J. Homotopy Relat. Struct. (2017) 12:433–446
DOI 10.1007/s40062-016-0134-2

Strong shape in categories enriched over groupoids

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Received: 5 February 2015 / Accepted: 3 May 2016 / Published online: 25 May 2016

Abstract For any pair of categories (C, K) enriched over the category Gpd of groupoids, it is possible to define a strong shape category $SSh(\mathsf{C}, \mathsf{K})$ in such a way that, for C the category of topological spaces and K its full subcategory of spaces having the homotopy type of absolute neighborhoods retracts for metric spaces, one obtains the strong shape category $SSh(\mathsf{Top})$, as defined by Mardešić. We also introduce a new category SS_{K} with the same objects as C and morphisms given by suitable pseudo-natural transformations into the category of groupoids. The main result is then that such a category SS_{K} is isomorphic to the strong shape category $SSh(\mathsf{C}, \mathsf{K})$, when C is also a proper model category.

Keywords Inverse system · Groupoid enriched category · Pseudo-natural transformation · Strong shape equivalence

Mathematics Subject Classification 55U35 · 55P55 · 18D20 · 18E35

Introduction

Strong shape theory is a modification of shape theory which is closer to homotopy theory, hence of a more geometric flavour. As a consequence, while shape theory very early had a satisfactory categorical interpretation (see [5,7,16]), the attempt to give

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Communicated by Walter Tholen.

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strong shape theory an abstract setting has been more complicated. Although some ideas of a strong shape theory, mostly related to compacta, were already in Christie [8], Porter [17], Quigley [19], it wasn't until 1976 that strong shape was rediscovered with the work of Edwards-Hastings [9], who generalized it to arbitrary topological spaces. In order to do this they first organized the category $Pro(\mathsf{Top})$ of inverse systems of spaces in a closed model category. The development of strong shape theory is in fact almost parallel to various attempts to define a homotopy theory for pro-categories, so that the two arguments are strongly related to each other. As for a categorical interpretation of strong shape theory one has to mention the work of Batanin [6] who adopted a 2-categorical point of view in his paper showing the connection of strong shape theory with a homotopy theory of simplicial distributors, linked to A_∞-categories. This paper has some connection with the present work although the point of view is essentially different.

In this paper we give another construction of the strong shape category working with categories enriched over the category of groupoids, also called ge-categories.

A map $\underline{X} \to \underline{Y}$ between inverse systems of topological spaces is called a level equivalence if, after a reindexing [16] to a common set of indexes for $\underline{X}$ and $\underline{Y}$, it can be represented by a natural transformation which is a homotopy equivalence at each level. A level equivalence in $Pro(\mathsf{Top})$ cannot be inverted in general, see, e.g., ([9], 2.5). Formally inverting the class of such level equivalences one obtains the homotopy category $Ho(Pro(\mathsf{Top}))$ [18], which is essential in order to define the strong shape category of topological spaces $SSh(\mathsf{Top})$ [15]. More recently Isaksen [13] has defined a (strict) model category structure on the pro-category $Pro(\mathsf{C})$, for C a proper model category, generalizing the construction of Edwards-Hastings.

If C is a ge-category, then inverse systems in C of type Λ (the index set) are 2-functors $\Lambda^{op} \to \mathsf{C}$ and such functors generate two ge-categories $[\Lambda^{op}, \mathsf{C}] \subset [\![\Lambda^{op}, \mathsf{C}]\!]$. The former has natural transformations and their modifications as morphisms and 2-cells, respectively, the latter is obtained by considering pseudo-natural transformations and their modifications. The key fact we use in the paper is that every level equivalence in $[\Lambda^{op}, \mathsf{C}]$ can be inverted in $[\![\Lambda^{op}, \mathsf{C}]\!]$ by a pseudo-natural transformation. Moreover, the inclusion $[\Lambda^{op}, \mathsf{C}] \subset [\![\Lambda^{op}, \mathsf{C}]\!]$ has a left 2-adjoint $[\![\Lambda^{op}, \mathsf{C}]\!] \to [\Lambda^{op}, \mathsf{C}]$, $F \mapsto F'$, and every pseudo-natural transformation of the form $F' \Rightarrow G$ is equivalent to an actual natural transformation. Such results allow us to define the strong shape category $SSh(\mathsf{C}, \mathsf{K})$ for every pair of ge-categories (C, K). If, moreover, C is a proper model category, then we introduce a new category SS_{K} with the same objects as C and morphisms given by suitable pseudo-natural transformations into the category of groupoids. The main result of the paper is the fact that $SSh(\mathsf{C}, \mathsf{K})$ and SS_{K} are isomorphic categories. In the case $\mathsf{C} = \mathsf{Top}$, the category of topological spaces, and $\mathsf{K} = \mathsf{ANR}$, the full subcategory of spaces having the homotopy type of absolute neighborhood retracts for metric spaces, $SSh(\mathsf{Top}, \mathsf{ANR}) = SSh(\mathsf{Top})$, the strong shape category of spaces as defined in [15], is isomorphic to SS_{ANR}.

Fundamental sources for shape and strong shape theory are the books [7, 15, 16].

