Certain maps preserving self-homotopy equivalences

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Abstract Let $\mathcal{E}(X)$ be the group of homotopy classes of self homotopy equivalences for a connected CW complex X . We consider two classes of maps, $\mathcal{E}$ -maps and co- $\mathcal{E}$ -maps. They are defined as the maps $X \rightarrow Y$ that induce homomorphisms $\mathcal{E}(X) \rightarrow \mathcal{E}(Y)$ and $\mathcal{E}(Y) \rightarrow \mathcal{E}(X)$, respectively. We give some rationalized examples related to spheres, Lie groups and homogeneous spaces by using Sullivan models. Furthermore, we introduce an $\mathcal{E}$ -equivalence relation between rationalized spaces $X_{\mathbb{Q}}$ and $Y_{\mathbb{Q}}$ as a geometric realization of an isomorphism $\mathcal{E}(X_{\mathbb{Q}}) \cong \mathcal{E}(Y_{\mathbb{Q}})$.

Keywords Self homotopy equivalence · $\mathcal{E}$ -Map · Co- $\mathcal{E}$ -map · Rational homotopy · Sullivan (minimal) model · Rational $\mathcal{E}$ -map · Rational co- $\mathcal{E}$ -map · Rationally $\mathcal{E}$ -equivalent

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1 Introduction

Needless to say, the based homotopy set $[X, Y]$ of based continuous maps from a based space X to a based space Y is a most interesting object in homotopy theory. In

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the following, all maps are based and we do not distinguish a homotopy class and the representative in a homotopy set. Let X be a connected CW complex with base point $*$ and let

$$\mathcal{E}(X) = \{[f] \in [X, X] \mid f : X \xrightarrow{\sim} X\}$$

be the group of homotopy classes of self-homotopy equivalences for X with the operation given by the composition of homotopy classes. This group is important and has been closely studied as part of homotopy theory (for example, see [2, 4–6, 15–17]).

It is clear that $\mathcal{E}(X) \cong \mathcal{E}(Y)$ as a group if $X \simeq Y$. One of the difficulties of its computation or evaluation may be based on the fact that $\mathcal{E}(\)$ is not functorial, i.e., there is no suitable induced map between $\mathcal{E}(X)$ and $\mathcal{E}(Y)$ for the map $f : X \rightarrow Y$ in general. However, recall that, for example, the injection $i_X : X \rightarrow X \times Y$ and the projection $p_Y : X \times Y \rightarrow Y$ induce the natural monomorphisms $\mathcal{E}(X) \rightarrow \mathcal{E}(X \times Y)$ and $\mathcal{E}(Y) \rightarrow \mathcal{E}(X \times Y)$, respectively.

Definition 1.1 We say a map $f : X \rightarrow Y$ is an $\mathcal{E}$ -map if there is a homomorphism $\phi_f : \mathcal{E}(X) \rightarrow \mathcal{E}(Y)$ such that

$$\begin{array}{ccc} X & \xrightarrow{g} & X \\ f \downarrow & & \downarrow f \\ Y & \xrightarrow{\phi_f(g)} & Y \end{array}$$

homotopically commutes for any element g of $\mathcal{E}(X)$. We say the map $f : X \rightarrow Y$ is a **co- $\mathcal{E}$ -map** if there is a homomorphism $\psi_f : \mathcal{E}(Y) \rightarrow \mathcal{E}(X)$ such that

$$\begin{array}{ccc} X & \xrightarrow{\psi_f(g)} & X \\ f \downarrow & & \downarrow f \\ Y & \xrightarrow{g} & Y \end{array}$$

homotopically commutes for any element g of $\mathcal{E}(Y)$.

Especially we consider the rationalized version of $\mathcal{E}$ -maps and co- $\mathcal{E}$ -maps by using Sullivan models [8, 9, 18]. Let $X_{\mathbb{Q}}$ be the rationalization of a nilpotent space X [12].

Definition 1.2 A map $f : X \rightarrow Y$ between nilpotent spaces is a **rational $\mathcal{E}$ -map** if the rationalization $f_{\mathbb{Q}} : X_{\mathbb{Q}} \rightarrow Y_{\mathbb{Q}}$ is an $\mathcal{E}$ -map. Similarly a map $f : X \rightarrow Y$ between nilpotent spaces is a **rational co- $\mathcal{E}$ -map** if the rationalization $f_{\mathbb{Q}} : X_{\mathbb{Q}} \rightarrow Y_{\mathbb{Q}}$ is a co- $\mathcal{E}$ -map.

Question 1.3 When is a map a (rational) $\mathcal{E}$ -map or a (rational) co- $\mathcal{E}$ -map?

Theorem 1.4 Let G be a compact connected Lie group and H be a connected closed sub-Lie group of G .

1. The inclusion $j : H \rightarrow G$ is a rational $\mathcal{E}$ -map if $\pi_*(j) \otimes \mathbb{Q}$ is injective.
2. For the homogeneous space G/H , the projection map $f : G \rightarrow G/H$ is a rational co- $\mathcal{E}$ -map.

The assumption $\mathcal{E}(X) \cong \mathcal{E}(Y)$ does not, in general, imply X and Y are homotopy equivalent spaces. Finally, we consider the question: *When is an isomorphism $\mathcal{E}(X_{\mathbb{Q}}) \cong \mathcal{E}(Y_{\mathbb{Q}})$ realized as a composition of rational $\mathcal{E}$ -maps and rational co- $\mathcal{E}$ -maps between $X_{\mathbb{Q}}$ and $Y_{\mathbb{Q}}$?*

Definition 1.5 We say that nilpotent spaces X and Y are **rationally $\mathcal{E}$ -equivalent** (denote as $X_{\mathbb{Q}} \sim_{\mathcal{E}} Y_{\mathbb{Q}}$) if there is a chain of *spherically injective* $\mathcal{E}$ -maps and *spherically injective* co- $\mathcal{E}$ -maps

$$X_{\mathbb{Q}} \xrightarrow{f_1} Z_1 \xleftarrow{f_2} \dots \xleftarrow{f_n} Z_n \xrightarrow{f_{n+1}} Y_{\mathbb{Q}}$$

(Z_i are rational spaces) such that an isomorphism $\mathcal{E}(X_{\mathbb{Q}}) \cong \mathcal{E}(Y_{\mathbb{Q}})$ is given by a composition of $n+1$ -isomorphisms $\{\phi_{f_i}\}_i$ and $\{\psi_{f_i}\}_i$, i.e., $\phi_{f_{n+1}} \circ \psi_{f_n} \circ \dots \circ \psi_{f_2} \circ \phi_{f_1} : \mathcal{E}(X_{\mathbb{Q}}) \xrightarrow{\cong} \mathcal{E}(Y_{\mathbb{Q}})$ or $\psi_{f_1} \circ \phi_{f_2} \circ \dots \circ \phi_{f_n} \circ \psi_{f_{n+1}} : \mathcal{E}(Y_{\mathbb{Q}}) \xrightarrow{\cong} \mathcal{E}(X_{\mathbb{Q}})$.

Remark 1.6 In this paper, we say that a map $f : X \rightarrow Y$ is *spherically injective* when $f_{\sharp}(u) \neq 0 \in \pi_*(Y)$ if $\text{hur}_X(u) \neq 0$ for $u \in \pi_*(X)$. Here $\text{hur}_X : \pi_*(X) \rightarrow H_*(X)$ is the Hurewicz homomorphism for a space X . Thus we have

(weakly) homotopy equivalent $\Rightarrow$ spherically injective $\Rightarrow$ homotopy non-trivial

If we admit the homotopy trivial maps as f_i , any isomorphism $\mathcal{E}(X_{\mathbb{Q}}) \cong \mathcal{E}(Y_{\mathbb{Q}})$ induces $X_{\mathbb{Q}} \sim_{\mathcal{E}} Y_{\mathbb{Q}}$ by the constant map $* : X_{\mathbb{Q}} \rightarrow Y_{\mathbb{Q}}$.

