triangularity, we obtain

$$\begin{aligned}
\pi^*(Sq_{4,2}) &= S_{4,2} \\
\pi^*(Sq_{5,1}) &= S_{5,1} + S_{3,3} + S_{2,4} + S_{2,3,1} + S_{1,4,1} + S_{3,1,2} + S_{2,2,2} \\
&\quad + S_{1,2,3} + S_{2,1,2,1} + S_{1,2,1,2} \\
\pi^*(Sq_6) &= S_6 + S_{2,4}.
\end{aligned}$$

5 Formula for the conjugation

Any connected commutative or co-commutative Hopf algebra has a unique conjugation χ satisfying

$$\chi(1) = 1, \quad \chi(xy) = \chi(y)\chi(x), \quad \chi^2(x) = x, \quad \sum x'\chi(x'') = 0,$$

where $\Delta(x) = \sum x' \otimes x''$ and $\deg(x) > 0$ [14]. The conjugation invariants in $\mathcal{A}_2^*$ is studied in [5] because it is relevant to the commutativity of ring spectra [1, Lecture 3]. The same problem in $\mathcal{F}_2^*$ has been also studied in [3,4]. Here we investigate them through our point of view.

Since π^* is a Hopf algebra homomorphism, we have $\pi^* \circ \chi_{\mathcal{A}_2^*} = \chi_{\mathcal{F}_2^*} \circ \pi^*$, where $\chi_{\mathcal{A}_2^*}$ and $\chi_{\mathcal{F}_2^*}$ denote the conjugation operations in $\mathcal{A}_2^*$ and $\mathcal{F}_2^*$ respectively. For the module basis S_I in $\mathcal{F}_2^*$, the conjugation $\chi_{\mathcal{F}_2^*}$ is calculated combinatorially.

Definition 2 The coarsening set $C(I)$ of a sequence $I = (i_1, \ldots, i_l)$ is defined recursively as

$$C(I) := \{(i_1, I'), (i_1 + i'_1, I'_2) \mid I' \in C((i_2, \ldots, i_l))\} \quad \text{and} \quad C((i)) = \{(i)\},$$

where I'_2 is the tail partial sequence $(i'_2, \ldots, i'_{l'})$ of $I' = (i'_1, i'_2, \ldots, i'_{l'})$.

Example 3 $C((a, b, c)) = \{(a, b, c), (a + b, c), (a, b + c), (a + b + c)\}$.

A formula for the conjugation operation in the dual Leibniz–Hopf algebra is given by Ehrenborg [6, Proposition 3.4]. We now give a simple proof for its mod 2 reduction.

Proposition 1

$$\chi_{\mathcal{F}_2^*}(S_I) = \sum_{I' \in C(I^{-1})} S_{I'},$$

where $I^{-1} = (i_l, \ldots, i_1)$ *is the reverse sequence of* $I = (i_1, \ldots, i_l)$.

Proof The conjugation is uniquely characterised by

$$\chi_{\mathcal{F}_2^*}(1) = 1, \quad \sum x' \chi_{\mathcal{F}_2^*}(x'') = 0,$$

where $\Delta(x) = \sum x' \otimes x''$ and $\deg(x) > 0$. We put $\chi'(S_I) = \sum_{I' \in C(I^{-1})} S_{I'}$ and show that it satisfies the above equations. It is obvious that $\chi'(1) = 1$. Since the co-product is given in (3), the second equation reads

$$\sum_{k=0}^{l} S_{i_1, \ldots, i_k} \chi'(S_{i_{k+1}, \ldots, i_n}) = 0 \qquad (\forall I = (i_1, i_2, \ldots, i_n)).$$

We regard an element of $\mathbb{F}_2\langle \mathcal{W} \rangle$ with a finite subset of $\mathcal{W}$ in the obvious way. We investigate relation between coarsening and the overlapping shuffle product. Define

$$C_k(I) = \sum_{I' \in C((I_{k+1})^{-1})} I' \cdot (i_1, \ldots, i_k).$$

We observe[2] that $C(I^{-1}) \subset C_1(I)$ and $C'_1(I) := C_1(I)\backslash C(I^{-1})$ consists of those sequences that i_1 appears to the left of i_2. In turn, $C'_1(I) \subset C_2(I)$ and $C'_2(I) := C_2(I)\backslash C'_1(I)$ consists of those sequences that i_2 appears to the left of i_3. Continuing similarly, we obtain

