Fiber-homotopy self-equivalences and a classification of fibrations in rational homotopy

Toshihiro Yamaguchi¹ · Shoji Yokura²

Received: 6 April 2015 / Accepted: 28 September 2015 / Published online: 17 October 2016
© Tbilisi Centre for Mathematical Sciences 2016

Abstract For a fibration $\xi : X \xrightarrow{i} E \xrightarrow{p} B$ we consider the image of the rationalized homotopy group homomorphism $\pi_*(\text{res } \xi)_{\mathbb{Q}} : \pi_*(\text{aut}_1 p)_{\mathbb{Q}} \rightarrow \pi_*(\text{aut}_1 X)_{\mathbb{Q}}$ obtained from the fibre-restricting map $\text{res } \xi : \text{aut}_1 p \rightarrow \text{aut}_1 X$. Then we consider a finite type classification of fibrations ξ with fibre X and base B . In particular, we measure the size of it by a rational homotopical invariant “ $\text{depth}_B X$ ” when X are certain homogeneous spaces and B are spheres.

Keywords Fibre-homotopy self-equivalences · Rational homotopy · Sullivan minimal model · Pure space · Derivation · Totally non-cohomologous to zero (tncz) · Halperin conjecture · Depth

1 Introduction

In this paper, a fibration $\xi : X \xrightarrow{i} E \xrightarrow{p} B$ is a Hurewicz fibration of simply connected spaces. Gottlieb posed the following problem in [12, §5]: *Which homotopy equivalences of X into itself can be extended to fibre homotopy equivalences of E into itself?*

Dedicated to Ronnie Brown for his eightieth birthday

Communicated by Tim Porter and George Janelidze.

✉ Toshihiro Yamaguchi
tyamag@kochi-u.ac.jp

Shoji Yokura
yokura@sci.kagoshima-u.ac.jp

¹ Faculty of Education, Kochi University, 2-5-1, Kochi 780-8520, Japan

² Department of Mathematics and Computer Science, Graduate school of Science and Engineering, Kagoshima University, 1-21-35, Korimoto, Kagoshima 890-0065, Japan

Let $\text{aut } X$ be the space of unpointed homotopy self-equivalences of X and let $\text{aut } p$ be the space of unpointed fibre-homotopy self-equivalences of p , i.e. the subspace of $\text{aut } E$ consisting of $g : E \rightarrow E$ satisfying $p \circ g = p$. Then the above Gottlieb's problem is nothing but the problem of determining the image of the "fiber-restricting" map

$$\text{res}\xi : \text{aut } p \rightarrow \text{aut } X,$$

which is defined by restricting the fiberwise map $f : E \rightarrow E$ to the fiber X ; $f|_X : X \rightarrow X$. This problem is replaced as follows: *Which homotopy classes of homotopy equivalences of X into itself can be extended to the homotopy class of fibre homotopy equivalences of E into itself?* In other words, it is the problem of determining the image of the π_0 -version of the map above:

$$\pi_0(\text{res}\xi) : \pi_0(\text{aut } p) \rightarrow \pi_0(\text{aut } X).$$

Here we note that in our previous work [17], using this map $\pi_0(\text{res}\xi) : \pi_0(\text{aut } p) \rightarrow \pi_0(\text{aut } X)$ for the base space $B = K(\mathbb{Z}, 2)$, we consider a certain poset structure on X .

It is quite natural to think of a more general π_n -version of the homomorphism above, i.e. for $n > 0$

$$\pi_n(\text{res}\xi) : \pi_n(\text{aut}_1 p) \rightarrow \pi_n(\text{aut}_1 X),$$

where $\text{aut}_1 p$ is the path component of $\text{aut } p$ containing $\text{id}_E : E \xrightarrow{\text{id}} E$ and similarly for $\text{aut}_1 X$.

In this paper, first, we discuss the surjectivity of the following rationalized homomorphisms of the total homotopy groups:

$$\pi_*(\text{res}\xi)_{\mathbb{Q}} : \pi_*(\text{aut}_1 p)_{\mathbb{Q}} \rightarrow \pi_*(\text{aut}_1 X)_{\mathbb{Q}}.$$

Here $\pi_*(\text{aut}_1 p)_{\mathbb{Q}} = \bigoplus_{n \geq 0} \pi_n(\text{aut}_1 p) \otimes \mathbb{Q}$ and $\pi_*(\text{aut}_1 X)_{\mathbb{Q}} = \bigoplus_{n \geq 0} \pi_n(\text{aut}_1 X) \otimes \mathbb{Q}$. (We denote $(-) \otimes \mathbb{Q}$ by $(-)_{\mathbb{Q}}$ using the suffix). This (homotopical) surjectivity problem is motivated by the following definition introduced by H. Samelson in early 1940's (cf. [23]). For a fibration $X \xrightarrow{i} E \xrightarrow{p} B$ the fiber X is *totally non-cohomologous to zero in E with respect to G* , where G is a field or the integers $\mathbb{Z}$, if the induced homomorphism $i^* : H^*(E, G) \rightarrow H^*(X, G)$ is surjective. This (cohomological) surjectivity problem has been studied by many people (e.g. [3, 4, 18, 20, 28] around 1950's and recently [2, 21], etc.). If G is the rationals $\mathbb{Q}$, we just simply say *totally non-cohomologous to zero (abbr., tncz)*. This is equivalent to the Leray–Serre spectral sequence of the fibration degenerating on the E_2 -term, and also it is well-known that $H^*(X, \mathbb{Q})$ has no negative degree derivations if and only if X is tncz for any fibration over a simply connected base space (equivalently, it holds over an odd-sphere [21]). It is a conjecture due to Halperin ([9]) that if X is an F_0 -space, i.e. it has positive Euler characteristic and $H^*(X, \mathbb{Q})$ and $\pi_*(X)_{\mathbb{Q}}$ are both finite dimensional (if we require only the second condition, then X is called *elliptic*), then X is tncz for any fibration.

Due to Meier [22], it is equivalent to $H^*(B\text{aut}_1 X; \mathbb{Q})$ being a $\mathbb{Q}$ -polynomial algebra for the Dold–Lashof classifying space $B\text{aut}_1 X$ [6]. In 1987, Shiga and Tezuka [30] showed that the Halperin conjecture is true when X is a homogeneous space G/H of rank $G = \text{rank } H$. In §3, we consider a correspondence between this condition and the surjectivity of $\pi_*(\text{res}\xi)_{\mathbb{Q}}$, for a more general space called a *pure* space (see Definition 3.1) defined by Sullivan *minimal model* [9] (see §2).

Next, we propose a kind of coarse way to classify fibrations using (subgroups of) $\pi_*(\text{aut}_1 X)_{\mathbb{Q}}$. Any fibration $\xi : X \rightarrow E \rightarrow B$ is a pull-back of the universal fibration with fiber $X : X \rightarrow B\text{aut}_1^* X \rightarrow B\text{aut}_1 X$, where $\text{aut}_1^* X$ is the subspace (also subgroup) of $\text{aut}_1 X$, consisting of base point preserving self-equivalence maps. Therefore the standard classification of fibrations with fiber X and base B is given by the homotopy set $[B, B\text{aut}_1 X]$. However it is in general hard to analyze. Refer [7, 29] for the rational classifications of fibrations using deformation.