Theorem 1.7 1. If X and Y are rationally homotopy equivalent, i.e., $X_{\mathbb{Q}} \simeq Y_{\mathbb{Q}}$, then $X_{\mathbb{Q}} \sim_{\mathcal{E}} Y_{\mathbb{Q}}$.

2. For any n , $S_{\mathbb{Q}}^2 \sim_{\mathcal{E}} \mathbb{C}P_{\mathbb{Q}}^n$ and $S_{\mathbb{Q}}^4 \sim_{\mathcal{E}} \mathbb{H}P_{\mathbb{Q}}^n$.
3. When m is even and n is odd, $(S^m \vee S^n)_{\mathbb{Q}} \sim_{\mathcal{E}} (S^m \times S^n)_{\mathbb{Q}}$ if and only if $m \neq n+1$.
4. For odd-integers $1 < m \leq n$, $(S^m \times S^n)_{\mathbb{Q}} \sim_{\mathcal{E}} E_{\mathbb{Q}} \sim_{\mathcal{E}} E'_{\mathbb{Q}}$ for non-trivial fibrations $S^{m+n-1} \rightarrow E \xrightarrow{p} S^m \times S^n$ and $S^{2m+n-2} \rightarrow E' \xrightarrow{p'} E$.
5. There are integers $m \neq n$ such that $S_{\mathbb{Q}}^m \sim_{\mathcal{E}} S_{\mathbb{Q}}^n$. For example, $S_{\mathbb{Q}}^{53} \sim_{\mathcal{E}} S_{\mathbb{Q}}^{67}$.

Remark 1.8 The proof of Theorem 1.7 (5) requires a rigid rational space X of [3], which induces $\mathcal{E}(X) = \{id_X\}$. For the total space Z_1 of a fibration

$$X \rightarrow Z_1 \rightarrow S_{\mathbb{Q}}^m \times S_{\mathbb{Q}}^n,$$

it is given by the sequence $S_{\mathbb{Q}}^m \xrightarrow{f_1} Z_1 \xleftarrow{f_2} S_{\mathbb{Q}}^n$ of an $\mathcal{E}$ -map f_1 with $\phi_{f_1} : \mathcal{E}(S_{\mathbb{Q}}^m) \cong \mathcal{E}(Z_1)$ and a co- $\mathcal{E}$ -map f_2 with $\psi_{f_2} : \mathcal{E}(Z_1) \cong \mathcal{E}(S_{\mathbb{Q}}^n)$. Because of the demand that

$\mathcal{E}(Z_1) \cong \mathbb{Q}^*$ ($\mathbb{Q}^* := \mathbb{Q} - 0$), we need suitable restrictions about the pair (m, n) and see that (53, 67) satisfies them in the proof. Of course, it depends on the model structure of X . So we may require more various types of rigid models for the proof of (5) in many cases of (m, n) . On the other hand, the authors cannot find an example that $S_{\mathcal{E}}^m \mathbb{Q} \approx S_{\mathcal{E}}^n \mathbb{Q}$ for some (m, n) .

Problem 1.9 If $\mathcal{E}(X) \cong \mathcal{E}(Y)$ for rational spaces X and Y , does it hold that $X \underset{\mathcal{E}}{\sim} Y$?

Remark 1.10 For rational spaces X, Y and Z , even if $Y \underset{\mathcal{E}}{\sim} Z$, it may not hold that $X \times Y \underset{\mathcal{E}}{\sim} X \times Z$. For example, when $X = S^5, Y = S^2$ and $Z = \mathbb{C}P^2, \mathcal{E}((X \times Y)_{\mathbb{Q}}) \cong \mathbb{Q}^* \times \mathbb{Q}^*$ but $\mathcal{E}((X \times Z)_{\mathbb{Q}})$ is isomorphic to the subgroup of lower triangular matrixes of $GL(2, \mathbb{Q})$.

In Sect. 2, we demonstrate the basic properties and provide examples in ordinary homotopy theory of $\mathcal{E}$ -maps and co- $\mathcal{E}$ -maps. In Sect. 3, we give some computations in rational homotopy theory using Sullivan minimal models.

2 Some properties

Recall that $[X, -]$ is the covariant functor from the category of spaces to the category of sets, wherse for a map $f : Y \rightarrow Z$, the map $f_*(g) : [X, Y] \rightarrow [X, Z]$ is given by $f_*(g) = f \circ g$. On the other hand, $[-, Z]$ is the contravariant functor. For the map $f : X \rightarrow Y$, the map $f^*(g) : [Y, Z] \rightarrow [X, Z]$ is given by $f^*(g) = g \circ f$. The following lemma holds from $\phi_f(g) \circ f = f \circ g$ and $f \circ \psi_f(g) = g \circ f$.

Lemma 2.1 A map $f : X \rightarrow Y$ is an $\mathcal{E}$ -map (or a co- $\mathcal{E}$ -map) if and only if there is a group homomorphism $\phi_f : \mathcal{E}(X) \rightarrow \mathcal{E}(Y)$ (or $\psi_f : \mathcal{E}(Y) \rightarrow \mathcal{E}(X)$) where the following diagrams

$$\begin{array}{ccc} [X, X] & \xrightarrow{f_*} & [X, Y] \xleftarrow{f^*} [Y, Y] \\ \uparrow \cup & & \uparrow \cup \\ \mathcal{E}(X) & \xrightarrow{\phi_f} & \mathcal{E}(Y) \end{array} \quad \begin{array}{ccc} [X, X] & \xrightarrow{f_*} & [X, Y] \xleftarrow{f^*} [Y, Y] \\ \uparrow \cup & & \uparrow \cup \\ \mathcal{E}(X) & \xleftarrow{\psi_f} & \mathcal{E}(Y) \end{array}$$

are commutative.

Of course, the maps ϕ_f and ψ_f may not be uniquely determined for a map f .

Lemma 2.2 1. If maps $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are $\mathcal{E}$ -maps, then $g \circ f : X \rightarrow Z$ is an $\mathcal{E}$ -map.

2. If f and g are co- $\mathcal{E}$ -maps, then $g \circ f$ is a co- $\mathcal{E}$ -map.

3. The constant map is both an $\mathcal{E}$ -map and a co- $\mathcal{E}$ -map.

4. A homotopy equivalence map is both an $\mathcal{E}$ -map and a co- $\mathcal{E}$ -map.

Proof 1. $\phi_{g \circ f}(h) := \phi_g \circ \phi_f(h)$ for $h \in \mathcal{E}(X)$.

2. $\psi_{g \circ f}(h) := \psi_f \circ \psi_g(h)$ for $h \in \mathcal{E}(Z)$.
3. It is sufficient to put $\phi_f = \psi_f = *$, i.e., $\phi_f(g) = id_Y$ and $\psi_f(g) = id_X$ for any g .
4. It is sufficient to put $\phi_f(h) := f \circ h \circ f^{-1}$ for $h \in \mathcal{E}(X)$ and $\psi_f(h) := f^{-1} \circ h \circ f$ for $h \in \mathcal{E}(Y)$, where f^{-1} is the homotopy inverse of f .

□

Definition 2.3 [11, Chapter 3] ([13]) Let $\alpha : X \rightarrow Y$ and $\beta : Z \rightarrow W$ be maps. $\Pi(\alpha, \beta)$ is the set of all homotopy classes of pairs $[f_1, f_2]$ such that

$$\begin{array}{ccc} X & \xrightarrow{f_1} & Z \\ \alpha \downarrow & & \downarrow \beta \\ Y & \xrightarrow{f_2} & W \end{array}$$

is homotopy commutative. Here a homotopy of (f_1, f_2) is just a pair of homotopies (f_{1t}, f_{2t}) such that $\beta f_{1t} = f_{2t} \alpha$. If $[f_1, f_2]$ has a two sided inverse in $\Pi(\alpha, \beta)$, we call $[f_1, f_2]$ a homotopy equivalence. If $\alpha = \beta$, we call $[f_1, f_2]$ a self-homotopy equivalence and denote the set of all self-homotopy equivalences by $\mathcal{E}(\alpha)$.