1 Background

1.1 ge-categories

A groupoid is a small category whose morphisms are all invertible. Gpd will denote the category of groupoids and their functors. Gpd is a complete and cocomplete category, in particular it is a symmetric, monoidal closed category, with tensor product the usual product of categories and unit object the groupoid e with only one object and one morphism. A category C is enriched over Gpd (hereafter called a ge-category) if every hom-set $Hom(X, Y)$ is the set of objects of a groupoid $\mathsf{Gpd}(X, Y)$ and the composition is a functor

$$\mathsf{Gpd}(X, Y) \times \mathsf{Gpd}(Y, Z) \to \mathsf{Gpd}(X, Z)$$

which respects identities, for all $X, Y, Z \in \mathsf{C}$. In other words a ge-category is a 2-category whose 2-cells are all invertible.

If C is a ge-category we call its 1-morphisms *maps* and its 2-cells *homotopies*, so

$$\alpha : f \simeq g : X \to Y$$

means that α is a homotopy connecting the maps $f, g : X \to Y$. Homotopies in C can be composed both vertically $\beta \cdot \alpha$ and horizontally $\gamma * \alpha$. We denote, for example, by f, both the map and the identity homotopy $1_f : f \simeq f$.

A map $f : X \to Y$ of C is called a *homotopy equivalence* if there are a map $g : Y \to X$ and homotopies $g \circ f \simeq 1_X$, $f \circ g \simeq 1_Y$.

Every ge-category C has a homotopy category denoted HoC: its quotient category with respect to the homotopy relation for maps. Alternatively HoC can be obtained as a localization Ho$\mathsf{C} = \mathsf{C}[\mathbb{W}^{-1}]$, where $\mathbb{W}$ is the class of homotopy equivalences [20].

1.2. Examples of ge-categories are:

- The category Top of topological spaces. The homotopies are the tracks [6] between continuous maps.
- Gpd itself is a ge-category: the homotopies are the natural isomorphisms of functors. A functor of groupoids is a homotopy equivalence iff it is an equivalence of categories.
- Every ordinary category can be thought of as a ge-category having only identity homotopies.

1.3. Let C be a given ge-category and let J be a small ordinary category, then every functor $F : \mathsf{J} \to \mathsf{C}$ is a 2-functor and every natural transformation $\tau : F \Rightarrow G : \mathsf{J} \to \mathsf{C}$ is a 2-natural transformation. There are two ge-categories with objects the functors from J to C:

- $[\mathsf{J}, \mathsf{C}]$, whose maps are the natural transformations and whose homotopies are their modifications,
- $[\![\mathsf{J}, \mathsf{C}]\!]$, whose maps are the pseudo-natural transformations and whose homotopies are their (coherent) modifications.

Recall that, for functors $F, G : \mathsf{J} \to \mathsf{C}$, a pseudo-natural transformation (called *psd-transformation*, for short) $\tau : F \Rightarrow G$ consists of

- maps $\tau_x : F(x) \to G(x)$ in C, for all $x \in \mathsf{J}$, together with
- homotopies $\tau_u : G(u) \circ \tau_x \simeq \alpha_y \circ F(u)$ in C, for $u : x \to y$ in J, in such a way that $\tau_{1_x} = 1_{\tau_x}$ and $\tau_{v \circ u} = [\tau_v * F(u)] \cdot [G(g) * \tau_u]$, for composable maps $x \xrightarrow{u} y \xrightarrow{v} z$. Moreover, for a homotopy $\alpha : u \simeq u' : x \to y$, one has $\tau_u \circ [\mathsf{G}(\alpha) * \tau_x] = [\tau_y * F(\alpha)] \circ \tau_{u'}$.

Given psd-transformations $\alpha, \beta : F \Rightarrow G$ a homotopy (modification) $\theta : \alpha \simeq \beta$ consists of homotopies $\theta_x : \alpha_x \simeq \beta_x$, for $x \in \mathsf{J}$, such that, given $u : x \to y$, then

$$\beta_u \circ [G(f) * \theta_x] = [\theta_y * F(u)] \circ \alpha_u.$$

1.4. A natural transformation or psd-transformation $\tau : F \Rightarrow G$ is called a *level equivalence* when, for each $x \in \mathsf{J}$, the map $\alpha_x : F(x) \to G(x)$ is a homotopy equivalence in C.

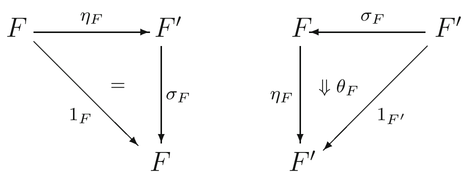
Every natural transformation of functors $\mathsf{J} \to \mathsf{C}$ is a pseudo-natural transformation and the inclusion 2-functor

$$E : [\mathsf{J}, \mathsf{C}] \to [\![\mathsf{J}, \mathsf{C}]\!],$$

is known to have a left 2-adjoint ([3,11]) denoted

$$(-)' : [\![\mathsf{J}, \mathsf{C}]\!] \to [\mathsf{J}, \mathsf{C}], \;\; F \mapsto F'.$$

The unit of the 2-adjunction η is levelwise given by pseudo-natural transformations $\eta_F : F \Rightarrow F'$. The components of the counit are (2)-natural transformations $\sigma_F : F' \to F$. It follows from the general theory of 2-monads ([2], §4) that the pseudo-natural transformations η_F and the (2)-natural transformations σ_F form an adjoint equivalence. In particular, there are diagrams



The homotopies $\theta_F : \sigma_F \circ \eta_F \Rightarrow 1_F$ are the (invertible) modifications providing the counit of the adjoint equivalence.