We observe that if a fibration ξ with fiber X and base B is fiber homotopic to the trivial fibration $\mathbb{I} : X \rightarrow B \times X \rightarrow B$, then $\text{Im}(\pi_*(\text{res}\xi)_{\mathbb{Q}}) = \text{Im}(\pi_*(\text{res}\mathbb{I})_{\mathbb{Q}}) = \pi_*(\text{aut}_1 X)_{\mathbb{Q}}$. Then we make

Definition Two fibrations ξ_1 and ξ_2 are equivalent if

$$\text{Im}(\pi_*(\text{res}\xi_1)_{\mathbb{Q}}) = \text{Im}(\pi_*(\text{res}\xi_2)_{\mathbb{Q}}).$$

This approach is motivated by the previous work [33] of the first named author. Using the base point b we have the evaluation map $\omega : \text{aut}_1 X \rightarrow X$ defined by $\omega(h) := h(b)$ for $h \in \text{aut}_1 X$, thus the above map $\pi_*(\text{res}\xi)_{\mathbb{Q}} : \pi_*(\text{aut}_1 p)_{\mathbb{Q}} \rightarrow \pi_*(\text{aut}_1 X)_{\mathbb{Q}}$ can be extended to the following sequence:

$$\pi_*(\omega \circ \text{res}\xi)_{\mathbb{Q}} : \pi_*(\text{aut}_1 p)_{\mathbb{Q}} \xrightarrow{\pi_*(\text{res}\xi)_{\mathbb{Q}}} \pi_*(\text{aut}_1 X)_{\mathbb{Q}} \xrightarrow{\pi_*(\omega)_{\mathbb{Q}}} \pi_*(X)_{\mathbb{Q}}.$$

Then we have $\text{Im}(\pi_*(\omega \circ \text{res}\xi)_{\mathbb{Q}}) \subset \text{Im}(\pi_*(\omega)_{\mathbb{Q}}) \subset \pi_*(X)_{\mathbb{Q}}$. Here $\text{Im}(\pi_n(\omega))$ is a well-known group called *the n -th Gottlieb group of X* , denoted by $G_n(X)$, introduced by Gottlieb [12, 13]. In this sense, $\text{Im}(\pi_n(\omega \circ \text{res}\xi))$ is named *the n -th fiber-restricted Gottlieb group of X with respect to the fibration ξ* and denoted by $G_n^{\xi}(X)$ in [33]. Two fibrations ξ_1 and ξ_2 are said to be *$\mathbb{Q}$ -Gottlieb-group-equivalent* provided that $G_*^{\xi_1}(X)_{\mathbb{Q}} = G_*^{\xi_2}(X)_{\mathbb{Q}}$. In [24, 33], a certain poset on X is introduced by using the inclusion of the fiber-restricted Gottlieb groups. This $\mathbb{Q}$ -Gottlieb-group-equivalence of fibrations is coarser because the Gottlieb group $G_n(X)_{\mathbb{Q}}$ is identified as a subgroup of $\pi_n(\text{aut}_1 X)_{\mathbb{Q}}$. See Example 5.1 that $G_*(X)_{\mathbb{Q}} = G_*^{\xi}(X)_{\mathbb{Q}}$ but $\pi_*(\text{aut}_1 X)_{\mathbb{Q}} \neq \text{Im}(\pi_*(\text{res}\xi)_{\mathbb{Q}})$.

In §4, we introduce a poset $\mathcal{P}_B(X)$ on X using the inclusions of the subgroups $\text{Im}(\pi_*(\text{res}\xi)_{\mathbb{Q}})$'s of $\pi_*(\text{aut}_1 X)_{\mathbb{Q}}$ where the total spaces are finite (see Definition 4.2), and consider $\text{depth}_B(X)$, which is the height of the Hasse diagram of $\mathcal{P}_B(X)$. Also $\text{depth}(X)$ is defined to be the maximum of $\text{depth}_B(X)$ where the base space B runs through all simply connected spaces.

From arguments given in §3 and §4 we deduce one of our results:

Theorem 1.1 *Let X be a homogeneous space G/H with $\text{rank}\pi_{\text{even}}(X) = 2$. Then the followings are equivalent.*

- (1) $H^*(\text{Baut}_1 X; \mathbb{Q})$ is a polynomial algebra.
- (2) $\pi_*(\text{res}\xi)_{\mathbb{Q}} : \pi_*(\text{aut}_1 p)_{\mathbb{Q}} \rightarrow \pi_*(\text{aut}_1 X)_{\mathbb{Q}}$ is surjective for any fibration $\xi : E \xrightarrow{p} S^n$ with fiber X .
- (3) $\text{depth}_{S^n}(X) = 0$ for any n .

Here we note that the space X satisfying (1)–(3) of Theorem 1.1 may not be an F_0 -space (i.e. $\text{rank} G = \text{rank} H$ in this case). See Example 3.5, in which there is a fibration ξ such that $i^* : H^*(E; \mathbb{Q}) \rightarrow H^*(X; \mathbb{Q})$ is not surjective (i.e. X is not tncz in E). Furthermore, in §5, we give some examples of $\text{depth}(X) > 0$. In particular, we see a finite type classification of fibrations in Example 5.5. In §2 we list some tools in order to obtain such results.

2 Preliminaries: Sullivan models and derivations

In this section we recall Sullivan models and related facts for later use.

We use the *Sullivan minimal model* $M(Y)$ of a simply connected space Y of finite type. It is a free $\mathbb{Q}$ -commutative differential graded algebra (abbr. CDGA) $(\Lambda V, d)$ with a $\mathbb{Q}$ -graded vector space $V = \bigoplus_{i \geq 1} V^i$ where $\dim V^i < \infty$ and a decomposable differential, i.e.

$$d(V^i) \subset (\Lambda^+ V \cdot \Lambda^+ V)^{i+1} \quad \text{and} \quad d \circ d = 0.$$

Here $\Lambda^+ V$ is the augmentation ideal of ΛV . We often denote $(\Lambda V, d)$ simply by ΛV . The degree of an element x of a graded algebra is denoted by $|x|$. Then we have

$$xy = (-1)^{|x||y|}yx \quad \text{and} \quad d(xy) = d(x)y + (-1)^{|x|}x d(y).$$

We note that $M(Y)$ determines the rational homotopy type $Y_{\mathbb{Q}}$ of Y . In particular, there are the following isomorphisms

$$\text{Hom}(V^i, \mathbb{Q}) \cong \pi_i(Y)_{\mathbb{Q}} \quad \text{and} \quad H^*(\Lambda V, d) \cong H^*(Y; \mathbb{Q})$$

See [9] for a general introduction and the standard notations.