Lemma 2.4 For a map $f : X \rightarrow Y$,

1. f is an $\mathcal{E}$ -map if and only if $h : \mathcal{E}(f) \rightarrow \mathcal{E}(X)$ given by $h[g_1, g_2] = [g_1]$ is an epimorphism with a section.
2. f is a co- $\mathcal{E}$ -map if and only if $h' : \mathcal{E}(f) \rightarrow \mathcal{E}(Y)$ given by $h'[g_1, g_2] = [g_2]$ is an epimorphism with a section.

Proof 1. Suppose that f is an $\mathcal{E}$ -map. Then we have a map $\phi_f : \mathcal{E}(X) \rightarrow \mathcal{E}(Y)$ such that $\phi_f(g) \circ f \simeq f \circ g$ for any $g \in \mathcal{E}(X)$. Thus we have $[g, \phi_f(g)] \in \mathcal{E}(f)$ and $h[g, \phi_f(g)] = [g]$ and h is epimorphic. Next suppose that h is an epimorphism. For any $[g] \in \mathcal{E}(X)$, we have $[g', g''] \in \mathcal{E}(f)$ such that $h[g', g''] = [g]$. So g is homotopic to g' . Since $[g', g''] \in \mathcal{E}(f)$, g' and g'' are homotopy equivalences and $g'' \circ f \simeq f \circ g'$. Thus we can define a map $\phi_f : \mathcal{E}(X) \rightarrow \mathcal{E}(Y)$ by $\phi_f(g) = \pi \circ s[g]$ where $\pi : \mathcal{E}(f) \rightarrow \mathcal{E}(Y)$ is the natural projection and s is the section of the assumption. Hence, f is an $\mathcal{E}$ -map.

2. Suppose that f is a co- $\mathcal{E}$ -map. Then we have a map $\psi_f : \mathcal{E}(Y) \rightarrow \mathcal{E}(X)$ such that $g \circ f \simeq f \circ \psi_f(g)$ for any $g \in \mathcal{E}(Y)$. So we have $[\psi_f(g), g] \in \mathcal{E}(f)$ and $h'[\psi_f(g), g] = [g]$. Thus h' is epimorphic. Next suppose that h' is an epimorphism. For any $[g] \in \mathcal{E}(Y)$, we have $[g', g''] \in \mathcal{E}(f)$ such that $h[g', g''] = [g]$ and thus g is homotopic to g'' . Since $[g', g''] \in \mathcal{E}(f)$, g' and g'' are homotopy equivalences and $g'' \circ f \simeq f \circ g'$. Then we can define a map $\psi_f : \mathcal{E}(Y) \rightarrow \mathcal{E}(X)$ by $\psi_f(g) = h \circ s'[g]$ for the section s' . Hence, f is a co- $\mathcal{E}$ -map.

□

Theorem 2.5 Let $\eta : S^3 \rightarrow S^2$ and $\nu : S^7 \rightarrow S^4$ be the Hopf fibrations with fibre S^1 and S^3 , respectively. Let $\epsilon_3 : S^{11} \rightarrow S^3$ be the generator of $\pi_{11}(S^3) \cong \mathbb{Z}_2$ [19]. Then

1. η is a co- $\mathcal{E}$ -map, but not an $\mathcal{E}$ -map,
2. ν is neither an $\mathcal{E}$ -map nor a co- $\mathcal{E}$ -map and
3. ϵ_3 is both an $\mathcal{E}$ -map and a co- $\mathcal{E}$ -map.

Proof 1. From [14, Example 4.2 (i)], we have $\Pi(\eta, \eta) = \{(k^2\iota_3, k\iota_2) \mid k \in \mathbb{Z}\}$ as a set. Therefore, we have a homotopy commutative diagram

$$\begin{array}{ccc} S^3 & \xrightarrow{k^2\iota_3} & S^3 \\ \eta \downarrow & & \downarrow \eta \\ S^2 & \xrightarrow{k\iota_2} & S^2 \end{array}$$

It is well known that $\mathcal{E}(S^n) = \{\iota_n, -\iota_n\} \cong \mathbb{Z}_2$. Since $(\iota_3, -\iota_2), (\iota_3, \iota_2) \in \Pi(\eta, \eta)$, η is a co- $\mathcal{E}$ -map. However, there is no map $f : S^2 \rightarrow S^2$ such that $(-\iota_3, f) \in \Pi(\eta, \eta)$. Thus η is not an $\mathcal{E}$ -map.

2. From [14, Example 4.2 (ii)], we have $\Pi(\nu, \nu) = \{(k^2\iota_7, k\iota_4) \mid k(k-1) \equiv 0 \pmod{8}\}$ as a set. Therefore, we have a homotopy commutative diagram

$$\begin{array}{ccc} S^7 & \xrightarrow{k^2\iota_7} & S^7 \\ \nu \downarrow & & \downarrow \nu \\ S^4 & \xrightarrow{k\iota_4} & S^4 \end{array}$$

Since there are no maps $f : S^7 \rightarrow S^7$ and $g : S^4 \rightarrow S^4$ such that $(f, -\iota_4), (-\iota_7, g) \in \Pi(\nu, \nu)$, ν is neither an $\mathcal{E}$ -map nor a co- $\mathcal{E}$ -map.

3. From [14, Example 4.2 (iv)], we have $\Pi(\epsilon_3, \epsilon_3) = \{(d+2s)\iota_{11}, d\iota_3 \mid d, s \in \mathbb{Z}\} \cong \mathbb{Z} \times \mathbb{Z}$ as a group. Therefore we have a homotopy commutative diagram

$$\begin{array}{ccc} S^{11} & \xrightarrow{(d+2s)\iota_{11}} & S^{11} \\ \epsilon_3 \downarrow & & \downarrow \epsilon_3 \\ S^3 & \xrightarrow{d\iota_3} & S^3 \end{array}$$

Since $(\iota_{11}, \iota_3), (-\iota_{11}, -\iota_3) \in \Pi(\epsilon_3, \epsilon_3)$, ϵ_3 is both an $\mathcal{E}$ -map and a co- $\mathcal{E}$ -map. $\square$

Example 2.6 1. Let $e : X \rightarrow \Omega\Sigma X$ be the adjoint of $id_{\Sigma X}$ from the one-to-one correspondence $[X, \Omega\Sigma X] \cong [\Sigma X, \Sigma X]$. We know that $e(x)(t) = \langle x, t \rangle$. Let f be a self homotopy equivalence on X , that is, $f \in \mathcal{E}(X)$ and let f' be a homotopy inverse of f . It is clear that the map $\Sigma f : \Sigma X \rightarrow \Sigma X$, $\Sigma f \langle x, t \rangle = \langle f(x), t \rangle$, is a homotopy equivalence with homotopy inverse $\Sigma f'$. Then we define a map $\tilde{f} : \Omega\Sigma X \rightarrow \Omega\Sigma X$ by $\tilde{f}(\alpha)(t) = \Sigma f(\alpha(t))$. Define another map $\tilde{f}' : \Omega\Sigma X \rightarrow \Omega\Sigma X$ by $\tilde{f}'(\alpha)(t) = \Sigma f'(\alpha(t))$. Clearly we have $\tilde{f} \circ \tilde{f}' \simeq id$ and $\tilde{f}' \circ \tilde{f} \simeq id$. Moreover we have $e(f(x))(t) = \langle f(x), t \rangle$ and $\tilde{f}(e(x))(t) = \Sigma f(e(x)(t)) =$

$\Sigma f \langle x, t \rangle = \langle f(x), t \rangle$. Therefore we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ e \downarrow & & \downarrow e \\ \Omega \Sigma X & \xrightarrow{\tilde{f}} & \Omega \Sigma X \end{array}$$

Thus $e : X \rightarrow \Omega \Sigma X$ is an $\mathcal{E}$ -map.

