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The free loop space homology of $(n-1)$-connected $2n$-manifolds

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Abstract Our goal in this paper is to compute the integral free loop space homology of $(n-1)$-connected $2n$-manifolds. We do this when $n \geq 2$ and $n \neq 2, 4, 8$, though the techniques here should cover a much wider range of manifolds. We also give partial information concerning the action of the Batalin–Vilkovisky operator.

Keywords String topology · Free loop space · Highly connected manifolds

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1 Introduction

Let $\mathcal{L}X = map(S^1, X)$ denote the free loop space on X. This space comes equipped with an action $\nu \colon S^1 \times \mathcal{L}X \longrightarrow \mathcal{L}X$ that rotates loops, and an induced degree 1 homomorphism

$$\Delta \colon H_*(\mathcal{L}X) \longrightarrow H_{*+1}(\mathcal{L}X)$$

known as the *BV-operator*, defined by setting $\Delta(a) = \nu_*([S^1] \otimes a)$. In addition Chas and Sullivan [9] constructed a pairing

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$$H_p(\mathcal{L}X) \otimes H_q(\mathcal{L}X) \longrightarrow H_{p+q-d}(\mathcal{L}X)$$

on a closed oriented d-manifold X that (together with the *BV-operator*) turns the shifted homology $\mathbb{H}_*(\mathcal{L}X) = H_{*+d}(\mathcal{L}X)$ into a Batalin–Vilkovisky (BV)-algebra.

Batalin–Vilkovisky algebras have been computed in only a few special cases. One of the more general results to date (due to Felix and Thomas [12]) states that over a field F of characteristic zero and 1-connected X, $\mathbb{H}_*(\mathcal{L}X; F)$ is isomorphic to a BV-algebra structure defined on the Hochschild cohomology $HH^*(C^*(X), C^*(X))$. Unfortunately, this theorem is generally not true for fields with nonzero characteristic [20]. Beyond these results, the BV-algebra over various coefficient rings has been completely determined for spheres [10,20,25], certain Stiefel manifolds [24], Lie groups [17], and projective spaces [10,16,22,27,28], using a mixture of techniques ranging from homotopy theoretic to geometric, as well as the well-known connections to Hochschild cohomology.

In this paper we focus on the free loop space homology of highly connected $2n$-manifolds, together with the action of the BV-operator. The coefficient ring R for homology and cohomology is assumed to be either any field, or the integers $\mathbb{Z}$, but we suppress it from notation most of the time. Fix $n \geq 2$, M a $(n-1)$-connected, closed, oriented $2n$-manifold with $H^n(M)$ of rank $m \geq 1$. Let

$$C = [c_{ij} = \langle a_i \cup a_j, [M]\rangle]$$

be the $m \times m$ matrix for the intersection form $H^n(M) \times H^n(M) \longrightarrow \mathbb{Z}$ with respect to some choice of basis $\{a_1, \ldots, a_m\}$ for $H^n(M)$ (we use the same notation for the dual basis of $H^n(M)$). This form is nonsingular, symmetric when n is even, and skew-symmetric when n is odd.

Denote $H^n(M)$ and $H^{2n}(M) \cong \mathbb{Z}$ by the free graded modules R-modules $A = R\{a_1, \ldots, a_m\}$ and $K = R\{[M]\}$, and the desuspension of A by $V = R\{u_1, \ldots, u_m\}$ with $|u_i| = n - 1$. Let

$$T(V) = R \oplus \bigoplus_{i \geq 1} V^{\otimes i}$$

be the free tensor algebra generated by V, and I be the two-sided ideal of the tensor algebra $T(V)$ generated by the following degree $2n-2$ element

$$\chi = \sum_{i<j} c_{ij}[u_i, u_j] + \sum_i c_{ii} u_i^2,$$

where $[x, y] = xy - (-1)^{|x||y|} yx$ denotes the graded Lie bracket in $T(V)$. Take the quotient algebra

$$U = \frac{T(V)}{I}$$

and the degree -1 maps of graded R-modules $d\colon A\otimes U \longrightarrow U$ and $d'\colon K\otimes U \longrightarrow A\otimes U$, which are given for any $y\in U$ by the formulas

$$d(a_i\otimes y) = [u_i, y]$$
$$d'([M]\otimes y) = \sum_{i,j} c_{ij}(a_j\otimes [u_i, y]).$$

