On the group structure of $[J(X), \Omega(Y)]$

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Abstract For a co- H -space X , we determine the group structure of $[J_n(X), \Omega(Y)]$ and hence of $[J(X), \Omega(Y)] = \varprojlim [J_n(X), \Omega(Y)]$, where $J_n(X)$ denotes the n -th stage of the James filtration of the James construction $J(X)$ of X . This is obtained by determining the co- H -structure of the reduced suspension $\Sigma J_n(X)$ (and of $\Sigma J(X)$). To capture the effect of $\pi_1(Y)$, we replace $J_n(X)$ by $J_n^*(X) = J_n(X) \sqcup *$ [respectively $J(X)$ by $J^*(X) = J(X) \sqcup *$] and determine the co- H -structure of $\Sigma J^*(X)$ [and of $\Sigma J^*(X)$].

Keywords Cohen groups · Torus homotopy groups · James construction · Co-multiplication

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Introduction

The *total Cohen group* $[J(X), \Omega(Y)]$, first introduced by Cohen [4] as an approach to the Barratt conjecture, plays an important role in classical homotopy theory and has found many connections with other areas of mathematics (see e.g., [4, 5, 18, 19]). Here, $J(X)$ denotes the James construction of a well pointed space X . Since $J(X) = \varinjlim J_n(X)$, where $J_1(X) = X \subseteq J_2(X) \subseteq \cdots \subseteq J_n(X) \subseteq J_{n+1}(X) \subseteq \cdots$ is the James filtration, it follows $[J(X), \Omega(Y)] \cong \varprojlim [J_n(X), \Omega(Y)]$ and one is interested in understanding the group structure of the *Cohen groups* $[J_n(X), \Omega(Y)]$. When $X = \mathbb{S}^1$, $[J(\mathbb{S}^1), \Omega(Y)]$ is, as a set, in 1–1 correspondence with the infinite direct product $\prod_{i \geq 2} \pi_i(Y)$ of the higher homotopy groups of Y . However, the group structure of $[J(\mathbb{S}^1), \Omega(Y)]$ is far from being abelian. Note that the suspension $\Sigma J_n(X)$ has the homotopy type of the suspension of the wedge $\Sigma \bigvee_{i=1}^n X^{(i)}$, where $X^{(i)}$ denotes the i -fold smash power of X . Using the usual co-multiplication on the suspension $\Sigma X^{(i)}$ for $i \geq 1$, it is easy to see that the groups $[\bigvee_{i=1}^n \Sigma X^{(i)}, Y]$ and $\prod_{i=1}^n [\Sigma X^{(i)}, Y]$ are isomorphic. Similarly, $\Sigma J(X)$ has the homotopy type of $\Sigma \bigvee_{i=1}^\infty X^{(i)}$, the suspension of the wedge. Then, the groups $[\bigvee_{i=1}^\infty \Sigma X^{(i)}, Y]$ and $\prod_{i=1}^\infty [\Sigma X^{(i)}, Y]$ are isomorphic. Similar to the case $X = \mathbb{S}^1$, the co- H -structure of $\Sigma J_n(X)$ [and hence of $\Sigma J(X)$] is different from the usual suspension co-multiplication in general.

In [14], we made use of the fact that $[J_n(\mathbb{S}^1), \Omega(Y)]$ can be embedded inside the Fox torus homotopy group $\tau_{n+1}(Y)$, which is completely determined by the homotopy groups $\{\pi_i(Y)\}_{i=2}^{n+1}$ and their Whitehead products. Similarly, in view of [13], we can embed $[J_n(X), \Omega(Y)]$ as subgroups of $[\Sigma(X^n \sqcup *), Y]$ so that the co- H -structure of $\Sigma J_n(X)$ is induced by the co- H -structure of $\hat{\mathfrak{F}}_{n+1}(X) = \Sigma(X^n \sqcup *)$. Strictly speaking, it was shown in [19, Chapter I] that the Cohen groups $[J_n(X), \Omega(Y)]$ are subgroups of $[X^n, \Omega(Y)]$ as the equalizer of face homomorphisms $d_i: [X^n, \Omega(Y)] \rightarrow [X^{n+1}, \Omega(Y)]$ for $i = 1, \dots, n$. Further, in view of [12], the groups $[X^n, \Omega(Y)]$ are closely related to the generalized Fox groups $[\hat{\mathfrak{F}}_{n+1}(X), Y]$. In fact, using the cofibration $\mathbb{S}^0 \rightarrow X^n \sqcup * \rightarrow X^n$, one obtains an isomorphism

$$[\hat{\mathfrak{F}}_{n+1}(X), Y] \cong [X^n, \Omega(Y)] \rtimes \pi_1(Y).$$

Thus, $[X^n, \Omega(Y)] \cong [\hat{\mathfrak{F}}_{n+1}(X), Y]$ provided Y is a simply connected space. When $X = \mathbb{S}^1$, $\Sigma J(\mathbb{S}^1) \simeq \mathbb{S}^2 \vee \mathbb{S}^3 \vee \cdots$ so that $[J(\mathbb{S}^1), \Omega(Y)]$ and $\prod_{i=2}^\infty \pi_i(Y)$ are isomorphic as sets. In order to incorporate the group $\pi_1(Y)$, we consider $J^*(\mathbb{S}^1) = J(\mathbb{S}^1) \sqcup *$ and $J_n^*(\mathbb{S}^1) = J_n(\mathbb{S}^1) \sqcup *$ [or more generally $J^*(X) = J(X) \sqcup *$ and $J_n^*(X) = J_n(X) \sqcup *$ for any space X].

In [9], Fox gave a complete description of the commutators in $\tau_k(Y)$ in terms of Whitehead products together with a combinatorial argument involving embeddings of $\pi_i(Y)$ into $\tau_k(Y)$ for $k > i$. This combinatorial counting argument was given an explicit

formula in [14] which allows us to determine the group structure of $[J_n(\mathbb{S}^1), \Omega(Y)]$ and hence of $[J(\mathbb{S}^1), \Omega(Y)]$.

In this paper, we generalize the arguments in [14] replacing $\mathbb{S}^1$ by a co- H -space X . In doing so, we follow [14] and determine the group structure of $[J_n(X), \Omega(Y)]$. This paper is organized as follows. In Sect. 1, we study the co- H -structure of the generalized Fox space $\hat{\mathfrak{F}}_n(X)$ via the generalized Whitehead products. In Sect. 2, we investigate the suspension co- H -structure on $\Sigma J_n^*(X)$. Theorem 2.6 describes this structure, provided X is a co- H -space. In Sect. 3, we make some computations of $[J_n(X^{(r)}), \Omega(Y)]$ for $r \geq 1$. In particular, we give an example (Example 3.3), where $[J(\mathbb{S}^r), \Omega(Y)]$ is not a subgroup of $[J(\mathbb{S}^1), \Omega(Y)]$. We also construct a space $\hat{Y}$ such that $\Omega(\hat{Y})$ is not homotopy commutative and $[J(\mathbb{S}^r), \Omega(\hat{Y})]$ is abelian for all $r \geq 1$ (Example 3.4 and Proposition 3.5).

1 Structure of $[\hat{\mathfrak{F}}_n(X), Y]$ and generalized Whitehead products

From now on, all spaces are assumed to be pointed with the homotopy type of CW -complexes and we do not distinguish between a map and its homotopy class. To examine the suspension co- H -structure on $\Sigma J_n^*(X)$ in the next section, we first recall from [12] such a structure on the generalized Fox space $\hat{\mathfrak{F}}_n(X) = \Sigma(X^{n-1} \sqcup *)$. It follows that $\hat{\mathfrak{F}}_1(X) = \mathbb{S}^1$ and $\hat{\mathfrak{F}}_n(\mathbb{S}^1) = F_n$, the Fox space.

Similar to F_n , the suspension co- H -structure on $\hat{\mathfrak{F}}_n(X)$ is given by the following split exact sequence

$$1 \rightarrow [\Sigma \mathcal{P}_{n-1}(X), Y] \rightarrow [\hat{\mathfrak{F}}_n(X), Y] \xrightarrow{\leftarrow} [\hat{\mathfrak{F}}_{n-1}(X), Y] \rightarrow 1$$

which can be deduced from [11, Theorem 3.1]. Here, $\mathcal{P}_n(X) = X^n / X^{n-1}$ is the n -fold pinched space of X .

Hence, $[\hat{\mathfrak{F}}_n(X), Y] \cong [\Sigma \mathcal{P}_{n-1}(X), Y] \rtimes [\hat{\mathfrak{F}}_{n-1}(X), Y]$ is the semi-direct product with respect to the natural action

$$[\hat{\mathfrak{F}}_{n-1}(X), Y] \times [\Sigma \mathcal{P}_{n-1}(X), Y] \rightarrow [\Sigma \mathcal{P}_{n-1}(X), Y].$$

In particular, for $Y = \hat{\mathfrak{F}}_{n-1}(X) \vee \Sigma \mathcal{P}_{n-1}(X)$, by the natural bijection $[\hat{\mathfrak{F}}_{n-1}(X), Y] \times [\Sigma \mathcal{P}_{n-1}(X), Y] \cong [\hat{\mathfrak{F}}_{n-1}(X) \vee \Sigma \mathcal{P}_{n-1}(X), Y]$, the identity map id_Y is sent to the corresponding co-action

$$\alpha_{n-1}: \Sigma \mathcal{P}_{n-1}(X) \rightarrow \hat{\mathfrak{F}}_{n-1}(X) \vee \Sigma \mathcal{P}_{n-1}(X).$$

Furthermore, the natural bijection $[\Sigma \mathcal{P}_{n-1}(X), Y] \rtimes [\hat{\mathfrak{F}}_{n-1}(X), Y] \cong [\hat{\mathfrak{F}}_n(X), Y]$ for any space Y yields a homotopy equivalence

$$\hat{\mathfrak{F}}_n(X) \simeq \hat{\mathfrak{F}}_{n-1}(X) \vee \Sigma \mathcal{P}_{n-1}(X).$$

Let

$$\hat{v}_n: \Sigma \hat{\mathfrak{F}}_n(X) \rightarrow \Sigma \hat{\mathfrak{F}}_n(X) \vee \Sigma \hat{\mathfrak{F}}_n(X)$$

and

$$v_n: \Sigma \mathcal{P}_n(X) \rightarrow \Sigma \mathcal{P}_n(X) \vee \Sigma \mathcal{P}_n(X)$$

be the suspension co- H -structure on $\hat{\mathfrak{F}}_n(X)$ and $\mathcal{P}_n(X)$ for $n \geq 1$, respectively.

