The category of equilogical spaces and the effective topos as homotopical quotients

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Abstract We show that the two models of an extensional version of Martin-Löf type theory, those given by the category of equilogical spaces and by the effective topos, are homotopical quotients of appropriate categories of 2-groupoids.

1 Introduction

The category of T_0 -spaces embeds fully in the category of equilogical spaces; the category of equilogical spaces is locally cartesian closed and the embedding functor preserves products and any exponential available in the original category. Thus the category of equilogical spaces provides a nice extension of the category of T_0 -spaces. The effective topos is the categorical rendering of Kleene's realizability model for intuitionistic logic, and is the first interesting example of a non-Grothendieck topos. We show that the category of equilogical spaces is the homotopical quotient of a category of groupoids, and that the effective topos is the homotopical quotient of a category of 2-groupoids of partitioned assemblies.

Groupoids are a main tool in algebraic topology, see [4] and groupoids were the first nontrivial models of the intensional version of Martin-Löf Type Theory in [12].

Dedicated to Ronald Brown on the occasion of his 80th birthday.

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Moreover in recent years the Univalent Foundations Program, see [19], has advocated a strong connection between algebraic topology and type theory.

Since both the category of equilogical spaces and the effective topos are models of an extensional version of Martin-Löf type theory, it is useful to find that each comes from the “extensionalization” of a model of intensional type theory and that such a process is actually a homotopical quotient. We should stop here to point out that the meaning we adopt for an homotopical quotient of a category is in line with a suggestion in [8] and is the more naive notion obtained from an interval-like object than that derived from a Quillen model category—the main reason is that one of the two example categories we study is neither complete nor cocomplete. So, as a homotopical quotient, we shall consider a category obtained as a quotient category from a category C with finite limits, as follows:

- there is a fixed *interval-like* object I , i.e. it has two global points $0 : T \rightarrow I$ and $1 : T \rightarrow I$ whose pushout

$$\begin{array}{ccc} T & \xrightarrow{1} & I \\ 0 \downarrow & & \downarrow 0' \\ I & \xrightarrow{1'} & I +_T I \end{array}$$

exists in C and is stable under products, an arrow $\gamma : I \rightarrow I +_T I$ and an arrow $\iota : I \rightarrow I$ such that the four arrows together with the unique arrow $! : I \rightarrow T$ form an *equivalence co-span* in C , i.e. the following diagrams commute

$$\begin{array}{ccc} T & \xrightarrow{0} I & \xleftarrow{1} T \\ & \searrow 1 & \swarrow 0 \\ & I & \end{array} \quad \begin{array}{ccc} T & \xrightarrow{0} I & \xleftarrow{1} T \\ & \searrow 0 & \swarrow 1 \\ & I & \\ & \searrow 0' & \swarrow 1' \\ & I +_T I & \end{array}$$

—note that there is also a necessarily commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{0} I & \xleftarrow{1} T \\ & \searrow \text{id}_T & \swarrow \text{id}_T \\ & T & \end{array}$$

since T is terminal—;

- two arrows $f, g : X \longrightarrow Y$ are identified in the quotient if there is an arrow $h : X \times I \longrightarrow Y$ such that the following diagram commute

$$\begin{array}{ccc}
 X & & \\
 \langle \text{id}_X, 0 \rangle \downarrow & \searrow f & \\
 X \times I & \xrightarrow{h} & Y \\
 \langle \text{id}_X, 1 \rangle \uparrow & \nearrow g & \\
 X & &
 \end{array}$$

The condition of the structure on I ensures that the identification is an equivalence relation on parallel arrows in $\mathcal{C}$.

It seems plausible that the categories we analyse in the following sustain suitable notions of fibrations, cofibrations and weak equivalences—in particular, that a map of the kind $\langle \text{id}_X, i \rangle : X \longrightarrow X \times I$, $i = 0, 1$, is a weak equivalence. But the categories are certainly not complete, nor cocomplete, and that prevents a direct comparison with standard homotopical quotients. It will be considered in future work.

We introduce the category of equilogical spaces in Sect. 2 and we recall one of the presentations of the effective topos in Sect. 3, reviewing properties which are needed in the following sections. In Sect. 4 we determine a category $\mathcal{A}$ of topological groupoids and an interval-like topological groupoid $\mathbb{I}$ such that the homotopical quotient of $\mathcal{A}$ determined by $\mathbb{I}$ is equivalent to the category of equilogical spaces. In Sect. 5 we produce a similar result for the effective topos using a category of 2-groupoids on partitioned assemblies.

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2 Equilogical spaces

Recall from [1, 18] that an **equilogical space** $\mathcal{E} = (S_{\mathcal{E}}, \tau_{\mathcal{E}}, \sim_{\mathcal{E}})$ consists of a T_0 -space $(S_{\mathcal{E}}, \tau_{\mathcal{E}})$ and an equivalence relation $\sim_{\mathcal{E}} \subseteq S_{\mathcal{E}} \times S_{\mathcal{E}}$ on the points of the space.

