2-track algebras and the Adams spectral sequence

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Abstract In previous work of the first author and Jibladze, the E_3 -term of the Adams spectral sequence was described as a secondary derived functor, defined via secondary chain complexes in a groupoid-enriched category. This led to computations of the E_3 -term using the algebra of secondary cohomology operations. In work with Blanc, an analogous description was provided for all higher terms E_r . In this paper, we introduce 2-track algebras and tertiary chain complexes, and we show that the E_4 -term of the Adams spectral sequence is a tertiary Ext group in this sense. This extends the work with Jibladze, while specializing the work with Blanc in a way that should be more amenable to computations.

Keywords Adams spectral sequence · Tertiary · Toda bracket · 2-track groupoid · Bigroupoid · Double groupoid

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Dedicated to Ronnie Brown on the occasion of his eightieth birthday.

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1 Introduction

A major problem in algebraic topology consists of computing homotopy classes of maps between spaces or spectra, notably the stable homotopy groups of spheres $\pi_*^S(S^0)$. One of the most useful tools for such computations is the Adams spectral sequence [1] (and its unstable analogues [8]), based on ordinary mod p cohomology. Given finite spectra X and Y , Adams constructed a spectral sequence of the form:

$$E_2^{s,t} = \text{Ext}_{\mathfrak{A}}^{s,t} (H^*(Y; \mathbb{F}_p), H^*(X; \mathbb{F}_p)) \Rightarrow [\Sigma^{t-s} X, Y_p^\wedge]$$

where $\mathfrak{A}$ is the mod p Steenrod algebra, consisting of primary stable mod p cohomology operations, and $Y_p^\wedge$ denotes the p -completion of Y . In particular, taking sphere spectra $X = Y = S^0$, one obtains a spectral sequence

$$E_2^{s,t} = \text{Ext}_{\mathfrak{A}}^{s,t} (\mathbb{F}_p, \mathbb{F}_p) \Rightarrow \pi_{t-s}^S(S^0)_p^\wedge$$

abutting to the p -completion of the stable homotopy groups of spheres. The differential d_r is determined by r th order cohomology operations [14]. In particular, secondary cohomology operations determine the differential d_2 and thus the E_3 -term. The algebra of secondary operations was studied in [2]. In [3], the first author and Jibladze developed secondary chain complexes and secondary derived functors, and showed that the Adams E_3 -term is given by secondary Ext groups of the secondary cohomology of X and Y . They used this in [5], along with the algebra of secondary operations, to construct an algorithm that computes the differential d_2 .

Primary operations in mod p cohomology are encoded by the homotopy category $\text{Ho}(\mathcal{K})$ of the Eilenberg–MacLane mapping theory $\mathcal{K}$, consisting of finite products of Eilenberg–MacLane spectra of the form $\Sigma^{n_1} H\mathbb{F}_p \times \cdots \times \Sigma^{n_k} H\mathbb{F}_p$. More generally, the n th Postnikov truncation $P_n\mathcal{K}$ of the Eilenberg–MacLane mapping theory encodes operations of order up to $n + 1$, which in turn determine the Adams differential d_{n+1} and thus the E_{n+2} -term [4]. However, $P_n\mathcal{K}$ contains too much information for practical purposes. In [6], the first author and Blanc extracted from $P_n\mathcal{K}$ the information needed in order to compute the Adams differential d_{n+1} . The resulting algebraic-combinatorial structure is called an *algebra of left n -cubical balls*.

In this paper, we specialize the work of [6] to the case $n = 2$. Our goal is to provide an alternate structure which encodes an algebra of left 2-cubical balls, but which is more algebraic in nature and better suited for computations. The combinatorial difficulties in an algebra of left n -cubical balls arise from triangulations of the sphere $S^{n-1} = \partial D^n$. In the special case $n = 2$, triangulations of the circle S^1 are easily described, unlike in the case $n > 2$. Our approach also extends the work in [3] from secondary chain complexes to tertiary chain complexes.

Organization and main results

We define the notion of 2-track algebra (Definition 5.1) and show that each 2-track algebra naturally determines an algebra of left 2-cubical balls (Theorem 9.3). Build-

ing on [6], we show that higher order resolutions always exist in a 2-track algebra (Theorem 8.7). We show that a suitable 2-track algebra related to the Eilenberg–MacLane mapping theory recovers the Adams spectral sequence up to the E_4 -term (Theorem 7.3). We show that the spectral sequence only depends on the weak equivalence class of the 2-track algebra (Theorem 7.5).

Remark 1.1 This last point is important in view of the strictification result for secondary cohomology operations: these can be encoded by a graded pair algebra B_* over $\mathbb{Z}/p^2$ [2, §5.5]. The secondary Ext groups of the E_3 -term turn out to be the usual Ext groups over B_* [5, Theorem 3.1.1], a key fact for computations. We conjecture that a similar strictification result holds for tertiary operations, i.e., in the case $n = 2$.

Appendix A explains why 2-track groupoids are not models for homotopy 2-types, and how to extract the underlying 2-track groupoid from a bigroupoid or a double groupoid.

2 Cubes and tracks in a space

In this section, we fix some notation and terminology regarding cubes and groupoids.

Definition 2.1 Let X be a topological space.

An n -**cube** in X is a map $a: I^n \rightarrow X$, where $I = [0, 1]$ is the unit interval. For example, a 0-cube in X is a point of X , and a 1-cube in X is a path in X .

An n -cube can be restricted to $(n-1)$ -cubes along the $2n$ faces of I^n . For $1 \leq i \leq n$, denote:

$$\begin{aligned} d_i^0(a) &= a \text{ restricted to } I \times I \times \cdots \times \overbrace{\{0\}}^i \times \cdots \times I \\ d_i^1(a) &= a \text{ restricted to } I \times I \times \cdots \times \overbrace{\{1\}}^i \times \cdots \times I. \end{aligned}$$

An n -**track** in X is a homotopy class, relative to the boundary ∂I^n , of an n -cube. If $a: I^n \rightarrow X$ is an n -cube in X , denote by $\{a\}$ the corresponding n -track in X , namely the homotopy class of a rel ∂I^n .

In particular, for $n = 1$, a 1-track $\{a\}$ is a path homotopy class, i.e., a morphism in the fundamental groupoid of X from $a(0)$ to $a(1)$. Let us fix our notation regarding groupoids. In this paper, we consider only *small* groupoids.

Notation 2.2 A **groupoid** is a (small) category in which every morphism is invertible. Denote the data of a groupoid by $G = (G_0, G_1, \delta_0, \delta_1, \text{id}^\square, \square, (-)^\square)$, where:

- $G_0 = \text{Ob}(G)$ is the set of objects of G .
- $G_1 = \text{Hom}(G)$ is the set of morphisms of G . The set of morphisms from x to y is denoted $G(x, y)$. We write $x \in G$ and $\deg(x) = 0$ for $x \in G_0$, and $\deg(x) = 1$ for $x \in G_1$.
- $\delta_0: G_1 \rightarrow G_0$ is the source map.

- $\delta_1 : G_1 \rightarrow G_0$ is the target map.
- $\text{id}^\square : G_0 \rightarrow G_1$ sends each object x to its corresponding identity morphism $\text{id}_x^\square$.
- $\square : G_1 \times_{G_0} G_1 \rightarrow G_1$ is composition in G .
- $f^\square : y \rightarrow x$ is the inverse of the morphism $f : x \rightarrow y$.

Groupoids form a category **Gpd**, where morphisms are functors between groupoids.

For any object $x \in G_0$, denote by $\text{Aut}_G(x) = G(x, x)$ the automorphism group of x .

Denote by $\text{Comp}(G) = \pi_0(G)$ the components of G , i.e., the set of isomorphism classes of objects $G_0/\sim$.

Denote the fundamental groupoid of a topological space X by $\Pi_{(1)}(X)$.

Definition 2.3 Let X be a pointed space, with basepoint $0 \in X$. The constant map $0 : I^n \rightarrow X$ with value $0 \in X$ is called the **trivial n -cube**.

A **left 1-cube** or **left path** in X is a map $a : I \rightarrow X$ satisfying $a(1) = 0$, that is, $d_1^1(a) = 0$, the trivial 0-cube. In other words, a is a path in X from a point $a(0)$ to the basepoint 0 . We denote $\delta a = a(0)$.

A **left 2-cube** in X is a map $\alpha : I^2 \rightarrow X$ satisfying $\alpha(1, t) = \alpha(t, 1) = 0$ for all $t \in I$, that is, $d_1^1(\alpha) = d_2^1(\alpha) = 0$, the trivial 1-cube.

More generally, a **left n -cube** in X is a map $\alpha : I^n \rightarrow X$ satisfying $\alpha(t_1, \dots, t_n) = 0$ whenever some coordinate satisfies $t_i = 1$. In other words, for all $1 \leq i \leq n$ we have $d_i^1(\alpha) = 0$, the trivial $(n - 1)$ -cube.

A **left n -track** in X is a homotopy class, relative to the boundary ∂I^n , of a left n -cube.

The equality $I^{m+n} = I^m \times I^n$ allows us to define an operation on cubes.

Definition 2.4 Let $\mu : X \times X' \rightarrow X''$ be a map, for example a composition map in a topologically enriched category $\mathcal{C}$. For $m, n \geq 0$, consider cubes

$$\begin{aligned} a : I^m &\rightarrow X \\ b : I^n &\rightarrow X'. \end{aligned}$$

The $\otimes$ -**composition** of a and b is the $(m + n)$ -cube $a \otimes b$ defined as the composite

$$a \otimes b : I^{m+n} = I^m \times I^n \xrightarrow{a \times b} X \times X' \xrightarrow{\mu} X''. \quad (2.1)$$

For $m = n$, the **pointwise composition** of a and b is the n -cube defined as the composite

$$ab : I^n \xrightarrow{(a,b)} X \times X' \xrightarrow{\mu} X''. \quad (2.2)$$

The pointwise composition is the restriction of the $\otimes$ -composition along the diagonal:

$$\begin{array}{ccc} I^n & \xrightarrow{\Delta} & I^n \times I^n \xrightarrow{a \otimes b} X'' \\ & \searrow & \uparrow \\ & & ab \end{array}$$

Remark 2.5 For $m = n = 0$, the 0-cube $x \otimes y = xy$ is the pointwise composition, which is the composition in the underlying category. For higher dimensions, there are still relations between the $\otimes$ -composition and the pointwise composition. In suggestive formulas, pointwise composition of paths is given by $(ab)(t) = a(t)b(t)$ for all $t \in I$, whereas the $\otimes$ -composition of paths is the 2-cube given by $(a \otimes b)(s, t) = a(s)b(t)$.

Assume moreover that μ satisfies

$$\mu(x, 0) = \mu(0, x') = 0$$

for the basepoints $0 \in X, 0 \in X', 0 \in X''$. For example, μ could be the composition map in a category $\mathcal{C}$ enriched in $(\mathbf{Top}_*, \wedge)$, the category of pointed topological spaces with the smash product as monoidal structure. If a and b are left cubes, then $a \otimes b$ and ab are also left cubes.