Certainly, $\hat{v}_1: \hat{\mathfrak{F}}_1(X) \rightarrow \hat{\mathfrak{F}}_1(X) \vee \hat{\mathfrak{F}}_1(X)$ is the suspension co- H -structure on $\hat{\mathfrak{F}}_1(X) = \mathbb{S}^1$. Given $\hat{v}_{n-1}: \Sigma \hat{\mathfrak{F}}_{n-1}(X) \rightarrow \Sigma \hat{\mathfrak{F}}_{n-1}(X) \vee \Sigma \hat{\mathfrak{F}}_{n-1}(X)$, consider

$$\hat{v}_n^1: \hat{\mathfrak{F}}_{n-1}(X) \xrightarrow{\hat{v}_{n-1}} \hat{\mathfrak{F}}_{n-1}(X) \vee \hat{\mathfrak{F}}_{n-1}(X) \hookrightarrow \hat{\mathfrak{F}}_n(X) \vee \hat{\mathfrak{F}}_n(X).$$

Next, let

$$\begin{aligned} \lambda_{n+1}: & (\hat{\mathfrak{F}}_{n-1}(X) \vee \Sigma \mathcal{P}_{n-1}(X)) \vee \Sigma \mathcal{P}_{n-1}(X) \\ & \equiv \Sigma \mathcal{P}_{n-1}(X) \vee (\hat{\mathfrak{F}}_{n-1}(X) \vee \Sigma \mathcal{P}_{n-1}(X)) \\ & \xrightarrow{\eta \vee \text{id}} (\hat{\mathfrak{F}}_{n-1}(X) \vee \Sigma \mathcal{P}_{n-1}(X)) \vee (\hat{\mathfrak{F}}_{n-1}(X) \vee \Sigma \mathcal{P}_{n-1}(X)), \end{aligned}$$

where $\equiv$ denotes the identification $A \vee B \equiv B \vee A$ and $\eta: \Sigma \mathcal{P}_{n-1}(X) \hookrightarrow \hat{\mathfrak{F}}_{n-1}(X) \vee \Sigma \mathcal{P}_{n-1}(X)$ is the inclusion map. Then, the map

$$\hat{v}_n^2: \Sigma \mathcal{P}_{n-1}(X) \rightarrow (\hat{\mathfrak{F}}_{n-1}(X) \vee \Sigma \mathcal{P}_{n-1}(X)) \vee (\hat{\mathfrak{F}}_{n-1}(X) \vee \Sigma \mathcal{P}_{n-1}(X))$$

is given by the following commutative diagram

$$\begin{array}{ccc} \Sigma \hat{\mathfrak{F}}_{n-1}(X) \vee \Sigma \mathcal{P}_{n-1}(X) & \xrightarrow{\alpha_{n-1} \vee \text{id}_{\Sigma \mathcal{P}_{n-1}(X)}} & (\hat{\mathfrak{F}}_{n-1}(X) \vee \Sigma \mathcal{P}_{n-1}(X)) \vee \Sigma \mathcal{P}_{n-1}(X) \\ \Sigma v_{n-1} \uparrow & & \downarrow \lambda_{n+1} \\ \Sigma \mathcal{P}_{n-1}(X) & \xrightarrow{\hat{v}_n^2} & (\hat{\mathfrak{F}}_{n-1}(X) \vee \Sigma \mathcal{P}_{n-1}(X)) \vee (\hat{\mathfrak{F}}_{n-1}(X) \vee \Sigma \mathcal{P}_{n-1}(X)). \end{array} \quad (1.1)$$

In view of [12, Theorem 2.2], the maps $\hat{v}_n^1$ and $\hat{v}_n^2$ lead to the suspension co- H -structure

$$\hat{v}_n: \hat{\mathfrak{F}}_n(X) \longrightarrow \hat{\mathfrak{F}}_n(X) \vee \hat{\mathfrak{F}}_n(X)$$

on the space $\hat{\mathfrak{F}}_n(X)$.

In [9], the group structure on the Fox torus homotopy groups $\tau_n(Y) = [F_n, Y]$ is described by means of the classical Whitehead products. Similarly, the group structure on the generalized Fox group $[\hat{\mathfrak{F}}_n(X), Y]$ can also be described by means of the generalized Whitehead products as follows.

We mimic below the work of Fox [9] on determining whether $\alpha \in \pi_k(Y)$, $\beta \in \pi_l(Y)$, when embedded in $\tau_r(Y)$ commute. First we let $\alpha \in [\Sigma X^{(k)}, Y]$ and $\beta \in$

$[\Sigma X^{(l)}, Y]$. The generalized Whitehead product

$$[-, -] : [\Sigma X^{(k)}, Y] \times [\Sigma X^{(l)}, Y] \rightarrow [\Sigma X^{(k+l)}, Y]$$

is given by the Whitehead map $\Sigma X^{(k+l)} \rightarrow \Sigma X^{(k)} \vee \Sigma X^{(l)}$ for $k + l \leq r + 1$. Following [9], we consider a $(k - 1)$ subset $\mathbf{a}$ and an $(l - 1)$ subset $\mathbf{b}$ of the set of indices $\{1, 2, \dots, r\}$ for some $r \geq k + l - 1$. The sets $\mathbf{a}$ and $\mathbf{b}$ determine two embeddings $[\Sigma X^{(i)}, Y] \rightarrow [\hat{\mathfrak{F}}_{r+1}(X), Y]$ for $i = k, l$. Denote by $\alpha^{\mathbf{a}}$ and $\beta^{\mathbf{b}}$ the corresponding images of α and β in $[\hat{\mathfrak{F}}_{r+1}(X), Y]$.

Proposition 1.1 *Let $X = \Sigma A$ be a suspension on a space A .*

(1) *If $\mathbf{a} \cap \mathbf{b} = \emptyset$ then $(\alpha^{\mathbf{a}}, \beta^{\mathbf{b}}) = (-1)^{w+|\mathbf{a}|-1}[\alpha, \beta]^{\mathbf{a} \cup \mathbf{b}}$.*

(2) *If $\mathbf{a} \cap \mathbf{b} \neq \emptyset$ then $(\alpha^{\mathbf{a}}, \beta^{\mathbf{b}}) = 1$.*

Here (x, y) denotes the commutator $xyx^{-1}y^{-1}$ of x and y and $w = \sum_{i \in \mathbf{a}, j \in \mathbf{b}} w_{i,j}$, where $w_{i,j} = 1$ if $j < i$ and $w_{i,j} = 0$ otherwise.

Proof (1) follows easily from the commutative diagram

$$\begin{array}{ccc} [\hat{\mathfrak{F}}_{k+1}(X), Y] \times [\hat{\mathfrak{F}}_{l+1}(X), Y] & \longrightarrow & [\hat{\mathfrak{F}}_{r+1}(X), Y] \\ \uparrow & & \uparrow \\ [\Sigma X^{(k)}, Y] \times [\Sigma X^{(l)}, Y] & \longrightarrow & [\Sigma X^{(k+l)}, Y]. \end{array} \quad (1.2)$$

(2): Let $\mathbf{a} \cap \mathbf{b} \neq \emptyset$ and consider two obvious homeomorphisms:

$$\chi_1 : A \times [1/6, 1/3]/\sim \longrightarrow X = \Sigma A$$

and

$$\chi_2 : A \times [2/3, 5/6]/\sim \longrightarrow X = \Sigma A.$$

Suppose $\alpha^{\mathbf{a}}$ and $\beta^{\mathbf{b}}$ are represented by maps $f, g : \hat{\mathfrak{F}}_{r+1}(X) \rightarrow Y$, respectively. For $m \in \mathbf{a} \cap \mathbf{b}$ and $x_m = [a, s] \in X = \Sigma A$, we define $H_1, H_2 : \hat{\mathfrak{F}}_{r+1}(X) \rightarrow Y$ given by:

$$\begin{aligned} & H_1([t, x_1, \dots, x_r]) \\ &= \begin{cases} f([2t, x_1, \dots, \chi_1([a, s]), \dots, x_r]), & \text{if } 1/6 \leq s \leq 1/3 \text{ and } 0 \leq t \leq 1/2; \\ g([2t - 1, x_1, \dots, \chi_2([a, s]), \dots, x_r]), & \text{if } 2/3 \leq s \leq 5/6 \text{ and } 1/2 \leq t \leq 1; \\ *, & \text{otherwise} \end{cases} \end{aligned}$$

and

$$\begin{aligned} & H_2([t, x_1, \dots, x_r]) \\ &= \begin{cases} g([2t, x_1, \dots, \chi_1([a, s]), \dots, x_r]), & \text{if } 1/6 \leq s \leq 1/3 \text{ and } 0 \leq t \leq 1/2; \\ f([2t - 1, x_1, \dots, \chi_2([a, s]), \dots, x_r]), & \text{if } 2/3 \leq s \leq 5/6 \text{ and } 1/2 \leq t \leq 1; \\ *, & \text{otherwise.} \end{cases} \end{aligned}$$

Then, H_1 represents $\alpha^{\mathbf{a}}\beta^{\mathbf{b}}$, and H_2 represents $\beta^{\mathbf{b}}\alpha^{\mathbf{a}}$. Because of a homotopy $H_1 \simeq H_2$, this leads to $(\alpha^{\mathbf{a}}, \beta^{\mathbf{b}}) = 1$ and the proof is complete. $\square$

Proposition 1.1 leads to:

Corollary 1.2 *Let X be a co- H -space.*

(1) *If $\mathbf{a} \cap \mathbf{b} = \emptyset$ then $(\alpha^{\mathbf{a}}, \beta^{\mathbf{b}}) = (-1)^{w+|\mathbf{a}|-1}[\alpha, \beta]^{\mathbf{a} \cup \mathbf{b}}$.*

(2) *If $\mathbf{a} \cap \mathbf{b} \neq \emptyset$ then $(\alpha^{\mathbf{a}}, \beta^{\mathbf{b}}) = 1$.*

Proof If X is a co- H -space then, in view of [10, Theorem 1.1], there is a coretraction map $\gamma_X: X \rightarrow \Sigma\Omega(X)$ of the natural projection map $p_X: \Sigma\Omega(X) \rightarrow X$, i.e., $p_X \circ \gamma_X \simeq \text{id}_X$. Here, p_X is the adjoint of the map Ωid_X .