If we apply the Jacobi identity to the summands $c_{ij}(a_j\otimes [u_i, y])$ in $d\circ d'(y)$ for $i<j$ (keeping in mind that $c_{ij}=(-1)^n c_{ji}$, $[u_i,[u_i,y]]=[u_i^2,y]$, and that products with χ are identified with zero in U), we see that $\operatorname{Im} d'\subseteq \ker d$, so we obtain a chain complex

$$0\longrightarrow K\otimes U \xrightarrow{d'} A\otimes U \xrightarrow{d} U \longrightarrow 0.$$

Now take the homology of this chain complex. That is, take the following graded R-modules:

$$\mathcal{Q}=\frac{U}{\operatorname{Im} d},\quad \mathcal{W}=\frac{\ker d}{\operatorname{Im} d'},\quad \mathcal{Z}=\ker d'.$$

One can think of $\mathcal{W}$ by first taking the R-submodule W' of $\Sigma^{-1}A\otimes T(V)\cong T(V)$ generated by elements that are invariant modulo I under graded cyclic permutations, that is, invariant after projecting to U. Then $\mathcal{W}$ is the projection of $\Sigma W'$ onto $(A\otimes U)/\operatorname{Im} d'$.

Our main result is that the homology of this chain complex is the integral free loop space homology of M under some conditions:

Theorem 1.1 *Suppose $n\geq 2$, $n\neq 2,4,8$, and $m\geq 1$. Then there exists an isomorphism of graded R-modules*

$$H_*(\mathcal{L}M)\cong \mathcal{Q}\oplus\mathcal{W}\oplus\mathcal{Z}.$$

The restriction away from 2, 4, and 8 traces back to an argument that we use to determine $H_*(\Omega M)$, which does not apply to situation where there are cup product squares equal to the fundamental class $[M]$, or $-[M]$. Failure of a degree placement argument to compute certain differentials is another reason that we restrict away from $n=2$.

We also determine the action of the BV-operator on $H_*(\mathcal{L}M;\mathbb{Q})$, in a sense, up-to-abelianization of U when $n>3$ is odd.

Consider the graded abelianization map $T(V)\xrightarrow{\eta} S(V)$, where $S(V)$ is the free graded symmetric algebra generated by V. Since $\eta(\chi)=0$, η factors through $U\xrightarrow{\eta} S(V)$. Also, consider the maps $A\otimes U\xrightarrow{\mathbb{1}_A\otimes\eta} A\otimes S(V)$ and $K\otimes U\xrightarrow{\mathbb{1}_K\otimes\eta} K\otimes S(V)$. Since $(\mathbb{1}_A\otimes\eta)\circ d'=0$ and $\eta\circ d=0$, then η and these two maps induce *abelianization* maps

$$\mathcal{Q} \xrightarrow{\eta_q} S(V),$$
$$\mathcal{W} \xrightarrow{\eta_w} A \otimes S(V),$$
$$\mathcal{Z} \xrightarrow{\eta_z} K \otimes S(V).$$

Theorem 1.2 *Let $n > 3$ be odd. The BV operator $\Delta\colon H_*(\mathcal{L}M; \mathbb{Q}) \longrightarrow H_{*+1}(\mathcal{L}M; \mathbb{Q})$ satisfies $\Delta(\mathcal{Q}) \subseteq \mathcal{W}$ and $\Delta(\mathcal{W}) \subseteq \mathcal{Z}$, and $\Delta(\mathcal{Z}) = \{0\}$. Moreover, the composite $\mathcal{Q} \xrightarrow{\Delta} \mathcal{W} \xrightarrow{\eta_w} A \otimes S(V)$ is given by*

$$\eta_w \circ \Delta(1 \otimes (u_{i_1} \ldots u_{i_k})) = \sum_{j=1}^{k} a_{i_j} \otimes (u_{i_1} \ldots u_{i_j - 1} u_{i_j + 1} \ldots u_{i_k}),$$