A **map** $[f] : \mathcal{E} \longrightarrow \mathcal{F}$ of **equilogical spaces** is an equivalence class of continuous functions $f : (S_{\mathcal{E}}, \tau_{\mathcal{E}}) \longrightarrow (S_{\mathcal{F}}, \tau_{\mathcal{F}})$ preserving the equivalence relations, *i.e.* if $x \sim_{\mathcal{E}} x'$, then $f(x) \sim_{\mathcal{F}} f(x')$ for all x and x' in $S_{\mathcal{E}}$. For two such continuous functions $f, g : (S_{\mathcal{E}}, \tau_{\mathcal{E}}) \longrightarrow (S_{\mathcal{F}}, \tau_{\mathcal{F}})$, one sets f **equivalent to** g when $f(x) \sim_{\mathcal{F}} g(x)$ for all $x \in S_{\mathcal{E}}$.

Composition of maps of equilogical spaces $[f] : \mathcal{E} \longrightarrow \mathcal{F}$ and $[g] : \mathcal{F} \longrightarrow \mathcal{G}$ is given on (any of) their continuous representatives: $[g] \circ [f] = [g \circ f]$.

The data above determine a category $\mathcal{E}qu$ of equilogical spaces. There is a full embedding

$$Y : \mathcal{Top}_0 \xrightarrow[\text{full}]{\quad} \mathcal{E}qu$$

which maps a T_0 -space (S, τ) to the equilogical space on (S, τ) with the diagonal relation, i.e. the equilogical space $(S, \tau, =_S)$.

The category $\mathcal{E}qu$ is a locally cartesian closed full extension of the category $\mathcal{Top}_0$ of T_0 -spaces. In fact, it is the intersection of two other locally cartesian closed full extensions of

$$\begin{array}{ccccc} \mathcal{Top}_0 & \xleftarrow{\perp} & \mathcal{E}qu & \xleftarrow{\perp} & (\mathcal{Top}_0)_{\text{ex}} \\ \uparrow \lrcorner \downarrow & & \uparrow \lrcorner \downarrow & & \uparrow \lrcorner \downarrow \\ \mathcal{Top}_c & \xleftarrow{\perp} & \mathcal{Top}_{\text{reg}} & \xleftarrow{\perp} & \mathcal{Top}_{\text{ex}} \end{array}$$

The exact completions $(\mathcal{Top}_0)_{\text{ex}}$ and $\mathcal{Top}_{\text{ex}}$ are pretoposes, while the regular completion $\mathcal{Top}_{\text{reg}}$ is a quasitopos, see [15].

The product of equilogical spaces $\mathcal{E} \times \mathcal{F}$ is computed as expected taking the topological product $(S_{\mathcal{E}}, \tau_{\mathcal{E}}) \times (S_{\mathcal{F}}, \tau_{\mathcal{F}})$ and the equivalence relation

$$\langle a, b \rangle \sim_{\mathcal{E} \times \mathcal{F}} \langle a', b' \rangle \quad \text{when } a \sim_{\mathcal{E}} a' \text{ and } b \sim_{\mathcal{F}} b'.$$

The projections to the factors are obvious.

The construction of the exponential $\mathcal{F}^{\mathcal{E}}$ is less direct and we refer the reader to the basic sources [1, 17, 18] as well as [2, 3].

It is useful for the purpose of this paper to point out the strong similarity between the presentation of $\mathcal{E}qu$ and that of $(\mathcal{Top}_0)_{\text{ex}}$. So recall from [5, 6, 8, 9] that the exact completion C_{ex} of a category C with finite limits is a quotient category of the full

subcategory $\text{ES}(C)$ of the category $C \xrightarrow{\bullet \rightarrow \bullet}$ of graphs in C on the equivalence spans.

Recall that a (directed) graph in C is a parallel pair $A_1 \xrightleftharpoons[d_2]{d_1} A_0$ of arrows of C

and a homomorphism from the graph $A_1 \xrightleftharpoons[d_2]{d_1} A_0$ to the graph $B_1 \xrightleftharpoons[e_2]{e_1} B_0$ is a pair

$(f_1 : A_1 \rightarrow B_1, f_0 : A_0 \rightarrow B_0)$ of arrows in C such that the following diagram commutes

$$\begin{array}{ccccc} A_0 & \xleftarrow{d_1} & A_1 & \xrightarrow{d_2} & A_0 \\ f_0 \downarrow & & f_1 \downarrow & & \downarrow f_0 \\ B_0 & \xleftarrow{e_1} & B_1 & \xrightarrow{e_2} & B_0. \end{array}$$