From ([3], Theorem 4.7) we record the following result

1.5. Any pseudo-natural transformation $F' \Rightarrow G$, where $F, G : \mathsf{J} \to \mathsf{C}$, is homotopic in $[\![\mathsf{J}, \mathsf{C}]\!]$ to a (2-) natural transformation.

1.6. From now on we will denote by

$$\mathsf{Gpd}(F, G) \quad \text{and} \quad \mathsf{GPD}(F, G)$$

the hom groupoids in $[\mathsf{J}, \mathsf{C}]$ and in $[\![\mathsf{J}, \mathsf{C}]\!]$, respectively .

Let $F : \mathsf{J} \to \mathsf{Gpd}$ be a given functor. Recall from [10] that there is an isomorphism of groupoids

$$\mathsf{Gpd}(X, \mathsf{GPD}(K_e, F)) \cong \mathsf{GPD}(K_X, F),$$

which exhibits the groupoid $\mathsf{GPD}(K_e, F)$ as the pseudo-limit (also the 2-limit, in this case) of the functor F. Here $K_X : \mathsf{J} \to \mathsf{Gpd}$ is the constant functor of value X. A dual argument holds for pseudo-colimits (2-colimits) in Gpd.

2 Maps and coherent maps of inverse systems

From now on let $\mathsf{J} = \Lambda^{op}$, being $(\Lambda, \leq)$ a cofinite, strongly directed set which we consider as a small category. We will be concerned with the ge-categories

$$[\Lambda^{op}, \mathsf{C}] \quad \text{and} \quad [\![\Lambda^{op}, \mathsf{C}]\!],$$

whose objects are the *inverse systems* in C of type Λ. Given such an inverse system $\underline{X} : \Lambda^{op} \to \mathsf{C}$, it is often useful to write explicitly

$$\underline{X} = (X_\lambda, x_{\lambda\lambda'}, \Lambda),$$

where $\underline{X}(\lambda) = X_\lambda$ and $\underline{X}(\lambda \leq \lambda') = x_{\lambda\lambda'} : X_{\lambda'} \to X_\lambda$.

If $f : M \to \Lambda$ is an increasing map of directed sets there is an inverse system of type M, $\underline{X}_f = \underline{X} \circ f^{op} : M^{op} \to \mathsf{C}$, given by $\underline{X}_f = (X_{f(\mu)}, x_{f(\mu)f(\mu')}, M)$. Here $M = (M, \leq)$ and f is considered as a functor.

2.1. The category $Pro(\mathsf{C})$.

Let $\underline{X} = (X_\lambda, x_{\lambda\lambda'}, \Lambda)$, and $\underline{Y} = (Y_\mu, y_{\mu,\mu'}, M)$ be inverse systems of type Λ and M, respectively. A *map of systems* $\underline{f} = (f, f_\mu) : \underline{X} \to \underline{Y}$ consists of

- an increasing map $f : M \to \Lambda$,
- a natural transformation $(f_\mu) : \underline{X}_f \to \underline{Y}$.

Let $\underline{f} = (f, f_\mu) : \underline{X} \to \underline{Y}$ and let $F : M \to \Lambda$ be an increasing map such that $f \leq F$. The *shift* of $\underline{f}$ by F is the map $(F; \overline{f}_\mu) : \underline{X}_F \to \underline{Y}$, where, $\overline{f}_\mu = f_\mu \circ x_{f(\mu)F(\mu)}$.

Two maps of systems $(f, f_\mu), (g, g_\mu) : \underline{X} \to \underline{Y}$ are *congruent* if they admit a common shift, that is there is an increasing map $F : M \to \Lambda$ such that $f, g \leq F$ and, for each $\mu \in M$, $f_\mu \circ x_{f(\mu),F(\mu)} = g_\mu \circ x_{g(\mu),F(\mu)}$.

Congruences of maps of systems are trivial modifications, so we can form the ge-category $Inv(\mathsf{C})$ whose objects, maps and homotopies are inverse systems, maps of systems and their congruences, respectively. Its homotopy category $Pro(\mathsf{C})$ is the category of inverse systems in C as defined by Grothendieck [12].

2.2. The category $\mathbb{P}\text{ro}(\mathsf{C})$.

The ge-category $Inv(\mathsf{C})$ is obtained by putting together the various ge-categories $[\Lambda^{op}, \mathsf{C}]$, for $(\Lambda, \leq)$ a cofinite, strongly directed set. If we consider instead the ge-categories $[\![\Lambda^{op}, \mathsf{C}]\!]$, we are led to the following definitions:

2.2.1. Let $\underline{X}$ and $\underline{Y}$ be as above. A *coherent map* of systems $\varphi = (f; f_\mu, f_{\mu\mu'}) : \underline{X} \to \underline{Y}$ consists of :

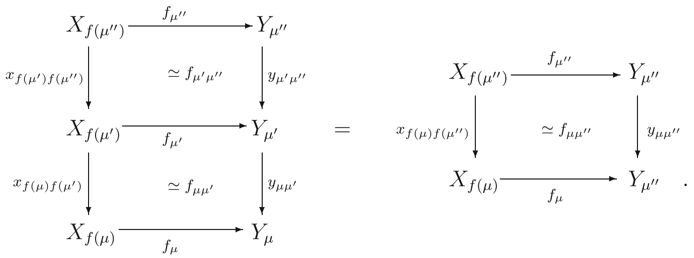
- an increasing map $f : M \to \Lambda$,
- a psd-transformation $(f_\mu, f_{\mu\mu'}) : \underline{X}_f \to \underline{Y}$.