- Let $\pi : \Sigma \Omega Y \rightarrow Y$ be the adjoint of $id_{\Omega Y}$ from the one-to-one correspondence $[\Sigma \Omega Y, Y] \cong [\Omega Y, \Omega Y]$. We know that $\pi \langle \alpha, t \rangle = \alpha(t)$. Let g be a self homotopy equivalence on Y , that is $g \in \mathcal{E}(Y)$ and let g' be a homotopy inverse of g . Then we define a map $\tilde{g} : \Sigma \Omega Y \rightarrow \Sigma \Omega Y$ by $\tilde{g} \langle \alpha, t \rangle = \langle g \circ \alpha, t \rangle$ and $\tilde{g}' : \Sigma \Omega Y \rightarrow \Sigma \Omega Y$ by $\tilde{g}' \langle \alpha, t \rangle = \langle g' \circ \alpha, t \rangle$. Clearly we have $\tilde{g} \circ \tilde{g}' \simeq id$ and $\tilde{g}' \circ \tilde{g} \simeq id$. Moreover we have $(\pi \circ \tilde{g}) \langle \alpha, t \rangle = \pi \langle g \circ \alpha, t \rangle = (g \circ \alpha)(t)$ and $(g \circ \pi) \langle \alpha, t \rangle = g(\alpha(t))$. Therefore we have a commutative diagram

$$\begin{array}{ccc} \Sigma \Omega Y & \xrightarrow{\tilde{g}} & \Sigma \Omega Y \\ \pi \downarrow & & \downarrow \pi \\ Y & \xrightarrow{g} & Y \end{array}$$

Therefore $\pi : \Sigma \Omega Y \rightarrow Y$ is a co- $\mathcal{E}$ -map.

Example 2.7 There is a natural homomorphism $\mathcal{E}(X_{(n)}) \rightarrow \mathcal{E}(X_{(n-1)})$ obtained by restricting the map to a lower Postnikov section [2, p. 27]. Thus the principal $K(\pi_n(X), n)$ -fibration $X_{(n)} \rightarrow X_{(n-1)}$ is an $\mathcal{E}$ -map. The restriction map $X \rightarrow X_{(n-1)}$ is also an $\mathcal{E}$ -map. On the other hand, for the n -skeleton $X^{(n)}$, the inclusions $X^{(n)} \rightarrow X^{(n+1)}$ and $X^{(n)} \rightarrow X$ are both co- $\mathcal{E}$ -maps.

Remark 2.8 Recall that a space X is said to be (homotopically) *rigid* when $\mathcal{E}(X) = \{id_X\}$ [3, 7]. If X is rigid, then every map $f : X \rightarrow Y$ is an $\mathcal{E}$ -map by $\phi_f(id_X) = id_Y$. If Y is rigid, then every map $f : X \rightarrow Y$ is a co- $\mathcal{E}$ -map by $\psi_f(id_Y) = id_X$. Also we can construct infinitely many examples of $\mathcal{E}$ -maps and co- $\mathcal{E}$ -maps by using the functor of [7, Remark 2.8].

3 Computations in Sullivan models

We assume that X is a nilpotent CW complex. Let $M(X) = (\Lambda V, d)$ be the Sullivan minimal model of X [18]. It is a free $\mathbb{Q}$ -commutative differential graded algebra (DGA) with a $\mathbb{Q}$ -graded vector space $V = \bigoplus_{i \geq 1} V^i$ where $\dim V^i < \infty$ and a decomposable differential; i.e., $d(V^i) \subset (\Lambda^+ V \cdot \Lambda^+ V)^{i+1}$ and $d \circ d = 0$. Here $\Lambda^+ V$ is the ideal of ΛV generated by elements of positive degree. The degree of a homogeneous element x of a graded algebra is denoted as $|x|$. Then $xy = (-1)^{|x||y|}yx$ and $d(xy) = d(x)y + (-1)^{|x|}x d(y)$. Note that $M(X)$ determines the rational homotopy

type of X . In particular, $H^*(\Lambda V, d) \cong H^*(X; \mathbb{Q})$ and $V^i \cong \text{Hom}(\pi_i(X), \mathbb{Q})$. Refer to [8] for details.

Let $\text{Aut}M$ be the group of DGA-automorphisms of a DGA M . For a nilpotent space X and a (not necessarily minimal) model $M(X)$, there is a group isomorphism

$$\mathcal{E}(X_{\mathbb{Q}}) \cong \mathcal{E}(M(X)),$$

where $\mathcal{E}(M(X)) = \text{Aut}M(X)/\sim$ is the group of self-DGA-homotopy equivalence classes of $M(X)$ [18]. Now recall about ‘‘DGA-homotopy’’ in [9]: In general, two maps $f : M(Y) \rightarrow M(X)$ and $g : M(Y) \rightarrow M(X)$ are DGA-homotopic (denote as $f \sim g$) if there is a DGA-map $H : M(Y) \rightarrow M(X) \otimes \Lambda(t, dt)$ such that $H|_{t=0, dt=0} = f$ and $H|_{t=1, dt=0} = g$. Here $|t| = 0$ and $|dt| = 1$ with $d(t) = dt$, $d(dt) = 0$.

The group $\mathcal{E}(M(X))$ does not depend on choosing a model of X . For example, the minimal model $M = M(S^{2n+1}) = (\Lambda w, 0)$ and a non-minimal model $M' = (\Lambda(y, w, v), D)$ with $|y| = 2$, $|w| = 2n + 1$, $|v| = 1$, $Dy = 0$, $Dw = y^{n+1}$ and $Dv = y$ are both models of S^{2n+1} . Obviously we have $\mathcal{E}(M) \cong \mathbb{Q}^*$ by $w \rightarrow aw$ for $a \in \mathbb{Q}^*$. On the other hand, in $\mathcal{E}(M')$ we can set $H(y) = cyt + cvdt$, $H(v) = cvt$ and $H(w) = aw + by^n v + b'y^n vt^{n+1}$ with $a, b, b', c \in \mathbb{Q}^*$. Then $a + b = 0$ and $b' = c^{n+1}$ from $D \circ H = H \circ D$. Thus two maps $f \in \text{Aut}M'$ given by $f(y) = f(v) = 0$, $f(w) = a(w - y^n v)$ and $g \in \text{Aut}M'$ given by $g(y) = cy$, $g(v) = cv$, $g(w) = a(w - y^n v) + c^{n+1}y^n v$ are DGA-homotopic. Hence we have $\mathcal{E}(M') = \mathcal{E}(M) \cong \mathbb{Q}^*$ as in Example 3.3(1) below.

Remark 3.1 From the universality of the localization [12], the rationalization map $l : X \rightarrow X_{\mathbb{Q}}$ is an $\mathcal{E}$ -map, but it is not a co- $\mathcal{E}$ -map in general. For example, when $X = S^3$, the elements f of $\mathcal{E}(M(X)) = \mathcal{E}(\Lambda(x), 0)$ with $f(x) = ax$ for $a \neq \pm 1 \in \mathbb{Q}^* = \mathbb{Q} - 0$ can not be realized as a homotopy equivalence of X .

The model of a map $f : X \rightarrow Y$ between nilpotent spaces is given by a relative model:

$$M(Y) = (\Lambda W, d_Y) \xrightarrow{i} (\Lambda W \otimes \Lambda V, D) \xrightarrow{q} (\Lambda V, \overline{D})$$

with $D|_{\Lambda W} = d_Y$ and the minimal model $(\Lambda V, \overline{D})$ of the homotopy fiber of f . It is well known that there is a quasi-isomorphism $M(X) \rightarrow (\Lambda W \otimes \Lambda V, D)$ [8]. Then Definition 1.2 is translated to

Definition 3.2 Let $f : X \rightarrow Y$ be a map between nilpotent spaces.