3 2-track groupoids

We now focus on left 2-tracks in a pointed space X , and observe that they form a groupoid. Define the groupoid $\Pi_{(2)}(X)$ with object set:

$$\Pi_{(2)}(X)_0 = \text{set of left 1-cubes in } X$$

and morphism set:

$$\Pi_{(2)}(X)_1 = \text{set of left 2-tracks in } X$$

where the source δ_0 and target δ_1 of a left 2-track $\alpha: I \times I \rightarrow X$ are given by restrictions

$$\delta_0(\alpha) = d_1^0(\alpha)$$

$$\delta_1(\alpha) = d_2^0(\alpha)$$

and note in particular $\delta\delta_0(\alpha) = \delta\delta_1(\alpha) = \alpha(0, 0)$. In other words, a morphism α from a to b looks like this:

$$\begin{array}{ccc} & 0 & \\ \uparrow a = \delta_0(\alpha) & \Downarrow \alpha & \uparrow 0 \\ \delta a = \delta b & \xrightarrow{b = \delta_1(\alpha)} & \end{array}$$

Remark 3.1 Up to reparametrization, a left 2-track $\alpha: a \Rightarrow b$ corresponds to a path homotopy from a to b , which can be visualized in a globular picture:

However, the $\otimes$ -composition will play an important role in this paper, which is why we adopt a cubical approach, rather than globular or simplicial.

Composition $\beta \square \alpha$ of left 2-tracks is described by the following picture:

Remark 3.2 To make this definition precise, let $\alpha: a \Rightarrow b$ and $\beta: b \Rightarrow c$ be left 2-tracks in X , i.e., composable morphisms in $\Pi_{(2)}(X)$. Choose representative maps $\tilde{\alpha}, \tilde{\beta}: I^2 \rightarrow X$. Consider the map $f_{\alpha, \beta}: [0, 1] \times [-1, 1] \rightarrow X$ pictured in (3.1). That is, define

$$f(s, t) = \begin{cases} \tilde{\alpha}(s, t) & \text{if } 0 \leq t \leq 1 \\ \tilde{\beta}(-t, s) & \text{if } -1 \leq t \leq 0. \end{cases}$$

Now consider the reparametrization map $w: I^2 \rightarrow [0, 1] \times [-1, 1]$ illustrated in this picture:

Explicitly, the restriction $w|_{\partial I^2}$ to the boundary is the piecewise linear map satisfying

$$\begin{cases} w(0, 0) = (0, 0) \\ w(0, 1) = (0, 1) \\ w(\frac{1}{2}, 1) = (1, 1) \\ w(1, 1) = (1, 0) \\ w(1, \frac{1}{2}) = (1, -1) \\ w(1, 0) = (0, -1) \end{cases}$$

and $w(x)$ is defined for points $x \in I^2$ in the interior as follows. Write $x = p(0, 0) + qy$ as a unique convex combination of $(0, 0)$ and a point y on the boundary ∂I^2 . Then define $w(x) = pw(0, 0) + qw(y) = qw(y)$. Finally, the composition $\beta \square \alpha: a \Rightarrow c$ is $\{f_{\alpha, \beta} \circ w\}$, the homotopy class of the composite

$$I^2 \xrightarrow{w} [0, 1] \times [-1, 1] \xrightarrow{f_{\alpha, \beta}} X$$

relative to the boundary ∂I^2 .

In other notation, we have inclusions $d_2^0: I^1 \hookrightarrow I^2$ as the bottom edge $I \times \{0\}$ and $d_1^0: I^1 \hookrightarrow I^2$ as the left edge $\{0\} \times I$, our w is a map $w: I^2 \rightarrow I^2 \cup_{I^1} I^2$, and $\beta \square \alpha$ is the homotopy class of the composite

$$I^2 \xrightarrow{w} I^2 \cup_{I^1} I^2 \xrightarrow{[\alpha \beta]} X.$$

Given a left path a in X , the identity of a in the groupoid $\Pi_{(2)}(X)$ is the left 2-track is pictured here:

More precisely, for points $x \in I^2$ in the interior, write $x = p(0, 0) + qy$ as a unique convex combination of $(0, 0)$ and a point y on the boundary ∂I^2 . Then define $\text{id}_a^\square(x) = a(q)$.

The inverse $\alpha^\square: b \Rightarrow a$ of a left 2-track $\alpha: a \Rightarrow b$ is the homotopy class of the composite $\alpha \circ T$, where $T: I^2 \rightarrow I^2$ is the map swapping the two coordinates: $T(x, y) = (y, x)$.

Lemma 3.3 *Given a pointed topological space X , the structure described above makes $\Pi_{(2)}(X)$ into a groupoid, called the **groupoid of left 2-tracks** in X .*

Proof Standard. □

Definition 3.4 A groupoid G is **abelian** if the group $\text{Aut}_G(x)$ is abelian for every object $x \in G_0$. The groupoid G is **strictly abelian** if it is pointed (with basepoint $0 \in G_0$), and is equipped with a family of isomorphisms

$$\psi_x: \text{Aut}_G(x) \xrightarrow{\cong} \text{Aut}_G(0)$$

indexed by all objects $x \in G_0$, such that the diagram

$$\begin{array}{ccc} \text{Aut}_G(y) & \xrightarrow{\varphi^f} & \text{Aut}_G(x) \\ & \searrow \psi_y & \downarrow \psi_x \\ & & \text{Aut}_G(0) \end{array} \quad (3.2)$$

commutes for every map $f: x \rightarrow y$ in G , where φ^f denotes the “change of basepoint” isomorphism

$$\begin{aligned} \varphi^f: \text{Aut}_G(y) &\xrightarrow{\cong} \text{Aut}_G(x) \\ \alpha &\mapsto \varphi^f(\alpha) = f^\square \square \alpha \square f. \end{aligned}$$

Remark 3.5 A strictly abelian groupoid is automatically abelian. Indeed, the compatibility condition (3.2) applied to automorphisms $f: 0 \rightarrow 0$ implies that conjugation $\varphi^f: \text{Aut}_G(0) \rightarrow \text{Aut}_G(0)$ is the identity.

Definition 3.6 A groupoid G is **pointed** if it has a chosen basepoint, i.e., an object $0 \in G_0$. Here 0 is an abuse of notation: the basepoint is not assumed to be an initial object for G .

The **star** of a pointed groupoid G is the set of all morphisms to the basepoint 0 , denoted by:

$$\text{Star}(G) = \{f \in G_1 \mid \delta_1(f) = 0\}.$$

For a morphism $f: x \rightarrow 0$ in $\text{Star}(G)$, we write $\delta f = \delta_0 f = x$.

If G has a basepoint $0 \in G_0$, then we take $\text{id}_0^\square \in G_1$ as basepoint for the set of morphisms G_1 and for $\text{Star}(G) \subseteq G_1$; we sometimes write $0 = \text{id}_0^\square$. Moreover, we take the component of the basepoint 0 as basepoint for $\text{Comp}(G)$, the set of components of G .

Proposition 3.7 $\Pi_{(2)}(X)$ is a strictly abelian groupoid, and it satisfies $\text{Comp } \Pi_{(2)}(X) \simeq \text{Star } \Pi_{(1)}(X)$.

Proof Let $a \in \Pi_{(2)}(X)_0$ be a left path in X . To any automorphism $\alpha: 0 \Rightarrow 0$ in $\Pi_{(2)}(X)$, one can associate the well-defined left 2-track indicated by the picture

which is a morphism $a \Rightarrow a$. This assignment defines a map $\text{Aut}_{\Pi_{(2)}(X)}(0) \rightarrow \text{Aut}_{\Pi_{(2)}(X)}(a)$ and is readily seen to be a group isomorphism, whose inverse we denote ψ_a . One readily checks that the family ψ_a is compatible with change-of-basepoint isomorphisms.

The set $\text{Comp } \Pi_{(2)}(X)$ is the set of left paths in X quotiented by the relation of being connected by a left 2-track. The set $\text{Star } \Pi_{(1)}(X)$ is the set of left paths in X quotiented by the relation of path homotopy. But two left paths are path-homotopic if and only if they are connected by a left 2-track. $\square$

The bijection $\text{Comp } \Pi_{(2)}(X) \simeq \text{Star } \Pi_{(1)}(X)$ is induced by taking the homotopy class of left 1-cubes. Consider the function $q: \Pi_{(2)}(X)_0 \rightarrow \Pi_{(1)}(X)_1$ which sends a left 1-cube to its left 1-track $q(a) = \{a\}$. Then the image of q is $\text{Star } \Pi_{(1)}(X) \subseteq \Pi_{(1)}(X)_1$ and q is constant on the components of $\Pi_{(2)}(X)_0$. We now introduce a definition based on those features of $\Pi_{(2)}(X)$.

Definition 3.8 A 2-track groupoid $G = (G_{(1)}, G_{(2)})$ consists of:

- Pointed groupoids $G_{(1)}$ and $G_{(2)}$, with $G_{(2)}$ strictly abelian.
- A pointed function $q: G_{(2)0} \rightarrow \text{Star } G_{(1)}$ which is constant on the components of $G_{(2)}$, and such that the induced function $q: \text{Comp } G_{(2)} \xrightarrow{\simeq} \text{Star } G_{(1)}$ is bijective.

We assign degrees to the following elements:

$$\deg(x) = \begin{cases} 0 & \text{if } x \in G_{(1)0} \\ 1 & \text{if } x \in G_{(2)0} \\ 2 & \text{if } x \in G_{(2)1} \end{cases}$$

and we write $x \in G$ in each case.

A **morphism of 2-track groupoids** $F: G \rightarrow G'$ consists of a pair of pointed functors

$$\begin{aligned} F_{(1)}: G_{(1)} &\rightarrow G'_{(1)} \\ F_{(2)}: G_{(2)} &\rightarrow G'_{(2)} \end{aligned}$$

satisfying the following two conditions.

1. (*Structural isomorphisms*) For every object $a \in G_{(2)0}$, the diagram

$$\begin{array}{ccc} \text{Aut}_{G_{(2)}}(a) & \xrightarrow{F_{(2)}} & \text{Aut}_{G'_{(2)}}(F_{(2)}a) \\ \psi_a \downarrow & & \downarrow \psi_{F_{(2)}a} \\ \text{Aut}_{G_{(2)}}(0) & \xrightarrow{F_{(2)}} & \text{Aut}_{G'_{(2)}}(0') \end{array}$$

commutes.

2. (*Quotient functions*) The diagram

$$\begin{array}{ccc} G_{(2)0} & \xrightarrow{F_{(2)}} & G'_{(2)0} \\ q \downarrow & & \downarrow q' \\ \text{Star } G_{(1)} & \xrightarrow{F_{(1)}} & \text{Star } G'_{(1)} \end{array}$$

commutes.

Let $\mathbf{Gpd}_{(1,2)}$ denote the category of 2-track groupoids.

Remark 3.9 If $\alpha: a \Rightarrow b$ is a left 2-track in a space, then the left paths a and b have the same starting point $\delta a = \delta b$. This condition is encoded in the definition of 2-track groupoid. Indeed, if $\alpha: a \Rightarrow b$ is a morphism in $G_{(2)}$, then $a, b \in G_{(2)0}$ belong to the same component of $G_{(2)}$. Thus, we have $q(a) = q(b) \in \text{Star } G_{(1)}$ and in particular $\delta q(a) = \delta q(b) \in G_{(1)0}$.