Let $A = (A^*, d_A)$ be a CDGA with $A^* = \bigoplus_{i \geq 0} A^i$, $A^0 = \mathbb{Q}$, $A^1 = 0$ and the augmentation $\epsilon : A \rightarrow \mathbb{Q}$. Define $\text{Der}_i A$ to be the vector space of derivations of A decreasing the degree by $i > 0$, where

$$\theta(xy) = \theta(x)y + (-1)^{i|x|}x\theta(y) \quad \text{for } \theta \in \text{Der}_i A.$$

Here $\text{Der}_1 A$ is required to be δ -cycles. We denote $\bigoplus_{i > 0} \text{Der}_i A$ by $\text{Der} A$. The boundary operator $\delta : \text{Der}_* A \rightarrow \text{Der}_{*-1} A$ is defined by

$$\delta(\sigma) = d_A \circ \sigma - (-1)^{|\sigma|} \sigma \circ d_A.$$

$\text{Der} A$ is a differential graded Lie algebra (DGL), where the Lie bracket is given by

$$[\sigma, \tau] := \sigma \circ \tau - (-1)^{|\sigma||\tau|} \tau \circ \sigma.$$

It is known [31, §11] that

$$\pi_{*+1}(\text{Baut}_1 Y) \otimes \mathbb{Q} = \pi_*(\text{aut}_1 Y) \otimes \mathbb{Q} \cong H_*(\text{Der} M(Y), \delta), \quad (2.1)$$

which is the case when the base space B is a point in Theorem 2.4 below. In this paper, following [31, p. 314], the symbol $(v, h) \in \text{Der}_{|v|-|h|}(\Lambda V)$ means the derivation sending an element $v \in V$ to $h \in \Lambda V$ and the other to zero. In particular $(v, 1) = v^*$. The differential is given by $\delta(v, h) = d \circ (v, h) - (-1)^{|v|-|h|} (v, h) \circ d$. Recall the notation of Tanré [32, p. 25]: $C^*(L, \partial) := (\Lambda s^{-1} \sharp L, D = d_1 + d_2)$ with

$$\langle d_1 s^{-1} z; s x \rangle = -\langle z; \partial x \rangle \quad \text{and}$$

$$\langle d_2 s^{-1} z; s x_1, s x_2 \rangle = (-1)^{|x_1|} \langle z; [x_1, x_2] \rangle$$

for a differential graded Lie algebra (L, ∂) . Here L is a positive graded vector space of finite type and $\sharp L$ means the dual space of L with $|sx| = |x| + 1$ and $|s^{-1}z| = |z| + 1$. (The symbols s and s^{-1} mean “shifting degree” respectively for L and $\sharp L$.)

Let X be an elliptic space. Then

Theorem 2.1 ([31, p.314]) *The DGA $C^*(\text{Der} M(X), \delta)$ is the Sullivan (non-minimal in general) model of $\text{Baut}_1 X$.*

This Theorem implies the isomorphism 2.1 above. Therefore we have

Corollary 2.2 *$H_{\text{even}}(\text{Der} M(X)) = 0$ if and only if $H^*(\text{Baut}_1 X; \mathbb{Q})$ is a polynomial algebra.*

Even if X is a Lie group of low rank, $H^*(\text{Baut}_1 X; \mathbb{Q})$ is not free in general (see Example 2.6 below).

Theorem 2.3 ([8]) *For $M(Y) = (\Lambda V, d)$ and the augmentation $\epsilon : M(Y) \rightarrow \mathbb{Q}$,*

$$G_n(Y)_{\mathbb{Q}} \cong \text{Im} (H_n(\epsilon_*) : H_n(\text{Der} M(Y)) \rightarrow \text{Hom}_n(V, \mathbb{Q}))$$

for $n > 0$.

Félix, Lupton and Smith showed

Theorem 2.4 ([10, Theorem 1.1]) *For a fibration $p : E \rightarrow B$ with E and B simply connected and E finite,*

$$\pi_*(\text{aut}_1 p)_{\mathbb{Q}} \cong H_*(\text{Der}_{\Lambda V}(\Lambda V \otimes \Lambda W), \delta),$$

where $(\Lambda V, d_B) \rightarrow (\Lambda V \otimes \Lambda W, D)$ is the Koszul–Sullivan model (abbr. KS-model) of p and $(\text{Der}_{\Lambda V}(\Lambda V \otimes \Lambda W), \delta)$ is the chain complex of derivations vanishing on ΛV .

Then we get

Proposition 2.5 ([33]) *For a fibration $X \rightarrow E \xrightarrow{p} B$, the map $\pi_n(\text{res}\xi)_{\mathbb{Q}} : \pi_n(\text{aut}_1(p))_{\mathbb{Q}} \rightarrow \pi_n(\text{aut}_1 X)_{\mathbb{Q}}$ is given by the restriction of derivations*

$$R_n : H_n(\text{Der}_{\Delta V}(\Delta V \otimes \Delta W), \delta) \rightarrow H_n(\text{Der}(\Delta W)).$$

Example 2.6 When X is the special unitary group $SU(3) \simeq_{\mathbb{Q}} S^3 \times S^5$ and $B = S^3$, there is a non-trivial fibration $X \rightarrow E \rightarrow B$ with $Dw_2 = vw_1$ and $Dw_1 = 0$ for $H^*(X; \mathbb{Q}) = \Delta(w_1, w_2)$ with $|w_1| = 3$ and $|w_2| = 5$ and $H^*(B; \mathbb{Q}) = \Delta v$ with $|v| = 3$. Then $\text{Im} R_* = \text{Im} R_2 \oplus \text{Im} R_5 = \mathbb{Q}\{(w_2, w_1), (w_2, 1)\}$. Notice that the element $(w_1, 1)$ is not a δ -cycle in $\text{Der}_{\Delta V}(\Delta V \otimes \Delta W)$ since $\delta(w_1, 1) = -(w_2, v)$. Thus $R_* = \oplus R_n$ is not surjective since $(w_1, 1)$ is not in $\text{Im} R_3$. From Theorem 2.1 the cohomology of $B\text{aut}_1 X$ is infinitely generated as

$$H^*(B\text{aut}_1 X; \mathbb{Q}) \cong \mathbb{Q}[a] \otimes \Delta(b_0, b_1, b_2, \dots) / (\{ab_i\}_i, \{b_i b_j\}_{i < j})$$

with $|a| = 4$ and $|b_i| = 3 + 6i$, which is not free.

3 Pure case

Definition 3.1 ([7]) An elliptic space X is said to be *pure* if $dW^{\text{even}} = 0$ and $dW^{\text{odd}} \subset \Delta W^{\text{even}}$ for $M(X) = (\Delta W, d)$.

For example, a simply connected homogeneous space G/H is a pure space. Of course, this class contains Lie groups. Also F_0 -spaces are pure. The following lemma is an explicit description to apply Proposition 2.5.

Lemma 3.2 *For a KS-model $M(B) = (\Delta V, d_B) \rightarrow (\Delta V \otimes \Delta W, D)$, the restriction $R_n : H_n(\text{Der}_{\Delta V}(\Delta V \otimes \Delta W), \delta) \rightarrow H_n(\text{Der}(\Delta W))$ is surjective if and only if there is an element ρ of $\text{Der}_n(\Delta W, \Delta^+ V \otimes \Delta W)$ such that $\theta + \rho$ is a non-exact δ -cycle for any non-exact δ -cycle θ of $\text{Der}_n(\Delta W)$.*

Proposition 3.3 *Let X be a pure space with $\text{rank} \pi_{\text{even}}(X) = 2$. Then the followings are equivalent:*

- (i) $H_{\text{even}}(\text{Der} M(X)) = 0$.
- (ii) $R_* : H_*(\text{Der}_{\Delta V}(\Delta V \otimes \Delta W)) \rightarrow H_*(\text{Der}(\Delta W)) = H_*(\text{Der} M(X))$ is surjective for any KS-model $M(S^n) = (\Delta V, d_B) \rightarrow (\Delta V \otimes \Delta W, D)$.

Proof Let $M(X) = (\Delta W, d) = (\Delta(x_1, x_2, y_1, \dots, y_m), d)$ with $|x_i|$ even and $|y_i|$ odd.