1. The map f is a rational $\mathcal{E}$ -map if and only if there is a homomorphism $\phi_f : \mathcal{E}(\Lambda W \otimes \Lambda V, D) \rightarrow \mathcal{E}(\Lambda W, d_Y)$ such that

$$\begin{array}{ccc} (\Lambda W \otimes \Lambda V, D) & \xrightarrow{g} & (\Lambda W \otimes \Lambda V, D) \\ \uparrow i & & \uparrow i \\ (\Lambda W, d_Y) & \xrightarrow{\phi_f(g)} & (\Lambda W, d_Y) \end{array}$$

is DGA-homotopy commutative for any element g of $\mathcal{E}(\Lambda W \otimes \Lambda V, D)$.

2. The map f is a rational co- $\mathcal{E}$ -map if and only if there is a homomorphism $\psi_f : \mathcal{E}(\Lambda W, d_Y) \rightarrow \mathcal{E}(\Lambda W \otimes \Lambda V, D)$ such that

$$\begin{array}{ccc} (\Lambda W \otimes \Lambda V, D) & \xrightarrow{\psi_f(g)} & (\Lambda W \otimes \Lambda V, D) \\ \uparrow i & & \uparrow i \\ (\Lambda W, d_Y) & \xrightarrow{g} & (\Lambda W, d_Y) \end{array}$$

is DGA-homotopy commutative for any element g of $\mathcal{E}(\Lambda W, d_Y)$.

Example 3.3 1. For the Hopf fibration $S^1 \rightarrow S^{2n+1} \xrightarrow{f} \mathbb{C}P^n$, the relative model is given by

$$(\Lambda(y, w), d_Y) \rightarrow (\Lambda(y, w, v), D) \rightarrow (\Lambda(v), 0)$$

with $|y| = 2$, $|w| = 2n + 1$, $|v| = 1$, $d_Y w = y^{n+1}$ and $Dv = y$. We can identify $\mathcal{E}(\mathbb{C}P^n_{\mathbb{Q}})$ as $\mathbb{Q}^*$ by $g(y) = ay$ and $g(w) = a^{n+1}w$ for $g \in \mathcal{E}(\mathbb{C}P^n_{\mathbb{Q}})$ and $a \in \mathbb{Q}^*$. Also we have $\mathcal{E}(S^{2n+1}_{\mathbb{Q}}) = \mathcal{E}(\Lambda(y, w, v), D) = \mathcal{E}(\Lambda w, 0) \cong \mathbb{Q}^*$. Then there is a homomorphism

$$\psi_f : \mathbb{Q}^* \cong \mathcal{E}(\mathbb{C}P^n_{\mathbb{Q}}) \rightarrow \mathcal{E}(S^{2n+1}_{\mathbb{Q}}) \cong \mathbb{Q}^*$$

which is given by $\psi_f(a) = a^{n+1}$ for $a \in \mathbb{Q}^*$. Thus f is a rational co- $\mathcal{E}$ -map, but it is not a rational $\mathcal{E}$ -map.

2. Let X be the pullback of the sphere bundle of the tangent bundle of S^{m+n} by the canonical degree 1 map $S^m \times S^n \rightarrow S^{m+n}$ for odd integers m and n . Then it is the total space of a fibration $S^{m+n-1} \rightarrow X \xrightarrow{f} S^m \times S^n$ whose model is

$$(\Lambda(w_1, w_2), 0) \rightarrow (\Lambda(w_1, w_2, u), D) \rightarrow (\Lambda(u), 0)$$

with $|w_1| = m$, $|w_2| = n$, $|u| = m + n - 1$ and $Du = w_1 w_2$ is both a rational $\mathcal{E}$ -map and a rational co- $\mathcal{E}$ -map.

3. The fibration $S^m \times S^{m+n-1} \rightarrow X \xrightarrow{f} S^n$ ($m \neq n$ are odd) whose model is

$$(\Lambda(w), 0) \rightarrow (\Lambda(w, v, u), D) \rightarrow (\Lambda(v, u), 0)$$

where $|w| = n$, $|v| = m$, $|u| = m + n - 1$ and $Du = wv$ with m, n odd is both a rational $\mathcal{E}$ -map and a rational co- $\mathcal{E}$ -map.

4. For the fibration $\mathbb{C}P^{n-1} \rightarrow \mathbb{C}P^{2n-1} \xrightarrow{f} S^{2n}$ given by

$$(\Lambda(y, w), d_Y) \rightarrow (\Lambda(y, w, x, v), D) \rightarrow (\Lambda(x, v), \overline{D})$$

with $|y| = 2n$, $|w| = 4n - 1$, $|x| = 2$, $|v| = 2n - 1$, $d_Y y = 0$, $d_Y w = y^2$, $Dx = \overline{D}x = 0$, $Dv = y - x^n$ and $\overline{D}v = x^n$, the map f is a rational $\mathcal{E}$ -map given by $\phi_f(a) = a^n$ for $a \in \mathbb{Q}^*$ but not a rational co- $\mathcal{E}$ -map.

Example 3.4 For an n -dimensional manifold X , the collapsing map of lower cells $f : X \rightarrow S^n$ is an $\mathcal{E}$ -map. Indeed, from the commutative diagram between cofibrations

$$\begin{array}{ccc} X^{(n-1)} & \xrightarrow{g|_{X^{(n-1)}}} & X^{(n-1)} \\ \downarrow & & \downarrow \\ X & \xrightarrow{g} & X \\ \downarrow f & & \downarrow f \\ S^n & \xrightarrow{\overline{g}} & S^n, \end{array}$$

we have $\phi_f(g) = \overline{g}$, but it is not a (rational) co- $\mathcal{E}$ -map in general. For example, the collapsing map of lower cells $f : X = \mathbb{C}P^n \rightarrow S^{2n} = Y$ induces a DGA-map

$$f^* : M(Y) = (\Lambda(y, w), d_Y) \rightarrow (\Lambda(x, v), d_X) = M(X)$$

with $d_Y w = y^2$, $d_X v = x^{n+1}$, $f^*(y) = x^n$ and $f^*(w) = x^{n-1}v$. The map f is a rational $\mathcal{E}$ -map by $\phi_f(a) = a^n$ for $a \in \mathbb{Q}^*$ but not a rational co- $\mathcal{E}$ -map. Indeed, for $g^*(y) = ay$ with $a \notin (\mathbb{Q}^*)^{\times m} := \mathbb{Q}^* \cdot \mathbb{Q}^* \cdots \mathbb{Q}^*$ (m -times), we cannot define $\psi_{f^*}(g^*)$.

Example 3.5 Let $\Omega Y = \text{map}((S^1, *), (Y, *))$ be the base point preserving the loop space of a simply connected space Y and $LY = \text{map}(S^1, Y)$, the free loop space of Y . We consider the evaluation map $f : LY \rightarrow Y$ with $f(\sigma) = \sigma(*)$. It is a co- $\mathcal{E}$ -map by $\psi_f(g)(h) = g \circ h$ for $g \in \mathcal{E}(Y)$. What is the (rational) homotopical condition of Y that allows f to be a (rational) $\mathcal{E}$ -map? According to [20], the relative model of the free loop fibration $\Omega Y \rightarrow LY \xrightarrow{f} Y$:

$$M(Y) = (\Lambda V, d) \rightarrow (\Lambda V \otimes \Lambda \overline{V}, D) \rightarrow (\Lambda \overline{V}, 0)$$

with $M(LY) = (\Lambda V \otimes \Lambda \overline{V}, D)$ is defined as follow: The graded vector space $\overline{V}$ satisfies $\overline{V}^n \cong V^{n-1}$ for $n > 0$ and denote by $s : V \rightarrow \overline{V}$ ($s(v) := \overline{v}$) this isomorphism of degree -1 . There is a unique extension of s into a derivation of algebra $s : \Lambda V \otimes \Lambda \overline{V} \rightarrow \Lambda V \otimes \Lambda \overline{V}$ such that $s(\overline{V}) = 0$. The differential D is given by $D(v) = d(v)$ for $v \in V$ and $D(\overline{v}) = -s \circ d(v)$ for $\overline{v} \in \overline{V}$.