Now, given $\alpha \in [\Sigma X^{(k)}, Y]$ and $\beta \in [\Sigma X^{(l)}, Y]$, consider $\alpha \Sigma(p_X)^{(k)} \in [\Sigma(\Sigma\Omega(X))^{(k)}, Y]$ and $\beta \Sigma(p_X)^{(l)} \in [\Sigma(\Sigma\Omega(X))^{(l)}, Y]$. Then, Proposition 1.1 yields:

(1) if $\mathbf{a} \cap \mathbf{b} = \emptyset$ then $((\alpha \Sigma(p_X)^{(k)})^{\mathbf{a}}, (\beta \Sigma(p_X)^{(l)})^{\mathbf{b}}) = (-1)^{w+|\mathbf{a}|-1}[\alpha \Sigma(p_X)^{(k)}, \beta \Sigma(p_X)^{(l)}]^{\mathbf{a} \cup \mathbf{b}}$;

(2) if $\mathbf{a} \cap \mathbf{b} \neq \emptyset$ then $((\alpha \Sigma(p_X)^{(k)})^{\mathbf{a}}, (\beta \Sigma(p_X)^{(l)})^{\mathbf{b}}) = 1$.

But,

$$((\alpha \Sigma(p_X)^{(k)})^{\mathbf{a}}, (\beta \Sigma(p_X)^{(l)})^{\mathbf{b}}) = (\alpha^{\mathbf{a}}, \beta^{\mathbf{b}}) \hat{\mathfrak{F}}_{r+1}(p_X)$$

and

$$[\alpha \Sigma(p_X)^{(k)}, \beta \Sigma(p_X)^{(l)}]^{\mathbf{a} \cup \mathbf{b}} = [\alpha, \beta]^{\mathbf{a} \cup \mathbf{b}} \hat{\mathfrak{F}}_{r+1}(p_X).$$

Here, $\hat{\mathfrak{F}}_{r+1}(p_X): \hat{\mathfrak{F}}_{r+1}(\Sigma\Omega(X)) \rightarrow \hat{\mathfrak{F}}_{r+1}(X)$ is the induced map. Thus, the relation $p_X \circ \gamma_X \simeq \text{id}_X$ completes the proof. $\square$

Based upon this proposition, we see that there are precisely $\binom{k+l-1}{l-1}$ such generalized Whitehead map $\Sigma X^{(k+l)} \rightarrow \Sigma X^{(k)} \vee \Sigma X^{(l)}$.

2 Group structure of $[J_n^*(X), \Omega(Y)]$

In this section, we introduce an extension $J^*(X)$ of $J(X)$ and we determine the co- H -structure of $\Sigma J^*(X)$ for a co- H -space X . First, we need to extend the results of [14] by replacing $\mathbb{S}^1$ with an arbitrary co- H -space X . Recall that Fox described in [9] the precise relationship between the Whitehead products when embedded in τ_n and the commutators in τ_n . This description is also valid in the more general setting as in Proposition 1.1 and Corollary 1.2. The number $(-1)^{w+|\mathbf{a}|-1}$, which is crucial in determining the structure of the generalized torus homotopy groups, depends on the parameters l and k and we shall call this the *Fox number* of the partition $\{\mathbf{a}, \mathbf{b}\}$.

For any integers l, k with $k > l > 0$, let

$$\phi(l, k) = \sum_{\mathbf{a}, |\mathbf{a}|=l} (-1)^{w+(\mathbf{a}|-1)}.$$

Moreover, for all $k \geq 0$, we let $\phi(k, k) = (-1)^k$ and $\phi(0, k) = 1$. We shall call ϕ the *Fox function*.

In [14], we found a closed form solution as follows:

Theorem 2.1 *For any integers l, k with $1 \leq l \leq k$, we have*

$$\phi(l, k) = \begin{cases} -\binom{\frac{k}{2}}{\frac{l}{2}}, & \text{if } l \text{ is even and } k \text{ is even;} \\ 0, & \text{if } l \text{ is odd and } k \text{ is even;} \\ \binom{\frac{k-1}{2}}{\frac{l-1}{2}}, & \text{if } l \text{ is odd and } k \text{ is odd;} \\ -\binom{\frac{k-1}{2}}{\frac{l}{2}}, & \text{if } l \text{ is even and } k \text{ is odd.} \end{cases}$$

To capture the group $\pi_1(Y)$, we consider $J_n^*(X) = J_n(X) \sqcup *$ for $n \geq 1$. Then, the groups $[J_n^*(X), \Omega(Y)]$ are subgroups of the group $[\hat{\mathfrak{F}}_{n+1}(X), Y]$ as the equalizer of face homomorphisms $d_i^*: [X^{n+1} \sqcup *, \Omega(Y)] \rightarrow [X^n \sqcup *, Y]$ for $i = 0, 1, \dots, n$. Notice that the splitting cofibration

$$\mathbb{S}^0 \rightarrow J_n^*(X) \rightarrow J_n(X),$$

in view of the Puppe exact sequence (see e.g., [17, Chapter III]), leads to an isomorphism $[\Sigma J_n^*(X), Y] \cong [\Sigma J_n(X), Y] \rtimes \pi_1(Y)$. Thus, $[\Sigma J_n^*(X), Y] \cong [\Sigma J_n(X), Y]$ provided Y is simply connected. For $J^*(X) = J(X) \sqcup *$, the group $[J^*(X), \Omega(Y)]$ is also the inverse limit of the tower $\{[J_n^*(X), \Omega(Y)]\}_{n \geq 1}$ and can be expressed in terms of $[J(X), \Omega(Y)]$ and $\pi_1(Y)$ via the action of the fundamental group $\pi_1(Y)$ on the group $[J(X), \Omega(Y)]$ as we illustrate in the next result.

Notice that an element $\theta \in \pi_1(Y)$ induces a map $\Omega(Y) \rightarrow \Omega(Y)$ given by $\alpha \mapsto \theta \alpha \theta^{-1}$ for $\alpha \in \Omega(Y)$. This gives rise to an action

$$\circ: \pi_1(Y) \times [W, \Omega(Y)] \longrightarrow [W, \Omega(Y)]$$

for any space W .

Proposition 2.2 *For any space X we have $[J^*(X), \Omega(Y)] \cong [J(X), \Omega(Y)] \rtimes \pi_1(Y)$, where the action of an element $\theta \in \pi_1(Y)$ on $(\alpha_1, \alpha_2, \dots, \alpha_n, \dots) \in [J(X), \Omega(Y)] \cong [\Sigma J(X), Y]$ is the sequence $(\theta \circ \alpha_1, \theta \circ \alpha_2, \dots, \theta \circ \alpha_n, \dots)$ with $\theta \circ \alpha_n$ given by the action of $\pi_1(Y)$ on $[\Sigma X^{(n)}, Y] \cong [X^{(n)}, \Omega(Y)]$ for all $n \geq 1$. In particular, if Y is simply connected then $[J^*(X), \Omega(Y)] \cong [J(X), \Omega(Y)]$.*

Proof Notice that an element $\theta \in \pi_1(Y)$ induces a map $\Omega(Y) \rightarrow \Omega(Y)$ given by $\alpha \mapsto \theta \alpha \theta^{-1}$ for $\alpha \in \Omega(Y)$. This gives rise to an action

$$\circ: \pi_1(Y) \times [W, \Omega(Y)] \longrightarrow [W, \Omega(Y)]$$

for any space W . Because $\Sigma J(X)$ has the homotopy type of $\Sigma \bigvee_{i=1}^{\infty} X^{(i)}$, we can fix a bijection $[J(X), \Omega(Y)] \cong [\Sigma J(X), Y] \cong \prod_{i=1}^{\infty} [X^{(i)}, \Omega(Y)]$. Then, identifying an element $\alpha \in [J(X), \Omega(Y)]$ with a sequence $(\alpha_1, \alpha_2, \dots, \alpha_n, \dots)$, we see that

$$\theta \circ (\alpha_1, \alpha_2, \dots, \alpha_n, \dots) = (\theta \circ \alpha_1, \theta \circ \alpha_2, \dots, \theta \circ \alpha_n, \dots)$$

for $\theta \in \pi_1(Y)$.