Definition 3.10 The **fundamental 2-track groupoid** of a pointed space X is

$$\Pi_{(1,2)}(X) := (\Pi_{(1)}(X), \Pi_{(2)}(X)).$$

This construction defines a functor $\Pi_{(1,2)}: \mathbf{Top}_* \rightarrow \mathbf{Gpd}_{(1,2)}$.

Remark 3.11 The grading on $\Pi_{(1,2)}(X)$ defined in Definition 3.8 corresponds to the dimension of the cubes. For $x \in \Pi_{(1,2)}(X)$, we have $\deg(x) = 0$ if x is a point in X , $\deg(x) = 1$ if x is a left path in X , and $\deg(x) = 2$ if x is a left 2-track in X . This 2-graded set is the left 2-cubical set $\text{Nul}_2(X)$ [6, Definition 1.9].

Definition 3.12 Given a 2-track groupoid G , its **homotopy groups** are

$$\pi_0 G = \text{Comp } G_{(1)}$$

$$\pi_1 G = \text{Aut}_{G_{(1)}}(0)$$

$$\pi_2 G = \text{Aut}_{G_{(2)}}(0).$$

Note that $\pi_0 G$ is a priori only a pointed set, $\pi_1 G$ is a group, and $\pi_2 G$ is an abelian group.

A morphism $F: G \rightarrow G'$ of 2-track groupoids is a **weak equivalence** if it induces an isomorphism on homotopy groups.

Remark 3.13 Let X be a topological space with basepoint $x_0 \in X$. Then the homotopy groups of its fundamental 2-track groupoid $G = \Pi_{(1,2)}(X, x_0)$ are the homotopy groups of the space $\pi_i G = \pi_i(X, x_0)$ for $i = 0, 1, 2$.

The following two lemmas are straightforward.

Lemma 3.14 $\mathbf{Gpd}_{(1,2)}$ has products, given by $G \times G' = (G_{(1)} \times G'_{(1)}, G_{(2)} \times G'_{(2)})$, and where the structural isomorphisms

$$\psi_{(x,x')}: \text{Aut}_{G_{(2)} \times G'_{(2)}}((x, x')) \xrightarrow{\cong} \text{Aut}_{G_{(2)} \times G'_{(2)}}((0, 0'))$$

are given by $\psi_x \times \psi_{x'}$, and the quotient function

$$\begin{array}{c} (G \times G')_{(2)0} = G_{(2)0} \times G'_{(2)0} \\ \downarrow q \times q' \\ \text{Star}(G \times G')_{(1)} = \text{Star } G_{(1)} \times \text{Star } G'_{(1)} \end{array}$$

is the product of the quotient functions for G and G' .

Lemma 3.15 *The fundamental 2-track groupoid preserves products:*

$$\Pi_{(1,2)}(X \times Y) \cong \Pi_{(1,2)}(X) \times \Pi_{(1,2)}(Y).$$

4 2-tracks in a topologically enriched category

Throughout this section, let $\mathcal{C}$ be a category enriched in $(\mathbf{Top}_*, \wedge)$. Explicitly:

- For any objects A and B of $\mathcal{C}$, there is a morphism space $\mathcal{C}(A, B)$ with basepoint denoted $0 \in \mathcal{C}(A, B)$.
- For any objects A, B , and C , there is a composition map

$$\mu: \mathcal{C}(B, C) \times \mathcal{C}(A, B) \rightarrow \mathcal{C}(A, C)$$

which is associative and unital.

- Composition satisfies

$$\mu(x, 0) = \mu(0, y) = 0$$

for all x and y .

We write $x \in \mathcal{C}$ if $x \in \mathcal{C}(A, B)$ for some objects A and B . For $x, y \in \mathcal{C}$, we write $xy = \mu(x, y)$ when x and y are composable, i.e., when the target of y is the source of x . From now on, whenever an expression such as xy or $x \otimes y$ appears, it is understood that x and y must be composable.

By Definition 2.4, we have the $\otimes$ -composition $x \otimes y$ for $x, y \in \Pi_{(1)}\mathcal{C}$ and $\deg(x) + \deg(y) \leq 1$. For $\deg(a) = \deg(b) = 1$, we have:

$$\begin{aligned} ab &= (a \otimes \delta_1 b) \square (\delta_0 a \otimes b) \\ &= (\delta_1 a \otimes b) \square (a \otimes \delta_0 b). \end{aligned}$$

This equation holds in any category enriched in groupoids, where ab denotes the (pointwise) composition. Note that for paths $\tilde{a}$ and $\tilde{b}$ representing a and b , the boundary of the 2-cube $\tilde{a} \otimes \tilde{b}$ corresponds to the equation.

Conversely, the $\otimes$ -composition in $\Pi_{(1)}\mathcal{C}$ is determined by the pointwise composition. For $\deg(x) = \deg(y) = 0$ and $\deg(a) = 1$, we have:

$$\begin{cases} x \otimes y = xy \\ x \otimes a = \text{id}_x \square a \\ a \otimes x = a \text{id}_x \square. \end{cases} \quad (4.1)$$

We now consider the 2-track groupoids $\Pi_{(1,2)}\mathcal{C}(A, B)$ of morphism spaces in $\mathcal{C}$, and we write $x \in \Pi_{(1,2)}\mathcal{C}$ if $x \in \Pi_{(1,2)}\mathcal{C}(A, B)$ for some objects A, B of $\mathcal{C}$. By Definition 2.4, composition in $\mathcal{C}$ induces the $\otimes$ -composition:

$$x \otimes y \in \Pi_{(1,2)}\mathcal{C}$$

if x and y satisfy $\deg(x) + \deg(y) \leq 2$. For $\deg(x) = \deg(y) = 1$, x and y are left paths, hence $x \otimes y$ is well-defined. The $\otimes$ -composition satisfies:

$$\deg(x \otimes y) = \deg(x) + \deg(y).$$

The $\otimes$ -composition is associative, since composition in $\mathcal{C}$ is associative. The identity elements $1_A \in \mathcal{C}(A, A)$ for $\mathcal{C}$ provide identity elements $1 = 1_A \in \Pi_{(1,2)}\mathcal{C}(A, A)$, with $\deg(1_A) = 0$, and $x \otimes 1 = x = 1 \otimes x$.

Let us describe the $\otimes$ -composition of left paths more explicitly. Given left paths a and b , then $a \otimes b$ is a 2-track from $\delta_0(a \otimes b) = (\delta a) \otimes b$ to $\delta_1(a \otimes b) = a \otimes (\delta b)$, as illustrated here:

$$\begin{array}{ccc} & \xrightarrow{0} & \\ \delta_0(a \otimes b) = \delta a \otimes b & \begin{array}{c} \swarrow \searrow \\ a \otimes b \end{array} & \xrightarrow{0} \\ & \xleftarrow{\delta_1(a \otimes b) = a \otimes \delta b} & \end{array}$$

Definition 4.1 The 2-track algebra associated to $\mathcal{C}$, denoted $(\Pi_{(1)}\mathcal{C}, \Pi_{(1,2)}\mathcal{C}, \square, \otimes)$, consists of the following data.

- $\Pi_{(1)}\mathcal{C}$ is the category enriched in pointed groupoids given by the fundamental groupoids $(\Pi_{(1)}\mathcal{C}(A, B), \square)$ of morphism spaces in $\mathcal{C}$, along with the $\otimes$ -composition, which determines (and is determined by) the composition in $\Pi_{(1)}\mathcal{C}$.
- $\Pi_{(1,2)}\mathcal{C}$ is given by the collection of fundamental 2-track groupoids $(\Pi_{(1,2)}\mathcal{C}(A, B), \square)$ together with the $\otimes$ -composition $x \otimes y$ for $x, y \in \Pi_{(1,2)}\mathcal{C}$ satisfying $\deg(x) + \deg(y) \leq 2$.

Proposition 4.2 Let $x, \alpha, \beta \in \Pi_{(1,2)}\mathcal{C}$ with $\deg(x) = 0$ and $\deg(\alpha) = \deg(\beta) = 2$. Then the following equations hold:

$$\begin{cases} x \otimes (\beta \square \alpha) = (x \otimes \beta) \square (x \otimes \alpha) \\ (\beta \square \alpha) \otimes x = (\beta \otimes x) \square (\alpha \otimes x). \end{cases}$$

Proof This follows from functoriality of $\Pi_{(2)}$ applied to the composition maps $\mu(x, -): \mathcal{C}(A, B) \rightarrow \mathcal{C}(A, C)$ and $\mu(-, x): \mathcal{C}(B, C) \rightarrow \mathcal{C}(A, C)$. $\square$

Proposition 4.3 Let $c, \alpha \in \Pi_{(1,2)}\mathcal{C}$ with $\deg(c) = 1$ and $\deg(\alpha) = 2$. Then the following equations hold:

$$\begin{cases} \delta_1 \alpha \otimes c = (\alpha \otimes \delta c) \square (\delta_0 \alpha \otimes c) \\ c \otimes \delta_0 \alpha = (c \otimes \delta_1 \alpha) \square (\delta c \otimes \alpha). \end{cases}$$

Proof Write $a = \delta_0 \alpha$ and $b = \delta_1 \alpha$, i.e., α is a left 2-track from a to b :

and note in particular $\delta a = \delta b$. Let $\tilde{\alpha}$ be a left 2-cube that represents α and consider the left 3-cube $\tilde{\alpha} \otimes c$:

Its boundary exhibits the equality of 2-tracks:

$$\begin{aligned} \text{top face} \square \text{right face} &= \text{front face} \\ (\alpha \otimes \delta c) \square (a \otimes c) &= b \otimes c \\ (\alpha \otimes \delta c) \square (\delta_0 \alpha \otimes c) &= \delta_1 \alpha \otimes c. \end{aligned}$$

Likewise, for the second equation, consider the left 3-cube $c \otimes \tilde{\alpha}$:

Its boundary exhibits the equality of 2-tracks:

$$\begin{aligned} \text{top face} \square \text{right face} &= \text{front face} \\ (c \otimes b) \square (\delta c \otimes \alpha) &= c \otimes a \\ (c \otimes \delta_1 \alpha) \square (\delta c \otimes \alpha) &= c \otimes \delta_0 \alpha. \end{aligned}$$

□

5 2-track algebras

We now collect the structure found in $(\Pi_{(1)}\mathcal{C}, \Pi_{(1,2)}\mathcal{C}, \square, \otimes)$ into the following definition.

Definition 5.1 A **2-track algebra** $\mathcal{A} = (\mathcal{A}_{(1)}, \mathcal{A}_{(1,2)}, \square, \otimes)$ consists of the following data.

- 1 A category $\mathcal{A}_{(1)}$ enriched in pointed groupoids, with the $\otimes$ -composition determined by Eq. (4.1).
- 2 A collection $\mathcal{A}_{(1,2)}$ of 2-track groupoids $(\mathcal{A}_{(1,2)}(A, B), \square)$ for all objects A, B of $\mathcal{A}_{(1)}$, such that the first groupoid in $\mathcal{A}_{(1,2)}(A, B)$ is equal to the pointed groupoid $\mathcal{A}_{(1)}(A, B)$.
- 3 For $x, y \in \mathcal{A}_{(1,2)}$, the $\otimes$ -composition $x \otimes y \in \mathcal{A}_{(1,2)}$ is defined. For $\deg(x) = 0$ and $\deg(y) = 1$, the following equations hold in $\mathcal{A}_{(1)}$:

$$\begin{cases} q(x \otimes y) = x \otimes q(y) \\ q(y \otimes x) = q(y) \otimes x. \end{cases}$$

The following equations are required to hold.