and $\mathcal{W} \xrightarrow{\Delta} \mathcal{Z} \xrightarrow{\eta_z} K \otimes S(V)$ is the restriction to ker *d of the map $(A \otimes U)/Im\, d' \longrightarrow S(A) \otimes S(V)$ given by*

$$a_i \otimes (u_{i_1} \ldots u_{i_k}) \mapsto \sum_{j=1}^{k} a_i a_{i_j} \otimes (u_{i_1} \ldots u_{i_j - 1} u_{i_j + 1} \ldots u_{i_k}),$$

where $[M] \in K$ is identified with $(\sum_{i \leq j} c_{ij} a_i a_j) \in S(A)$.

Berglund and Borjeson [6] have subsequently computed the free loop space homology of highly connected manifolds (including the ones considered in this paper) using different techniques. They also give a description of the action of the BV-operator and the Chas–Sullivan loop product. With a bit of effort it is likely that the spectral sequence methods in this paper can be extended to cover many of the highly connected manifolds in [6]. For example, the based loop space homology of highly connected manifolds is largely known [5], and this is one of the main ingredients that we use here. On the other hand, we do not know whether a complete description of the Chas–Sullivan loop product and BV-operator is possible using our approach—one difficulty being extension issues in the Cohen–Jones–Yan spectral sequence [10] when computing the loop product, together with a seeming incompatibility between the BV-operator and the Serre spectral sequence of a free loop fibration.

We should mention that there are sources of applications for the above calculations that go beyond the classical question: *are there infinitely many geometrically distinct periodic geodesics on a Riemannian manifold M?* For example, detailed information about the Betti numbers of $\mathcal{L}M$ reflects more detailed information about the number of geodesics of variable length. See [2,3,6,13] for details.

2 A useful lemma

Take a fibration sequence $F \xrightarrow{i} X \xrightarrow{f} B$ with B simply-connected. Recall the induced homotopy fibration sequence

$$\Omega B \xrightarrow{\vartheta} F \xrightarrow{i} X \tag{1}$$

is a *principal* homotopy fibration. Namely, there is a homotopy associative H-space structure on the homotopy fiber ΩB together with a left action

$$\theta\colon \Omega B \times F \longrightarrow F$$

that fits into a homotopy commutative square

$$\begin{array}{ccc} \Omega B \times \Omega B & \xrightarrow{\mathbb{1}\times\vartheta} & \Omega B \times F \\ \downarrow{\scriptstyle mult.} & & \downarrow{\scriptstyle \theta} \\ \Omega B & \xrightarrow{\vartheta} & F. \end{array}$$

In our case the H-space multiplication $mult.$ on ΩB is taken as the one defined by composing loops, and the action θ is defined by applying the homotopy lifting property to loops in B.

By a result of Moore [21], the homology Serre spectral sequence ξ of a principal fibration such as (1) has a left $H_*(\Omega B)$-module induced by the associated action θ. Namely, there is a left action $H_*(\Omega B) \otimes \xi^r_{i,j} \longrightarrow \xi^r_{i,j+*}$ reducing to the Pontrjagin multiplication on $\xi^2_{0,*} \cong H_*(\Omega B)$ and differentials respect this action. Most of the effort in computing differentials is therefore reduced to determining those emanating from the degree 0 horizontal line.

Since fibrations are characterized by the homotopy lifting property, one might also expect θ to have a direct bearing on the homology Serre spectral sequence for our original fibration f. This was exploited by McCleary [19], where he used a result of Brown [8] and Shih [23] to give a computation of the free loop space homology of certain low rank Stiefel manifolds. The following proposition strengthens the result in [8,23] by doing away with an assumption about certain elements being trangressive. The proof is moreover fairly simple. Let

$$\mathcal{E} = \{\mathcal{E}^r, \delta^r\}$$

denote the homology Serre spectral sequence for f, and

$$E = \{E^r, d^r\}$$

the homology Serre spectral sequence for the path-loop fibration sequence $\Omega B \xrightarrow{\subset} \mathcal{P}B \xrightarrow{ev_1} B$.