An **equivalence span** is a graph $A_1 \begin{smallmatrix} \xrightarrow{d_1} \\ \xrightarrow{d_2} \end{smallmatrix} A_0$ in $\mathcal{C}$ which is reflexive, symmetric, and endowed with a compatible operation on pairs of consecutive arcs, i.e. there are arrows $r : A_0 \rightarrow A_1$, $s : A_1 \rightarrow A_1$, and $t : A_1 \times_{A_0} A_1 \rightarrow A_1$, where

$$\begin{array}{ccc} A_1 \times_{A_0} A_1 & \xrightarrow{d'_2} & A_1 \\ d'_1 \downarrow & & \downarrow d_1 \\ A_1 & \xrightarrow{d_2} & A_0 \end{array}$$

is a pullback in $\mathcal{C}$, such that the following diagrams commute:

$$\begin{array}{ccccc} & A_0 & & A_1 & \\ \text{id}_{A_0} \swarrow & \downarrow r & \searrow \text{id}_{A_0} & & \\ A_0 & \xleftarrow{d_1} A_1 & \xrightarrow{d_2} & A_0 & \\ & & & & \\ & A_1 \times_{A_0} A_1 & & A_1 & \\ d'_1 \swarrow & \downarrow t & \searrow d'_2 & & \\ A_1 & & A_1 & & \\ d_1 \swarrow & & \searrow d_2 & & \\ A_0 & \xleftarrow{d_1} A_1 & \xrightarrow{d_2} & A_0 & \end{array}$$

The quotient category $\mathcal{C}_{\text{ex}}$ is obtained by identifying homomorphisms (f_1, f_0) and (g_1, g_0) from $A_1 \begin{smallmatrix} \xrightarrow{d_1} \\ \xrightarrow{d_2} \end{smallmatrix} A_0$ to $B_1 \begin{smallmatrix} \xrightarrow{e_1} \\ \xrightarrow{e_2} \end{smallmatrix} B_0$ if there is an arrow $h : A_0 \rightarrow B_1$ such that

$$\begin{array}{ccccc} & A_0 & & & \\ f_0 \swarrow & \downarrow h & \searrow g_0 & & \\ B_0 & \xleftarrow{e_1} B_1 & \xrightarrow{e_2} & B_0 & \end{array}$$

—nothing is asked of the other component.

The following proposition makes the similarity explicit.

Proposition 1 *The category $\mathcal{E}q_{\text{u}}$ is equivalent to the full subcategory $\mathcal{A}$ of $(\text{Top}_0)_{\text{ex}}$ on those equivalence spans $A_1 \begin{smallmatrix} \xrightarrow{d_1} \\ \xrightarrow{d_2} \end{smallmatrix} A_0$ of topological spaces and continuous maps such that the pair $\langle d_1, d_2 \rangle : A_1 \rightarrow A_0 \times A_0$ is a subspace inclusion.*

Proof Consider an equivalence span $A = A_1 \underset{d_2}{\overset{d_1}{\rightrightarrows}} A_0$ of topological spaces and continuous maps such that the pair $\langle d_1, d_2 \rangle : A_1 \longrightarrow A_0 \times A_0$ is a subspace inclusion. Note that the functions r , s and t requested by the definition of equivalence span are unique, and determine that the subset $|A_1|$ of pairs of points of $|A_0|$ is an equivalence relation. Write $F(A)$ for the equilogical space which consists of the topological space A_0 and the equivalence relation $|A_1|$.

For a homomorphism (f_1, f_0) between two such equivalence spans, the component f_1 is uniquely determined by the other data as the restriction of the pair $\langle f_0, f_0 \rangle$, and ensures that f_0 is a representative of a map of equilogical spaces. Moreover, in the quotient category $(\text{Top}_0)_{\text{ex}}$, the homomorphism (f_1, f_0) is identified with (g_1, g_0) precisely when $\langle f(x), g(x) \rangle$ is in A_1 for all points x in A_0 .

Thus the assignment $F([f_1, f_0]) = [f_0]$ is well defined, and determines a functor from $\mathcal{A}$ to $\mathcal{Equ}$ which is full and faithful.

To see that F is also bijective on objects, suppose $\mathcal{E} = (S_{\mathcal{E}}, \tau_{\mathcal{E}}, \sim_{\mathcal{E}})$ is an equilogical space. Consider the subspace topology $\sigma_{\mathcal{E}}$ on $\sim_{\mathcal{E}} \subseteq S_{\mathcal{E}} \times S_{\mathcal{E}}$ and the graph of topological spaces

$$(\sim_{\mathcal{E}}, \sigma_{\mathcal{E}}) \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} (S_{\mathcal{E}}, \tau_{\mathcal{E}})$$

induced by the two projections. It is easy to check that it is an equivalence span and, by construction, the pair $\langle \pi_1, \pi_2 \rangle : (\sim_{\mathcal{E}}, \sigma_{\mathcal{E}}) \longrightarrow (S_{\mathcal{E}}, \tau_{\mathcal{E}}) \times (S_{\mathcal{E}}, \tau_{\mathcal{E}})$ is a subspace inclusion. It is obvious that the functor F takes that equivalence span of $\mathcal{A}$ to the equilogical space $\mathcal{E}$. $\square$

In the following, we shall refer to an equivalence span $A_1 \underset{d_2}{\overset{d_1}{\rightrightarrows}} A_0$ of topological spaces and continuous maps such that the pair $\langle d_1, d_2 \rangle : A_1 \longrightarrow A_0$ is a subspace inclusion as a **spatial** equivalence span.