On the other hand, the Puppe exact sequence for the cofibration $\mathbb{S}^0 \rightarrow J^*(X) \rightarrow J(X)$ leads to the following splitting short exact sequence

$$1 \rightarrow [J(X), \Omega(Y)] \rightarrow [J^*(X), \Omega(Y)] \rightarrow \pi_1(Y) \rightarrow 1$$

and consequently, to an isomorphism

$$[J^*(X), \Omega(Y)] \cong [J(X), \Omega(Y)] \rtimes \pi_1(Y),$$

where the induced action of $\pi_1(Y)$ on the group $[J(X), \Omega(Y)]$ coincides with that given above as claimed. $\square$

Remark 2.3 If $\theta \in \pi_1(Y)$ and $\alpha \in [W, \Omega(Y)] \cong [\Sigma(W), Y]$ then $\theta \circ \alpha = [\theta, \alpha]\alpha$ for the generalized Whitehead product $[\theta, \alpha]$.

Next, we compare the groups $[J^*(X), \Omega(Y)]$ and $[J^*(X), \Omega(Y')]$ for a covering map $p: Y' \rightarrow Y$.

Theorem 2.4 Let $p: Y' \rightarrow Y$ be a covering of a space Y and X a path connected space. Then, there is a bijection

$$p_*: [J(X), \Omega(Y')] \xrightarrow{\cong} [J(X), \Omega(Y)]$$

and a short exact sequence

$$1 \rightarrow [J^*(X), \Omega(Y')] \rightarrow [J^*(X), \Omega(Y)] \rightarrow \pi_1(Y)/p_*\pi_1(Y') \rightarrow \star$$

of pointed sets.

In particular, if $p: Y' \rightarrow Y$ is a regular covering then there is a short exact sequence

$$1 \rightarrow [J^*(X), \Omega(Y')] \xrightarrow{p_*} [J^*(X), \Omega(Y)] \rightarrow \pi_1(Y)/p_*\pi_1(Y') \rightarrow 1$$

of groups.

Proof The dual of the Puppe exact sequence for the covering $p: Y' \rightarrow Y$ is

$$\begin{aligned} \cdots \rightarrow [\Sigma J(X), \pi_1(Y)/p_*\pi_1(Y')] &\rightarrow [\Sigma J(X), Y'] \xrightarrow{p_*} [\Sigma J(X), Y] \rightarrow \\ [J(X), \pi_1(Y)/p_*\pi_1(\tilde{Y})] &\rightarrow [J(X), Y'] \xrightarrow{p_*} [J(X), Y]. \end{aligned}$$

Certainly, $\pi_0(\Sigma J(X)) = *$ and, because X is path connected, $\pi_0(J(X)) \cong \pi_1(\Sigma(X)) = 1$, hence we get $[\Sigma J(X), \pi_1(Y)/p_*\pi_1(Y')] = 1$ and $[J(X), \pi_1(Y)/p_*\pi_1(Y)] = *$. Consequently, the sequence above leads to a bijection

$$p_* : [J(X), \Omega(Y')] \xrightarrow{\cong} [J(X), \Omega(Y)].$$

Then, the isomorphisms $[\Sigma J^*(X), Y] \cong [J(X), \Omega(Y)] \rtimes \pi_1(Y)$ and $[\Sigma J^*(X), Y'] \cong [J(X), \Omega(Y')] \rtimes \pi_1(Y')$ determined by Proposition 2.2 and the canonical map $[J(X), \Omega(Y)] \rtimes \pi_1(Y) \rightarrow \pi_1(Y)/p_*\pi_1(Y')$ yield a short exact sequence

$$1 \rightarrow [J^*(X), \Omega(Y')] \xrightarrow{p_*} [J^*(X), \Omega(Y)] \rightarrow \pi_1(Y)/p_*\pi_1(Y') \rightarrow *$$

of pointed sets.

Now, when $p: Y' \rightarrow Y$ is a regular we have the following short exact sequence

$$1 \rightarrow [J^*(X), \Omega(Y')] \xrightarrow{p_*} [J^*(X), \Omega(Y)] \rightarrow \pi_1(Y)/p_*\pi_1(Y') \rightarrow 1$$

of groups and the proof is complete. $\square$

The group structure of $[J_n^*(X), \Omega(Y)]$ is equivalent to the co- H -structure on $\Sigma J_n^*(X)$. Notice that the cofibration

$$X \xrightarrow{j} \mathcal{P}_n(X) = X^n/X^{n-1} \xrightarrow{q} X \wedge X^{n-1}$$

has a retraction $p: \mathcal{P}_n(X) \rightarrow X$. Hence, the map

$$\Sigma p + \Sigma q : \Sigma \mathcal{P}_n(X) \xrightarrow{\cong} \Sigma X \vee \Sigma X \wedge X^{n-1}$$

is a homotopy equivalence for $n \geq 1$. Because $\Sigma(X_1 \times X_2) \simeq \Sigma X_1 \vee \Sigma X_2 \vee \Sigma(X_1 \wedge X_2)$ for pointed spaces X_1 and X_2 , by an inductive argument, we derive

$$\Sigma \mathcal{P}_n(X) \simeq \bigvee_{k=1}^n \binom{n-1}{k-1} \Sigma X^{(k)} \quad (2.1)$$

for $n \geq 1$, where $X^{(k)}$ states for the k -th smash power of X .

Notice that

$$\Sigma J_n^*(X) \simeq \bigvee_{k=0}^n \Sigma X^{(k)} \text{ and } \hat{\mathfrak{F}}_{n+1}(X) \simeq \bigvee_{k=0}^n \binom{n}{k} \Sigma X^{(k)}, \quad (2.2)$$

where $X^{(0)} = \mathbb{S}^0$. Now, write $p_n: \hat{\mathfrak{F}}_{n+1}(X) \rightarrow \Sigma J_n(X)$ for the suspension of the projection map $X^n \sqcup * \rightarrow J_n^*(X)$. In view of [13], this leads to a monomorphism of groups

$$[\Sigma J_n^*(X), Y] \hookrightarrow [\hat{\mathfrak{F}}_{n+1}(X), Y].$$

Thus, the suspension co- H -structure on $\Sigma J_n^*(X)$ is induced by those on $\hat{\mathfrak{F}}_{n+1}(X)$. Similarly to $\hat{\mathfrak{F}}_{n+1}(X)$, the suspension co- H -structure

$$\bar{v}_n : \Sigma J_n^*(X) \rightarrow \Sigma J_n^*(X) \vee \Sigma J_n^*(X)$$

is also described inductively.

Because $\Sigma J_1^*(X) = \hat{\mathfrak{F}}_2(X)$, the co- H -structure $\bar{v}_1 = v_2 : J_1^*(X) \rightarrow J_1^*(X) \vee J_1^*(X)$. Given the co- H -structure $\bar{v}_n : \Sigma J_n^*(X) \rightarrow \Sigma J_n^*(X) \vee \Sigma J_n^*(X)$ write $\Sigma J_{n+1}^*(X) \simeq \Sigma J_n^*(X) \vee \Sigma X^{(n+1)}$ and $\hat{\mathfrak{F}}_{n+2}(X) \simeq \hat{\mathfrak{F}}_{n+1}(X) \vee \Sigma \mathcal{P}_{n+1}(X)$. Then, one can easily verify that the composite maps

$$\bar{v}_{n+1}^1 : \Sigma J_n^*(X) \xrightarrow{\bar{v}_n} \Sigma J_n^*(X) \vee \Sigma J_n^*(X) \hookrightarrow \Sigma J_{n+1}^*(X) \vee \Sigma J_{n+1}^*(X)$$

and

$$\begin{aligned} \bar{v}_{n+1}^2 : \Sigma X^{(n+1)} &\hookrightarrow \Sigma \mathcal{P}_{n+1}(X) \xrightarrow{\hat{v}_{n+2}^2} (\hat{\mathfrak{F}}_{n+1}(X) \vee \Sigma \mathcal{P}_{n+1}(X)) \vee (\hat{\mathfrak{F}}_{n+1}(X) \\ &\vee \Sigma \mathcal{P}_{n+1}(X)) \xrightarrow{p_{n+1} \vee p_{n+1}} \Sigma J_{n+1}^*(X) \vee \Sigma J_{n+1}^*(X) \end{aligned}$$

lead to the suspension co- H -structure

$$\bar{v}_{n+1} = \bar{v}_{n+1}^1 \vee \bar{v}_{n+1}^2 : \Sigma J_{n+1}^*(X) \longrightarrow \Sigma J_{n+1}^*(X) \vee \Sigma J_{n+1}^*(X)$$

on the space $J_{n+1}^*(X)$.

Therefore, the monomorphism of groups $[J_n^*(X), \Omega(Y)] \hookrightarrow [\hat{\mathfrak{F}}_{n+1}(X), Y]$ must determine some strict connection between those products and the suspension co- H -structure on $\Sigma J_n^*(X)$ studied above.

First, we recall the recent work [2] on co- H -structures on a wedge of spheres $X = \bigvee_{i=1}^t \mathbb{S}^{n_i}$. Write $k_i : \mathbb{S}^{n_i} \hookrightarrow X$ for the inclusion maps with $i = 1, \dots, t$. Further, set $\iota_j : X \hookrightarrow X \vee X$ for the inclusion, and $p_j : X \vee X \rightarrow X$ for the projection maps with $j = 1, 2$.

Proposition 2.5 ([2, Lemma 3.3]) *Let $\varphi : X \rightarrow X \vee X$ be a co-action. Then, φ is a co- H -structure on X if and only if $\varphi k_i = \iota_1 k_i + \iota_2 k_i + P_i$, where $P_i : \mathbb{S}^{n_i} \rightarrow X \vee X$ has the property $p_1 P_i = 0 = p_2 P_i$ for $i = 1, \dots, t$.*

We mimic Proposition 2.5, to examine the suspension co- H -structure on $J_n^*(X)$. Fixing the decomposition (2.2), we write $k_i : \Sigma X^{(i)} \hookrightarrow \Sigma J_n^*(X)$ for the inclusion maps with $i = 0, \dots, n$. Further, set $\iota_j : J_n^*(X) \hookrightarrow J_n^*(X) \vee J_n^*(X)$ for the inclusion and $p_j : J_n^*(X) \vee J_n^*(X) \rightarrow J_n^*(X)$ for the projection maps with $j = 1, 2$.