It is worth to explicitate that the psd-transformation $(f_\mu, f_{\mu\mu'}) : \underline{X}_f \to \underline{Y} : M^{op} \to \mathsf{C}$ consists of

- maps $f_\mu : X_{f(\mu)} \to Y_\mu$, for all $\mu \in M$,
- homotopies $f_{\mu\mu'} : y_{\mu\mu'} \circ f_{\mu'} \simeq f_\mu \circ x_{f(\mu)f(\mu')}$ in C, for all $\mu \leq \mu'$ in M, in such a way that $\tau_{1_j} = 1_{\tau_j}$ and, for $\mu \leq \mu' \leq \mu''$ in M,

$$f_{\mu\mu''} = [f_{\mu\mu'} * x_{f(\mu')f(\mu'')}] \cdot [y_{\mu\mu'} * f_{\mu'\mu''}]$$

as in



2.2.2. Let $\varphi = (f; f_\mu, f_{\mu\mu'}) : \underline{X} \to \underline{Y}$ be a coherent map of systems and let $F : M \to \Lambda$ be an increasing map such that $f \leq F$. The *coherent shift* of φ by F is the coherent map $\varphi_F = (F; \overline{f}_\mu, \overline{f}_{\mu\mu'}) : \underline{X}_F \to \underline{Y}$, where, $\overline{f}_\mu = f_\mu \circ x_{f(\mu)F(\mu)}$ and $\overline{f}_{\mu\mu'} = f_{\mu\mu'} * x_{f(\mu')F(\mu')}$.

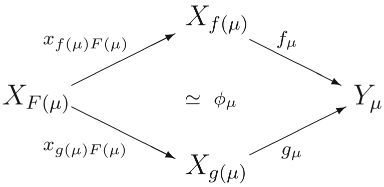
If $\varphi' = (f'; f'_\mu, f'_{\mu\mu'})$ is another coherent map $\underline{X} \to \underline{Y}$, a *homotopy* (coherent modification) $(F, \Phi) : \varphi \simeq \varphi'$ consists of :

- an increasing map $F : M \to \Lambda$ such that $f, f' \leq F$,
- a modification of psd-transformations

$$\Phi : (\overline{f}_\mu, \overline{f}_{\mu\mu'}) \simeq (\overline{f}'_\mu, \overline{f}'_{\mu\mu'}) : \underline{X}_F \to \underline{Y},$$

between their coherent shifts by F. It follows that Φ is family of homotopies of C,

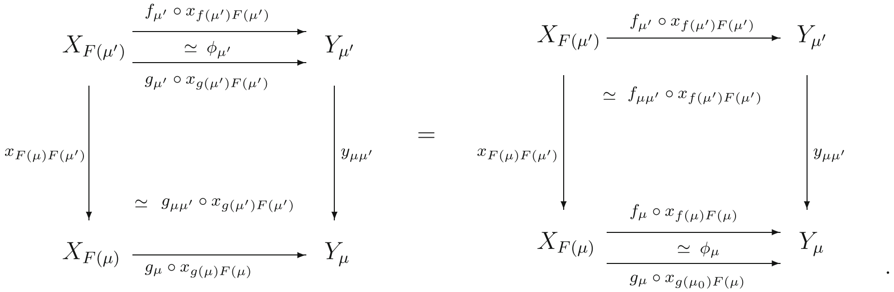
$$\phi_\mu : f_\mu \circ x_{f(\mu)F(\mu)} \simeq g_\mu \circ x_{g(\mu)F(\mu)}, \ \mu \in M,$$



such that

$$(g_{\mu\mu'} * x_{F(\mu')g(\mu')}) \cdot (y_{\mu\mu'} * \phi_{\mu'}) = (\phi_\mu * x_{F(\mu)F(\mu')}) \cdot (f_{\mu\mu'} * x_{f(\mu')F(\mu')}),$$

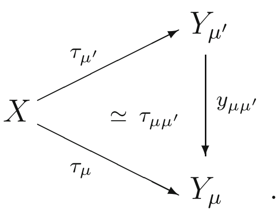
as in



The data above define the ge-category $\mathbb{I}\text{nv}(\mathsf{C})$ with objects the inverse systems in C, coherent maps and their homotopies. The homotopy category of $\mathbb{I}\text{nv}(\mathsf{C})$, denoted $\mathbb{P}\text{ro}(\mathsf{C})$, is studied in a forthcoming paper [21]. It is clear that there are inclusion functors $Inv(\mathsf{C}) \to \mathbb{I}\text{nv}(\mathsf{C})$ and $Pro(\mathsf{C}) \to \mathbb{P}\text{ro}(\mathsf{C})$.

For $\underline{X} \in \mathbb{I}\text{nv}(\mathsf{C})$ and $Y \in \mathsf{C}$, one has $\mathbb{I}\text{nv}(\mathsf{C})(\underline{X}, Y) = Inv(\mathsf{C})(\underline{X}, Y)$, that is, a coherent map $\varphi : \underline{X} \to Y$ just amounts to choose a morphism $f_\lambda : X_\lambda \to Y$. Given another coherent map $\psi : \underline{X} \to Y$ corresponding to $g_{\lambda'} : X_{\lambda'} \to Y$, a coherent modification between them is obtained taking an index $\lambda_0 \geq \lambda, \lambda'$ and a homotopy $f_\lambda \circ x_{\lambda\lambda_0} \simeq g_{\lambda'} \circ x_{\lambda'\lambda_0}$.