First, we show $(ii) \Rightarrow (i)$. Suppose $H_{\text{even}}(\text{Der} M(X)) \neq 0$. Then, since X is pure, there is an index i such that

$$[(y_i, f) + \dots] \in H_*(\text{Der} M(X))$$

for some odd-degree element $f \in \mathbb{Q}[x_1, x_2] \otimes \Lambda^1(y_1, \dots, y_m)$. Then we can construct a KS-model $(\Lambda v, 0) \rightarrow (\Lambda v \otimes \Lambda W, D)$ where $|v| = |y_i| - |f| + 1$ is odd and

$$Dy_i = dy_i + fv$$

(cf. [29]). Note that fv is not cohomologous to an element of ΛW . (It is the KS-model of a rationally non-trivial fibration over an odd-sphere.) When f contains y_j ($j < i$), it follows from Lemma 3.2 that the element $[(y_j, 1)]$ of $H_*(\text{Der}M(X))$ is not in the image of R_* . Because we have

$$\delta(y_j, 1) = \pm(y_i, \partial f / \partial y_j \cdot v) + \dots$$

in $\text{Der}_{\Lambda v}(\Lambda v \otimes \Lambda W)$, in which $\partial f / \partial y_j \cdot v$ is not D -exact. Then there is no derivation ρ of $\text{Der}(\Lambda W, \Lambda^+ v \otimes \Lambda W)$ such that $(y_j, 1) + \rho$ is a δ -cycle.

Next, we show (i) $\Rightarrow$ (ii). Since X is pure, if $H_{\text{even}}(\text{Der}M(X)) = 0$, then we get

$$H_*(\text{Der}M(X)) = H_*(\text{Der}(\Lambda W)) = \bigoplus_{i=1}^n \mathbb{Q}[(y_i, f_{ij}); f_{ij} \in \mathbb{Q}[x_1, x_2]].$$

Let $V = \mathbb{Q}\{v\}$ with $d_B v = 0$ when n is odd and $\mathbb{Q}\{v, z\}$ with $d_B z = v^2$ and $d_B v = 0$ when n is even. Then for any KS-model $(\Lambda V, d_B) \rightarrow (\Lambda V \otimes \Lambda W, D)$, we can assume that the differential D satisfies

$$Dx_i = 0 \text{ and } Dy_i = dy_i + \sum_j a_{ij} f_{ij} v \quad (|v| = n : \text{even})$$

$$Dx_i = 0 \text{ and } Dy_i = dy_i \quad (|v| = n : \text{odd})$$

for some $a_{ij} \in \mathbb{Q}$ since $\dim W^{\text{even}} = 2$. Thus it follows from Lemma 3.2 that R_* is surjective, because each element (y_i, f_{ij}) of $\text{Der}(\Lambda W)$ is just a non-exact δ -cycle in $\text{Der}_{\Lambda V}(\Lambda V \otimes \Lambda W)$. $\square$

Theorem 1.1 (1) $\Leftrightarrow$ (2) follows from the geometrical realization of Proposition 3.3 as

Theorem 3.4 *For a pure space X with $\text{rank} \pi_{\text{even}}(X) = 2$, $H^*(\text{Baut}_1 X; \mathbb{Q})$ is a polynomial algebra if and only if $\pi_*(\text{res} \xi)_{\mathbb{Q}} : \pi_*(\text{aut}_1 p)_{\mathbb{Q}} \rightarrow \pi_*(\text{aut}_1 X)_{\mathbb{Q}}$ is surjective for any fibration $\xi : E \xrightarrow{p} S^n$ with fiber X .*

Proof It follows from Corollary 2.2 that Proposition 3.3 (i) is equivalent to $H^*(\text{Baut}_1 X; \mathbb{Q})$ being a polynomial algebra, and it follows from Proposition 2.5 that Proposition 3.3 (ii) is equivalent to $\pi_*(\text{res} \xi)_{\mathbb{Q}} : \pi_*(\text{aut}_1 p)_{\mathbb{Q}} \rightarrow \pi_*(\text{aut}_1 X)_{\mathbb{Q}}$ being surjective. $\square$

When X is not an F_0 -space, even if $H^*(\text{Baut}_1 X; \mathbb{Q})$ is a polynomial algebra, there are non-tncz fibrations ξ . For example:

Example 3.5 Let $X = SU(2) \times SU(2) \times SU(2)/T^2$ ($T^2 = S^1 \times S^1$), which is a homogeneous space. Due to [26], there are two types of two-torus subgroups $T^2 \subset SU(2) \times SU(2) \times SU(2)$ such that

$$M(X) = (\Lambda(x_1, x_2, y_1, y_2, y_3), d) \text{ with } |x_1| = |x_2| = 2, |y_1| = |y_2| = |y_3| = 3$$

and

- (1) $dx_1 = dx_2 = 0, dy_1 = x_1^2, dy_2 = x_2^2, dy_3 = 0$ (formal [9]) and
 (2) $dx_1 = dx_2 = 0, dy_1 = x_1^2, dy_2 = x_2^2, dy_3 = x_1x_2$ (non-formal).

In the case (1) there is a fibration $\xi : X \rightarrow E_1 \rightarrow S^4$ such that $E_1 \simeq_{\mathbb{Q}} S^2 \times S^2 \times S^7$ whose KS-model is given by $Dy_1 = dy_1, Dy_2 = dy_2$ and $Dy_3 = v$ for $H^*(S^4; \mathbb{Q}) = \mathbb{Q}[v]/(v^2)$.

In the case (2) there is a fibration $\xi : X \rightarrow E_2 \rightarrow S^4$ with KS-model given by $Dy_3 = x_1x_2 + v$. Then we have $H^*(E_2; \mathbb{Q}) = \mathbb{Q}[x_1, x_2]/(x_1^2, x_2^2)$.

In both cases, $i^* : H^*(E_j; \mathbb{Q}) \rightarrow H^*(X; \mathbb{Q})$ are not surjective, since in the case (1) the element $[y_3]$ of $H^3(X; \mathbb{Q})$ is not contained in $\text{Im} i^*$ and in the case (2) the element $[x_1y_3 - x_2y_1]$ of $H^5(X; \mathbb{Q})$ is not contained in $\text{Im} i^*$.

On the other hand, in both cases, we have that

$$H^*(\text{Baut}_1 X; \mathbb{Q}) = \mathbb{Q}[u_1, u_2, u_3, u_4, u_5, u_6, u_7]$$

with $|u_1| = |u_2| = |u_3| = |u_4| = 2$ and $|u_5| = |u_6| = |u_7| = 4$, since we have that

$$\pi_*(\text{aut}_1 X)_{\mathbb{Q}} = \mathbb{Q}\{(y_1, x_2), (y_2, x_k), (y_3, x_1), (y_3, x_2), (y_1, 1), (y_2, 1), (y_3, 1)\},$$

in which $k = 1$ in the case (1) and $k = 2$ in the case (2). Thus it follows from Theorem 3.4 (Theorem 1.1) that $\pi_*(\text{res}(\xi))_{\mathbb{Q}}$ is surjective.