$$C(I^{-1}) = \sum_{k=1}^{l} C_k(I).$$

It follows that

$$\chi'(S_I) = \sum_{k=1}^{l} \sum_{I' \in C((I_{k+1})^{-1})} S_{(i_1,\ldots,i_k)} \cdot S_{I'} = \sum_{k=0}^{l} S_{i_1,\ldots,i_k} \chi'(S_{i_{k+1},\ldots,i_n}) - \chi'(S_I)$$

and $\sum_{k=0}^{l} S_{i_1,\ldots,i_k} \chi'(S_{i_{k+1},\ldots,i_n}) = 0.$ □

We give another formula for $\chi_{\mathcal{F}_2^*}(S_I)$.

Definition 3 For a sequence $a_1, a_2, \ldots, a_n$, the *set of ordered block partitions* $\mathcal{P}(a_1, a_2, \ldots, a_n)$ consists of elements of the form

$$\beta = ((a_1, a_2, \ldots, a_{i_1})|(a_{i_1+1}, \ldots, a_{i_2})|\ldots|(a_{i_{l-1}+1}, \ldots, a_{i_l})),$$

where $1 \le i_1 < i_2 < \cdots < i_l = n$. We denote $l(\beta) = l$ and $\beta(k) = (a_{i_{k-1}+1}, \ldots, a_{i_k})$. Or inductively, we can define

$$\mathcal{P}(a_1, a_2, \ldots, a_n) = \bigcup_{k=1}^{n} \left\{ ((a_1, \ldots, a_k)|\beta) \,\middle|\, \beta \in \mathcal{P}(a_{k+1}, a_{k+2}, \ldots, a_n) \right\}. \quad (7)$$

Theorem 2

$$\chi_{\mathcal{F}_2^*}(S_I) = \sum_{\beta \in \mathcal{P}(I)} \prod_{k=1}^{l(\beta)} S_{\beta(k)}.$$

Proof Let $I = (a_1, a_2, \ldots, a_n)$. Put

$$\chi'(S_{a_1,a_2,\ldots,a_n}) = \sum_{\beta \in \mathcal{P}(a_1,a_2,\ldots,a_n)} \prod_{k=1}^{l(\beta)} S_{\beta(k)}$$

[2] Here, we deal with sequences symbolically so that we avoid cancellations like $(i_3+i_2, i_1)+(i_3+i_1, i_2) = 0$ when $i_1 = i_2$.

and we check that

$$\chi'(1) = 1, \quad \sum_{k=0}^{n} S_{a_1,\ldots,a_k}\chi'(S_{a_{k+1},\ldots,a_n}) = 0.$$

Then, by the uniqueness of the conjugation, we have $\chi_{\mathcal{F}_2^*} = \chi'$. The first assertion is trivial. For the second, observe that by (7)

$$\begin{aligned}\chi'(S_{a_1,a_2,\ldots,a_n}) &= \sum_{k=1}^{n} S_{a_1,\ldots,a_k}\left(\sum_{\beta\in\mathcal{P}(a_{k+1},a_{k+2},\ldots,a_n)}\prod_{j=1}^{l(\beta)} S_{\beta(j)}\right)\\ &= \sum_{k=1}^{n} S_{a_1,\ldots,a_k}\chi'(S_{a_{k+1},\ldots,a_n}).\end{aligned}$$

Hence, we have

$$\begin{aligned}\sum_{k=0}^{n} S_{a_1,\ldots,a_k}\chi'(S_{a_{k+1},\ldots,a_n}) &= \chi'(S_{a_1,a_2,\ldots,a_n}) + \sum_{k=1}^{n} S_{a_1,\ldots,a_k}\chi'(S_{a_{k+1},\ldots,a_n})\\ &= 2\chi'(S_{a_1,a_2,\ldots,a_n})\\ &= 0.\end{aligned}$$

□

Example 4

$$\begin{aligned}\chi_{\mathcal{F}_2^*}(S_{1,2,3}) &= S_{3,2,1} + S_{5,1} + S_{3,3} + S_6\\ &= S_{1,2,3} + S_1S_{2,3} + S_{1,2}S_3 + S_1S_2S_3.\end{aligned}$$

The first line is computed by Proposition 1, and the second by Theorem 2.