If every DGA-isomorphism g of $(\Lambda V \otimes \Lambda \overline{V}, D)$ satisfies $g|_{\Lambda V} \in \mathcal{E}(\Lambda V, d)$, then $f(M(f))$ is a rational $\mathcal{E}$ -map by $\phi_f(g) = g|_{\Lambda V}$.

1. When $Y = S^n$, we observe that the map f is a rational $\mathcal{E}$ -map. If n is even, $M(S^n) = (\Lambda(x, y), d)$ with $|x| = n$, $|y| = 2n + 1$, $dx = 0$ and $dy = x^2$. For example, when $n = 2$, note that there is no DGA-map $g(x) = x + \overline{y}$.

2. When $Y = S^m \times S^n$ for odd integers $m < n$, the map f is a rational $\mathcal{E}$ -map if and only if $m - 1$ is not a divisor of $n - 1$. Indeed, let $M(S^m \times S^n) = (\Lambda(x, y), 0)$. When $n - 1 = a(m - 1)$ for an integer $a > 1$, there is a DGA-isomorphism $g : (\Lambda(x, y, \bar{x}, \bar{y}), 0) \rightarrow (\Lambda(x, y, \bar{x}, \bar{y}), 0)$ with $g(x) = x$, $g(\bar{x}) = \bar{x}$, $g(\bar{y}) = \bar{y}$ and $g(y) = y + \bar{x}^{a-1}x$. Then f cannot be a rational $\mathcal{E}$ -map. When $n - 1 \neq a(m - 1)$ for any a , a self-map g is given by $g(x) = x$ and $g(y) = y$ from the degree reason.

Proof of Theorem 1.4. (1) Note that $\pi_*(j)_{\mathbb{Q}}$ is injective if and only if the model of $j : H \rightarrow G$ is given as the projection $M(G) \cong (\Lambda(v_1, \dots, v_k, u_1, \dots, u_l), 0) \rightarrow (\Lambda(v_1, \dots, v_k), 0) \cong M(H)$ after a suitable basis change. Then we can define as $\phi_j(g) = g \otimes 1_{\Lambda(u_1, \dots, u_l)}$ for any $g \in \mathcal{E}(\Lambda(v_1, \dots, v_k), 0)$. $\square$

For the n -dimensional unitary group $U(n)$, $M(U(n)) = M(S^1 \times \dots \times S^{2n-1}) = (\Lambda(v_1, \dots, v_n), 0)$ with $|v_i| = 2i - 1$. For the n -dimensional special unitary group $SU(n)$, $M(SU(n)) = (\Lambda(v_1, \dots, v_{n-1}), 0)$ with $|v_i| = 2i + 1$. For the n -dimensional symplectic group $Sp(n)$, $M(Sp(n)) = (\Lambda(v_1, \dots, v_n), 0)$ with $|v_i| = 4i - 1$.

Example 3.6 In general, for a connected closed sub-Lie group H of a compact connected Lie group G , the inclusion $j : H \rightarrow G$, is not a rational $\mathcal{E}$ -map. For example, the blockwise inclusion $j : SU(3) \times SU(3) \rightarrow SU(6)$ is not. Indeed, $M(SU(3) \times SU(3)) = (\Lambda(u_1, w_1, u_2, w_2), 0)$ with $|u_1| = |w_1| = 3$, $|u_2| = |w_2| = 5$ and $M(SU(6)) = (\Lambda(v_1, v_2, v_3, v_4, v_5), 0)$ with $|v_i| = 2i + 1$. $M(j)(v_i) = u_i + w_i$ for $i = 1, 2$. Then we cannot define $\phi_j(g)$ for $g \in \mathcal{E}(\Lambda(u_1, w_1, u_2, w_2), 0)$ when $g(u_i) = u_i$, $g(w_i) = -w_i$ for example.

Lemma 3.7 Let $X = S^{a_1} \times \dots \times S^{a_m} \times Y$ and $Y = S^{b_1} \times \dots \times S^{b_n}$ for odd-integers $a_1 \leq \dots \leq a_m \leq b_1 \leq \dots \leq b_n$. Then the second factor projection map $f : X \rightarrow Y$ is a rational $\mathcal{E}$ -map if and only if there are no subsets $\{i_1, \dots, i_k\}$ of $\{1, \dots, m\}$ and $\{j_1, \dots, j_k\}$ of $\{1, \dots, n\}$ with $b_k = a_{i_1} + \dots + a_{i_k} + b_{j_1} + \dots + b_{j_k}$ for $k = 1, \dots, n$.

Proof Put $M(X) = (\Lambda(x_1, \dots, x_m, y_1, \dots, y_n), 0)$ and $M(Y) = (\Lambda(y_1, \dots, y_n), 0)$ with $|x_i| = a_i$ and $|y_i| = b_i$. If $b_k = a_{i_1} + \dots + a_{i_k} + b_{j_1} + \dots + b_{j_k}$, there is a map $g \in \mathcal{E}(M(X))$ such that

$$g(x_i) = x_i \quad (i \leq m), \quad g(y_i) = y_i \quad (i \neq k), \quad g(y_k) = y_k + x_{i_1} \cdots x_{i_k} y_{j_1} \cdots y_{j_k}$$

and $M(f)(y_i) = y_i$ for all i . Then we can not have a DGA-homotopy commutative diagram

$$\begin{array}{ccc} (\Lambda(x_1, \dots, x_m, y_1, \dots, y_n), 0) & \xrightarrow{g} & (\Lambda(x_1, \dots, x_m, y_1, \dots, y_n), 0) \\ \uparrow M(f) & & \uparrow M(f) \\ (\Lambda(y_1, \dots, y_n), 0) & \xrightarrow{\phi_f(g)} & (\Lambda(y_1, \dots, y_n), 0). \end{array}$$

If $b_k \neq a_{i_1} + \dots + a_{i_k} + b_{j_1} + \dots + b_{j_k}$ for any k and index set, we can put

$$\phi_f(g) = g|_{\Lambda(y_1, \dots, y_n)}$$

in the diagram for any map $g \in \mathcal{E}(M(X))$. $\square$

- Theorem 3.8** 1. When $2 < m < n$, the natural projection $p_{n,m} : U(n) \rightarrow U(n)/U(m)$ is a rational $\mathcal{E}$ -map if and only if $n < 5$.
 2. When $2 < m < n$, the natural projection $p_{n,m} : SU(n) \rightarrow SU(n)/SU(m)$ is a rational $\mathcal{E}$ -map if and only if $n < 8$.

Lemma 3.9 Let $X = S^{a_1} \times \cdots \times S^{a_m}$ and $Y = X \times S^{b_1} \times \cdots \times S^{b_n}$ for odd-integers $a_1 \leq \cdots \leq a_m \leq b_1 \leq \cdots \leq b_n$. Then the first factor inclusion map $f : X \rightarrow Y$ is a rational co- $\mathcal{E}$ -map if and only if there is no subset $\{i_1, \dots, i_k\}$ of $\{1, \dots, m\}$ with $b_k = a_{i_1} + \cdots + a_{i_k}$ for $k = 1, \dots, n$.

Proof Let $M(X) = (\Lambda(x_1, \dots, x_m), 0)$ and $M(Y) = (\Lambda(x_1, \dots, x_m, y_1, \dots, y_n), 0)$ with $|x_i| = a_i$ and $|y_i| = b_i$. If $b_k = a_{i_1} + \cdots + a_{i_k}$, there is a map $g \in \mathcal{E}(M(Y))$ such that

$$g(x_i) = x_i \quad (i \leq m), \quad g(y_i) = y_i \quad (i \neq k), \quad g(y_k) = y_k + x_{i_1} \cdots x_{i_k}$$

and $M(f)(x_i) = x_i$ and $M(f)(y_i) = 0$ for all i . Then we cannot have a DGA-homotopy commutative diagram

$$\begin{array}{ccc} (\Lambda(x_1, \dots, x_m), 0) & \xrightarrow{\psi_f(g)} & (\Lambda(x_1, \dots, x_m), 0) \\ M(f) \uparrow & & \uparrow M(f) \\ (\Lambda(x_1, \dots, x_m, y_1, \dots, y_n), 0) & \xrightarrow{g} & (\Lambda(x_1, \dots, x_m, y_1, \dots, y_n), 0). \end{array}$$

If $b_k \neq a_{i_1} + \cdots + a_{i_k}$ for any k and $\{i_1, \dots, i_k\}$, we can put

$$\psi_f(g) = g|_{\Lambda(x_1, \dots, x_m)}$$

in the diagram for any map $g \in \mathcal{E}(M(Y))$. $\square$

From Lemma 3.9, we have the following.