An element of $\mathbb{I}\text{nv}(\mathsf{C})(X, \underline{Y})$ is a coherent cone $\tau = (\tau_\mu, \tau_{\mu\mu'}) : X \to \underline{Y}$



2.3. Let now K be a full ge-subcategory of C. If $\underline{X} = (X_\mu, x_{\mu\mu'}, M)$ is an inverse system in C, for every inverse system $\underline{A} = (A_\lambda, a_{\lambda\lambda'}, \Lambda)$ in K, applying the functor $\mathsf{GPD}(\underline{X}, -) : \mathsf{K} \to \mathsf{Gpd}$ we obtain an inverse system in Gpd

$$(\mathsf{GPD}(\underline{X}, A_\lambda), \mathsf{GPD}(\underline{X}, a_{\lambda\lambda'}), \Lambda).$$

From 1.5 it follows that the pseudo-limit (2-limit) of such an inverse system is the groupoid $\mathsf{GPD}(\underline{X}, \underline{A})$, then

$$\mathsf{GPD}(\underline{X}, \underline{A}) = \text{psd-}\varprojlim_{\lambda} \mathsf{GPD}(\underline{X}, A_\lambda).$$

By a dual argument, if $\underline{X} = (X_\mu, x_{\mu\mu'}, M)$, it follows

$$\mathsf{GPD}(\underline{X}, \underline{A}) = \text{psd-}\varprojlim_{\lambda} \text{psd-}\varinjlim_{\mu} \mathsf{Gpd}(X_\mu, A_\lambda).$$

3 Strong shape

3.1. The homotopy category of $Pro(\mathsf{C})$.

In this section let us consider a proper model ge-category C. Following [9,18] a related (strict) model structure has been defined in $Pro(\mathsf{C})$ by Isaksen [13], so that the resulting homotopy category $Ho(Pro(\mathsf{C}))$ is obtained by localizing $Pro(\mathsf{C})$ at the class of level homotopy equivalences. The category Top of topological spaces, with its Strom structure (homotopy equivalences, Hurewicz cofibrations), is a proper model category, hence the construction in [13] gives the same homotopy category $Ho(Pro(\mathsf{Top}))$ as defined in [9,18].

Since every morphism $\phi : \underline{X} \to \underline{Y}$ in $Ho(Pro(\mathsf{C}))$ is an alternating composition of congruence classes of left-pointing level homotopy equivalences and right-pointing (classes of) maps of systems, it follows that ϕ always induces a pseudo-natural transformation $\phi^* : \mathsf{GPD}(\underline{Y}, -) \Rightarrow \mathsf{GPD}(\underline{X}, -)$. In particular, for a map of systems $\underline{f} : \underline{Z} \to \underline{Y}$ and a level equivalence $\sigma : \underline{Z} \to \underline{X}$, the composition $[\sigma]^{-1} \circ [\underline{f}]$ is a morphism in $Ho(Pro(\mathsf{C}))$ which induces the psd-transformation

$$\mathsf{GPD}(\underline{Y}, -) \overset{\underline{f}^*}{\Longrightarrow} \mathsf{GPD}(\underline{Z}, -) \overset{\sigma^{*-1}}{\Longrightarrow} \mathsf{GPD}(\underline{X}, -)\ .$$

3.2. The strong shape category of a pair (C, K).

Let K be a full ge-subcategory of C. A morphism $f : X \to Y$ in C is called a *strong shape equivalence* for the pair (C, K) if it induces a natural equivalence of functors

$$f^* : \mathsf{Gpd}(Y, -) \Rightarrow \mathsf{Gpd}(X, -) : \mathsf{K} \to \mathsf{Gpd},$$

that is $f^*_A : \mathsf{Gpd}(Y, A) \to \mathsf{Gpd}(X, A)$ is an equivalence of groupoids for all $A \in \mathsf{K}$.

Theorem 3.1 *Let $f : X \to Y$ be a continuous map. The following are equivalent:*

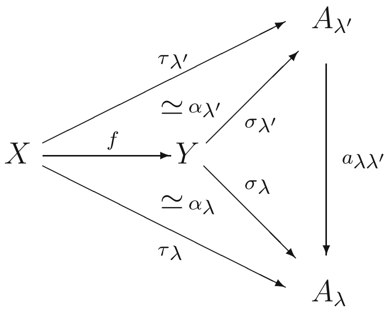
(a) *f is a strong shape equivalence,*

(b) $f_{\underline{A}}^* : \mathsf{GPD}(Y, \underline{A}) \to \mathsf{GPD}(X, \underline{A})$ *is an equivalence of groupoids for all* $\underline{A} \in Pro(\mathsf{K})$.