3 The effective topos

The effective topos $\mathcal{Eff}$ was introduced in [10, 13]. It was shown in [16] that $\mathcal{Eff}$ is (equivalent to) the exact completion of the category $\mathcal{PAsm}$ of partitioned assemblies, see [7].

A **partitioned assembly** is a function $\xi : X \longrightarrow \mathbb{N}$; a **map** $\begin{array}{ccc} X & & Y \\ \downarrow \xi & \xrightarrow{f} & \downarrow \zeta \\ \mathbb{N} & & \mathbb{N} \end{array}$ **of**

partitioned assemblies is a function $f : X \longrightarrow Y$ such that there is a partial recursive function $\phi : \mathbb{N} \rightarrow \mathbb{N}$ such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \xi \downarrow & & \downarrow \zeta \\ \mathbb{N} & \xrightarrow{\phi} & \mathbb{N}. \end{array}$$

In order to make sure that the exact completion introduced in Sect. 2 can be applied to the category $\mathcal{P}A\mathcal{S}m$ we recall how finite limits can be obtained in that category.

The product of two partitioned assemblies is obtained by adopting some particular recursive encoding $\langle\langle n, m \rangle\rangle$ of pairs of numbers; the product partitioned assembly of

$$\begin{array}{cc} X & Y \\ \xi \downarrow & \zeta \downarrow \\ \mathbb{N} & \mathbb{N} \end{array} \text{ is the function}$$

$$(x, y) \mapsto \langle\langle \xi(x), \zeta(y) \rangle\rangle : X \times Y \longrightarrow \mathbb{N}$$

with obvious projections.

$$\text{The equalizer of } \begin{array}{ccc} X & & Y \\ \downarrow \xi & \xrightarrow[g]{f} & \downarrow \zeta \\ \mathbb{N} & & \mathbb{N} \end{array} \text{ is the partitioned assembly } \xi|_E : E \longrightarrow \mathbb{N} \text{ where}$$

$$E := \{x \in \mathbb{N} \mid f(x) = g(x)\} \text{ with the obvious inclusion into } \begin{array}{c} X \\ \xi \downarrow \\ \mathbb{N} \end{array}.$$

The next result will be useful in the following.

Lemma 1 *Every equivalence span*

$$\begin{array}{ccc} A_1 & & A_0 \\ \downarrow \alpha_1 & \xrightarrow[d_2]{d_1} & \alpha_0 \downarrow \\ \mathbb{N} & & \mathbb{N} \end{array}$$

in $\mathcal{P}A\mathcal{S}m_{\text{ex}}$ is isomorphic to one of the form

$$\begin{array}{ccc} E & & A_0 \\ \downarrow \epsilon & \xrightarrow[e_2]{e_1} & \alpha_0 \downarrow \\ \mathbb{N} & & \mathbb{N} \end{array}$$

such that the triple $\langle e_1, e_2, \epsilon \rangle$ is monic.

Proof Consider an arbitrary equivalence span

$$\begin{array}{ccc} A_1 & & A_0 \\ \downarrow \alpha_1 & \xrightarrow[d_2]{d_1} & \alpha_0 \downarrow \\ \mathbb{N} & & \mathbb{N} \end{array}$$

in $\mathcal{P}A\mathit{sm}_{\text{ex}}$. So there are two partial recursive functions ϕ_1 and ϕ_2 such that the following diagram commutes

$$\begin{array}{ccccc} A_0 & \xleftarrow{d_1} & A_1 & \xrightarrow{d_2} & A_0 \\ \alpha_0 \downarrow & & \alpha_1 \downarrow & & \downarrow \alpha_0 \\ \mathbb{N} & \xleftarrow{\phi_1} & \mathbb{N} & \xrightarrow{\phi_2} & \mathbb{N}. \end{array}$$

Take E to be the image of the function $\langle d_1, d_2, \alpha_1 \rangle : A_1 \rightarrow A_0 \times A_0 \times \mathbb{N}$, let $f : A_1 \rightarrow E$ be the factoring surjection, and let $\epsilon := \pi_3 \upharpoonright_E : E \rightarrow \mathbb{N}$. Let

$e_1, e_2 : \downarrow \epsilon \rightrightarrows \alpha_0 \downarrow$ be the first and second projection respectively. It is easy to see that it is an equivalence span.