4 Poset and depth

Definition 4.1 Let ξ and η be two fibrations with fiber X and base B . They are called $\mathbb{Q}$ -fiber restriction equivalent (abbr. $\mathbb{Q}$ -f.r.equivalent) if the images of the homomorphisms $\pi_*(\text{res}\xi)_{\mathbb{Q}}$ and $\pi_*(\text{res}\eta)_{\mathbb{Q}}$ are identical:

$$\text{Im}(\pi_*(\text{res}\xi)_{\mathbb{Q}}) = \text{Im}(\pi_*(\text{res}\eta)_{\mathbb{Q}}) \subset \pi_*(\text{aut}_1 X)_{\mathbb{Q}}.$$

Definition 4.2 For simply connected spaces X and B , we define the poset $\mathcal{P}_B(X)$ to be the set of $\mathbb{Q}$ -f.r.equivalence classes, denoted by double bracket $[[\xi]]$, of fibrations ξ 's with fiber X and base B :

$$\mathcal{P}_B(X) := \left(\{ [[\xi]] \mid \xi : X \rightarrow E \rightarrow B \text{ where } E \text{ is finite} \}, \leq \right)$$

in which

$$[[\xi]] \leq [[\eta]] \iff \text{Im}(\pi_n(\text{res}\xi))_{\mathbb{Q}} \subset \text{Im}(\pi_n(\text{res}\eta))_{\mathbb{Q}}, \forall n > 0.$$

Then we define the depth of X over B as

$$\text{depth}_B(X) := \max\{n \mid [[\xi_0]] > [[\xi_1]] > \cdots > [[\xi_n]]\}.$$

In particular, we denote $\text{depth}_B(X) = -1$ when $\mathcal{P}_B(X) = \phi$. (For example, $\mathcal{P}_{K(\mathbb{Z}, 2)}(S^{2n}) = \phi$, since E cannot be finite in any fibration $S^{2n} \rightarrow E \rightarrow K(\mathbb{Z}, 2)$.) Furthermore,

$$\text{depth}(X) := \sup \{\text{depth}_B(X) \mid B \text{ is simply connected}\} (\geq 0).$$

Lemma 4.3 *It holds that $\text{depth}_B(X \times Y) \geq \max\{\text{depth}_B(X), \text{depth}_B(Y)\}$. In particular, $\text{depth}_B(X \times Y) \geq \text{depth}_B(X) + \text{depth}_B(Y)$ when $B = S^{2n+1}$.*

Proof The former is obvious since $M(X \times Y) = M(X) \otimes M(Y)$. The latter holds since the KS-model of any fibration over S^{2n+1} satisfies that $DW \subset \Lambda v \otimes \Lambda^+ W$ for $M(S^{2n+1}) = (\Lambda v, 0)$. Then we can construct the KS-model $(\Lambda v, 0) \rightarrow (\Lambda v \otimes \Lambda W \otimes \Lambda U, D'')$ with $D''|_W = D$ and $D''|_U = D'$ for any KS-models $(\Lambda v, 0) \rightarrow (\Lambda v \otimes \Lambda W, D)$ and $(\Lambda v, 0) \rightarrow (\Lambda v \otimes \Lambda U, D')$. $\square$

Let $[[\mathbb{I}]]$ denote the $\mathbb{Q}$ -f.r.equivalence class of the trivial fibration $\mathbb{I}$ with fiber X and finite base space B , thus $\text{Im}(\pi_*(\text{res } \mathbb{I}))_{\mathbb{Q}} = \pi_*(\text{aut}_1 X)_{\mathbb{Q}}$. Hence for any fibration ξ we have that $[[\xi]] \leq [[\mathbb{I}]]$, i.e., $[[\mathbb{I}]]$ is the maximal point of the poset $\mathcal{P}_B(X)$. Hence the following lemma is clear.

Lemma 4.4 $\pi_*(\text{res } \xi)_{\mathbb{Q}}$ is surjective for any fibration $\xi : X \rightarrow E \rightarrow B$ over a finite base B if and only if $\text{depth}_B(X) = 0$.

Hence we deduce the equivalence (2) $\Leftrightarrow$ (3) of Theorem 1.1.

Even if $\text{rank } G > \text{rank } H$, it may hold that $\text{depth}(G/H) = 0$. For example, see Example 3.5, in which $\text{rank } G = 3 > 2 = \text{rank } H$. Furthermore we have obviously

Lemma 4.5 *When $X \simeq_{\mathbb{Q}} S^{n_1} \times \cdots \times S^{n_m}$ with all n_i odd, $\text{depth}(X) = 0$ if and only if $n_1 = \cdots = n_m$. In particular, $\text{depth}(S^n) = 0$ for any $n > 1$.*

See §5 for the examples such that $n_1, \dots, n_m$ are not equal.

Remark 4.6 Let $\mathcal{F}_B(X) := \{\text{fibration } \xi \text{ over } B \text{ with fibre } X\} / \simeq_{fhe}$, where $\simeq_{fhe}$ denotes fibre-homotopy equivalence [5]. When B is finite, there is the following surjection

$$\Phi : \mathcal{F}_B(X) \rightarrow \mathcal{P}_B(X),$$

given by $\Phi([\xi]) = [[\xi]]$. Of course, it is not bijective in general. For example, $\Phi^{-1}([[\xi]])$ can be infinite (see Example 4.7 below).

Example 4.7 There are fibrations $S^3 \rightarrow E_a \rightarrow B$ with infinitely many different rational homotopy type E_a over B with

$$M(B) = (\Lambda(v_1, v_2, v), d) \quad \text{with} \quad |v_2| = |v_2| = 2, |v| = 3, dv_1 = dv_2 = 0, dv = v_1 v_2.$$

When a fibration ξ_a for $a \in \mathbb{Q}^* = \mathbb{Q} - \{0\}$ is given by the KS-model

$$(\Lambda(v_1, v_2, v), d) \rightarrow (\Lambda(v_1, v_2, v, w), D_a) \quad \text{with} \quad |w| = 3, \\ D_a v = v_1 v_2, D_a w = v_1^2 + a v_2^2,$$

E_a 's are all rational-homotopically different for each $a \in \mathbb{Q}^*/(\mathbb{Q}^*)^2$, although ξ_a 's are all $\mathbb{Q}$ -f.r.equivalent to each other, since $\pi_*(\text{aut}_1 S^3)_{\mathbb{Q}} = \text{Im} \pi_*(\text{res} \xi_a)_{\mathbb{Q}} \cong \mathbb{Q}$ for all a .

Remark 4.8 Recall that the depth of the rational Gottlieb poset of X over B in [33]. Denote it as $\text{depth}_B^G(X)$ in this paper. From the proof of Proposition 3.3 with Theorem 2.3, we can deduce that $\text{depth}_{S^n}(X) = 0$ for any n if and only if $\text{depth}_{S^n}^G(X) = 0$ for any n when X is pure with $\text{rank} \pi_{\text{even}}(X) = 2$.

5 Examples

In this section, all KS-models are realized as fibrations in which the total spaces are finite.

As we recalled the definition of the fibre restricted Gottlieb group $G_*^{\xi}(X)$ for a fibration ξ ([33]) in the introduction, even if $G_*(X)_{\mathbb{Q}} = G_*^{\xi}(X)_{\mathbb{Q}}$, it of course may happen that $\pi_*(\text{aut}_1 X)_{\mathbb{Q}} \neq \text{Im} \pi_*(\text{res} \xi)_{\mathbb{Q}}$, as shown in the following example:

Example 5.1 Let $M(X) = (\Lambda(w_1, w_2, w_3, w_4, w), d)$ with

$$dw_i = 0, dw = w_1 w_2 w_3 w_4, |w_1| = |w_2| = 3, |w_3| = 5, |w_4| = 7 \quad \text{and} \quad |w| = 17.$$

Then, if $Dw_3 = v w_1$ (and $D = d$ for the others) for a KS-model

$$(\Lambda v, 0) \rightarrow (\Lambda v \otimes \Lambda(w_1, w_2, w_3, w_4, w), D) \quad \text{with} \quad |v| = 3,$$

it follows from Lemma 3.2 that the element (w_4, w_3) of $H_2(\text{Der} M(X))$ is not contained in $\text{Im} R_2$. Thus from Proposition 2.5 we get the inequality $\pi_*(\text{aut}_1 X)_{\mathbb{Q}} \neq \text{Im} \pi_*(\text{res} \xi)_{\mathbb{Q}}$. On the other hand it follows from Theorem 2.3 that $G_*(X)_{\mathbb{Q}} = G_*^{\xi}(X)_{\mathbb{Q}} \cong \mathbb{Q}\{(w, 1)\}$.