Theorem 2 can be thought of as a generalisation of Milnor's conjugation formula in $\mathcal{A}_2^*$. To see this, we first show a small lemma:

Lemma 4

$$\bar{\xi}_n^{2^m} = (S_{2^{n-1},2^{n-2},\ldots,2^0})^{2^m} = S_{2^{n+m-1},2^{n+m-2},\ldots,2^m}.$$

Proof This is a direct consequence of Corollary 1 combined with Lemma 2. □

Corollary 3 [13, Lemma 10]

$$\chi_{\mathcal{A}_2^*}(\xi_n) = \sum_{\alpha}\prod_{k=1}^{l(\alpha)} \xi_{\alpha(k)}^{2^{\sigma(k)}}, \tag{8}$$

where $\alpha = (\alpha(1)|\alpha(2)|\ldots|\alpha(l(\alpha))$ runs through all the compositions of the integer n and $\sigma(k) = \sum_{j=1}^{k-1}\alpha(j)$.

Proof We apply the injection π^* to the both sides of (8) and show that they coincide. For the left-hand side, by Lemma 4 we have

$$\pi^*(\chi_{\mathcal{A}_2^*}(\xi_n)) = \chi_{\mathcal{F}_2^*}(\pi^*(\xi_n)) = \chi_{\mathcal{F}_2^*}(S_{2^{n-1},2^{n-2},\dots,2^0}).$$

Since $\pi^*(\xi_{\alpha(k)}^{2^{\sigma(k)}}) = S_{2^{\alpha(k)+\sigma(k)-1},\dots,2^{\sigma(k)}}$ by Lemma 4, we see

$$\pi^*\left(\prod_{k=1}^{l(\alpha)} \xi_{\alpha(k)}^{2^{\sigma(k)}}\right) = S_{2^{n-1},\dots,2^{n-\alpha(l(\alpha))}} \cdot S_{2^{n-1-\alpha(l(\alpha))},\dots,2^{n-\alpha(l(\alpha))-\alpha(l(\alpha)-1)}} \cdots S_{2^{\alpha(1)-1},\dots,2^0}.$$

So when α ranges over all compositions of n, we get all the ordered block partitions of the sequence $2^{n-1}, 2^{n-2}, \dots, 2^0$. The assertion follows from Theorem 2. □

6 Duality between $\mathcal{F}_2$ and $\mathcal{F}_2^*$

In the previous section we discussed how to compute the conjugation in $\mathcal{F}_2^*$. Here, we relate the conjugation in $\mathcal{F}_2$ with that in $\mathcal{F}_2^*$ by using a self-duality of $\mathcal{W}$. Denote $I \preceq I'$ if $I \in C(I')$. We think of $I \in \mathcal{W}$ as a string of 1's separated by '+' and commas; $(\underbrace{1+1+\cdots+1}_{i_1}, \underbrace{1+1+\cdots+1}_{i_2}, \dots, \underbrace{1+1+\cdots+1}_{i_l})$.

Definition 4 We define the dual $\bar{I} \in \mathcal{W}$ of I by switching + and the commas.

Example 5 For $I = (1, 3, 2) = (1, 1+1+1, 1+1)$, its dual is

$$\bar{I} = (1+1, 1, 1+1, 1) = (2, 1, 2, 1).$$

It is easily seen that $\bar{\bar{I}} = I$ and $I \preceq I' \Leftrightarrow \bar{I} \succeq \bar{I}'$. Extend the duality to one between $\mathcal{F}_2$ and $\mathcal{F}_2^*$ by

$$D(S^I) = S_{\bar{I}}, \quad D^{-1}(S_I) = S^{\bar{I}}.$$

Theorem 3 *We have $D \circ \chi_{\mathcal{F}_2} = \chi_{\mathcal{F}_2^*} \circ D$. In particular, $f \in \mathcal{F}_2$ is a conjugation invariant if and only if so is $\bar{f} \in \mathcal{F}_2^*$.*

Proof We compute

$$\begin{aligned} D^{-1} \circ \chi_{\mathcal{F}_2^*} \circ D(S^I) &= D^{-1}\chi_{\mathcal{F}_2^*}(S_{\bar{I}}) = D^{-1}\left(\sum_{I' \preceq (\bar{I})^{-1}} S_{I'}\right) = D^{-1}\left(\sum_{\bar{I}' \succeq I^{-1}} S_{I'}\right) \\ &= \sum_{\bar{I}' \succeq I^{-1}} S^{\bar{I}'} = \sum_{I' \succeq I^{-1}} S^{I'}. \end{aligned}$$