Clearly f gives rise to a map of partitioned assemblies $\downarrow \alpha_1 \xrightarrow{f} \downarrow \epsilon$ since there is a commutative diagram

$$\begin{array}{ccc} A_1 & \xrightarrow{f} & E \\ \alpha_1 \downarrow & & \downarrow \epsilon \\ \mathbb{N} & \xrightarrow{\text{id}_{\mathbb{N}}} & \mathbb{N}. \end{array}$$

Moreover any section $s : E \rightarrow A_1$ of f (as a function of sets) is a map of partitioned assemblies $\downarrow \epsilon \xrightarrow{s} \downarrow \alpha_1$ and a section of $\downarrow \alpha_1 \xrightarrow{f} \downarrow \epsilon$ in $\mathcal{P}A\mathit{sm}$. Thus an appeal to the axiom of choice yields the conclusion. $\square$

Remark 1 Note that the axiom of choice was used in a crucial way in 1 to determine an equivalence span of the required form and the requested isomorphism, but the proof that

$$\downarrow \epsilon \begin{array}{c} \xrightarrow{e_1} \\ \xrightarrow{e_2} \end{array} \alpha_0 \downarrow$$

is an equivalence span does not require the use of the axiom of choice.

We conclude this brief review of the effective topos recalling a diagram of functors considered by Aurelio Carboni in [5] which shows how similar the situation is to that of topological spaces. Write $\mathcal{P}A\mathit{sm}_0$ for the full subcategory of $\mathcal{P}A\mathit{sm}$ on those partitioned assemblies which are 1–1 (functions). This is clearly equivalent to the category $\mathcal{PR}$

whose objects are subsets of $\mathbb{N}$ and whose arrows are restriction of partial recursive functions between those, total on the domain.

$$\begin{array}{ccccc}
 \mathcal{P}A\mathcal{S}m_0 & \hookrightarrow & \mathcal{P}E\mathcal{R}_C & \xleftarrow{\perp} & (\mathcal{P}A\mathcal{S}m_0)_{\text{ex}} \\
 \uparrow \downarrow \scriptstyle{-1} & & \uparrow \downarrow \scriptstyle{-1} & & \uparrow \downarrow \scriptstyle{-1} \\
 \mathcal{P}A\mathcal{S}m & \hookrightarrow & \mathcal{P}A\mathcal{S}m_{\text{reg}} & \xleftarrow{\perp} & \mathcal{P}A\mathcal{S}m_{\text{ex}}
 \end{array}$$

In the diagram of full subcategories of $\mathcal{E}ff$, the exact completion $\mathcal{P}A\mathcal{S}m_{\text{ex}}$ is itself the effective topos; $\mathcal{P}A\mathcal{S}m_{\text{reg}}$ is the full subcategory of $\mathcal{E}ff$ on the $\neg\neg$ -separated objects; $(\mathcal{P}A\mathcal{S}m_0)_{\text{ex}}$ is the full subcategory of $\mathcal{E}ff$ on the discrete objects—i.e. subquotients of the natural number object of $\mathcal{E}ff$, see [11]—; and $\mathcal{P}E\mathcal{R}$ is the intersection of the last two, the full subcategory of $\mathcal{E}ff$ on the $\neg\neg$ -separated subquotients of the natural number object of $\mathcal{E}ff$, also known as “partial equivalence relations on $\mathbb{N}$ ”, see [14]. As is shown in [5], this last is not the regular completion of $\mathcal{P}\mathcal{R} \equiv \mathcal{P}A\mathcal{S}m_0$. A similar remark applies to $\mathcal{E}qu$ and $(\mathcal{T}op_0)_{\text{reg}}$ which are not equivalent—this corrects a hastily mistaken, happily irrelevant statement in [3].

4 Groupoids

Consider a category $\mathcal{C}$ with pullbacks. A **groupoid** $\mathbb{G}$ in $\mathcal{C}$ is a graph $G_1 \xrightleftharpoons[d_2]{d_1} G_0$ of objects and arrows in $\mathcal{C}$ together with three more arrows

$$i : G_0 \longrightarrow G_1 \quad c : G_1 \times_{G_0} G_1 \longrightarrow G_1 \quad s : G_1 \longrightarrow G_1$$

where

$$\begin{array}{ccc}
 G_1 \times_{G_0} G_1 & \xrightarrow{d'_2} & G_1 \\
 d'_1 \downarrow & & \downarrow d_1 \\
 G_1 & \xrightarrow{d_2} & G_0
 \end{array}$$

is a pullback in $\mathcal{C}$, such that

- the graph $G_1 \xrightleftharpoons[d_2]{d_1} G_0$ with i and c is a category object in $\mathcal{C}$,
- s is an involution which makes every arrow an isomorphism.

The notions of **functor** of groupoids in $\mathcal{C}$ is obvious as well as that of **natural transformation**. It is straightforward to check that a functor between groupoids preserves the involution which makes every arrow an isomorphism.

We have already available a large number of examples as follows from the next property.