1. (*Associativity*) $\otimes$ is associative: $(x \otimes y) \otimes z = x \otimes (y \otimes z)$.
2. (*Units*) The units $1 \in \mathcal{A}_{(1)}$, with $\deg(1_A) = 0$, serve as units for $\otimes$, i.e., satisfy $x \otimes 1 = x = 1 \otimes x$ for all $x \in \mathcal{A}_{(1,2)}$.
3. (*Pointedness*) $\otimes$ satisfies $x \otimes 0 = 0$ and $0 \otimes y = 0$.
4. For $x, y, \alpha, \beta \in \mathcal{A}_{(1,2)}$ with $\deg(x) = \deg(y) = 0$ and $\deg(\alpha) = \deg(\beta) = 2$, we have:

$$\begin{cases} \delta_i(x \otimes \alpha \otimes y) = x \otimes (\delta_i \alpha) \otimes y & \text{for } i = 0, 1 \\ x \otimes (\beta \square \alpha) \otimes y = (x \otimes \beta \otimes y) \square (x \otimes \alpha \otimes y) \end{cases}$$

5. For $a, b \in \mathcal{A}_{(1,2)}$ with $\deg(a) = \deg(b) = 1$, we have:

$$\begin{cases} \delta_0(a \otimes b) = \delta a \otimes b \\ \delta_1(a \otimes b) = a \otimes \delta b. \end{cases}$$

6. For $c, \alpha \in \mathcal{A}_{(1,2)}$ with $\deg(c) = 1$ and $\deg(\alpha) = 2$, we have:

$$\begin{cases} \delta_1 \alpha \otimes c = (\alpha \otimes \delta c) \square (\delta_0 \alpha \otimes c) \\ c \otimes \delta_0 \alpha = (c \otimes \delta_1 \alpha) \square (\delta c \otimes \alpha). \end{cases}$$

Definition 5.2 A morphism of 2-track algebras $F: \mathcal{A} \rightarrow \mathcal{B}$ consists of the following.

1. A functor $F_{(1)}: \mathcal{A}_{(1)} \rightarrow \mathcal{B}_{(1)}$ enriched in pointed groupoids.
2. A collection $F_{(1,2)}$ of morphisms of 2-track groupoids

$$F_{(1,2)}(A, B): \mathcal{A}_{(1,2)}(A, B) \rightarrow \mathcal{B}_{(1,2)}(FA, FB)$$

for all objects A, B of $\mathcal{A}$, such that $F_{(1,2)}(A, B)$ restricted to the first groupoid in $\mathcal{A}_{(1,2)}(A, B)$ is the functor $F_{(1)}(A, B): \mathcal{A}_{(1)}(A, B) \rightarrow \mathcal{B}_{(1)}(FA, FB)$.

3. (*Compatibility with $\otimes$*) F commutes with $\otimes$:

$$F(x \otimes y) = Fx \otimes Fy.$$

Denote by $\mathbf{Alg}_{(1,2)}$ the category of 2-track algebras.

Definition 5.3 Let $\mathcal{A}$ be a 2-track algebra. The underlying **homotopy category** of $\mathcal{A}$ is the homotopy category of the underlying track category $\mathcal{A}_{(1)}$, denoted

$$\pi_0 \mathcal{A} := \pi_0 \mathcal{A}_{(1)} = \text{Comp } \mathcal{A}_{(1)}.$$

We say that $\mathcal{A}$ is **based** on the category $\pi_0 \mathcal{A}$.

Definition 5.4 A morphism of 2-track algebras $F: \mathcal{A} \rightarrow \mathcal{B}$ is a **weak equivalence** (or *Dwyer–Kan equivalence*) if the following conditions hold:

1. For all objects A and B of $\mathcal{A}$, the morphism

$$F_{(1,2)}: \mathcal{A}_{(1,2)}(A, B) \rightarrow \mathcal{B}_{(1,2)}(FA, FB)$$

is a weak equivalence of 2-track groupoids (Definition 3.12).

2. The induced functor $\pi_0 F: \pi_0 \mathcal{A} \rightarrow \pi_0 \mathcal{B}$ is an equivalence of categories.

6 Higher order chain complexes

In this section, we construct tertiary chain complexes, extending the work of [3] on secondary chain complexes. We will follow the treatment therein.

Definition 6.1 A **chain complex** (A, d) in a pointed category $\mathbf{A}$ is a sequence of objects and morphisms

$$\cdots \longrightarrow A_{n+1} \xrightarrow{d_n} A_n \xrightarrow{d_{n-1}} A_{n-1} \longrightarrow \cdots$$

in $\mathbf{A}$ satisfying $d_{n-1}d_n = 0$ for all $n \in \mathbb{Z}$. The map d is called the *differential*.

A **chain map** $f: (A, d) \rightarrow (A', d')$ between chain complexes is a sequence of morphisms $f_n: A_n \rightarrow A'_n$ commuting with the differentials:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & A_{n+1} & \xrightarrow{d_n} & A_n & \xrightarrow{d_{n-1}} & A_{n-1} \longrightarrow \cdots \\ & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} \\ \cdots & \longrightarrow & A'_{n+1} & \xrightarrow{d'_n} & A'_n & \xrightarrow{d'_{n-1}} & A'_{n-1} \longrightarrow \cdots \end{array}$$

i.e., satisfying $f_n d_n = d'_n f_{n+1}$ for all $n \in \mathbb{Z}$.

Definition 6.2 [3, Definition 2.6] Let $\mathbf{B}$ be a category enriched in pointed groupoids. A **secondary pre-chain complex** (A, d, γ) in $\mathbf{B}$ is a diagram of the form:

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \curvearrowright & & \curvearrowright & & \\ \cdots & \longrightarrow & A_{n+2} & \xrightarrow{d_{n+1}} & A_{n+1} & \xrightarrow{d_n} & A_n \xrightarrow{d_{n-1}} A_{n-1} \longrightarrow \cdots \\ & & \Downarrow & & \Downarrow \gamma_{n-1} & & \\ & & 0 & & 0 & & \end{array}$$

More precisely, the data consists of a sequence of objects A_n and maps $d_n: A_{n+1} \rightarrow A_n$, together with left tracks $\gamma_n: d_n d_{n+1} \Rightarrow 0$ for all $n \in \mathbb{Z}$.

(A, d, γ) is a **secondary chain complex** if moreover for each $n \in \mathbb{Z}$, the tracks

$$d_{n-1}d_nd_{n+1} \xrightarrow{d_{n-1} \otimes \gamma_n} d_{n-1}0 \xrightarrow{\text{id}_0^\square} 0$$

and

$$d_{n-1}d_nd_{n+1} \xrightarrow{\gamma_{n-1} \otimes d_{n+1}} 0d_{n+1} \xrightarrow{\text{id}_0^\square} 0$$

coincide. In other words, the track

$$\mathcal{O}(\gamma_{n-1}, \gamma_n) := (\gamma_{n-1} \otimes d_{n+1}) \square (d_{n-1} \otimes \gamma_n)^\square : 0 \Rightarrow 0$$

in the groupoid $\mathbf{B}(A_{n+2}, A_{n-1})$ is the identity track of 0.

We say that the secondary pre-chain complex (A, d, γ) is **based** on the chain complex $(A, \{d\})$ in the homotopy category $\pi_0 \mathbf{B}$.

Remark 6.3 One can show that the notion of secondary (pre-)chain complex in $\mathbf{B}$ coincides with the notion of 1st order (pre-)chain complex in $\text{Nul}_1 \mathbf{B}$ described in [6, §4, c.f. Example 12.3].

Definition 6.4 A **tertiary pre-chain complex** (A, d, δ, ξ) in a 2-track algebra $\mathcal{A}$ is a sequence of objects A_n and maps $d_n: A_{n+1} \rightarrow A_n$ in the category $\mathcal{A}_{(1)0}$, together with left paths $\gamma_n: d_nd_{n+1} \rightarrow 0$ in $\mathcal{A}_{(1,2)}$, as illustrated in the diagram

along with left 2-tracks $\xi_n: \gamma_n \otimes d_{n+2} \Rightarrow d_n \otimes \gamma_{n+1}$ in $\mathcal{A}_{(1,2)}$, for all $n \in \mathbb{Z}$.

(A, d, γ, ξ) is a **tertiary chain complex** if moreover for each $n \in \mathbb{Z}$, the left 2-track:

$$d_{n-1} \otimes \gamma_n \otimes d_{n+2} \xrightarrow{d_{n-1} \otimes \xi_n} d_{n-1}d_n \otimes \gamma_{n+1} \xrightarrow{\gamma_{n-1} \otimes \gamma_{n+1}} \gamma_{n-1} \otimes d_{n+1}d_{n+2} \xrightarrow{\xi_{n-1} \otimes d_{n+2}} d_{n-1} \otimes \gamma_n \otimes d_{n+2}$$

is the identity of $d_{n-1} \otimes \gamma_n \otimes d_{n+2}$ in the groupoid $\mathcal{A}_{(2)}(A_{n+3}, A_{n-1})$. In other words, the element:

$$\begin{aligned} \mathcal{O}(\xi_{n-1}, \xi_n) &:= \psi_{d_{n-1} \otimes \gamma_n \otimes d_{n+2}} ((\xi_{n-1} \otimes d_{n+2}) \square (\gamma_{n-1} \otimes \gamma_{n+1}) \square (d_{n-1} \otimes \xi_n)) \\ &\in \pi_2 \mathcal{A}_{(1,2)}(A_{n+3}, A_{n-1}) \end{aligned}$$

is trivial. Here, ψ is the structural isomorphism in the 2-track groupoid $\mathcal{A}_{(1,2)}(A_{n+3}, A_{n-1})$, as in Definitions 3.4 and 3.8.

We say that the tertiary pre-chain complex (A, d, γ, ξ) is **based** on the chain complex $(A, \{d\})$ in the homotopy category $\pi_0\mathcal{A}$.

6.1 Toda brackets of length 3 and 4

Let $\mathcal{C}$ be a category enriched in $(\mathbf{Top}_*, \wedge)$. Let $\pi_0\mathcal{C}$ be the category of path components of $\mathcal{C}$ (applied to each mapping space) and let

$$Y_0 \xleftarrow{y_1} Y_1 \xleftarrow{y_2} Y_2 \xleftarrow{y_3} Y_3 \xleftarrow{y_4} Y_4$$

be a diagram in $\pi_0\mathcal{C}$ satisfying $y_1y_2 = 0$, $y_2y_3 = 0$, and $y_3y_4 = 0$. Choose maps x_i in $\mathcal{C}$ representing y_i . Then there exist left 1-cubes a, b, c as in the diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & \swarrow & & \searrow & \swarrow & & \searrow \\ & & a \uparrow & & c \uparrow & & \\ Y_0 & \xleftarrow{x_1} & Y_1 & \xleftarrow{x_2} & Y_2 & \xleftarrow{x_3} & Y_3 & \xleftarrow{x_4} & Y_4 \\ & & & & \downarrow b & & & & \\ & & & & 0 & & & & \end{array}$$

Definition 6.5 The **Toda bracket** of length 3, denoted $\langle y_1, y_2, y_3 \rangle \subseteq \pi_1\mathcal{C}(Y_3, Y_0)$, is the set of all elements in $\text{Aut}(0) = \pi_1\mathcal{C}(Y_3, Y_0)$ of the form

$$\mathcal{O}(a, b) := (a \otimes x_3) \square (x_1 \otimes b) \square$$

as above.