Now let X be a co- H -space. Then, the co- H -structure on ΣX is co-commutative and, by means of decompositions (2.1) and (2.2), so are the co- H -structures on $\mathcal{P}_n(X)$ and $\Sigma J_n(X)$ as well. Hence, we are in position to state our main theorem (compared to [14, Theorem 9.2]).

Theorem 2.6 *Let X be a co- H -space. Then, the suspension co- H -structure*

$$\bar{v}_n : \Sigma J_n^*(X) \rightarrow \Sigma J_n^*(X) \vee \Sigma J_n^*(X)$$

for $n \geq 1$ is given as follows:

$$\bar{v}_n k_i = \iota_1 k_i + \iota_2 k_i + P_i, \text{ where } P_i = \sum_{l=0}^i \phi(i-l, i-1) P_{l,i} \text{ with}$$

$$P_{l,i} : \Sigma X^{(i)} \rightarrow \Sigma X^{(i-l)} \vee \Sigma X^{(l)} \xrightarrow{k_i - l \vee k_l} \Sigma J_n^*(X) \vee \Sigma J_n^*(X)$$

determined by the generalized Whitehead map $\Sigma X^{(i)} \rightarrow \Sigma X^{(i-l)} \vee \Sigma X^{(l)}$ for $i = 0, \dots, n$ and $l = 0, \dots, i$, where ϕ is the Fox function.

Proof We proceed inductively on $n \geq 1$. Since the space $J_n^*(X)$ is a CW -complex, by the Cellular Approximation Theorem, $\bar{v}_n \simeq \bar{v}'_n : \Sigma J_n^*(X) \rightarrow \Sigma J_n^*(X) \vee \Sigma J_n^*(X)$, where $\bar{v}'_n$ is cellular. Hence, from now on all, we may assume that $\bar{v}_n$ is a cellular map.

If $n = 1$ then $P_0 = 0$ and, by the definition, $\bar{v}_1 k_0 = \iota_1 k_0 + \iota_2 k_0$.

Because $J_n^*(X) \subseteq J_{n+1}^*(X)$, so for dimensional reasons, we derive that the restriction $\bar{v}_{n+1}|_{J_n^*(X)} = \bar{v}_n$. Now we analyse the map $\bar{v}_{n+1} k_{n+2} : \Sigma X^{(n+1)} \rightarrow J_{n+1}^*(X) \vee J_{n+1}^*(X)$ defined by the composition

$$\begin{aligned} \bar{v}_{n+1}^2 : \Sigma X^{(n+1)} &\hookrightarrow \Sigma \mathcal{P}_{n+1}(X) \xrightarrow{v_{n+2}^2} (\hat{\mathfrak{F}}_{n+1}(X) \vee \Sigma \mathcal{P}_{n+1}(X)) \vee (\hat{\mathfrak{F}}_{n+1}(X) \\ &\vee \Sigma \mathcal{P}_{n+1}(X)) \xrightarrow{p_{n+1} \vee p_{n+1}} \Sigma J_{n+1}^*(X) \vee \Sigma J_{n+1}^*(X). \end{aligned}$$

In view of decompositions (2.1) and (2.2), we have

$$p_{n+1}|_{\Sigma \mathcal{P}_{n+1}(X)} : \Sigma \mathcal{P}_{n+1}(X) \simeq \bigvee_{k=1}^{n+1} \binom{n}{k-1} \Sigma X^{(k)} \rightarrow \bigvee_{k=1}^{n+1} \Sigma X^{(k)}.$$

Because v_{n+2}^2 is defined by means of the co-action $\alpha_{n+1} : \Sigma \mathcal{P}_{n+1}(X) \rightarrow \hat{\mathfrak{F}}_{n+1}(X) \vee \Sigma \mathcal{P}_{n+1}(X)$, Corollary 1.2 leads to $\bar{v}_{n+1} k_{n+1} - \iota_1 k_{n+1} - \iota_2 k_{n+1} = P_{n+1} = \sum_{i=0}^{n+1} \phi(n+1, i) P_{n+1,i}$ with $P_{n+1,i} : \Sigma X^{(n+1)} \rightarrow \Sigma X^{(n+1-i)} \vee \Sigma X^{(i)} \hookrightarrow J_{n+1}^*(X) \vee J_{n+1}^*(X)$ determined by the generalized Whitehead map $\Sigma X^{(n+1)} \rightarrow \Sigma X^{(n+1-i)} \vee \Sigma X^{(i)}$ and the proof is complete. $\square$

When Y is a W -space, that is, all Whitehead products vanish, it has been shown in [13, Corollary 3.6] that the Cohen groups are isomorphic to the direct product of the higher homotopy groups of Y so that they are abelian, in particular.

As a consequence of Theorem 2.6 the following holds:

Corollary 2.7 *Let Y be a path connected pointed space such that the generalized Whitehead products $[f, g] : \Sigma X^{(k+l)} \rightarrow Y$ for $f : \Sigma X^{(k)} \rightarrow Y$ and $g : \Sigma X^{(l)} \rightarrow Y$ vanish with k, l to be even. Then, the Cohen groups $[J_n(X), \Omega(Y)]$ and $[J(X), \Omega(Y)]$ are abelian for all $n \geq 1$. In particular, $[J_n(\mathbb{S}^1), \Omega(Y)]$ and $[J(\mathbb{S}^1), \Omega(Y)]$ are abelian for $n \geq 1$ provided Y is a path connected pointed space with $\pi_k(Y) = 0$ for k to be odd.*

Proof Let the generalized Whitehead products $[f, g]: \Sigma X^{(k+l)} \rightarrow Y$ for $f: \Sigma X^{(k)} \rightarrow Y$ and $g: \Sigma X^{(l)} \rightarrow Y$ vanish with k, l to be even. Then Theorem 2.1 on properties of the Fox function ϕ yields that the group structure on $[J_n^*(X), Y]$ determined by the co- H -structure

$$\bar{v}_n: \Sigma J_n^*(X) \rightarrow \Sigma J_n^*(X) \vee \Sigma J_n^*(X)$$

for $n \geq 1$ as stated in Theorem 2.6 is commutative and the assertions follow from Proposition 2.2. $\square$

Further, we make use of [13, Theorem 2.1] to deduce:

Corollary 2.8 *Let X be a co- H -space. If generalized Whitehead products $\Sigma Y^{(k+l)} \rightarrow \Sigma Y^{(k)} \vee \Sigma Y^{(l)}$ vanish with $k + l \leq n - 1$ then: (1) $[J_{n-1}(X), \Omega(Y)] \cong \Pi_{k=1}^{n-1}[X^{(k)}, \Omega(Y)]$ and the extension*

$$1 \longrightarrow [X^{(n)}, \Omega(Y)] \longrightarrow [J_n(X), \Omega(Y)] \longrightarrow [J_{n-1}(X), \Omega(Y)] \longrightarrow 1$$

splits;

(2) $[J_n(X), \Omega(Y)] \cong \Pi_{k=1}^n[X^{(k)}, \Omega(Y)]$ if n is even. In particular, if Y is n -connected then $[J_{2n}(\mathbb{S}^1), \Omega(Y)] \cong \Pi_{k=1}^{2n+1} \pi_k(Y)$.

3 Computations for $X^{(r)}$

In [14], we investigated when the group $[J_n(\mathbb{S}^1), \Omega(Y)]$ (and hence $[J(\mathbb{S}^1), \Omega(Y)]$) is abelian. Based upon our generalization in the previous sections, we now study when two elements in $[J(X^{(r)}), \Omega(Y)]$ commute, where X is a co- H -space and $X^{(r)}$ is the r -fold smash product of X . In particular, when $X = \mathbb{S}^1$ then $[J(X^{(r)}), \Omega(Y)] = [J(\mathbb{S}^r), \Omega(Y)]$.

We make use of the ϕ function of Sect. 2 to determine whether $\alpha \in [\Sigma X^{(rn)}, Y]$ and $\beta \in [\Sigma X^{(rm)}, Y]$ commute in the group $[J(X^{(r)}), \Omega(Y)]$. Denote by $\otimes_r$ the group multiplication in $[J(X^{(r)}), \Omega(Y)]$ provided X is a co- H -space.

First, recall that

$$\Sigma J_n(X^{(r)}) \simeq \Sigma \left(\bigvee_{i=1}^n X^{(ir)} \right).$$

Then $[J_n(X^{(r)}), \Omega(Y)]$, as a set, is isomorphic to the direct product $\prod_{i=1}^n [\Sigma X^{(ir)}, Y]$. Take $\alpha \in [\Sigma X^{(nr)}, Y]$, $\beta \in [\Sigma X^{(mr)}, Y]$.