Proposition 2 Let $G_1 \rightrightarrows^{d_1}_{d_2} G_0$ be a graph in $\mathcal{C}$ with arrows $r : G_0 \rightarrow G_1$, $t : G_1 \times_{G_0} G_1 \rightarrow G_1$, and $s : G_1 \rightarrow G_1$ such that the diagrams

$$\begin{array}{ccc}
 & G_0 & \\
 \text{id}_{G_0} \swarrow & & \searrow \text{id}_{G_0} \\
 G_0 & \xleftarrow{d_1} G_1 \xrightarrow{d_2} & G_0
 \end{array}
 \quad
 \begin{array}{ccc}
 & G_1 & \\
 d_2 \swarrow & & \searrow d_1 \\
 G_0 & \xleftarrow{d_1} G_1 \xrightarrow{d_2} & G_0
 \end{array}$$

$$\begin{array}{ccc}
 & G_1 \times_{G_0} G_1 & \\
 d'_1 \swarrow & & \searrow d'_2 \\
 G_1 & & G_1 \\
 d_1 \swarrow & & \searrow d_2 \\
 G_0 & \xleftarrow{d_1} G_1 \xrightarrow{d_2} & G_0
 \end{array}$$

commute. If the pair $G_1 \rightrightarrows^{d_1}_{d_2} G_0$ is jointly monic, then

(i) the structure given by

$$\begin{array}{ccc}
 & & r \\
 & & \curvearrowright \\
 G_2 & \xrightarrow{t} & G_1 \rightrightarrows^{d_1}_{d_2} G_0 \\
 & & \curvearrowleft s
 \end{array}$$

is a groupoid $\mathbb{G}$ in $\mathcal{C}$,

(ii) for any groupoid $\mathbb{H}$ in $\mathcal{C}$, a graph-homomorphism from the underlying graph

$$H_1 \rightrightarrows^{e_1}_{e_2} H_0 \text{ of } \mathbb{H} \text{ to } G_1 \rightrightarrows^{d_1}_{d_2} G_0 \text{ is also a functor from } \mathbb{H} \text{ to } \mathbb{G},$$

(iii) for any groupoid $\mathbb{H}$ in $\mathcal{C}$, let (f_1, f_0) and (g_1, g_0) be functors from the groupoid $\mathbb{H}$ to the groupoid $\mathbb{G}$. Then an arrow $a : H_0 \rightarrow G_1$ such that

$$\begin{array}{ccc}
 & H_0 & \\
 f_0 \swarrow & & \searrow g_0 \\
 G_0 & \xleftarrow{d_1} G_1 \xrightarrow{d_2} & G_0
 \end{array}$$

is a natural transformation from (f_1, f_0) to (g_1, g_0) .

Proof Straightforward. □

Corollary 1 Every subspatial equivalence span is a groupoid in $\mathcal{Top}_0$. Every representative of an arrow in $\mathcal{A}$ is a functor between the groupoids.

Consider the interval-like groupoid

$$\mathbb{I}: = \begin{array}{ccc} & \curvearrowright & \\ & \bullet & \\ & \curvearrowleft & \end{array} \quad \begin{array}{ccc} & \curvearrowright & \\ & \bullet & \\ & \curvearrowleft & \end{array}$$

with the discrete topology. A natural transformation as in 2(iii) is the same as a functor $\mathbb{H} \times \mathbb{I} \longrightarrow \mathbb{G}$. Thanks to 1, we may rephrase Corollary 1 as follows.

Theorem 1 *The category $\mathcal{E}qu$ of equiological spaces is the homotopical quotient of the category $\mathcal{A}$ of topological groupoids.*

5 2-Groupoids

A similar case can be made for the effective topos. We prove in the following that it is the homotopical quotient of a category of higher groupoids in $\mathcal{P}\mathcal{A}sm$.

Consider a category $\mathcal{C}$ with pullbacks. A **2-groupoid** $\mathbb{G}$ in $\mathcal{C}$ is a 2-graph

$$\begin{array}{ccc} G_2 & \begin{array}{c} \xrightarrow{d_{21}} \\ \xrightarrow{d_{22}} \end{array} & G_1 \\ & \searrow \begin{array}{c} d_{11}d_{21}=d_{11}d_{22} \\ d_{12}d_{21}=d_{12}d_{22} \end{array} & \downarrow \begin{array}{c} d_{12} \\ d_{11} \end{array} \\ & & G_0 \end{array}$$

of objects and arrows in $\mathcal{C}$ together with arrows

$$\begin{array}{lll} i_1 : G_0 \longrightarrow G_1 & c_1 : G_1 \times_{G_0} G_1 \longrightarrow G_1 & s_1 : G_1 \longrightarrow G_1 \\ i_2 : G_1 \longrightarrow G_2 & c_2 : G_2 \times_{G_1} G_2 \longrightarrow G_2 & s_2 : G_2 \longrightarrow G_2 \\ & cc'_2 : G_2 \times_{G_0} G_2 \longrightarrow G_2 & q : G_1 \longrightarrow G_2 \end{array}$$