Example 5.2 Let us recall that the exceptional simple Lie group E_6 has the following rational homotopy type:

$$E_6 \simeq_{\mathbb{Q}} S^3 \times S^9 \times S^{11} \times S^{15} \times S^{17} \times S^{23}.$$

Then the three depths $\text{depth}_{S^3}(E_6)$, $\text{depth}_{S^4}(E_6)$ and $\text{depth}_{S^5}(E_6)$ are all different as follows:

- (1) $\text{depth}_{S^3}(E_6) = 2$
- (2) $\text{depth}_{S^4}(E_6) = 3$
- (3) $\text{depth}_{S^5}(E_6) = 1$.

Indeed, let $M(X) = (\Lambda(w_1, w_2, w_3, w_4, w_5, w_6), 0)$ with

$$|w_1| = 3, |w_2| = 9, |w_3| = 11, |w_4| = 15, |w_5| = 17, |w_6| = 23$$

and $H^*(S^n; \mathbb{Q}) = \Lambda(v)/(v^2)$. Then (1), (2) and (3) are given by the following chains of KS-models with the maximal length:

- (1) $[[\mathbb{I}]] > [[\xi_1]] > [[\xi_2]]$ with
 $\xi_1 : Dw_1 = Dw_2 = Dw_3 = Dw_4 = 0, Dw_5 = vw_4, Dw_6 = 0$
 $\xi_2 : Dw_1 = Dw_2 = 0, Dw_3 = vw_2, Dw_4 = 0, Dw_5 = vw_4, Dw_6 = 0,$
- (2) $[[\mathbb{I}]] > [[\xi_1]] > [[\xi_2]] > [[\xi_3]]$ with
 $\xi_1 : Dw_1 = Dw_2 = Dw_3 = 0, Dw_4 = vw_1w_2, Dw_5 = Dw_6 = 0$
 $\xi_2 : Dw_1 = Dw_2 = Dw_3 = 0, Dw_4 = vw_1w_2, Dw_5 = vw_1w_3, Dw_6 = 0$
 $\xi_3 : Dw_1 = Dw_2 = Dw_3 = 0, Dw_4 = vw_1w_2, Dw_5 = vw_1w_3, Dw_6 = vw_1w_5$ and
- (3) $[[\mathbb{I}]] > [[\xi_1]]$ with
 $\xi_1 : Dw_1 = Dw_2 = Dw_3 = 0, Dw_4 = vw_3, Dw_5 = Dw_6 = 0.$

Example 5.3 As to the special unitary group $SU(n)$, for $n > 1$, we have

$$\text{depth}_{S^3}(SU(n)) \geq n - 1.$$

Indeed, let $M(SU(n)) = (\Lambda(w_1, w_2, \dots, w_{n-1}), 0)$ with

$$|w_1| = 3, |w_2| = 5, \dots, |w_{n-1}| = 2n - 1$$

and $M(S^3) = (\Lambda(v), 0)$. When ξ_k ($k > 1$) is given by the KS-model

$$(\Lambda(v), 0) \rightarrow (\Lambda(v, w_1, w_2, \dots, w_{n-1}), D_k)$$

with the differential D_k defined by:

$$D_k w_1 = 0, D_k w_2 = vw_1, \dots, D_k w_k = vw_{k-1}, D_k w_{k+1} = \dots = D_k w_{n-1} = 0,$$

then we have that $\text{Im} \pi_i(\text{res}(\xi_k))_{\mathbb{Q}} = \text{Der}_i(\Lambda(w_k, \dots, w_{n-1}))$ for all i . Thus we have

$$[[\xi_{n-1}]] < [[\xi_{n-2}]] < \dots < [[\xi_2]] < [[\mathbb{I}]].$$

Example 5.4 (1) Let X be a homogeneous space with $X = SU(6)/SU(3) \times SU(3)$. Then we have $\text{depth}(X) \neq 0$. Especially

$$\text{depth}_{S^3}(X) \geq 1.$$

Indeed, let $M(X) = (\Lambda(w_1, w_2, w_3, w_4, w_5), d)$ with $|w_1| = 4, |w_2| = 6, |w_3| = 7, |w_4| = 9, |w_5| = 11, dw_1 = dw_2 = 0, dw_3 = w_1^2, dw_4 = w_1w_2$ and $dw_5 = w_2^2$ (see [15]). Then, it follows from the isomorphism (2.1) in §2 that

$$\pi_*(\text{aut}_1 X)_{\mathbb{Q}} \cong H_*(\text{Der} M(X))$$

$$\cong \mathbb{Q}\{(w_5, w_1), (w_5, w_2), (w_3, w_1), (w_3, w_2), (w_3, 1), (w_4, 1), (w_5, 1), \\ 2(w_5, w_4) + (w_4, w_3) + (w_2, w_1)\}.$$

Let ξ be a fibration given by the KS-model with differential D defined by:

$$Dw_1 = 0, Dw_2 = vw_1, Dw_3 = w_1^2, Dw_4 = w_1w_2 - vw_3, Dw_5 = w_2^2 - 2vw_4.$$

Then it follows from Proposition 2.5 that

$$\mathrm{Im}\pi_*(\mathrm{res}(\xi))_{\mathbb{Q}} \cong \mathbb{Q}\{(w_5, w_1), (w_5, 1), 2(w_5, w_4) + (w_4, w_3) + (w_2, w_1)\}.$$

We recall that

$$M(\mathrm{Baut}_1 X) = (\Lambda(v_2, v_3, v_4, v_6, v_8, v'_{10}, v_{12}), d)$$

with $|v_n| = |v'_n| = n$, $dv_2 = dv_3 = dv_8 = 0$, $dv_4 = v_2v_3$, $dv_6 = v_4v_3$, $dv'_8 = v_6v_3$, $dv_{10} = v_8v_3$ and $dv_{12} = v_{10}v_3$ [25]. In particular, $H^*(\mathrm{Baut}_1 X; \mathbb{Q})$ is not free.

- (2) Let X be a homogeneous space with $X = SU(7)/SU(3) \times SU(4)$. From [1, Proposition 2.8], $M(X) = (\Lambda(w_1, w_2, w_3, w_4, w_5), d)$ with $|w_1| = 4$, $|w_2| = 6$, $|w_3| = 9$, $|w_4| = 11$, $|w_5| = 13$ and $dw_1 = dw_2 = 0$, $dw_3 = w_1w_2$, $dw_4 = w_1^3 - w_2^2$, $dw_5 = 0$ (formal). Then $H^*(\mathrm{Baut}_1 X; \mathbb{Q})$ is a polynomial algebra [25]. Therefore $\mathrm{depth}_{S^n}(X) = 0$ for any n from Theorem 1.1.
- (3) Let X be a homogeneous space with $X = SU(8)/SU(3) \times SU(5)$. From [1, Appendix A], we see that $M(X) = (\Lambda(w_1, w_2, w_3, w_4, w_5), d)$ with $|w_1| = 4$, $|w_2| = 6$, $|w_3| = 11$, $|w_4| = 13$, $|w_5| = 15$ and $dw_1 = dw_2 = 0$, $dw_3 = w_1^3 - w_2^2$, $dw_4 = w_1^2w_2$, $dw_5 = w_1w_2^2$ (non-formal). Then $H^*(\mathrm{Baut}_1 X; \mathbb{Q})$ is a polynomial algebra [25]. Therefore $\mathrm{depth}_{S^n}(X) = 0$ for any n from Theorem 1.1.