If there are non-trivial perturbation $P_{l,i}$ determined by the generalized Whitehead product $[\alpha, \beta]$ then $l = m$ and $i - l = n$. In other words, $P_{l,i} = P_{m,n+m}$. It follows that $P_{n+m} = \phi(n, n + m - 1)P_{m,n+m}$. Now, we have

$$\begin{aligned} \alpha \otimes_r \beta &= \alpha + \beta + \phi(n, n + m - 1)[\alpha, \beta] \\ \text{and } \beta \otimes_r \alpha &= \alpha + \beta + \phi(m, n + m - 1)[\beta, \alpha]. \end{aligned}$$

The anti-commutativity property [1, Proposition 3.3] of the generalized Whitehead product asserts that $[f, g] = -(\Sigma\sigma)^*[g, f]$ for $f: \Sigma A \rightarrow Z$ and $g: \Sigma B \rightarrow Z$, where $\sigma: A \wedge B \rightarrow B \wedge A$ is induced by the transposition $A \times B \rightarrow B \times A$ interchanging the coordinates. By applying successive transpositions, it follows that $[\beta, \alpha] = (-1)^{(nr+1)(mr+1)}[\alpha, \beta]$. Now, it suffices to compare $\phi(n, n+m-1)$ with $(-1)^{(nr+1)(mr+1)}\phi(m, n+m-1)$. Let

$$\Delta(n, m) = |\phi(n, n+m-1) - (-1)^{(nr+1)(mr+1)}\phi(m, n+m-1)|. \quad (3.1)$$

We have the following results (compare with [14, Theorem 4.4]).

Proposition 3.1 *Let $\alpha \in [\Sigma X^{(nr)}, Y]$ and $\beta \in [\Sigma X^{(mr)}, Y]$ for an even integer r . Suppose the generalized Whitehead product $[\alpha, \beta] \neq 0$ and has order k . Then:*

- *if $k = \infty$ then $\alpha \otimes_r \beta \neq \beta \otimes_r \alpha \in [J(X^{(r)}), \Omega(Y)]$;*
- *if $k < \infty$ then the product $\alpha \otimes_r \beta \in [J(X^{(r)}), \Omega(Y)]$ coincides with $\beta \otimes_r \alpha$ if and only if k divides $\Delta(n, m)$,*

where

$$\Delta(n, m) = \begin{cases} \binom{\frac{n+m}{2}}{\frac{n}{2}}, & \text{if } m \text{ is even and } n \text{ is even;} \\ 2 \binom{\frac{n+m-2}{2}}{\frac{n-1}{2}}, & \text{if } m \text{ and } n \text{ are both odd;} \\ \binom{\frac{n+m-1}{2}}{\frac{n}{2}}, & \text{otherwise.} \end{cases}$$

Proof Since r is even, (3.1) becomes $\Delta(n, m) = |\phi(n, n+m-1) + \phi(m, n+m-1)|$. In the case when both m and n are even, we have

$$\begin{aligned} \Delta(n, m) &= \binom{\frac{n+m-2}{2}}{\frac{n}{2}} + \binom{\frac{n+m-2}{2}}{\frac{m}{2}} \\ &= \binom{\frac{n+m-2}{2}}{\frac{n}{2}} + \binom{\frac{n+m-2}{2}}{\frac{n}{2} - 1} \\ &= \binom{\frac{n+m}{2}}{\frac{n}{2}} \quad \text{by Pascal's triangle.} \end{aligned}$$

A straightforward calculation then yields the values of $\Delta(n, m)$ as stated. By the definition of $\Delta(n, m)$, α and β commute in $[J(X^{(r)}), \Omega(Y)]$ if and only if $\Delta(n, m)[\alpha, \beta] = 0$. Since $\Delta(n, m) \neq 0$, the first assertion follows when $k = \infty$. The second assertion follows from the value of $\Delta(n, m)$. $\square$

Next, we consider $X^{(r)}$ for an odd r . The following result and Corollary 1.2 show that [14, Theorem 4.4], which is for $X = \mathbb{S}^1$, can be generalized for a co- H -space X and is valid for all odd r .

Proposition 3.2 *Let $\alpha \in [\Sigma X^{(nr)}, Y]$ and $\beta \in [\Sigma X^{(mr)}, Y]$ for an odd integer r . Suppose the generalized Whitehead product $[\alpha, \beta] \neq 0$ and has order k . Then:*

- if $k = \infty$ then the product $\alpha \circledast_r \beta \in [J(X^{(r)}), \Omega(Y)]$ coincides with $\beta \circledast_r \alpha$ if and only if both n and m are odd;
- if $k < \infty$ then the product $\alpha \circledast_r \beta \in [J(X^{(r)}), \Omega(Y)]$ coincides with $\beta \circledast_r \alpha$ if and only if both n and m are odd or

$$k \text{ divides } \begin{cases} \left(\begin{array}{c} \frac{n+m-1}{2} \\ m \\ 2 \end{array} \right) & \text{if } m \text{ is even and } n \text{ is odd;} \\ \left(\begin{array}{c} \frac{n+m-1}{2} \\ m \\ 2 \end{array} \right) & \text{if } m \text{ is odd and } n \text{ is even;} \\ \left(\begin{array}{c} \frac{n+m}{2} \\ n \\ 2 \end{array} \right) & \text{if } m \text{ and } n \text{ are both even.} \end{cases}$$

Proof Since r is odd, $\Delta(n, m) = \phi(n, n+m-1) + \phi(m, n+m-1)$ when both n and m are even and $\Delta(n, m) = \phi(n, n+m-1) - \phi(m, n+m-1)$ otherwise. A straightforward computation using Theorem 2.1 and the same argument in the previous result for the case when both n and m are even yield

$$\Delta(n, m) = \begin{cases} \left(\begin{array}{c} \frac{n+m}{2} \\ n \\ 2 \end{array} \right), & \text{if } m \text{ and } n \text{ are both even;} \\ \left(\begin{array}{c} \frac{n+m-1}{2} \\ m \\ 2 \end{array} \right), & \text{if } m \text{ is even and } n \text{ is odd;} \\ \left(\begin{array}{c} \frac{n+m-1}{2} \\ n \\ 2 \end{array} \right), & \text{if } m \text{ is odd and } n \text{ is even;} \\ 0, & \text{otherwise.} \end{cases}$$

The assertions follow from the values of $\Delta(n, m)$. $\square$

For any integer $r \geq 1$, we can identify $[J(X^{(r)}), \Omega(Y)]$ with $\prod_{i=1}^{\infty} [\Sigma X^{(ir)}, Y]$ as sets. Thus, $[J(X^{(r)}), \Omega(Y)]$ is a subset of $[J(X), \Omega(Y)]$. It is natural to ask if $[J(X^{(r)}), \Omega(Y)]$ is a subgroup of $[J(X), \Omega(Y)]$. Certainly, this is the case if Y is a space with no nontrivial generalized Whitehead products. In particular, if $X = \mathbb{S}^1$ and Y is a W -space then $[J(\mathbb{S}^r), \Omega(Y)]$ is also a subgroup of $[J(\mathbb{S}^1), \Omega(Y)]$.

For pointed spaces A and B , the commutator $[f, g]$ of $f: \Sigma A \rightarrow Y, g: \Sigma B \rightarrow Y$ which gives the generalized Whitehead product lies in $[\Sigma(A \times B), Y]$, which is isomorphic to $[A \times B, \Omega(Y)]$. Hence, if $\Omega(Y)$ is a homotopy commutative space then this commutator $[f, g]$ vanishes (cf. [1, Proposition 5.4]) which means that all generalized Whitehead products vanish. If Y itself is an H -space then $\Omega(Y)$ is homotopy commutative. However, the converse is not true. In fact, Stasheff [16, Theorem 1.18] proved that $\Omega(\mathbb{C}P^3)$ is homotopy commutative. On the other hand, it follows from Borel's theorem on the cohomology ring of an H -space that $\mathbb{C}P^3$ is not an H -space.

Given $\alpha \in [\Sigma X^{(nr)}, Y]$ and $\beta \in [\Sigma X^{(mr)}, Y]$, we have two products as follows:

$$\alpha \circledast_r \beta = \alpha + \beta + \phi(n, n+m-1)[\alpha, \beta] \in [J(X^{(r)}), \Omega(Y)]$$

and

$$\alpha \circledast_1 \beta = \alpha + \beta + \phi(nr, nr+mr-1)[\alpha, \beta] \in [J(X), \Omega(Y)].$$

Now, we investigate when $\alpha \circledast_r \beta$ coincides with $\alpha \circledast_1 \beta$ so we need to determine when $D_{1,r}(n, m) := \phi(n, n+m-1) - \phi(nr, nr+mr-1)$ is divisible by k , the order of the generalized Whitehead product $[\alpha, \beta]$.

It follows from Theorem 2.1 that

$$\phi(n, n+m-1) = \begin{cases} \left(\frac{n+m-2}{\frac{n}{2}}\right), & \text{if } m \text{ and } n \text{ are both even;} \\ 0, & \text{if } n \text{ is odd and } m \text{ is even;} \\ \left(\frac{n+m-2}{\frac{n-1}{2}}\right), & \text{if } m \text{ and } n \text{ are both odd;} \\ -\left(\frac{n+m-1}{\frac{n}{2}}\right), & \text{if } n \text{ is even and } m \text{ is odd.} \end{cases}$$

Similarly, if r is odd then we have

$$\phi(nr, nr+mr-1) = \begin{cases} -\left(\frac{nr+mr-2}{\frac{nr}{2}}\right), & \text{if } m \text{ and } n \text{ are both even;} \\ 0, & \text{if } n \text{ is odd and } m \text{ is even;} \\ \left(\frac{nr+mr-2}{\frac{nr-1}{2}}\right), & \text{if } m \text{ and } n \text{ are both odd;} \\ -\left(\frac{nr+mr-1}{\frac{nr}{2}}\right), & \text{if } n \text{ is even and } m \text{ is odd} \end{cases}$$

and

$$\phi(nr, nr+mr-1) = -\left(\frac{nr+mr-2}{\frac{nr}{2}}\right)$$

when r is even.

Recall from elementary number theory, a theorem of Lucas (see e.g., [8]) states that for any prime p ,

$$\binom{py+j}{px+i} \equiv \binom{j}{i} \binom{y}{x} \pmod{p},$$

where x, y, i, j are non-negative integers, $i < p$, and $j < p$.