Put $\chi'(S^I) = \sum_{I' \succeq I^{-1}} S^{I'}$. Then, one can check $\chi'(1) = 1$ and $\sum x'\chi'(x'') = 0$ for $\Delta x = \sum x' \otimes x''$ as in Proposition 1. Hence, by the uniqueness of the conjugation, we have $\chi' = \chi_{\mathcal{F}_2}$. □

Example 6 $f = S^{1,1,2} + S^{2,1,1} + S^{1,1,1,1}$ is a $\chi_{\mathcal{F}_2}$-invariant, whilst $D(f) = S_{3,1} + S_{1,3} + S_4$ is a $\chi_{\mathcal{F}_2^*}$-invariant.

Remark 1 The sub-module of the conjugation invariants in $\mathcal{F}_2$ is $\ker(\chi_{\mathcal{F}_2} - 1)$ and that in $\mathcal{F}_2^*$ is $\ker(\chi_{\mathcal{F}_2^*} - 1)$. The conjugations in $\mathcal{F}_2$ and $\mathcal{F}_2^*$ are dual to each other, and hence, the linear map $\chi_{\mathcal{F}_2} - 1$ is transpose to $\chi_{\mathcal{F}_2^*} - 1$ with the kernel of same dimension [3]. Theorem 3 gives more information by specifying an explicit correspondence between their elements.

Acknowledgements We would like to thank Stephen Theriault, Martin Crossley, and Carmen Rovi for their comments on the earlier version of this paper.

References

1. Adams, J.F.: Lectures on generalised cohomology. Springer Lect. Notes Math. **99**, 1–138 (1969)
2. Crossley, M.D.: Some Hopf algebras of words. Glasgow Math J. **48**, 575–582 (2006)
3. Crossley, M.D., Turgay, N.D.: Conjugation invariants in the Leibniz Hopf algebra. J. Pure Appl. Algebra **217**, 2247–2254 (2013)
4. Crossley, M.D., Turgay, N.D.: Conjugation invariants in the mod 2 dual Leibniz–Hopf algebra. Commun. Algebra **41**, 3261–3266 (2013)
5. Crossley, M.D., Whitehouse, S.: On conjugation invariants in the dual Steenrod algebra. Proc. AMS. **128**, 2809–2818 (2000)
6. Ehrenborg, R.: On posets and Hopf algebras. Adv. Math. **119**, 1–25 (1996)
7. Gelfand, I.M., Krob, D., Lascoux, A., Leclerc, B., Retakh, V.S., Thibon, J.-Y.: Noncommutative symmetric functions. Adv. Math. **112**, 218–348 (1995)
8. Hazewinkel, M.: Generalized overlapping shuffle algebras. J. Math. Sci. N Y **106**, 3168–3186 (2001)
9. Hazewinkel, M.: The algebra of quasi-symmetric functions is free over the integers. Adv. Math. **164**, 283–300 (2001)
10. Hazewinkel, M.: Symmetric functions, noncommutative symmetric functions, and quasisymmetric functions. Monodromy and differential equations. Acta Appl. Math. **75**, 55–83 (2003)
11. Hazewinkel, M.: Explicit polynomial generators for the ring of quasisymmetric functions over the integers. Acta Appl. Math **109**, 39–44 (2010)
12. Malvenuto, C., Reutenauer, C.: Duality between quasi-symmetric functions and the solomon descent algebra. J. Algebra **177**, 967–982 (1995)
13. Milnor, J.: The Steenrod algebra and its dual. Ann. Math. **67**, 150–171 (1958)
14. Milnor, J., Moore, J.: On the structure of Hopf algebras. Ann. Math. **81**, 211–264 (1965)
15. Monks, K.G.: Change of basis, monomial relations, and the P_t^s bases for the Steenrod algebra. J. Pure Appl. Algebra **125**, 235–260 (1998)
16. Kaji, S.: Maple script for conjugation and product in the (dual) Leibniz–Hopf algebra and Steenrod algebra. http://www.skaji.org/code
17. Solomon, L.: A Mackey formula in the group ring of a Coxeter group. J. Algebra **41**, 255–268 (1976)
18. Steenrod, N.E., Epstein, D.B.A.: Cohomology Operations. Princeton Univ. Press, Princeton (1962)
19. Turgay, N.D.: An alternative approach to the Adem relations in the mod 2 Steenrod algebra. Turkish J. Math. **38**(5), 924–934 (2014)