Assume now that we can choose left 2-tracks $\alpha: a \otimes x_3 \Rightarrow x_1 \otimes b$ and $\beta: b \otimes x_4 \Rightarrow x_2 \otimes c$ in $\Pi_{(1,2)}\mathcal{C}$. Then the composite of left 2-tracks

$$(\alpha \otimes x_4) \square (a \otimes c) \square (x_1 \otimes \beta)$$

is an element of $\text{Aut}(x_1 \otimes b \otimes x_4)$, to which we apply the structural isomorphism

$$\psi_{x_1 \otimes b \otimes x_4}: \text{Aut}(x_1 \otimes b \otimes x_4) \xrightarrow{\cong} \pi_2\mathcal{C}(Y_4, Y_0).$$

The set of all such elements is the **Toda bracket** of length 4, denoted $\langle y_1, y_2, y_3, y_4 \rangle \subseteq \pi_2\mathcal{C}(Y_4, Y_0)$.

Note that the existence of α , resp. β , implies that the bracket $\langle y_1, y_2, y_3 \rangle$, resp. $\langle y_2, y_3, y_4 \rangle$ contains the zero element.

Remark 6.6 For a secondary pre-chain complex (A, d, γ) , we have

$$\mathcal{O}(\gamma_{n-1}, \gamma_n) \in \langle d_{n-1}, d_n, d_{n+1} \rangle$$

for every $n \in \mathbb{Z}$. Likewise, for a tertiary pre-chain complex (A, d, γ, ξ) , we have

$$\mathcal{O}(\xi_{n-1}, \xi_n) \in \langle d_{n-1}, d_n, d_{n+1}, d_{n+2} \rangle$$

for every $n \in \mathbb{Z}$. However, the vanishing of these Toda brackets does not guarantee the existence of a tertiary chain complex based on the chain complex $(A, \{d\})$. In a secondary chain complex (A, d, γ) , these Toda brackets vanish in a *compatible* way, that is, the equations $\mathcal{O}(\gamma_{n-1}, \gamma_n) = 0$ and $\mathcal{O}(\gamma_n, \gamma_{n+1}) = 0$ involve the same left track $\gamma_n: d_n d_{n+1} \Rightarrow 0$.

7 The Adams differential d_3

Let **Spec** denote the topologically enriched category of spectra and mapping spaces between them. More precisely, start from a simplicial (or topological) model category of spectra, like that of Bousfield–Friedlander [9, §2], or symmetric spectra or orthogonal spectra [13], and take **Spec** to be the full subcategory of fibrant–cofibrant objects; c.f. [6, Example 7.3].

Let $H := H\mathbb{F}_p$ be the Eilenberg–MacLane spectrum for the prime p and let $\mathfrak{A} = H^*H$ denote the mod p Steenrod algebra. Consider the collection **EM** of all mod p generalized Eilenberg–MacLane spectra that are bounded below and of finite type, i.e., degreewise finite products $A = \prod_i \Sigma^{n_i} H$ with $n_i \in \mathbb{Z}$ and $n_i \geq N$ for some integer N for all i . Since the product is degreewise finite, the natural map $\bigvee_i \Sigma^{n_i} H \rightarrow \prod_i \Sigma^{n_i} H$ is an equivalence, so that the mod p cohomology H^*A is a free $\mathfrak{A}$ -module. Moreover, the cohomology functor restricted to the full subcategory of **Spec** with objects **EM** yields an equivalence of categories in the diagram:

$$\begin{array}{ccc} \pi_0 \mathbf{Spec}^{\text{op}} & \xrightarrow{H^*} & \mathbf{Mod}_{\mathfrak{A}} \\ \uparrow & & \uparrow \\ \pi_0 \mathbf{EM}^{\text{op}} & \xrightarrow[\cong]{H^*} & \mathbf{Mod}_{\mathfrak{A}}^{\text{fin}} \end{array}$$

where $\mathbf{Mod}_{\mathfrak{A}}^{\text{fin}}$ denotes the full subcategory consisting of free $\mathfrak{A}$ -modules which are bounded below and of finite type.

Given spectra Y and X , consider the Adams spectral sequence:

$$E_2^{s,t} = \text{Ext}_{\mathfrak{A}}^{s,t}(H^*X, H^*Y) \Rightarrow \left[\Sigma^{t-s} Y, X_p^\wedge \right].$$

Assume that Y is a finite spectrum and X is a connective spectrum of finite type, i.e., X is equivalent to a CW-spectrum with finitely many cells in each dimension and no cells

below a certain dimension. Then the mod p cohomology H^*X is an $\mathfrak{A}$ -module which is bounded below and degreewise finitely generated (as an $\mathfrak{A}$ -module, or equivalently, as an $\mathbb{F}_p$ -vector space). Choose a free resolution of H^*X as an $\mathfrak{A}$ -module:

$$\cdots \longrightarrow F_2 \xrightarrow{e_1} F_1 \xrightarrow{e_0} F_0 \xrightarrow{\lambda} H^*X$$

where each F_i is a free $\mathfrak{A}$ -module of finite type and bounded below. This diagram can be realized as the cohomology of a diagram in the stable homotopy category $\pi_0\mathbf{Spec}$:

$$\cdots \longleftarrow A_2 \xleftarrow{d_1} A_1 \xleftarrow{d_0} A_0 \xleftarrow{\epsilon} A_{-1} = X$$

with each A_i in $\mathbf{EM}$ (for $i \geq 0$) and satisfying $H^*A_i \cong F_i$. We consider this diagram as a diagram in the opposite category $\pi_0\mathbf{Spec}^{\mathrm{op}}$ of the form:

$$\cdots \longrightarrow A_2 \xrightarrow{d_1} A_1 \xrightarrow{d_0} A_0 \xrightarrow{\epsilon} A_{-1} = X$$

Since $A_\bullet \rightarrow X$ is an $\mathbf{EM}$ -resolution of X in $\pi_0\mathbf{Spec}^{\mathrm{op}}$, there exists a tertiary chain complex (A, d, γ, ξ) in $\Pi_{(1,2)}\mathbf{Spec}^{\mathrm{op}}$ based on the resolution $A_\bullet \rightarrow X$, by Theorem 8.7.

Notation 7.1 Given spectra X and Y , let $\mathbf{EM}\{X, Y\}$ denote the topologically enriched subcategory of $\mathbf{Spec}$ consisting of all spectra in $\mathbf{EM}$ and mapping spaces between them, along with the objects X and Y , with the mapping spaces $\mathbf{Spec}(X, A)$ and $\mathbf{Spec}(Y, A)$ for all A in $\mathbf{EM}$; c.f. [3, Remark 4.3], [6, Remark 7.5]. We consider the 2-track algebra $\Pi_{(1,2)}\mathbf{EM}\{X, Y\}^{\mathrm{op}}$, or any 2-track algebra $\mathcal{A}$ weakly equivalent to it. In the following construction, everything will take place within $\Pi_{(1,2)}\mathbf{EM}\{X, Y\}^{\mathrm{op}}$, but we will write $\Pi_{(1,2)}\mathbf{Spec}^{\mathrm{op}}$ for notational convenience.

Start with a class in the E_2 -term:

$$x \in E_2^{s,t} = \mathrm{Ext}_{\mathfrak{A}}^{s,t}(H^*X, H^*Y) = \mathrm{Ext}_{\mathfrak{A}}^{s,0}(H^*X, \Sigma^t H^*Y)$$

represented by a cocycle $x': F_s \rightarrow \Sigma^t H^*Y$, i.e., a map of $\mathfrak{A}$ -modules satisfying $x'd_s = 0$. Realize x' as the cohomology of a map $x'': A_s \rightarrow \Sigma^t Y$ in $\mathbf{Spec}^{\mathrm{op}}$. The equation $x'd_s = 0$ means that $x''d_s$ is null-homotopic; let $\gamma: x''d_s \rightarrow 0$ be a null-homotopy. Consider the diagram in $\mathbf{Spec}^{\mathrm{op}}$:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & A_{s+2} & \xrightarrow{d_{s+1}} & A_{s+1} & \xrightarrow{d_s} & A_s & \xrightarrow{d_{s-1}} & A_{s-1} & \longrightarrow & \cdots & \longrightarrow & A_0 & \xrightarrow{\epsilon} & X \\ & & & & & & \downarrow x'' & & & & & & & & \\ & & & & & & \Sigma^t Y & & & & & & & & \end{array}$$

Now consider the underlying secondary pre-chain complex in $\Pi_{(1)}\mathbf{Spec}^{\text{op}}$:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & \curvearrowright & & \curvearrowright & & & \\
 \cdots & \longrightarrow & A_{s+3} & \xrightarrow{d_{s+2}} & A_{s+2} & \xrightarrow{d_{s+1}} & A_{s+1} & \xrightarrow{d_s} & A_s & \xrightarrow{x''} & \Sigma^t Y \\
 & & & & \downarrow \gamma_{s+1} & & \uparrow \gamma_s & & \downarrow \gamma & & \\
 & & & & 0 & & 0 & & 0 & &
 \end{array} \quad (7.1)$$

in which the obstructions $\mathcal{O}(\gamma_i, \gamma_{i+1})$ are trivial, for $i \geq s$.

Theorem 7.2 *The obstruction $\mathcal{O}(\gamma, \gamma_s) \in \pi_1 \mathbf{Spec}^{\text{op}}(A_{s+2}, \Sigma^t Y) = \pi_0 \mathbf{Spec}^{\text{op}}(A_{s+2}, \Sigma^{t+1} Y)$ is a (co)cycle and does not depend on the choices, up to (co)boundaries, and thus defines an element:*

$$d_{(2)}(x) \in \text{Ext}_{\mathfrak{A}}^{s+2, t+1}(H^* X, H^* Y).$$

Moreover, this function

$$d_{(2)}: \text{Ext}_{\mathfrak{A}}^{s, t}(H^* X, H^* Y) \rightarrow \text{Ext}_{\mathfrak{A}}^{s+2, t+1}(H^* X, H^* Y)$$

is the Adams differential d_2 .

Proof This is [3, Theorems 4.2 and 7.3], or the case $n = 1, m = 3$ of [6, Theorem 15.11].