Example 5.5 (1) Let $X_1 = SU(6) \simeq_{\mathbb{Q}} S^3 \times S^5 \times S^7 \times S^9 \times S^{11}$, $B = K(\mathbb{Z}, 2)$ and let $M(X_1) = (\Lambda W, 0) = (\Lambda(w_1, w_2, w_3, w_4, w_5), 0)$ with

$$|w_1| = 3, |w_2| = 5, |w_3| = 7, |w_4| = 9, |w_5| = 11$$

and $M(B) = (\mathbb{Q}[t], 0)$ with $|t| = 2$.

Then the equivalence classes of $\mathcal{P}_B(X_1)$ are given by the following KS-models:

- (a) $Dw_1 = Dw_2 = Dw_3 = Dw_4 = 0$, $Dw_5 = t^6$
- (b) $Dw_1 = Dw_2 = Dw_3 = Dw_4 = 0$, $Dw_5 = w_1w_3t + t^6$
- (c) $Dw_1 = Dw_2 = Dw_3 = Dw_4 = 0$, $Dw_5 = w_1w_2t^3 + t^6$
- (d) $Dw_1 = Dw_2 = Dw_3 = 0$, $Dw_4 = w_1w_2t$, $Dw_5 = t^6$
- (e) $Dw_1 = Dw_2 = Dw_3 = 0$, $Dw_4 = w_1w_2t$, $Dw_5 = w_1w_3t + t^6$

Futhermore the bases of $\text{Im}(R_* : H_*(\text{Der}_{\mathbb{Q}[t]}(\mathbb{Q}[t] \otimes \Lambda W)) \rightarrow H_*(\text{Der}(\Lambda W)))$ are given by the following table:

n	(a)	(b)	(c)	(d)	(e)
1	$(w_5, w_1 w_3)$	$(w_5, w_1 w_3)$	$(w_5, w_1 w_3)$	$(w_5, w_1 w_3)$	$(w_5, w_1 w_3)$
2	$w_{5,4}, w_{4,3},$ $w_{3,2}, w_{2,1}$	$w_{5,4}, w_{4,3},$ $w_{2,1}$	$w_{5,4}, w_{4,3}, w_{3,2},$ $w_{2,1}$	$w_{4,3}, w_{3,2},$ $w_{2,1}$	$w_{4,3}, w_{3,2},$ $w_{2,1}$
3	$(w_1, 1),$ $(w_5, w_1 w_2)$	$(w_5, w_1 w_2)$	$(w_5, w_1 w_2)$	$(w_5, w_1 w_2)$	$(w_5, w_1 w_2)$
4	$w_{5,3},$ $w_{4,2}, w_{3,1}$	$w_{5,3}, w_{4,2},$ $w_{3,1}$	$w_{5,3},$ $w_{4,2}, w_{3,1}$	$w_{5,3}, w_{4,2},$ $w_{3,1}$	$w_{5,3}, w_{4,2},$ $w_{3,1}$
5	$(w_2, 1)$	$(w_2, 1)$			
6	$(w_5, w_2),$ (w_4, w_1)	$(w_5, w_2),$ (w_4, w_1)	$(w_5, w_2),$ (w_4, w_1)	$(w_5, w_2),$ (w_4, w_1)	$(w_5, w_2),$ (w_4, w_1)
7	$(w_3, 1)$		$(w_3, 1)$	$(w_3, 1)$	
8	(w_5, w_1)	(w_5, w_1)	(w_5, w_1)	(w_5, w_1)	(w_5, w_1)
9	$(w_4, 1)$	$(w_4, 1)$	$(w_4, 1)$	$(w_4, 1)$	$(w_4, 1)$
10					
11	$(w_5, 1)$	$(w_5, 1)$	$(w_5, 1)$	$(w_5, 1)$	$(w_5, 1)$

where $w_{i,j} := (w_i, w_j)$

- (2) Let $X_2 = S^3 \times S^5 \times S^9 \times S^{19}$, $B = K(\mathbb{Z}, 2)$ and let $M(X_2) = (\Lambda W, 0) = (\Lambda(w_1, w_2, w_3, w_4), 0)$ with

$$|w_1| = 3, |w_2| = 5, |w_3| = 9, |w_4| = 19.$$

From the degree reason, we can set as

$$Dw_1 = Dw_2 = 0, Dw_3 = a_1 w_1 w_2 t + a_2 t^5,$$

$$Dw_4 = a_3 w_1 w_2 t^6 + a_4 w_1 w_3 t^4 + a_5 w_2 w_3 t^3 + a_6 t^{10}$$

for $a_1, \dots, a_6 \in \mathbb{Q}$, in general. Then the equivalence classes of $\mathcal{P}_B(X_2)$ are given by the following KS-models:

- (a) $Dw_1 = Dw_2 = Dw_3 = 0, Dw_4 = t^{10}$
- (b) $Dw_1 = Dw_2 = Dw_3 = 0, Dw_4 = w_2 w_3 t^3 + t^{10}$
- (c) $Dw_1 = Dw_2 = Dw_3 = 0, Dw_4 = w_1 w_3 t^4 + t^{10}$
- (d) $Dw_1 = Dw_2 = Dw_4 = 0, Dw_3 = w_1 w_2 t, Dw_4 = t^{10}$
- (e) $Dw_1 = Dw_2 = 0, Dw_3 = w_1 w_2 t + t^5, Dw_4 = 0$
- (f) $Dw_1 = Dw_2 = 0, Dw_3 = w_1 w_2 t, Dw_4 = w_1 w_3 t^4 + t^{10}$
- (g) $Dw_1 = Dw_2 = 0, Dw_3 = w_1 w_2 t, Dw_4 = w_2 w_3 t^3 + t^{10}$

Futhermore the bases of $\text{Im}(R_* : H_*(\text{Der}_{\mathbb{Q}[t]}(\mathbb{Q}[t] \otimes \Lambda W)) \rightarrow H_*(\text{Der}(\Lambda W)))$ are given by the following table:

Their Hasse diagrams are given as follows:

n	(a)	(b)	(c)	(d)
1	$(w_3, w_1 w_2)$	$(w_3, w_1 w_2)$	$(w_3, w_1 w_2)$	$(w_3, w_1 w_2)$
2	$(w_4, w_{123}), (w_2, w_1)$	(w_4, w_{123})	$(w_4, w_{123}), (w_2, w_1)$	$(w_4, w_{123}), (w_2, w_1)$
3	$(w_1, 1)$	$(w_1, 1)$		
4	(w_3, w_2)	(w_3, w_2)	(w_3, w_2)	(w_3, w_2)
5	$(w_2, 1), (w_4, w_2 w_3)$	$(w_4, w_2 w_3)$	$(w_2, 1), (w_4, w_2 w_3)$	$(w_4, w_2 w_3)$
6	(w_3, w_1)		(w_3, w_1)	(w_3, w_1)
7	$(w_4, w_1 w_3)$	$(w_4, w_1 w_3)$	$(w_4, w_1 w_3)$	$(w_4, w_1 w_3)$
9	$(w_3, 1)$			$(w_3, 1)$
10	(w_4, w_3)	(w_4, w_3)	(w_4, w_3)	
11	$(w_4, w_1 w_2)$	$(w_4, w_1 w_2)$	$(w_4, w_1 w_2)$	$(w_4, w_1 w_2)$
14	(w_4, w_2)	(w_4, w_2)	(w_4, w_2)	(w_4, w_2)
16	(w_4, w_1)	(w_4, w_1)	(w_4, w_1)	(w_4, w_1)
19	$(w_4, 1)$	$(w_4, 1)$	$(w_4, 1)$	$(w_4, 1)$

n	(e)	(f)	(g)
1	$(w_3, w_1 w_2)$	$(w_3, w_1 w_2)$	$(w_3, w_1 w_2)$
2	(w_2, w_1)	$(w_4, w_{123}), (w_2, w_1)$	(w_2, w_{123})
3			
4	(w_3, w_2)	(w_3, w_2)	(w_3, w_2)
5		$(w_4, w_2 w_3)$	$(w_4, w_2 w_3)$
6	(w_3, w_1)	(w_3, w_1)	(w_3, w_1)
7		$(w_4, w_1 w_3)$	$(w_4, w_1 w_3)$
9	$(w_3, 1)$		
10			
11	$(w_4, w_1 w_2)$	$(w_4, w_1 w_2)$	$(w_4, w_1 w_2)$
14	(w_4, w_2)	(w_4, w_2)	(w_4, w_2)
16	(w_4, w_1)	(w_4, w_1)	(w_4, w_1)
19	$(w_4, 1)$	$(w_4, 1)$	$(w_4, 1)$

where $w_{123} := w_1 w_2 w_3$

Thus we have that

$$\text{depth}_{K(\mathbb{Z}, 2)}(X_1) = \text{depth}_{K(\mathbb{Z}, 2)}(X_2) = 3.$$

However, we have that

$$r_0(X_1) = 5 > 4 = r_0(X_2),$$

where $r_0(X_i)$ means the rational toral rank of X_i [16]. From the above results, we see that there are rationally 5 types of almost free S^1 -actions on spaces which are rationally equivalent to X_1 and 7 types of almost free S^1 -actions on spaces which are rationally equivalent to X_2 .

Acknowledgements The second author has been supported in part by JSPS KAKENHI Grant Number 23244008, 24540085, 16H03936.

References

1. Amann, M.: Non-formal homogeneous spaces. *Math. Z.* **274**, 1299–1325 (2013)
2. Amann, M., Kapovitch, V.: On fibrations with formal elliptic fibers. *Adv. Math.* **231**, 2048–2068 (2012)
3. Borel, A.: Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts. *Ann. Math.* **57**, 115–207 (1953)
4. Borel, A.: Cohomologie des espaces localement compacts d'après. J. Leray. Springer, Lecture Notes in Math **2** (1964)
5. Buijs, U., Smith, S.B.: Rational homotopy type of the classifying space for fibrewise self-equivalences. *Proc. Am. Math. Soc.* **141**, 2153–2167 (2013)
6. Dold, A., Lashof, R.: Principal quasi-fibrations and fibre homotopy equivalence of bundles. III. *J. Math.* **3**, 285–305 (1959)
7. Félix, Y.: Dénombrement des types de K-homotopie théorie de la déformation. *Mem. Soc. Math. Fr.* **3**, 1–49 (1980)
8. Félix, Y., Halperin, S.: Rational LS category and its applications. *Trans. Am. Math. Soc.* **273**, 1–38 (1982)
9. Félix, Y., Halperin, S., Thomas, J.C.: Rational homotopy theory, vol. 205. Springer, Berlin (2001)
10. Félix, Y., Lupton, G., Smith, S.B.: The rational homotopy type of the space of self-equivalences of a fibration. *Homol Homot Appl* **12**(2), 371–400 (2010)
11. Félix, Y., Oprea, J.: Rational homotopy of gauge groups. *Proc. Am. Math. Soc.* **137**, 1519–1527 (2009)
12. Gottlieb, D.H.: On fibre spaces and the evaluation map. *Ann. Math.* **87**, 42–55 (1968)
13. Gottlieb, H.: Evaluation subgroups of homotopy groups. *Am. J. Math.* **91**, 729–756 (1969)
14. Gottlieb, H.: Applications of bundle map theory. *Trans. Am. Math. Soc.* **171**, 23–50 (1972)
15. Greub, W., Halperin, S., Vanstone, R.: Connection, curvature and cohomology III. Academic press, Dublin (1976)
16. Halperin, S.: Rational homotopy and torus actions. *Lond. Math. Soc. Lecture note series*, vol. 93, pp. 293–306. Cambridge Univ. Press, Cambridge (1985)
17. Hamada, K., Yamaguchi, T., Yokura, S.: C-symplectic poset structure on a simply connected space. *Topol. Appl.* **161**, 107–126 (2014)
18. Hirsch, G.: Un isomorphisme attaché aux structures fibré. *C. R. Acad. Sci. Paris* **227**, 1328–1330 (1948)
19. Hilton, P., Mislin, G., Roitberg, J.: Localization of nilpotent groups and spaces. *N.-Holl. Math. Stud.* **15** (1975)
20. Leray, J.: L'homologie d'un espace fibré dont la fibre est connexe. *J. Math. Pures Appl.* **29**, 169–213 (1950)
21. Lupton, G.: Variations on a conjecture of Halperin. *Homotopy and geometry*, vol. 45, pp. 115–135. Banach Centre Publications, Warszawa (1998)
22. Meier, W.: Rational universal fibrations and flag manifolds. *Math. Ann.* **258**, 329–340 (1983)
23. McCleary, J.: A user's guide to spectral sequences. Cambridge Stud. Adv. Math., vol. 58, 2nd edn. Cambridge Univ. Press, Cambridge (2001)
24. Nishinobu, H., Yamaguchi, T.: Certain examples of posets of rational Gottlieb subgroups. *Kochi Math. J.* **10**, 77–94 (2015)

25. Nishinobu, H., Yamaguchi, T.: Sullivan minimal models of classifying spaces for non-formal spaces of small rank. *Topol. Appl.* **196**, 290–307 (2015)
26. Parent, P.-E.: Formal and non-formal homogeneous spaces of small rank. Thesis, Univ. of Ottawa (1996)
27. Salvatore, P.: Rational homotopy nilpotency of self-equivalences. *Topology and its Appl.* **77**, 37–50 (1997)
28. Serre, J.P.: Homologie singulière des espaces fibrés. *Appl. Ann. Math.* **54**, 425–505 (1951)
29. Schlessinger, M., Stasheff, J.: Deformation theory and rational homotopy type. [arXiv:1211.1647v1](https://arxiv.org/abs/1211.1647v1)
30. Shiga, H., Tezuka, M.: Rational fibrations, homogeneous spaces with positive Euler characteristics and Jacobians. *Ann. Inst. Fourier (Grenoble)* **37**, 81–106 (1987)
31. Sullivan, D.: Infinitesimal computations in topology. *I.H.E.S.* **47**, 269–331 (1978)
32. Tanré, D.: Homotopie Rationnelle: Modèles de Chen, Quillen, Sullivan. Lecture Note in Math. Springer, Berlin (1983)
33. Yamaguchi, T.: A fibre-restricted Gottlieb group and its rational realization problem. *Topol. Appl.* **196**, 1060–1076 (2015)