where

$$\begin{array}{ccc} G_1 \times_{G_0} G_1 \longrightarrow G_1 & G_2 \times_{G_1} G_2 \longrightarrow G_2 & G_2 \times_{G_0} G_2 \longrightarrow G_2 \\ \downarrow & \downarrow & \downarrow \\ G_1 \xrightarrow{d_{12}} G_0 & G_2 \xrightarrow{d_{22}} G_1 & G_2 \xrightarrow{d_{12}d_{22}} G_0 \end{array}$$

are pullbacks in $\mathcal{C}$, such that

- the 2-graph $G_2 \begin{array}{c} \xrightarrow{d_{21}} \\ \xrightarrow{d_{22}} \end{array} G_1 \begin{array}{c} \xrightarrow{d_{11}} \\ \xrightarrow{d_{12}} \end{array} G_0$ with i_1, c_1, i_2, c_2, c'_2 is a 2-category object in $\mathcal{C}$,
- s_1 is an involution which makes every 1-arrow an equivalence via the pair of arrows given by q ,
- s_2 is an involution which makes every 2-arrow an iso.

The notions of **2-functor** of 2-groupoids in $\mathcal{C}$ is obvious as well as that of **2-transformation**.

Consider the 2-category $\text{Grpd}(\mathcal{P}\mathcal{A}sm)$ of 2-groupoids in $\mathcal{P}\mathcal{A}sm$ with 2-functors and 2-transformations. Clearly, the underlying graph of a 2-groupoid G of $\mathcal{P}\mathcal{A}sm$ is an equivalence span in $\mathcal{P}\mathcal{A}sm$, thus an object of $\mathcal{E}ff$. This extends directly to a functor $U : \text{Grpd}(\mathcal{P}\mathcal{A}sm) \longrightarrow \mathcal{E}ff$.

Theorem 2 *The functor $U : \text{Grpd}(\mathcal{P}\mathcal{A}sm) \longrightarrow \mathcal{E}ff$ is essentially surjective.*

Proof Consider an object in $\mathcal{E}ff$, by 1 we can assume without loss of generality that it

is an equivalence span
$$\begin{array}{ccc} A_1 & & A_0 \\ \downarrow \alpha_1 & \xrightarrow[a_2]{a_1} & \downarrow \alpha_0 \\ \mathbb{N} & & \mathbb{N} \end{array}$$
 in $\mathcal{P}\mathcal{A}sm$ such that the triple $\langle a_1, a_2, \alpha \rangle$ is

monic. Take the free dagger category on that graph in $\mathcal{P}\mathcal{A}sm$ —by a **dagger category** we mean a category together with an involutive contravariant functor which is the identity

on objects. It consists of $A_0 \downarrow$ as objects of objects. The object of 1-arrows is $A^\wedge \downarrow$

where $A^\wedge$ consists of the zigzag paths in the graph $A_1 \xrightarrow[a_2]{a_1} A_0$. By a zigzag path in the graph we mean a list which is either of the form $\langle x \rangle$ where $x \in A_0$ or

$$\langle x_0, e_1, i_1, x_1, e_2, i_2, x_2, \dots, x_n, e_{n+1}, i_{n+1}, x_{n+1} \rangle,$$

where

- $x_\ell \in A_0$ for $0 \leq \ell \leq n+1$,
- $e_\ell \in A_1$ for $1 \leq \ell \leq n+1$,
- $i_\ell \in \{0, 1\}$ for $1 \leq \ell \leq n+1$,
- for $0 \leq \ell \leq n$, if $i_\ell = 0$, then $\langle x_\ell, x_{\ell+1}, e_{\ell+1} \rangle \in A_1$,
- for $0 \leq \ell \leq n$, if $i_\ell = 1$, then $\langle x_{\ell+1}, x_\ell, e_{\ell+1} \rangle \in A_1$.

Intuitively, if one considers a triple $\langle x, x', e \rangle \in A_1$ as an edge e from the source x to the target x' in the graph $A_1 \xrightarrow[a_2]{a_1} A_0$, then the zigzag

$$\langle x_0, e_1, i_1, x_1, e_2, i_2, x_2, \dots, x_n, e_{n+1}, i_{n+1}, x_{n+1} \rangle$$

is a mixed-directional path of edges from the vertex x_0 to the vertex x_{n+1} where each edge e_ℓ between x_ℓ and $x_{\ell+1}$ is marked with either 0 or 1: if the mark is 0, e_ℓ goes from x_ℓ to $x_{\ell+1}$ in the original graph; if the mark is 1, e_ℓ goes from $x_{\ell+1}$ to x_ℓ . The function $\alpha^\wedge$ is defined by mapping a zigzag to the encoding of the list of its numerical components:

$$\begin{aligned} \alpha^\wedge(\langle x \rangle) &:= \langle 0, \alpha_0(x) \rangle \\ \alpha^\wedge(\langle x_0, e_1, i_1, \dots, x_n, e_{n+1}, i_{n+1}, x_{n+1} \rangle) \\ &:= \langle n+1, \langle \alpha^\wedge(\langle x_0, e_1, i_1, \dots, x_n \rangle), \langle \langle e_{n+1}, i_{n+1} \rangle, \alpha_0(x_{n+1}) \rangle \rangle \rangle. \end{aligned}$$