Here we used the natural isomorphism:

$$\text{Ext}_{\pi_0 \mathbf{EM}^{\text{op}}}^{i, j}(H^* X, H^* Y) \cong \text{Ext}_{\mathfrak{A}}^{i, j}(H^* X, H^* Y)$$

where the left-hand side is defined as in Example 8.4. Using the equivalence of categories $H^*: \pi_0 \mathbf{EM}^{\text{op}} \xrightarrow{\cong} \mathbf{Mod}_{\mathfrak{A}}^{\text{fin}}$, this natural isomorphism follows from the natural isomorphisms:

$$\begin{aligned}
 \pi_0 \mathbf{Spec}^{\text{op}}(A_{s+2}, \Sigma^{t+1} Y) &= \text{Hom}_{\mathfrak{A}}(F_{s+2}, H^* \Sigma^{t+1} Y) \\
 &= \text{Hom}_{\mathfrak{A}}(F_{s+2}, \Sigma^{t+1} H^* Y).
 \end{aligned}$$

Cocycles modulo coboundaries in this group are precisely $\text{Ext}_{\mathfrak{A}}^{s+2, t+1}(H^* X, H^* Y)$. $\square$

Now assume that $d_2(x) = 0$ holds, so that x survives to the E_3 -term. Since the obstruction

$$\mathcal{O}(\gamma, \gamma_s) = (\gamma \otimes d_{s+1}) \square (x'' \otimes \gamma_s)^{\square}$$

vanishes, one can choose a left 2-track $\xi: \gamma \otimes d_{s+1} \Rightarrow x'' \otimes \gamma_s$, which makes (7.1) into a tertiary pre-chain complex in $\Pi_{(1,2)}\mathbf{Spec}^{\text{op}}$. Since (A, d, γ, ξ) was a tertiary chain complex to begin with, the obstructions $\mathcal{O}(\xi_i, \xi_{i+1})$ are trivial, for $i \geq s$.

Theorem 7.3 *The obstruction $\mathcal{O}(\xi, \xi_s) \in \pi_2\mathbf{Spec}^{\text{op}}(A_{s+3}, \Sigma^t Y) = \pi_0\mathbf{Spec}^{\text{op}}(A_{s+3}, \Sigma^{t+2} Y)$ is a (co)cycle and does not depend on the choices up to (co)boundaries, and thus defines an element:*

$$d_{(3)}(x) \in E_3^{s+3, t+2}(X, Y).$$

Moreover, this function

$$d_{(3)}: E_3^{s, t}(X, Y) \rightarrow E_3^{s+3, t+2}(X, Y)$$

is the Adams differential d_3 .

Proof This is the case $n = 2, m = 4$ of [6, Theorem 15.11]. More precisely, by Theorem 9.3, the tertiary chain complex (A, d, γ, ξ) in $\Pi_{(1,2)}\mathbf{Spec}^{\text{op}}$ yields a 2nd order chain complex in $\text{Nul}_2\mathbf{Spec}^{\text{op}}$ based on the same **EM**-resolution $A_\bullet \rightarrow X$ in $\pi_0\mathbf{Spec}^{\text{op}}$. The construction of $d_{(3)}$ above corresponds to the construction d_3 in [6, Definition 15.8]. $\square$

Remark 7.4 The groups $E_3^{s, t}(X, Y)$ are an instance of the secondary Ext groups defined in [3, §4]. Likewise, the next term $E_4^{s, t}(X, Y) = \ker d_{(3)}/\text{im } d_{(3)}$ is a higher order Ext group as defined in [6, §15].

Theorem 7.5 *A weak equivalence of 2-track algebras induces an isomorphism of higher Ext groups, compatible with the differential $d_{(3)}$. More precisely, let $F: \mathcal{A} \rightarrow \mathcal{A}'$ be a weak equivalence between 2-track algebras $\mathcal{A}$ and $\mathcal{A}'$ which are weakly equivalent to $\Pi_{(1,2)}\mathbf{EM}\{X, Y\}^{\text{op}}$. Then F induces isomorphisms $E_{3, \mathcal{A}}^{s, t}(X, Y) \xrightarrow{\cong} E_{3, \mathcal{A}'}^{s, t}(FX, FY)$ making the diagram*

$$\begin{array}{ccc} E_{3, \mathcal{A}}^{s, t}(X, Y) & \xrightarrow{d_{(3), \mathcal{A}}} & E_{3, \mathcal{A}}^{s+3, t+2}(X, Y) \\ \cong \downarrow & & \cong \downarrow \\ E_{3, \mathcal{A}'}^{s, t}(FX, FY) & \xrightarrow{d_{(3), \mathcal{A}'}} & E_{3, \mathcal{A}'}^{s+3, t+2}(FX, FY) \end{array}$$

commute. Here the additional subscript $\mathcal{A}$ or $\mathcal{A}'$ denotes the ambient 2-track category in which the secondary Ext groups and the differential are defined.

Proof This follows from the case $n = 2$ of [6, Theorem 15.9], or an adaptation of the proof of [3, Theorem 5.1]. $\square$

8 Higher order resolutions

In this section, we specialize some results of [6] about higher order resolutions to the case $n = 2$. We use the fact that a 2-track algebra has an underlying algebra of left 2-cubical balls, which is the topic of Sect. 9.

First, we recall some background on *relative* homological algebra; more details can be found in [3, §1].

Definition 8.1 Let $\mathbf{A}$ be an additive category and $\mathbf{a} \subseteq \mathbf{A}$ a full additive subcategory.

1. A chain complex (A, d) is **a-exact** if for every object X of $\mathbf{a}$ the chain complex $\mathrm{Hom}_{\mathbf{A}}(X, A_{\bullet})$ is an exact sequence of abelian groups.
2. A chain map $f: (A, d) \rightarrow (A', d')$ is an **a-equivalence** if for every object X of $\mathbf{a}$, the chain map $\mathrm{Hom}_{\mathbf{A}}(X, f)$ is a quasi-isomorphism.
3. For an object A of $\mathbf{A}$, an **A-augmented chain complex** $A_{\bullet}^{\epsilon}$ is a chain complex of the form

$$\cdots \longrightarrow A_1 \xrightarrow{d_0} A_0 \xrightarrow{\epsilon} A \longrightarrow 0 \longrightarrow \cdots$$

i.e., with $A_{-1} = A$ and $A_n = 0$ for $n < -1$. Such a complex can be viewed as a chain map $\epsilon: A_{\bullet} \rightarrow A$ where A is a chain complex concentrated in degree 0. The map $\epsilon = d_{-1}$ is called the **augmentation**.

4. An **a-resolution** of A is an A -augmented chain complex $A_{\bullet}^{\epsilon}$ which is **a-exact** and such that for all $n \geq 0$, the object A_n belongs to $\mathbf{a}$. In other words, an **a-resolution** of A is a chain complex $A_{\bullet}$ in $\mathbf{a}$ together with an **a-equivalence** $\epsilon: A_{\bullet} \rightarrow A$.

Example 8.2 Consider the category $\mathbf{A} = \mathbf{Mod}_R$ of R -modules for some ring R , and the subcategory $\mathbf{a}$ of free (or projective) R -modules. This recovers the usual homological algebra of R -modules.

Definition 8.3 Let $\mathcal{A}$ be an abelian category and $F: \mathbf{A} \rightarrow \mathcal{A}$ an additive functor. The **a-relative left derived functors** of F are the functors $L_n^{\mathbf{a}}F: \mathbf{A} \rightarrow \mathcal{A}$ for $n \geq 0$ defined by

$$(L_n^{\mathbf{a}}F)A = H_n(F(A_{\bullet}))$$

where $A_{\bullet} \rightarrow A$ is any **a-resolution** of A .

Likewise, if $F: \mathbf{A}^{\mathrm{op}} \rightarrow \mathcal{A}$ is a contravariant additive functor, its **a-relative right derived functors** of F are defined by

$$(R_{\mathbf{a}}^n F)A = H^n(F(A_{\bullet})).$$

Example 8.4 The **a-relative Ext** groups are given by

$$\mathrm{Ext}_{\mathbf{a}}^n(A, B) := (R_{\mathbf{a}}^n \mathrm{Hom}_{\mathbf{A}}(-, B))(A) = H^n \mathrm{Hom}_{\mathbf{A}}(A_{\bullet}, B).$$

Proposition 8.5 (Correction of 1-tracks) *Let $\mathbf{B}$ be a category enriched in pointed groupoids, such that its homotopy category $\pi_0\mathbf{B}$ is additive. Let $\mathbf{a} \subseteq \pi_0\mathbf{B}$ be a full additive subcategory. Let (A, d, γ) be a secondary pre-chain complex in $\mathbf{B}$ based on an $\mathbf{a}$ -resolution $A_\bullet \rightarrow X$ of an object X in $\pi_0\mathbf{B}$. Then there exists a secondary chain complex (A, d, γ') in $\mathbf{B}$ with the same objects A_i and differentials d_i . In particular (A, d, γ') is also based on the $\mathbf{a}$ -resolution $A_\bullet \rightarrow X$.*

Proof This follows from an adaptation of the proof of [3, Lemma 2.14], or the case $n = 1$ of [6, Theorem 13.2]. $\square$

Proposition 8.6 (Correction of 2-tracks) *Let $\mathcal{A}$ be a 2-track algebra such that its homotopy category $\pi_0\mathcal{A}$ is additive. Let $\mathbf{a} \subseteq \pi_0\mathcal{A}$ be a full additive subcategory. Let (A, d, γ, ξ) be a tertiary pre-chain complex in $\mathcal{A}$ based on an $\mathbf{a}$ -resolution $A_\bullet \rightarrow X$ of an object X in $\pi_0\mathcal{A}$. Then there exists a tertiary chain complex (A, d, γ, ξ') in $\mathcal{A}$ with the same objects A_i , differentials d_i , and left paths γ_i . In particular, (A, d, γ, ξ') is also based on the $\mathbf{a}$ -resolution $A_\bullet \rightarrow X$.*

Proof This follows from the case $n = 2$ of [6, Theorem 13.2]. $\square$

Theorem 8.7 (Resolution Theorem) *Let $\mathcal{A}$ be a 2-track algebra such that its homotopy category $\pi_0\mathcal{A}$ is additive. Let $\mathbf{a} \subseteq \pi_0\mathcal{A}$ be a full additive subcategory. Let $A_\bullet \rightarrow X$ be an $\mathbf{a}$ -resolution in $\pi_0\mathcal{A}$. Then there exists a tertiary chain complex in $\mathcal{A}$ based on the resolution $A_\bullet \rightarrow X$.*

Proof This follows from the resolution theorems [6, Theorems 8.2 and 14.5]. $\square$

9 Algebras of left 2-cubical balls

Proposition 9.1 *Every left cubical ball of dimension 2 is equivalent to C_k for some $k \geq 2$, where $C_k = B_1 \cup \dots \cup B_k$ is the left cubical ball of dimension 2 consisting of k closed 2-cells going cyclically around the vertex 0, with one common 1-cell e_i between successive 2-cells B_i and B_{i+1} , where by convention $B_{k+1} := B_1$.*

See Fig. 1, which is taken from [6, Figure 3].

Proof Let B be a left cubical ball of dimension 2. For each closed 2-cell B_i , equipped with its homeomorphism $h_i: I^2 \xrightarrow{\cong} B_i$, the faces $\partial_1^1 B_i$ and $\partial_2^1 B_i$ are required to be 1-cells of the boundary $\partial B \cong S^1$, while the faces $\partial_1^0 B_i$ and $\partial_2^0 B_i$ are not in ∂B , and therefore must be faces of some other 2-cells. In other words, we have $\partial_1^0 B_i = \partial_1^0 B_j$ or $\partial_1^0 B_i = \partial_2^0 B_j$ for some other 2-cell B_j , in fact a unique B_j , because B is homeomorphic to a 2-disk.