Suppose r is an odd prime. Then,

$$\binom{\frac{r+mr-2}{2}}{\frac{nr}{2}} = \binom{\frac{(n+m-2)r}{2} + \frac{2r-2}{2}}{\frac{nr}{2} + 0} \equiv \binom{\frac{n+m-2}{2}}{\frac{n}{2}} \binom{\frac{2r-2}{2}}{0} \pmod{r}.$$

Thus, when n and m are even, $D_{1,r}(n, m) \equiv 0 \pmod{r}$. A similar argument yields that

$$D_{1,r}(n, m) \equiv \begin{cases} \binom{\frac{n+m-2}{2}}{\frac{n-1}{2}} \cdot \left(1 - \binom{r-1}{\frac{r-1}{2}}\right) \pmod{r}, & \text{if } m \text{ and } n \text{ are odd;} \\ 0 \pmod{r}, & \text{otherwise.} \end{cases}$$

Now suppose r is an even prime, that is, $r = 2$. Then

$$\phi(nr, nr + mr - 1) = \phi(2n, 2n + 2m - 1) = -\binom{n+m-1}{n}.$$

Note that

$$\begin{aligned} \phi(2n, 2n + 2m - 1) &= -\binom{n+m-1}{n} \\ &= \binom{\frac{(n+m-2)}{2} \cdot 2 + 1}{\frac{n}{2} \cdot 2 + 0} \equiv \phi(n, n+m-1) \pmod{2} \quad \text{if } n, m \text{ are both even;} \\ &= \binom{\frac{(n+m-2)}{2} \cdot 2 + 1}{\frac{n-1}{2} \cdot 2 + 1} \equiv \phi(n, n+m-1) \pmod{2} \quad \text{if } n, m \text{ are both odd;} \\ &= \binom{\frac{(n+m-1)}{2} \cdot 2 + 0}{\frac{n}{2} \cdot 2 + 0} \equiv \phi(n, n+m-1) \pmod{2} \quad \text{if } n \text{ is even and } m \text{ is odd;} \\ &= \binom{\frac{(n+m-1)}{2} \cdot 2 + 0}{\frac{n-1}{2} \cdot 2 + 1} \equiv 0 \equiv \phi(n, n+m-1) \pmod{2} \quad \text{if } n \text{ is odd and } m \text{ is even.} \end{aligned}$$

It follows that $D_{1,2}(n, m) \equiv 0 \pmod{2}$.

The next example shows that for the sphere $\mathbb{S}^r$ with $r > 1$, the set $[J(\mathbb{S}^r), \Omega(Y)]$ is *not* a subgroup of $[J(\mathbb{S}^1), \Omega(Y)]$ in general.

Example 3.3 Let $r = 3$, $X = \mathbb{S}^3$, $Y = \mathbb{S}^{10}$, $m = n = 3$ and $\alpha = \beta = \iota_{10} \in \pi_{10}(\mathbb{S}^{10})$. Since Y is an even sphere, the classical Whitehead product $[\iota_{10}, \iota_{10}]$ has infinite order. We should point out that the generalized Whitehead product coincides with the classical one when X is a sphere. Thus in our case, the discussion above regarding $D_{1,r}(n, m)$ applies. Note that

$$D_{1,3}(3, 3) = \phi(3, 5) - \phi(9, 17) = -68 \neq 0.$$

It follows that, as elements in $[J(\mathbb{S}^1), \Omega(\mathbb{S}^{10})]$, the products $\iota_{10} \otimes_1 \iota_{10}$ and $\iota_{10} \otimes_3 \iota_{10}$ are different. Thus, $[J(\mathbb{S}^3), \Omega(\mathbb{S}^{10})]$ cannot be a subgroup of $[J(\mathbb{S}^1), \Omega(\mathbb{S}^{10})]$.

We have shown in [13, Example 3.7] that $[J(\mathbb{S}^1), \Omega(Y)]$ is abelian for $Y = \mathbb{S}^2$ or $Y = \mathbb{R}P^2$ even though the space $Y = \mathbb{S}^2$ or $\mathbb{R}P^2$ has nontrivial Whitehead products and thus $\Omega(Y)$ is not homotopy commutative. We end this paper with a construction of a space $\hat{Y}$ for which $[J(\mathbb{S}^r), \Omega(\hat{Y})]$ is abelian for all $r \geq 1$ while $\hat{Y}$ has nontrivial Whitehead products and hence $\Omega(\hat{Y})$ is not homotopy commutative.

Example 3.4 Let $\widehat{\mathbb{S}^{26}}$ be the complex by attaching higher dimensional cells to $\mathbb{S}^{26}$ such that $\pi_i(\widehat{\mathbb{S}^{26}}) = \pi_i(\mathbb{S}^{26})$ for $i \leq 54$ and $\pi_i(\widehat{\mathbb{S}^{26}}) = 0$ for $i \geq 55$. Equivalently it has the same homotopy type as the 54-th stage in the Postnikov tower of the sphere $\mathbb{S}^{26}$ and note that $[\iota_{26}, \iota_{26}] \in \pi_{51}(\mathbb{S}^{26})$ has infinite order for ι_{26} the identity map on the sphere $\mathbb{S}^{26}$.

For n not a power of 2, necessary and sufficient conditions on n were given in [14, Proposition 4.9] for which $[J_{4n+1}(\mathbb{S}^1), \Omega(\mathbb{S}^{2n})]$ is abelian. In particular, $n = 13$ satisfies the conditions of [14, Proposition 4.9]. Thus, the group $[J_{53}(\mathbb{S}^1), \Omega(\mathbb{S}^{26})]$ is abelian. By the construction of $\widehat{\mathbb{S}^{26}}$, we have that

$$[J(\mathbb{S}^1), \Omega(\widehat{\mathbb{S}^{26}})] \cong [J_{53}(\mathbb{S}^1), \Omega(\widehat{\mathbb{S}^{26}})] \cong [J_{53}(\mathbb{S}^1), \Omega(\mathbb{S}^{26})]$$

is abelian as well.

Now, let $\alpha \in \pi_{k+1}(\widehat{\mathbb{S}^{26}})$ and $\beta \in \pi_{l+1}(\widehat{\mathbb{S}^{26}})$. Then the Whitehead product $[\alpha, \beta] = 0$ unless

$$(k, l) \in \{(25, 25), (25, 26), (25, 27), (25, 28), (26, 25), (26, 26), (26, 27), (27, 25), (27, 26)\}.$$

By anti-commutativity of the Whitehead product, we may assume that $k \leq l$ so that we need only to consider the set

$$\Gamma_1 = \{(25, 25), (25, 26), (25, 27), (25, 28), (26, 26), (26, 27)\}.$$

For $r \geq 1$, we have $\Sigma J(\mathbb{S}^r) \simeq \mathbb{S}^{r+1} \vee \mathbb{S}^{2r+1} \vee \mathbb{S}^{3r+1} \vee \dots$ and $[J(\mathbb{S}^r), \Omega(\widehat{\mathbb{S}^{26}})] = \prod_{i=1}^{\infty} \pi_{ir+1}(\widehat{\mathbb{S}^{26}})$ as sets. Suppose $\alpha \in \pi_{kr+1}(\widehat{\mathbb{S}^{26}})$ and $\beta \in \pi_{lr+1}(\widehat{\mathbb{S}^{26}})$ and let $\Gamma_r = \{(kr, lr) \mid (kr, lr) \in \Gamma_1\}$.

If $r = 2$ then $\Gamma_2 = \{(26, 26)\}$. It follows that $\alpha, \beta \in \pi_{27}(\widehat{\mathbb{S}^{26}}) \cong \mathbb{Z}_2$ and thus $[\alpha, \beta] = [\beta, \alpha]$ and $[J(\mathbb{S}^2), \Omega(\widehat{\mathbb{S}^{26}})]$ is abelian by Proposition 3.1.

For $r = 5$, we have $\Gamma_5 = \{(25, 25)\}$. In this case, both α and β are even dimensional so that $[\alpha, \beta] = [\beta, \alpha]$ and $[J(\mathbb{S}^5), \Omega(\widehat{\mathbb{S}^{26}})]$ is abelian by Proposition 3.2.

For all other values of r , we have $\Gamma_r = \emptyset$. Since there are no Whitehead products in $\pi_i(\widehat{\mathbb{S}^{26}})$ for $i \geq 55$, therefore we can conclude that $[J(\mathbb{S}^r), \Omega(\widehat{\mathbb{S}^{26}})]$ is abelian for all $r \geq 1$.

Following the example above, consider the space $\widehat{\mathbb{S}^{2n}}$ such that $\pi_i(\widehat{\mathbb{S}^{2n}}) = \pi_i(\mathbb{S}^{2n})$ for $i \leq 4n+2$ and $\pi_i(\widehat{\mathbb{S}^{2n}}) = 0$ for $i \geq 4n+3$. Let $\alpha \in \pi_{kr+1}(\widehat{\mathbb{S}^{2n}})$ and $\beta \in \pi_{lr+1}(\widehat{\mathbb{S}^{2n}})$.

For any $r > 1$, let

$$\Gamma_r = \{(kr, lr) \mid (kr, lr) \in \Gamma_1\},$$

where

$$\Gamma_1 = \{(2n-1, 2n-1), (2n-1, 2n), (2n-1, 2n+1), (2n-1, 2n+2), \\ (2n, 2n), (2n, 2n+1)\}.$$

For $r = 1$, if $n \neq 2^\ell$ for any $\ell > 0$ and n satisfies the following two conditions:

$$\begin{aligned} \text{(A)} \quad n &\neq 2^a - 1 \text{ for any } a \text{ and } n \neq 2^b + 2^a - 1 \text{ for any } b > a \geq 0; \\ \text{(B)} \quad n &\equiv 0 \pmod{3} \text{ or } n \equiv 1 \pmod{3} \end{aligned} \quad (3.2)$$

then $[J_{4n+1}(\mathbb{S}^1), \Omega(\mathbb{S}^{2n})]$ is abelian by [14, Proposition 4.9] and hence by the same argument as in Example 3.4 that $[J(\mathbb{S}^1), \Omega(\widehat{\mathbb{S}^{2n}})]$ is also abelian.