The structure of dagger category in $\mathcal{P}\mathcal{A}sm$ is obvious, changing each i_ℓ with $\sigma(i_\ell)$ where $\sigma : \{0, 1\} \longrightarrow \{0, 1\}$ swaps 0 with 1. The object of 2-arrows $\alpha^- \downarrow$ is formed by taking the total relation on each 1-homset, where $A^\mathbb{N} := A^\wedge \times_{A_0} A^\wedge$. Explicitly, $A^\mathbb{N}$ consists of all pairs of zigzags

$$\langle \langle x_0, e, \dots, x_n \rangle, \langle x_0, e', \dots, x_n \rangle \rangle$$

between each two given vertices x and x' ; clearly all 2-diagrams commute as there is at most one 2-arrow from an 1-arrow to another. In this way, the dagger functor becomes the involution which makes every 1-arrow an equivalence. It is easy to see that that gives a 2-groupoid on the given span in $\mathcal{P}\mathcal{A}sm$ and that the functor U takes it

$$\text{to a span which is isomorphic to } \begin{array}{ccc} A_1 & & A_0 \\ \downarrow \alpha_1 & \xrightarrow[e_2]{e_1} & \alpha_0 \downarrow \\ \mathbb{N} & & \mathbb{N} \end{array}.$$

We shall refer to a 2-groupoid like that produced in the proof of 2 as a **numeric** 2-groupoid as all edges are denoted by numbers. More precisely, it is a 2-groupoid $\mathbb{G}$ in $\mathcal{P}\mathcal{A}sm$ such that its underlying category in $\mathcal{P}\mathcal{A}sm$

$$G_1 \xrightarrow[d_{12}]{d_{11}} G_0$$

is a free dagger category and $\mathbb{G}$ embeds, fully at level 2, into the 2-groupoid

$$G_0 \times G_0 \times \mathbb{N} \times \mathbb{N} \xrightarrow[\pi_{124}]{\pi_{123}} G_0 \times G_0 \times \mathbb{N} \xrightarrow[\pi_2]{\pi_1} G_0$$

where π_{123} and π_{124} are the projections deleting the fourth and third component, respectively.

Theorem 3 *The functor $U : \text{Grpd}(\mathcal{P}\mathcal{A}sm) \longrightarrow \mathcal{E}ff$ restricts to a homotopical quotient of the full subcategory $\mathcal{N}$ on the numeric 2-groupoids.*

Proof Suppose that $\mathbb{G}$ and $\mathbb{H}$ are numeric groupoids. Since $\mathbb{G}$ is a free dagger category and all 2-diagrams commute in $\mathbb{H}$, it is easy to see that every arrow $[f] : U(\mathbb{G}) \longrightarrow U(\mathbb{H})$ in $\mathcal{E}ff$ has a representative which is a 2-functor $F : \mathbb{G} \longrightarrow \mathbb{H}$.

To see that the functor $U : \text{Grpd}(\mathcal{P}\mathcal{A}sm) \longrightarrow \mathcal{E}ff$ restricted to $\mathcal{N}$ is indeed a homotopical quotient, consider the interval-like groupoid $\mathbb{I}$: it is the free dagger category on the graph in $\mathcal{P}\mathcal{A}sm$ on $T + T$ with two (disjoint) nodes and a single edge u connecting one with the other, with all possible 2-arrows. It is clearly a numeric 2-groupoid. Consider now two functors $F, F' : \mathbb{G} \longrightarrow \mathbb{H}$ such that $U(F) = U(F')$; in other words, there is a map $k : G_0 \longrightarrow H_1$ in $\mathcal{P}\mathcal{A}sm$ such that

$$F_0 = d_{11}^{\mathbb{H}} \circ k \quad \text{and} \quad F'_0 = d_{12}^{\mathbb{H}} \circ k.$$

Note that the 1-category underlying the 2-groupoid $\mathbb{G} \times \mathbb{I}$ is a retract of a free dagger category. Using k to act on the generating arrow of $\mathbb{I}$ as follows

$$\begin{array}{ccc} \langle x, 0 \rangle & \longmapsto & F_0(x) \\ \downarrow \langle \langle x \rangle, u \rangle & \longmapsto & k(x) \downarrow \\ \langle x, 1 \rangle & \longmapsto & F'_0(x) \end{array}$$

by freeness it is easy to obtain a functor $K : \mathbb{G} \times \mathbb{I} \longrightarrow \mathbb{H}$ which gives a homotopy from F to F' .

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