Pick any 2-cell of B and call it B_1 . Then the face $e_1 := \partial_2^0 B_1$ appears as a face of exactly one other 2-cell, which we call B_2 . The remaining face e_2 of B_2 appears as a face of exactly one other 2-cell, which we call B_3 . Repeating this process, we list distinct 2-cells $B_1, \dots, B_k$, and B_{k+1} is one of the previously labeled 2-cells. Then B_{k+1} must be B_1 , with $e_k = \partial_1^0 B_1$, since a 1-cell cannot appear as a common face of three 2-cells. Finally, this process exhausts all 2-cells, because all 2-cells share the common vertex 0, which has a neighborhood homeomorphic to an open 2-disk. $\square$

Fig. 1 The left cubical balls C_2 , C_3 , and C_4

Fig. 2 A left 2-cubical ball

Proposition 9.2 A left 2-cubical ball [6, Definition 10.1] in a pointed space X corresponds to a circular chain of composable left 2-tracks:

$$a = a_0 \xrightarrow{\alpha_1^{\epsilon_1}} a_1 \xrightarrow{\alpha_2^{\epsilon_2}} \cdots \rightarrow a_{k-1} \xrightarrow{\alpha_k^{\epsilon_k}} a_k = a$$

where the sign $\epsilon_i = \pm 1$ is the orientation of the 2-cells in the left cubical ball [6, Definition 10.8]. Moreover, such an expression $(\alpha_1, \dots, \alpha_k)$ of a left 2-cubical ball is unique up to cyclic permutation of the k left 2-tracks α_i . For example, $(\alpha_1, \alpha_2, \dots, \alpha_k)$ and $(\alpha_2, \dots, \alpha_k, \alpha_1)$ represent the same left 2-cubical ball. See Fig. 2.

Proof By our convention for the $\square$ -composition, a left 2-track α defines a morphism between left paths $\alpha: d_1^0 \alpha \Rightarrow d_2^0 \alpha$. The gluing condition for a left 2-cubical ball $(\alpha_1, \dots, \alpha_k)$ based on a left cubical ball $B = B_1 \cup \cdots \cup B_k$ as in Proposition 9.1 is that the restrictions $\alpha_i|_{e_i}$ and $\alpha_{i+1}|_{e_i}$ agree on the common edge $e_i \subset B_i \cap B_{i+1}$. This is the composability condition for $\alpha_{i+1}^{\epsilon_{i+1}} \square \alpha_i^{\epsilon_i}$. Indeed, up to a global sign, the sign of B_i is

$$\epsilon_i = \begin{cases} +1 & \text{if } e_i = \partial_2^0 B_i \\ -1 & \text{if } e_i = \partial_1^0 B_i \end{cases}$$

so that we have $\alpha_i^{\epsilon_i}: \alpha_i|_{e_{i-1}} \Rightarrow \alpha_i|_{e_i}$ and we may take $a_i = \alpha_i|_{e_i}$. $\square$

Theorem 9.3 1. A 2-track algebra $\mathcal{A}$ yields an algebra of left 2-cubical balls [6, Definition 11.1] in the following way. Consider the system $\Theta(\mathcal{A}) := ((\mathcal{A}_{(1,2)}, \otimes), \pi_0 \mathcal{A}, D, \mathcal{O})$, where:

- $(\mathcal{A}_{(1,2)}, \otimes)$ is the underlying 2-graded category of $\mathcal{T}$ (described in Definition 5.1).
- $\pi_0 \mathcal{A}$ is the homotopy category of $\mathcal{A}$.
- $q: (\mathcal{A})^0 = \mathcal{A}_{(1)0} \twoheadrightarrow \pi_0 \mathcal{A}$ is the canonical quotient functor.
- $D: (\pi_0 \mathcal{A})^{\text{op}} \times \pi_0 \mathcal{A} \rightarrow \mathbf{Ab}$ is the functor defined by $D(A, B) = \pi_2 \mathcal{A}_{(1,2)}(A, B)$.
- The obstruction operator $\mathcal{O}$ is obtained by concatenating the corresponding left 2-tracks and using the structural isomorphisms ψ of the mapping 2-track groupoid:

$$\begin{aligned} \mathcal{O}_B(\alpha_1, \alpha_2, \dots, \alpha_k) &= \psi_a(\alpha_k^{\epsilon_k} \square \dots \square \alpha_2^{\epsilon_2} \square \alpha_1^{\epsilon_1}) \in \text{Aut}_{\mathcal{A}_{(2)}(A, B)}(0) \\ &= \pi_2 \mathcal{A}_{(1,2)}(A, B) \end{aligned}$$

where we denoted $a = \delta_0 \alpha_1 = \delta_1 \alpha_k$.

- Given a category $\mathcal{C}$ enriched in pointed spaces, $\Theta(\Pi_{(1,2)} \mathcal{C})$ is the algebra of left 2-cubical balls

$$(\text{Nul}_2 \mathcal{C}, \pi_0 \mathcal{C}, \pi_2 \mathcal{C}(-, -), \mathcal{O})$$

described in [6, §11].

- The construction Θ sends a tertiary pre-chain complex (A, d, δ, ξ) in $\mathcal{A}$ to a 2nd order pre-chain complex in $\Theta(\mathcal{A})$, in the sense of [6, Definition 11.4]. Moreover, (A, d, δ, ξ) is a tertiary chain complex if and only if the corresponding 2nd order pre-chain complex in $\Theta(\mathcal{A})$ is a 2nd order chain complex.

Proof Let us check that the obstruction operator $\mathcal{O}$ is well-defined. By Proposition 9.2, the only ambiguity is the starting left 1-cube a_i in the composition. Two such compositions are conjugate in the groupoid $\mathcal{A}_{(2)}(A, B)$:

$$\begin{aligned} &\alpha_{i-1}^{\epsilon_{i-1}} \square \dots \square \alpha_2^{\epsilon_2} \square \alpha_1^{\epsilon_1} \square \alpha_k^{\epsilon_k} \square \dots \square \alpha_{i+1}^{\epsilon_{i+1}} \square \alpha_i^{\epsilon_i} \\ &= (\alpha_{i-1}^{\epsilon_{i-1}} \square \dots \square \alpha_1^{\epsilon_1}) \square \alpha_k^{\epsilon_k} \square \dots \square \alpha_{i+1}^{\epsilon_{i+1}} \square \alpha_i^{\epsilon_i} \square \dots \square \alpha_1^{\epsilon_1} \square (\alpha_{i-1}^{\epsilon_{i-1}} \square \dots \square \alpha_1^{\epsilon_1})^{\square} \\ &= \beta^{\square} \square \alpha_k^{\epsilon_k} \square \dots \square \alpha_1^{\epsilon_1} \square \beta \end{aligned}$$

with $\beta = (\alpha_{i-1}^{\epsilon_{i-1}} \square \dots \square \alpha_1^{\epsilon_1})^{\square}: a_i \Rightarrow a_0$. Since $\mathcal{A}_{(2)}(A, B)$ is a strictly abelian groupoid, we have the commutative diagram:

$$\begin{array}{ccc} \text{Aut}(a_0) & \xrightarrow{\varphi^\beta} & \text{Aut}(a_i) \\ & \searrow \psi_{a_0} & \downarrow \psi_{a_i} \\ & & \text{Aut}(0) \end{array}$$

so that $\mathcal{O}_B(\alpha_1, \dots, \alpha_k)$ is well-defined.

The remaining properties listed in [6, Definition 11.1] are straightforward verifications. $\square$

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Appendix A: Models for homotopy 2-types

Recall that the left n -cubical set $\text{Nul}_n(X)$ of a pointed space X depends only on the n -type $P_n X$ of X [6, §1]. In particular the fundamental 2-track groupoid $\Pi_{(1,2)}(X)$ depends only on the 2-type $P_2 X$ of X . There are various algebraic models for homotopy 2-types in the literature, using 2-dimensional categorical structures. Let us mention the weak 2-groupoids of [15], the bigroupoids of [12], the double groupoids of [10], the two-typical double groupoids of [7], and the double groupoids with filling condition of [11].

In contrast, 2-track groupoids are *not* models for homotopy 2-types, not even of connected homotopy 2-types. In the application we are pursuing, the functor $\Pi_{(1,2)}$ will be applied to topological abelian groups, hence products of Eilenberg–MacLane spaces. We are *not* trying to encode the homotopy 2-type of the Eilenberg–MacLane mapping theory, but rather as little information as needed in order to compute the Adams differential d_3 .

The fundamental 2-track groupoid $\Pi_{(1,2)}(X)$ encodes the 1-type of X , via the fundamental groupoid $\Pi_{(1)}(X)$. Moreover, as noted in Remark 3.13, it also encodes the homotopy group $\pi_2(X)$. However, it fails to encode the $\pi_1(X)$ -action on $\pi_2(X)$, as we will show below.

A.1 Connected 2-track groupoids

Recall that a category $\mathcal{C}$ is called *skeletal* if any isomorphic objects are equal. A *skeleton* of $\mathcal{C}$ is a full subcategory on a collection consisting of one representative object in each isomorphism class of objects of $\mathcal{C}$. Every groupoid is equivalent to a disjoint union of groups, that is, a coproduct of single-object groupoids. The inclusion $\text{sk} G \xrightarrow{\sim} G$ of a skeleton of G provides such an equivalence. A similar construction yields the following statement for 2-track groupoids.

Lemma A.1 *Let $G = (G_{(1)}, G_{(2)})$ be a 2-track groupoid.*

1. *There is a weak equivalence of 2-track groupoids $\text{sk}_{(1)} G \xrightarrow{\sim} G$ where the first groupoid of $\text{sk}_{(1)} G$ is skeletal.*
2. *If G is connected and $G_{(1)}$ is skeletal, then there is a weak equivalence of 2-track groupoids $\text{sk}_{(2)} G \xrightarrow{\sim} G$ where both groupoids of $\text{sk}_{(2)} G$ are skeletal.*

In particular, if G is connected, then $\text{sk}_{(2)} \text{sk}_{(1)} G \xrightarrow{\sim} \text{sk}_{(1)} G \xrightarrow{\sim} G$ is a weak equivalence between G and a 2-track groupoid whose constituent groupoids are both skeletal.

Lemma A.2 *Let G and G' be connected 2-track groupoids whose constituent groupoids are skeletal. If there are isomorphisms of homotopy groups $\varphi_1: \pi_1 G \simeq \pi_1 G'$ and $\varphi_2: \pi_2 G \simeq \pi_2 G'$, then there is a weak equivalence $\varphi: G \xrightarrow{\sim} G'$.*

Proof Since $G_{(1)}$ and $G'_{(1)}$ are skeletal, they are in fact groups, and the group isomorphism φ_1 is an isomorphism of groupoids $\varphi_{(1)}: G_{(1)} \xrightarrow{\cong} G'_{(1)}$.