Proof One direction is trivial. Let $\underline{A} = (A_\lambda, a_{\lambda\lambda'}, \Lambda)$ be an inverse system in K. Since f is a strong shape equivalence, for all $\lambda \in \Lambda$ there are equivalences of groupoids

$$f_{A_\lambda}^* : \mathsf{Gpd}(Y, A_\lambda) \to \mathsf{Gpd}(X, A_\lambda).$$

Let $\tau = (\tau_\lambda, \tau_{\lambda\lambda'}) : X \to \underline{A}$ be a coherent cone. For each $\lambda \in \Lambda$ we get a map $\sigma_\lambda : Y \to A_\lambda$ and a homotopy $\alpha_\lambda : \sigma_\lambda f \simeq \tau_\lambda$. For $\lambda \leq \lambda'$ in Λ there is a diagram



then

$$\alpha_\lambda^{-1} * \tau_{\lambda\lambda'} * \alpha_{\lambda'} : a_{\lambda\lambda'} \circ \sigma_{\lambda'} \circ f \simeq a_{\lambda\lambda'} \circ \tau_{\lambda'} \simeq \tau_\lambda \simeq \sigma_\lambda \circ f,$$

where α_λ^{-1} is the inverse homotopy of α_λ. Since $f_{A_\lambda}^*$ is an equivalence, there is a unique homotopy $\sigma_{\lambda\lambda'} : a_{\lambda\lambda'} \circ \sigma_{\lambda'} \simeq \sigma_\lambda$ such that $\sigma_{\lambda\lambda'} * f = \alpha_\lambda^{-1} * \tau_{\lambda\lambda'} * \alpha_{\lambda'}$. It follows that $\sigma = (\sigma_\lambda, \sigma_{\lambda\lambda'}) : Y \to \underline{A}$ is a coherent cone such that $\sigma \circ f \simeq \tau$, in fact for $\lambda \leq \lambda'$ in Λ, from the previous diagram it is clear that

$$\alpha_\lambda \cdot (\sigma_{\lambda\lambda'} * f) \cdot \alpha_\lambda^{-1} = \tau_{\lambda\lambda'},$$

so that $\sigma \circ f$ and τ are coherently homotopic coherent cones, in other words the functor $f_{\underline{A}}^*$ is essentially surjective.

It remains to show that, given coherent cones $\sigma, \delta : Y \to \underline{A}$, every homotopy $\theta : \sigma \circ f \simeq \delta \circ f$ comes from a unique homotopy $\sigma \simeq \delta$. If $\theta = \{\theta_\lambda : \sigma_\lambda \circ f \simeq \delta_\lambda \circ f \mid \lambda \in \Lambda\}$, then the assertion follows by the very definition of strong shape equivalence. □

The notion of strong shape equivalence can be extended to maps of systems in C by saying that $\underline{f} = (f, f_\mu) : \underline{X} \to \underline{Y}$ is a strong shape equivalence in $Pro(\mathsf{C})$ if

$$\underline{f}^* : \mathsf{Gpd}(\underline{Y}, -) \Rightarrow \mathsf{Gpd}(\underline{X}, -) : \mathsf{K} \to \mathsf{Gpd}$$

is a natural equivalence.

Theorem 3.2 *Let $\underline{f} = (f, f_\mu) : \underline{X} \to \underline{Y}$ be a map of systems. The following are equivalent:*

(*a*) *$\underline{f}$ is a strong shape equivalence in $Pro(\mathsf{C})$,*

(*b*) *$\underline{f}^*_{\underline{A}} : \mathsf{GPD}(\underline{Y}, \underline{A}) \to \mathsf{GPD}(\underline{X}, \underline{A})$ is an equivalence of groupoids for all inverse systems $\underline{A} \in Pro(\mathsf{K})$.*

Proof Note that, assuming

$$\underline{f}^*_{A_\lambda} : \mathsf{GPD}(\underline{Y}, A_\lambda) \to \mathsf{GPD}(\underline{X}, A_\lambda)$$

to be an equivalence for all $\lambda \in \Lambda$, then the assertion follows from 2.4 since the pseudo-limit of a diagram of equivalences is an equivalence. □

Definition 3.3 A map of system $\underline{p} : X \to \underline{X}$ is called a *strong expansion* for X if it is a strong shape equivalence in $Pro(\mathsf{C})$. If $\underline{X} \in Pro(\mathsf{K})$, then $\underline{p}$ is a *strong* $\mathsf{K} - expansion$.

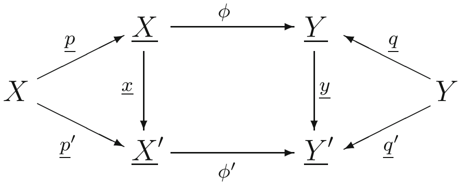
Given $\underline{p} : X \to \underline{X}$ and $\underline{p}' : X \to \underline{X}'$ different strong K-expansions for $X \in \mathsf{C}$, from the previous theorem it follows that there is a unique homotopy equivalence $\underline{x} : \underline{X} \to \underline{X}'$ in $Pro(\mathsf{K})$ such that $\underline{p}' \simeq \underline{x} \circ \underline{p}$.

In the following we assume for the pair (C, K) the property that each $X \in \mathsf{C}$ admits a strong K-expansion. This is the case of the the pair (Top, ANR), where $\mathsf{ANR} \subset \mathsf{Top}$ is the full subcategory of spaces having the homotopy type of absolute neighborhood retracts for metric spaces [15].