- Theorem 3.10** 1. When $2 < m < n$, the natural inclusion map $i_{m,n} : U(m) \rightarrow U(n)$ is a rational co- $\mathcal{E}$ -map if and only if $n < 5$.
 2. When $2 < m < n$, the natural inclusion map $i_{m,n} : SU(m) \rightarrow SU(n)$ is a rational co- $\mathcal{E}$ -map if and only if $n < 8$.
 3. When $m \leq 4$, the natural inclusion map $i_{m,n} : Sp(m) \rightarrow Sp(n)$ is a rational co- $\mathcal{E}$ -map for any $m \leq n$. When $4 < m < n$, the natural inclusion map $i_{m,n} : Sp(m) \rightarrow Sp(n)$ is a rational co- $\mathcal{E}$ -map if and only if $n < 14$.

Proof (3) For $S = \{3, 7, 11, 15, 19, 23, 27, 31, 35, 39, 43, 47, 51, 55, \dots\}$, there are no integers $a, b, c, d \in S$ with $a < b < c < d$ satisfying the equation $a + b + c = d$ since

$$(4i - 1) + (4j - 1) + (4k - 1) = 4(i + j + k) - 3 \neq 4l - 1$$

for any $i, j, k, l \in \mathbb{N}$. On the other hand, $3+7+11+15+(19+4i) = 55+4i = |v_{14+i}|$ for $i \geq 0$. $\square$

For a connected closed sub-Lie group H of a compact connected Lie group G with inclusion $j : H \rightarrow G$, there is the induced map $Bj : BH \rightarrow BG$ between the classifying spaces. It induces a map $Bj^* : M(BG) = (\Lambda V_{BG}, 0) = (\mathbb{Q}[x_1, \dots, x_k], 0) \rightarrow (\Lambda V_{BH}, 0) = M(BH)$ between the models. Here $|x_i|$ are even and $\text{rank } G = k$. Let $V_G^n = V_{BG}^{n+1}$ by corresponding y_i to x_i with $|y_i| = |x_i| - 1$.

Lemma 3.11 ([8, Proposition 15.16]) *The (non-minimal) model of G/H is given as $(\Lambda V_{BH} \otimes \Lambda V_G, d)$ where $dx_i = 0$ and $dy_i = Bj^*(x_i)$ for $i = 1, \dots, k$.*

Proof of Theorem 1.4. (2) For $f : G \rightarrow G/H$, $M(f)$ is given by the projection $(\Lambda V_{BH} \otimes \Lambda V_G, d) \rightarrow (\Lambda V_G, 0)$ sending elements of ΛV_{BH} to zero from Lemma 3.11. Thus we can define $\psi_f(g)$ for any $g \in \mathcal{E}(\Lambda V_{BH} \otimes \Lambda V_G, d)$ by $\psi_f(g) = \bar{g}$ because $g(x_i) \in \mathbb{Q}[x_1, \dots, x_k]$. $\square$

Example 3.12 Let X be a G -space for a Lie group G . When is the orbit map $f : X \rightarrow X/G$ a rational co- $\mathcal{E}$ -map? Let $X = S^2 \times S^3$, where $M(S^2 \times S^3) = (\Lambda(x, y, z), d)$ with $dx = dz = 0$ and $dy = x^2$ of $|x| = 2, |y| = |z| = 3$. There are free S^1 -actions on X where $M(X/S^1) = M(ES^1 \times_{S^1} X) = (\Lambda(t, x, y, z), D)$ for $M(BS^1) = (\mathbb{Q}[t], 0)$ with $|t| = 2$ [1, 10]. If the Borel space of a S^1 -action has the model with $Dx = Dt = 0$, $Dy = x^2$ and $Dz = t^2$ (it is given by a free action on S^3), f is not a rational co- $\mathcal{E}$ -map. Indeed, we can not define $\psi_f(g)$ for the DGA-map g with $g(x) = t, g(t) = x, g(y) = z$ and $g(z) = y$. But if a S^1 -action has the model with $Dy = x^2 + at^2$ and $Dz = xt$ for $a \notin \mathbb{Q}^*/(\mathbb{Q}^*)^2$, the orbit map f is a rational co- $\mathcal{E}$ -map.

Remark 3.13 Even if a map f is an $\mathcal{E}$ -map, it may not be a rational $\mathcal{E}$ -map. Recall an example of Arkowitz and Lupton [3]: Let Y be a rational (non-universal) space such that $M(Y) = (\Lambda(x_1, x_2, y_1, y_2, y_3, z), d)$ with $|x_1| = 10, |x_2| = 12, |y_1| = 41, |y_2| = 43, |y_3| = 45, |z| = 119$,

$$dx_1 = dx_2 = 0, \quad dy_1 = x_1^3 x_2, \quad dy_2 = x_1^2 x_2^2, \quad dy_3 = x_1 x_2^3 \quad \text{and}$$

$$dz = x_2(y_1 x_2 - x_1 y_2)(y_2 x_2 - x_1 y_3) + x_1^{12} + x_2^{10}.$$

Then $\mathcal{E}(Y) = \{g_1, g_2\} (\cong \{1, -1\})$ where $g_1 = id_Y$ and g_2 is given by

$$g_2(x_1) = x_1, \quad g_2(x_2) = -x_2, \quad g_2(y_1) = -y_1,$$

$$g_2(y_2) = y_2, \quad g_2(y_3) = -y_3, \quad g_2(z) = z$$

[3, Example 5.2]. Consider the 12-dimensional homotopy generator $f : S^{12} \rightarrow Y$ corresponding to x_2 . It is an $\mathcal{E}$ -map by the homotopy commutative diagram:

$$\begin{array}{ccc}
 S^{12} & \xrightarrow{\pm 1} & S^{12} \\
 f \downarrow & & \downarrow f \\
 Y & \xrightarrow{\phi_f(\pm 1)} & Y
 \end{array}$$

where $\phi_f : \mathcal{E}(S^{12}) = \{\pm 1\} \cong \mathcal{E}(Y)$ by $\phi_f(1) = g_1$ and $\phi_f(-1) = g_2$. But it is not a rational $\mathcal{E}$ -map. Because there is no map $\phi_f : \mathcal{E}(M(S^{12})) = \mathcal{E}(\Lambda(u, v), d) \rightarrow \mathcal{E}(M(Y))$ when $a \neq \pm 1 \in \mathbb{Q}^*$, i.e., there is no homotopy commutative diagram:

$$\begin{array}{ccc}
 (\Lambda(u, v), d) & \xrightarrow{\times a} & (\Lambda(u, v), d) \\
 M(f) \uparrow & & \uparrow M(f) \\
 M(Y) & \xrightarrow{\phi_f(\times a)} & M(Y)
 \end{array}$$

where $M(f)(x_2) = u$, $M(f)(z) = u^8 v$ and $M(f)$ sends the others to zero. Here $M(S^{12}) = (\Lambda(u, v), d)$ with $|u| = 12$, $|v| = 23$, $du = 0$, $dv = u^2$ and $\times a(u) = au$, $\times a(v) = a^2 v$.

Proof of Theorem 1.7 1. It is obvious from the definition.