Now we define a functor $\varphi_{(2)}: G_{(2)} \rightarrow G'_{(2)}$. On objects, it is given by the composite

$$\begin{aligned} G_{(2)0} = \text{Comp } G_{(2)} &\xrightarrow[\cong]{q} \text{Star } G_{(1)} = G_{(1)}(0, 0) = \pi_1 G \longrightarrow \\ &\xrightarrow[\cong]{\varphi_1} \pi_1 G' = G'_{(1)}(0, 0) = \text{Star } G'_{(1)} \xleftarrow[\cong]{q} \text{Comp } G'_{(2)} = G'_{(2)0} \end{aligned}$$

which is a bijection. On morphisms, $\varphi_{(2)}$ is defined as follows. We have $G_{(2)}(a, b) = \emptyset$ when $a \neq b$, so there is nothing to define then. On the automorphisms of an object $a \in G_{(2)0}$, define $\varphi_{(2)}$ as the composite

$$\begin{aligned} G_{(2)}(a, a) = \text{Aut}_{G_{(2)}}(a) &\xrightarrow[\cong]{\psi_a} \text{Aut}_{G_{(2)}}(0) = \pi_2 G \longrightarrow \\ &\xrightarrow[\cong]{\varphi_2} \pi_2 G' = \text{Aut}_{G'_{(2)}}(0') \xleftarrow[\cong]{\psi'_{\varphi(a)}} \text{Aut}_{G'_{(2)}}(\varphi(a)) = G'_{(2)}(\varphi(a), \varphi(a)). \end{aligned}$$

Then $\varphi_{(2)}$ is a functor and commutes with the structural isomorphisms, by construction. Thus $\varphi = (\varphi_{(1)}, \varphi_{(2)}): G \rightarrow G'$ is a morphism of 2-track groupoids, and is moreover a weak equivalence. $\square$

Corollary A.3 *Let G and G' be connected 2-track groupoids with isomorphic homotopy groups $\pi_i G \cong \pi_i G'$ for $i = 1, 2$. Then G and G' are weakly equivalent, i.e., there is a zigzag of weak equivalences between them.*

Proof Consider the zigzag of weak equivalences

$$\begin{array}{ccc} & G & \\ \sim \uparrow & & \uparrow \sim \\ & \text{sk}_{(1)} G & \\ \sim \uparrow & & \uparrow \sim \\ & \text{sk}_{(2)} \text{sk}_{(1)} G & \xrightarrow[\sim]{\varphi} \text{sk}_{(2)} \text{sk}_{(1)} G' \\ & & \uparrow \sim \\ & & \text{sk}_{(1)} G' \\ & & \uparrow \sim \\ & & G' \end{array}$$

where the bottom morphism φ is obtained from Lemma A.2. $\square$

By Remark 3.13, the functor $\Pi_{(1,2)}: \mathbf{Top}_* \rightarrow \mathbf{Gpd}_{(1,2)}$ induces a functor

$$\Pi_{(1,2)}: \text{Ho}(\mathbf{connected 2-Types}) \rightarrow \text{Ho}(\mathbf{Gpd}_{(1,2)}) \quad (\text{A.1})$$

where the left-hand side denotes the homotopy category of connected 2-types (localized with respect to weak homotopy equivalences), and the right-hand side denotes the localization with respect to weak equivalences, as in Definition 3.12.

Proposition A.4 *The functor $\Pi_{(1,2)}$ in (A.1) is not an equivalence of categories.*

Proof Let X and Y be connected 2-types with isomorphic homotopy groups π_1 and π_2 , but distinct π_1 -actions on π_2 . Then X and Y are not weakly equivalent, but $\Pi_{(1,2)}(X)$ and $\Pi_{(1,2)}(Y)$ are weakly equivalent, by Corollary A.3. $\square$

A.2 Comparison to bigroupoids

Any algebraic model for (pointed) homotopy 2-types has an underlying 2-track groupoid. Using the globular description in Remark 3.1, the most direct comparison is to the bigroupoids of [12]. A *pointed* bigroupoid (resp. double groupoid) will mean one equipped with a chosen object, here denoted x_0 to emphasize that it is unrelated to the algebraic structure of the bigroupoid.

Proposition A.5 *Let $\Pi_2^{\text{BiGpd}}(X)$ denote the homotopy bigroupoid of a space X constructed in [12], where it was denoted $\Pi_2(X)$.*

1. *There is a forgetful functor U from pointed bigroupoids to 2-track groupoids.*
2. *For a pointed space X , there is a natural isomorphism of 2-track groupoids $\Pi_{(1,2)}(X) \cong U\Pi_2^{\text{BiGpd}}(X)$.*

Proof Let B be a bigroupoid. We construct a 2-track groupoid UB as follows. The first constituent groupoid of UB is the underlying groupoid of B

$$UB_{(1)} := \pi_0 B$$

obtained by taking the components of each mapping groupoid $B(x, y)$. The second constituent groupoid of UB is a coproduct of mapping groupoids

$$UB_{(2)} := \coprod_{x \in \text{Ob}(B)} B(x, x_0).$$

The quotient function $q: UB_{(2)0} \rightarrow \text{Star } UB_{(1)}$ is induced by the natural quotient maps $\text{Ob}(B(x, x_0)) \rightarrow \pi_0 B(x, x_0)$. To define the structural isomorphisms

$$\psi_a: \text{Aut}(a) \xrightarrow{\cong} \text{Aut}(c_{x_0})$$

for objects $a \in UB_{(2)0}$, which are 1-morphisms to the basepoint $a: x \rightarrow x_0$, consider the diagram

where $\lambda: c_{x_0} \bullet a \Rightarrow a$ is the *left identity* coherence 2-isomorphism, $\bullet$ denotes composition of 1-morphisms, and c_{x_0} is the identity 1-morphism of the object x_0 . (We kept our notation $\square$ for composition of 2-morphisms.) The inverse $\psi_a^{-1}: \text{Aut}(c_{x_0}) \rightarrow \text{Aut}(a)$ is defined by going from top to bottom in the diagram, namely

$$\psi_a^{-1}(\alpha) = \lambda \square (\alpha \bullet \text{id}_a^{\square}) \square \lambda^{\square}.$$

One readily checks that UB is a 2-track groupoid, that this construction U is functorial, and that $U\Pi_2^{\text{BiGpd}}(X)$ is naturally isomorphic to $\Pi_{(1,2)}(X)$ as 2-track groupoids. $\square$

A.3 Comparison to double groupoids

The homotopy double groupoid $\rho_2^{\square}(X)$ from [10] is a cubical construction. Following the terminology therein, *double groupoid* will be shorthand for *edge symmetric double groupoid with connection*.

Let us recall the geometric idea behind $\rho_2^{\square}(X)$. A path $a: I \rightarrow X$ has an underlying *semitrack* $\langle a \rangle$, defined as its equivalence class with respect to *thin* homotopy rel ∂I . A semitrack $\langle a \rangle$ in turn has an underlying track $\{a\}$. A square $u: I^2 \rightarrow X$ has an underlying 2-track $\{u\}$. A 2-track $\{u\}$ in turn has an underlying equivalence class $\{u\}_T$ with respect to *cubically thin homotopy*, i.e., a homotopy whose restriction to the boundary ∂I^2 is thin (not necessarily stationary). The homotopy double groupoid $\rho_2^{\square}(X)$ encodes semitracks $\langle a \rangle$ in X and 2-tracks $\{u\}_T$ up to cubically thin homotopy.

Proposition A.6 *Let $\rho_2^{\square}(X)$ denote the homotopy double groupoid of a space X constructed in [10].*

1. *There is a forgetful functor U from pointed double groupoids to 2-track groupoids.*
2. *For a pointed space X , there is a natural weak equivalence of 2-track groupoids*

$$\Pi_{(1,2)}(X) \xrightarrow{\sim} U\rho_2^{\square}(X).$$

Proof We adopt the notation of [10], including that compositions in a double groupoid are written in diagrammatic order, i.e., $a + b$ denotes the composition $x \xrightarrow{a} y \xrightarrow{b} z$. However, we keep our graphical convention for the two axes:

Let D be a double groupoid, whose data is represented in the diagram of sets

along with connections $\Gamma^-, \Gamma^+ : D_1 \rightarrow D_2$. Two 1-morphisms $a, b \in D_1$ with same endpoints $\partial_1^-(a) = \partial_1^-(b) = x$, $\partial_1^+(a) = \partial_1^+(b) = y$ are called *homotopic* if there exists a 2-morphism $u \in D_2$ satisfying $\partial_2^-(u) = a$, $\partial_2^+(u) = b$, $\partial_1^-(u) = \epsilon(x)$, $\partial_1^+(u) = \epsilon(y)$. We write $a \sim b$ if a and b are homotopic.

We now define the underlying 2-track groupoid UD . The first constituent groupoid $UD_{(1)}$ has object set D_0 and morphism set $D_1 / \sim$, with groupoid structure inherited from the groupoid (D_0, D_1) . The second constituent groupoid $UD_{(2)}$ has object set

$$UD_{(2)0} := \{a \in D_1 \mid \partial_1^+(a) = x_0\}.$$

A morphism in $UD_{(2)}$ from a to b is an element $u \in D_2$ satisfying $\partial_1^-(u) = a$, $\partial_2^-(u) = b$, $\partial_1^+(u) = \epsilon(x_0)$, $\partial_2^+(u) = \epsilon(x_0)$, as illustrated here:

Composition in $UD_{(2)}$ is defined as follows. Given 1-morphisms $a, b, c : x \rightarrow x_0$ in D_1 and morphisms $u : a \Rightarrow b$ and $v : b \Rightarrow c$ in $UD_{(2)}$, their composition $v \square u : a \Rightarrow c$ is defined by

$$\begin{aligned}
 v \square u &= (\Gamma^+(b) +_2 u) +_1 (v +_2 \odot_{x_0}) \\
 &= (\Gamma^+(b) +_2 u) +_1 v \\
 &= (\Gamma^+(b) +_1 v) +_2 u
 \end{aligned}$$

as illustrated here:

The identity morphisms in $UD_{(2)}$ are given by $\text{id}_a^\square = \Gamma^-(a)$. The inverse of $u: a \Rightarrow b$ is given by

$$\begin{aligned}
 u^\square &= ((-1)\Gamma^+(b) +_2 (-1)u) +_1 (\epsilon_2(a) +_2 \Gamma^-(a)) \\
 &= ((-1)\Gamma^+(b) +_2 (-1)u) +_1 \Gamma^-(a)
 \end{aligned}$$

as illustrated here:

The structural isomorphisms $\psi_a^{-1}: \text{Aut}(\epsilon(x_0)) \rightarrow \text{Aut}(a)$ are defined by

$$\begin{aligned}
 \psi_a^{-1}(u) &= (\Gamma^-(a) +_2 \odot_{x_0}) +_1 (\odot_{x_0} +_2 u) \\
 &= \Gamma^-(a) +_1 u \\
 &= \Gamma^-(a) +_2 u
 \end{aligned}$$

as illustrated here:

The quotient function $q: UD_{(2)0} \twoheadrightarrow \text{Star } UD_{(1)}$ is induced by the quotient function $D_1 \rightarrow D_1/\sim$. One readily checks that UD is a 2-track groupoid, and that this construction U is functorial.

For a pointed space X , define a comparison map $\Pi_{(1,2)}(X) \rightarrow U\rho_2^\square(X)$ which is an isomorphism on $\Pi_{(1)}(X)$, and which quotients out the thin homotopy relation between left paths in X and the cubically thin homotopy relation between left 2-tracks. This defines a natural weak equivalence of 2-track groupoids. $\square$

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