Definition 3.4 Given $X, Y \in \mathsf{C}$, a *strong shape morphism* $X \to Y$ is determined by a triple $(\underline{p}, \underline{q}, \phi)$ as in the diagram

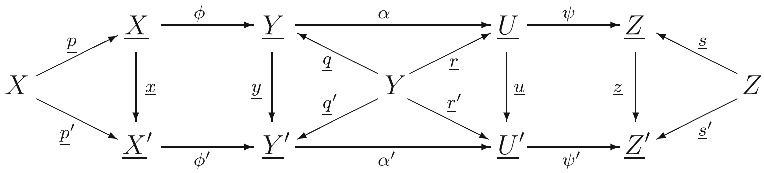
$$X \xrightarrow{\ \underline{p}\ } \underline{X} \xrightarrow{\ \phi\ } \underline{Y} \xleftarrow{\ \underline{q}\ } Y$$

where $\underline{p}$ and $\underline{q}$ are strong K-expansions and ϕ is a morphism in $Pro(\mathsf{C})$. Two such triples $(\underline{p}, \underline{q}, \phi)$ and $(\underline{p}', \underline{q}', \phi')$ are declared to be equivalent when there are homotopy equivalences $\underline{x} : \underline{X} \to \underline{X}'$ and $\underline{y} : \underline{Y} \to \underline{Y}'$ which make the following diagram commutative up to homotopy



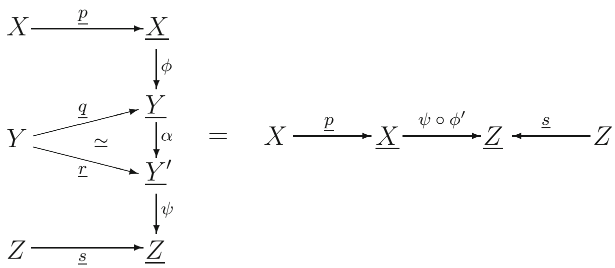
Let now $(\underline{p}, \underline{q}, \phi) : X \to Y$ and $(\underline{r}, \underline{s}, \psi) : Y \to Z$ two triples, with $\underline{r} : Y \to \underline{U}$ and $\underline{s} : Z \to \underline{Z}$. Since $\underline{q}$ and $\underline{r}$ are both strong K-expansions for Y, there is a unique homotopy equivalence $\alpha : \underline{Y} \to \underline{U}$ such that $\alpha \circ \underline{q} \simeq \underline{r}$. We define the composition of the two triples to be $(\underline{p}, \underline{s}, \psi \circ \alpha \circ \phi) : X \to Z$.

Such a composition is coherent with the equivalence relation previously defined. This is shown by the following diagram: in fact, for $(\underline{p}, \underline{q}, \phi) \sim (\underline{p}', \underline{q}', \phi') : X \to Y$ and $(\underline{r}, \underline{s}, \psi) \sim (\underline{r}', \underline{s}', \psi') : Y \to Z$, consider $(\underline{p}, \underline{s}, \psi \circ \alpha \circ \phi)$, $(\underline{p}', \underline{s}', \psi' \circ \alpha' \circ \phi') : X \to Z$



then $(\underline{p}, \underline{s}, \psi \circ \alpha \circ \phi) \sim (\underline{p}', \underline{s}', \psi' \circ \alpha' \circ \phi')$.

Let us note that a triple $(\underline{p}, \underline{q}, \phi)$ is equivalent to one of the form $(\underline{p}, \underline{r}, \phi')$ where $\phi' = \alpha \circ \phi$ is obtained by the mapping property of the strong K-expansion $\underline{q}$. Then, to compose $(\underline{p}, \underline{q}, \phi) : X \to Y$ and $(\underline{r}, \underline{s}, \psi) : Y \to Z$ one may as well compose the triples $(\underline{p}, \underline{r}, \phi') : X \to Y$ and $(\underline{r}, \underline{s}, \psi) : Y \to Z$ to obtain a diagram



Objects of C and their strong shape morphisms give the *strong shape category* $SSh(\mathsf{C}, \mathsf{K})$ of the pair (C, K).

Such a definition of $SSh(\mathsf{C}, \mathsf{K})$ is based on the construction of the strong shape category of topological spaces $SSh(\mathsf{Top}) = SSh(\mathsf{Top}, \mathsf{ANR})$ as given in ([15], 8.2, see also 4.38).

3.7. Let SS_{K} denote the category having the same objects as C and where a morphism $\tau : X \to Y$ is (a homotopy class of) a psd-transformation

$$\tau : \mathsf{Gpd}(Y, -) \Rightarrow \mathsf{Gpd}(X, -) : \mathsf{K} \to \mathsf{Gpd}.$$

Consider the correspondence

$$\gamma : SSh(\mathsf{C}, \mathsf{K}) \to SS_{\mathsf{K}}$$

which is the identity on objects and sends every strong shape morphism

$$[(\underline{p}, \underline{q}, \phi)] : X \to Y$$

to the psd-transformation obtained as follows : consider the composition

$$\mathsf{Gpd}(Y,-) \overset{(\underline{q}^*)^{-1}}{\Longrightarrow} \mathsf{GPD}(\underline{Y},-) \overset{\phi^*}{\Longrightarrow} \mathsf{GPD}(\underline{X},-) \overset{\underline{p}^*}{\Longrightarrow} \mathsf{Gpd}(X,-)$$

where $(\underline{q}^*)^{-1} : \mathsf{Gpd}(Y,-) \Rightarrow \mathsf{Gpd}(\underline{Y},-)$ is the pseudo-natural transformation inverse to $\underline{q}^*$: $\mathsf{GPD}(\underline{Y},-) \Rightarrow \mathsf{Gpd}(Y,-)$. $\underline{q}^*$ is a level equivalence, hence it is invertible in $[\![\mathsf{K},\mathsf{Gpd}]\!]$. Define $\gamma(\underline{p},\underline{q},\phi)) = \underline{p}^* \circ \phi^* \circ (\underline{q}^*)^{-1}$.