2. As (non-graded) DGAs, $M(\mathbb{C}P^n) \cong M(\mathbb{H}P^n) \cong (\Lambda(x, y), d)$ where $dx = 0$ and $dy = x^{n+1}$. Therefore the inclusions $S^2 \rightarrow \mathbb{C}P^n$ and $S^4 \rightarrow \mathbb{H}P^n$ induce $\mathcal{E}(S^2_{\mathbb{Q}}) \cong \mathcal{E}(\mathbb{C}P^n_{\mathbb{Q}}) (\cong \mathbb{Q}^* := \mathbb{Q} - 0)$ and $\mathcal{E}(S^4_{\mathbb{Q}}) \cong \mathcal{E}(\mathbb{H}P^n_{\mathbb{Q}}) (\cong \mathbb{Q}^*)$, respectively.
3. Suppose that m is even and n is odd. Then $M(S^m \times S^n) = (\Lambda(x, y, z), d)$ where $dx = dy = 0$ and $dz = x^2$ with $|x| = m$, $|y| = n$ and $|z| = 2m - 1$.

If $m - 1 = n$, we have $|z| = |xy|$. Then any element of $\mathcal{E}(\Lambda(x, y, z), d)$ is given as

$$x \rightarrow ax, \quad y \rightarrow by, \quad z \rightarrow a^2 z + cxy$$

for some $a, b \in \mathbb{Q}^*$ and $c \in \mathbb{Q}$. The same is not true for $\mathcal{E}((S^m \vee S^n)_{\mathbb{Q}}) = \mathcal{E}((\Lambda(x, y, z, \dots), d)$ since $[xy] = 0$ in $H^*(S^m \vee S^n; \mathbb{Q}) = \mathbb{Q}[x] \otimes \Lambda(y)/(x^2, xy)$, i.e., $c = 0$.

If $2m - 1 = n$, we have $|z| = |y|$. Then any element of $\mathcal{E}(\Lambda(x, y, z), d)$ is given as

$$x \rightarrow ax, \quad y \rightarrow by, \quad z \rightarrow a^2 z + cy$$

for some $a, b \in \mathbb{Q}^*$ and $c \in \mathbb{Q}$. The same is true for $\mathcal{E}((S^m \vee S^n)_{\mathbb{Q}}) = \mathcal{E}((\Lambda(x, y, z, \dots), d)$, which is also checked by using its Quillen model [6, 8].

In the other case, $c = 0$ for $S^m \vee S^n$ and $S^m \times S^n$.

Thus we have the following table:

$\mathcal{E}((S^m \vee S^n)_{\mathbb{Q}})$	$\mathcal{E}((S^m \times S^n)_{\mathbb{Q}})$	$m:\text{even}, n:\text{odd}$
$\mathbb{Q}^* \times \mathbb{Q}^*$	$\mathbb{Q}^* \times \mathbb{Q}^* \times \mathbb{Q}$	$m - 1 = n$
$\mathbb{Q}^* \times \mathbb{Q}^* \times \mathbb{Q}$	$\mathbb{Q}^* \times \mathbb{Q}^* \times \mathbb{Q}$	$2m - 1 = n$
$\mathbb{Q}^* \times \mathbb{Q}^*$	$\mathbb{Q}^* \times \mathbb{Q}^*$	other

Hence the inclusion $f : S^m \vee S^n \rightarrow S^m \times S^n$ induces $\phi_f : \mathcal{E}((S^m \vee S^n)_{\mathbb{Q}}) \cong \mathcal{E}((S^m \times S^n)_{\mathbb{Q}})$ if and only if $m \neq n + 1$.

- Since the fibrations are non-trivial, the models of the total spaces are uniquely determined as $M(E) \cong (\Lambda(x, y, v_1), d)$ with $|x| = m, |y| = n, dx = dy = 0$ and $dv_1 = xy$ and $M(E') \cong (\Lambda(x, y, v_1, v_2), d)$ with $dx = dy = 0, dv_1 = xy, dv_2 = xv_1$. Then $\psi_{p'_{\mathbb{Q}}} \circ \psi_{p_{\mathbb{Q}}} : \mathcal{E}(\Lambda(x, y), 0) \cong \mathcal{E}(\Lambda(x, y, v_1), d) \cong \mathcal{E}(\Lambda(x, y, v_1, v_2), d)$ from degree reasons.
- Recall the rigid model of Arkowitz and Lupton [3]: Let

$$M = (\Lambda(x_1, x_2, y_1, y_2, y_3, z), d)$$

with given by $|x_1| = 8, |x_2| = 10, |y_1| = 33, |y_2| = 35, |y_3| = 37$ and $|z| = 119$,

$$dx_1 = dx_2 = 0, dy_1 = x_1^3 x_2, dy_2 = x_1^2 x_2^2, dy_3 = x_1 x_2^3 \quad \text{and}$$

$$dz = \alpha := x_1^4 (y_1 x_2 - x_1 y_2) (y_2 x_2 - x_1 y_3) + x_1^{15} + x_2^{12}.$$

(Note that the degrees of elements are determined by the differential d). Then $\mathcal{E}(M) = \{id_M\}$ [3, Example 5.1]. Now define a Hirsch extension [9] by M as

$$(\Lambda(v, w), 0) \rightarrow (\Lambda(v, w) \otimes M, d') =: M'$$

where $|v|$ and $|w|$ are suitable odd integers with $|v| \neq |w|, |v| + |w| = 120$ and $d'v = d'w = 0, d'z = \alpha + vw$ ($d' = d$ for the other elements). For example, put $|v| = 53$ and $|w| = 67$. Then any DGA-automorphism h of M' is DGA-homotopic to a map with no unipotent part. Indeed, let

$$h(z) = z + f_1 v + f_2 w, \quad h(v) = av, \quad h(w) = bw + f_3 v$$

where $a, b \in \mathbb{Q}^*$ with $ab = 1$ and $f_1, f_2, f_3 \in (x_1, x_2)$, which is the ideal generated by x_1, x_2 . Then, since $|f_1| = 66, |f_2| = 52$ and $|f_3| = 14$, we have

$$f_1 = k_1 x_1^2 x_2^5 + l_1 x_1^7 x_2, \quad f_2 = k_2 x_1^4 x_2^2, \quad f_3 = 0$$

for $k_1, l_1, k_2 \in \mathbb{Q}$. Thus f_1 and f_2 are d' -exact cocycles. Therefore h is DGA-homotopic to the map with $f_1 = f_2 (= f_3) = 0$ [3, 9]. Hence any element $h \in \mathcal{E}(M')$ is determined by $h(v) = av$ and $h(w) = bw$ for $a, b \in \mathbb{Q}^*$ such that $ab = 1$ (since $h|_M \sim id_M$). Thus we obtain

$$\mathcal{E}(M') = \{(a, b) \in \mathbb{Q}^* \times \mathbb{Q}^* \mid ab = 1\} \cong \mathbb{Q}^*.$$

Therefore the DGA-surjections $(\Lambda v, 0) \xleftarrow{f^*} M' \xrightarrow{g^*} (\Lambda w, 0)$ (spherically injective maps $f : S_{\mathbb{Q}}^{|v|} \rightarrow ||M'||$ and $g : S_{\mathbb{Q}}^{|w|} \rightarrow ||M'||$, which are their geometric realizations) with $f^*(M) = g^*(M) = 0$, $f^*(v) = v$, $g^*(w) = w$ and $f^*(w) = g^*(v) = 0$ induce

$$(\mathbb{Q}^* \cong) \mathcal{E}(S_{\mathbb{Q}}^{|v|}) = \mathcal{E}(\Lambda v, 0) \underset{\phi_f}{\cong} \mathcal{E}(M') \underset{\psi_g}{\cong} \mathcal{E}(\Lambda w, 0) = \mathcal{E}(S_{\mathbb{Q}}^{|w|})$$

with $\psi_g \phi_f(a) = a^{-1}$ for $a \in \mathbb{Q}^*$. Thus we have $S_{\mathbb{Q}}^{|v|} \underset{\mathcal{E}}{\sim} S_{\mathbb{Q}}^{|w|}$.

□

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