For $r = 2$, we get $\Gamma_2 = \{(2n, 2n)\}$ so $\alpha, \beta \in \pi_{2n+1}(\widehat{\mathbb{S}^{2n}}) = \pi_{2n+1}(\mathbb{S}^{2n}) \cong \mathbb{Z}_2$ if $n > 1$ or $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$ if $n = 1$. In either case, we have $[\alpha, \beta] = [\beta, \alpha]$. It follows from Proposition 3.1 that $[J(\mathbb{S}^2), \Omega(\widehat{\mathbb{S}^{2n}})]$ is abelian.

For $r > 3$, we have $\Gamma_r = \{(2n, 2n)\}$ provided $r \mid 2n$ or $\Gamma_r = \{(2n-1, 2n-1)\}$ provided $r \mid 2n-1$ or $\Gamma_r = \emptyset$ otherwise. If $r \mid 2n$ then $\alpha, \beta \in \pi_{2n+1}(\widehat{\mathbb{S}^{2n}}) \cong \mathbb{Z}_2$. If $r \mid (2n-1)$ then both α and β are even dimensional. Thus, in all cases, we conclude that $[\alpha, \beta] = [\beta, \alpha]$ and hence $[J(\mathbb{S}^r), \Omega(\widehat{\mathbb{S}^{2n}})]$ is abelian by Proposition 3.1.

For $r = 3$, we have $\Gamma_3 = \{(2n, 2n)\}$ provided $3 \mid n$ or $\Gamma_3 = \{(2n-1, 2n-1), (2n-1, 2n+2)\}$ provided $3 \mid (2n-1)$, or $\Gamma_3 = \emptyset$ otherwise. If n is an integer satisfying condition (B) above, then either $3 \mid n$ in which case $\alpha, \beta \in \pi_{2n+1}(\mathbb{S}^{2n}) \cong \mathbb{Z}_2$ so that $[\alpha, \beta] = [\beta, \alpha]$, or $n \equiv 1 \pmod{3}$ in which case $3 \nmid (2n-1)$ so that $\Gamma_3 = \emptyset$. Thus, we conclude that if n satisfies (B) or equivalently $n \not\equiv 2 \pmod{3}$ then $[J(\mathbb{S}^3), \Omega(\widehat{\mathbb{S}^{2n}})]$ is abelian by Proposition 3.1.

Now, we consider the case where $n \equiv 2 \pmod{3}$ and $r = 3$. In this case, $3 \mid (2n-1)$ and $\Gamma_3 = \{(2n-1, 2n-1), (2n-1, 2n+2)\}$. Thus, either $\alpha, \beta \in \pi_{2n}(\widehat{\mathbb{S}^{2n}})$ in which case $[\alpha, \beta] = [\beta, \alpha]$ or $\alpha \in \pi_{2n}(\widehat{\mathbb{S}^{2n}})$ and $\beta \in \pi_{2n+3}(\widehat{\mathbb{S}^{2n}})$. For the latter case, without loss of generality, we may assume that $\alpha = \iota_{2n}$ and $\beta = \nu_{2n}$, where ν_{2n} is a generator of $\pi_{2n+3}(\mathbb{S}^{2n})$ with $n \geq 2$ and consider the Whitehead product $[\iota_{2n}, \nu_{2n}]$. According to [15, (1.32)], the order $\sharp[\iota_{2n}, \nu_{2n}] = 12$ if n is odd or 24 if n is even.

Since $3 \mid (2n-1)$, we have $2n-1 = 3\lambda$ and $2n+2 = 3(\lambda+1)$ for some odd integer λ . By Proposition 3.2 (with $r = 3$), the difference

$$\Delta(\lambda, \lambda+1) = \binom{\lambda}{\frac{\lambda+1}{2}} = \binom{\lambda}{\frac{\lambda-1}{2}} = \binom{2\gamma+1}{\gamma},$$

where $\gamma = \frac{\lambda-1}{2}$. Rewriting Δ , we get

$$\Delta(\lambda, \lambda+1) = \binom{2\gamma+1}{\gamma} = \frac{2\gamma+1}{\gamma+1} \binom{2\gamma}{\gamma} = (2\gamma+1)C_\gamma,$$

where C_γ is the γ -th Catalan number and $n = 3\gamma + 2$. It follows that α and β commute as elements of $[J(\mathbb{S}^3), \Omega(\widehat{\mathbb{S}^{2n}})]$ if and only if $\#[\iota_{2n}, v_{2n}] \mid \Delta(\lambda, \lambda + 1) = (2\gamma + 1)C_\gamma$.

Case I: n odd or γ odd.

It follows that $12 \mid (2\gamma + 1)C_\gamma$ if and only if $4 \mid C_\gamma$ and either $3 \mid C_\gamma$ or $3 \mid (2\gamma + 1)$. By [7, Theorem 2.3] for the divisibility of C_γ by 4, it holds $4 \mid C_\gamma$ if and only if γ satisfies conditions (A) and (B) of (3.2). By [6, Theorem 5.2] for the divisibility of C_γ by 3, we have $3 \mid C_\gamma$ if and only if $\gamma \equiv 2 \pmod{3}$. The condition $3 \mid (2\gamma + 1)$ is equivalent to $\gamma \equiv 1 \pmod{3}$. Thus, we have just shown, using Proposition 3.2, that in this case (γ is odd) that α and β commute if and only if γ satisfies (3.2) and $3 \nmid \gamma$.

Case II: n even or γ even.

It follows that $24 \mid (2\gamma + 1)C_\gamma$ if and only if $8 \mid C_\gamma$ and either $3 \mid C_\gamma$ or $3 \mid (2\gamma + 1)$. By [7, Theorem 4.2] for the divisibility of C_γ by 8, we have $8 \mid C_\gamma$ if and only if γ satisfies the following conditions:

- (1) $\gamma \neq 0, 1$;
 - (2) $\gamma \neq 2^a + 2^{a+1} - 1$ for any $a \geq 0$;
 - (3) $\gamma \neq 2^a + 2^b + 2^c - 1$ for any $c > b > a \geq 0$;
 - (4) $\gamma \neq 2^a - 1$ for any $a \geq 2$;
 - (5) $\gamma \neq 2^a + 2^b - 1$ for any $(b - 2) \geq a \geq 0$.
- (3.3)

Thus, we have just shown, using Proposition 3.1, that in this case (γ is even) that α and β commute if and only if γ satisfies the conditions of (3.3) and $3 \nmid \gamma$.

Summarizing the discussion above, we have the following:

Proposition 3.5 *Let $\widehat{\mathbb{S}^{2n}}$ be the complex as defined above. Then $[J(\mathbb{S}^r), \Omega(\widehat{\mathbb{S}^{2n}})]$ is abelian if one of the following conditions is satisfied:*

- (i) $r \neq 1, 3$;
- (ii) n satisfies (B) of (3.2) and $r = 3$;
- (iii) $n \neq 2^\ell$ for any $\ell > 0$, n satisfies (3.2) and $r = 1$;
- (iv) $n = 3\gamma + 2$ is odd, γ satisfies (3.2) and $r = 3$;
- (v) $n = 3\gamma + 2$ is even, γ satisfies (3.3) and $r = 3$.

Remark 3.6 Observe that the universal covering $\mathbb{S}^{2n} \rightarrow \mathbb{R}P^{2n}$ gives rise to the universal covering $\widehat{\mathbb{S}^{2n}} \rightarrow \widehat{\mathbb{R}P^{2n}}$ which, by Theorem 2.4, induces an isomorphism

$$[J(X), \Omega(\widehat{\mathbb{S}^{2n}})] \xrightarrow{\cong} [J(X), \Omega(\widehat{\mathbb{R}P^{2n}})]$$

for any path connected space X . Hence, a result similar to Proposition 3.5 can be obtained for $[J(\mathbb{S}^r), \Omega(\widehat{\mathbb{R}P^{2n}})]$ using the results above and Proposition 2.2.

Recall that the action of $\pi_1(\mathbb{R}P^{2n})$ on $\pi_{2n}(\mathbb{R}P^{2n}) \cong \mathbb{Z}$ is multiplication by -1 . More generally, a formula for the Whitehead product $[\gamma_{2n}\alpha, i_{\mathbb{R}}]$ with $\alpha \in \pi_m(\mathbb{S}^{2n})$ and $\pi_1(\mathbb{R}P^{2n}) = \langle i_{\mathbb{R}} \rangle$, is known (see e.g., [3] or [15, Lemma 2.4]), where $i_{\mathbb{R}}: \mathbb{R}P^1 \hookrightarrow \mathbb{R}P^{2n}$ is the standard inclusion and $\gamma_{2n}: \widehat{\mathbb{S}^{2n}} \rightarrow \mathbb{R}P^{2n}$ is the quotient map. This implies the action of $\pi_1(\mathbb{R}P^{2n})$ on $\pi_m(\widehat{\mathbb{R}P^{2n}})$ for $m \geq 1$. Now, it is straightforward

to compute

$$[J^*(\mathbb{S}^r), \widehat{\Omega(\mathbb{R}P^{2n})}] \cong [J(\mathbb{S}^r), \widehat{\Omega(\mathbb{R}P^{2n})}] \rtimes \pi_1(\mathbb{R}P^{2n})$$

for all $r \geq 1$.

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