It is clear that for equivalent triples $(\underline{p},\underline{q},\phi)$ and $(\underline{p}',\underline{q}',\phi')$ one has $\gamma[(\underline{p},\underline{q},\phi)] = \gamma[(\underline{p}',\underline{q}',\phi')]$.

Theorem 3.5 *$\gamma : SSh(\mathsf{C},\mathsf{K}) \to SS_{\mathsf{K}}$ is an isomorphism of categories.*

Proof First of all γ is a functor. The identity strong shape morphism represented by the triple $(\underline{p},\underline{p},1_{\underline{X}}) : X \to X$ goes to the identity pseudo-natural transformation $\mathsf{Gpd}(X,-) \Rightarrow \mathsf{Gpd}(X,-)$. Let $(\underline{p},\underline{q},\phi) : X \to Y$ and $(\underline{r},\underline{s},\psi) : Y \to Z$ and consider their composition (3.6)

$$X \xrightarrow{\underline{p}} \underline{X} \xrightarrow{\phi'} \underline{Y}' \xrightarrow{\psi} \underline{Z} \xleftarrow{\underline{s}} Z \ .$$

One has

$$\gamma[(\underline{r},\underline{s},\psi)\circ(\underline{p},\underline{q},\phi)] = \gamma[(\underline{r},\underline{s},\psi)\circ(\underline{p},\underline{r},\phi')] = \gamma[(\underline{p},\underline{s},\psi\circ\phi')]$$
$$= \underline{p}^*\circ\psi^*\circ\phi'^*\circ(\underline{s}^*)^{-1}$$

and

$$\gamma[(\underline{r},\underline{s},\psi)]\circ\gamma[(\underline{p},\underline{q},\phi)] = \gamma[(\underline{r},\underline{s},\psi)]\circ\gamma[(\underline{p},\underline{r},\phi')]$$
$$= (\underline{p}^*\circ\psi^*\circ(\underline{q}^*)^{-1})\circ(\underline{q}^*\circ\phi'^*\circ(\underline{r}^*)^{-1}),$$

hence the equality.

Let now $\tau : \mathsf{Gpd}(Y,-) \Rightarrow \mathsf{Gpd}(X,-)$ be a psd-transformation and let $\underline{p} : X \to \underline{X}$ and $\underline{q} : Y \to \underline{Y}$ be strong K−expansions. Consider

$$\mathsf{GPD}(\underline{Y},-) \overset{\underline{q}^*}{\Longrightarrow} \mathsf{Gpd}(Y,-) \overset{\tau}{\Longrightarrow} \mathsf{Gpd}(X,-) \overset{(\underline{p}^*)^{-1}}{\Longrightarrow} \mathsf{GPD}(\underline{X},-) \overset{\sigma^*_{\underline{X}}}{\Longrightarrow} \mathsf{GPD}(\underline{X}',-)$$

and note that, being $\sigma^*_{\underline{X}}$ a level homotopy equivalence (1.3), then $\underline{X}'$ is an inverse system in K. By the pseudo Yoneda Lemma [10] there is a unique coherent map of systems $\underline{f} : \underline{X}' \to \underline{X}$ that induces the pseudo-natural transformation

$$\sigma^*_{\underline{X}} \circ (\underline{p}^*)^{-1} \circ \tau \circ \underline{q}^*,$$

moreover by 1.4 we may assume up to coherent homotopy that $\underline{f}$ is actually a map of systems. Consider the diagram

$$X \xrightarrow{\underline{p}} \underline{X} \xrightarrow{[\sigma_{\underline{X}}]^{-1}} \underline{X}' \xrightarrow{[\underline{f}]} \underline{Y} \xleftarrow{\underline{q}} Y$$

which gives a strong shape morphism $[(\underline{p}, \underline{q}, [\underline{f}] \circ [\sigma_{\underline{X}}]^{-1})] : X \to Y$ such that

$$\gamma[(\underline{p}, \underline{q}, [\underline{f}] \circ [\sigma_{\underline{X}}]^{-1})] = \tau.$$

In fact

$$\begin{aligned}\gamma[(\underline{p}, \underline{q}, [\underline{f}] \circ [\sigma_{\underline{X}}]^{-1})] &= \underline{p}^* \circ \sigma_{\underline{X}}^{*-1} \circ \underline{f}^* \circ (\underline{q}^*)^{-1} \\ &= \underline{p}^* \circ \sigma_{\underline{X}}^{*-1} \circ \sigma_{\underline{X}}^* \circ (\underline{p}^*)^{-1} \tau \circ \underline{q}^* \circ (\underline{q}^*)^{-1} = \tau.\end{aligned}$$

□

Corollary 3.6 *The strong shape category of topological spaces $SSh(\mathsf{Top})$ [15] is isomorphic to SS_{ANR}, so that a strong shape morphism $\tau : X \to Y$ of topological spaces is represented by a psd-transformation*

$$\tau : \mathsf{Gpd}(Y, -) \Rightarrow \mathsf{Gpd}(X, -) : \mathsf{ANR} \to \mathsf{Gpd}.$$

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