Proposition 2.1 *Suppose $H_*(B)$ and $H_*(F)$ are torsion free. Given $z \in H_*(B)$, and $\sum_i x_i \otimes v_i \in E^2_{*,*} \cong H_*(B) \otimes H_*(\Omega B)$, suppose $d^s(z \otimes 1) = d^s(\sum_i x_i \otimes v_i) = 0$ in $E^s_{*,*}$ for $2 \leq s < r$, and*

$$d^r(z \otimes 1) = \sum_i x_i \otimes v_i.$$

Then given $z \otimes y \in \mathcal{E}^2_{*,*} \cong H_*(B) \otimes H_*(F)$ *for any* $y \in H_*(F)$, *for each* $2 \leq s < r$ *we have*

$$\delta^s(z \otimes y) = \delta^s\left(\sum_i x_i \otimes \theta_*(v_i \otimes y)\right) = 0$$

and

$$\delta^r(z \otimes y) = \sum_i x_i \otimes \theta_*(v_i \otimes y).$$

Proof First recall the following well-known property (which is essentially the homotopy lifting property in disguise). Let $P^{ev_0,f} \subseteq map([0,1], B) \times X$ be the pullback of $X \xrightarrow{f} B$ and the evaluation map $map([0,1], B) \xrightarrow{ev_0} B$, where $ev_t(\omega) = \omega(t)$. Now consider the map $\bar{f}\colon map([0,1], X) \longrightarrow P^{ev_0,f}$ defined by $\bar{f}(\omega) = (f \circ \omega, \omega(0))$. Then a surjection f is a fibration if and only if there exists a map $g\colon P^{ev_0,f} \longrightarrow map([0,1], X)$ such that $\bar{f} \circ g = \mathbb{1}\colon P^{ev_0,f} \longrightarrow P^{ev_0,f}$.

Take the inclusion $\phi\colon \mathcal{P}B \times F \longrightarrow P^{ev_0,f}$ given by $\phi(\omega, a) = (\omega, a)$, and take the the composite

$$\bar{\theta}\colon (\mathcal{P}B \times F) \xrightarrow{\phi} P^{ev_0,f} \xrightarrow{g} map([0,1], X) \xrightarrow{ev_1} X.$$

Let the fibration sequence

$$\Omega B \times F \xrightarrow{\subset \times \mathbb{1}} \mathcal{P}B \times F \xrightarrow{ev_1 \times *} B \times * \tag{2}$$

be the product of the path-loop fibration sequence $\Omega B \xrightarrow{\subset} \mathcal{P}B \xrightarrow{ev_1} B$ and the trivial fibration sequence $F \xrightarrow{\mathbb{1}} F \xrightarrow{*} *$. Let $E = \{E^s, d^s\}$ and $\mathring{E} = \{\mathring{E}^s, \mathring{d}^s\}$ be the homology Serre spectral sequences for the path-loop and trivial fibration respectively, and $\hat{E} = \{\hat{E}^s, \hat{d}^s\}$ be the homology spectral sequence for their product (2). Define a differential $d^s_{\otimes}\colon E^s \otimes \mathring{E}^s \longrightarrow E^s \otimes \mathring{E}^s$ by $\hat{d}^s(a \otimes b) = (d^s(a) \otimes b) + (-1)^{|a|}(a \otimes \mathring{d}^s(b))$. Since $H_*(F)$ is torsion-free, $\hat{E}^s = E^s \otimes \mathring{E}^s$ and $\hat{d}^s = d^s_{\otimes}$ (see [7,14]). In our case $\mathring{d} = 0$, so we have

$$\hat{d}^s(a \otimes b) = d^s(a) \otimes b$$

for any $a \in E^s$ and $b \in \mathring{E}^s$. One can easily check that the following diagram of fibration sequences commutes

$$\begin{array}{ccccc} \Omega B \times F & \xrightarrow{\subset \times \mathbb{1}} & \mathcal{P}B \times F & \xrightarrow{ev_1 \times *} & B \times * \\ \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \bar{\theta}} & & \| \\ F & \xrightarrow{i} & X & \xrightarrow{f} & B, \end{array} \tag{3}$$

with our action θ being in fact the restriction of $\bar{\theta}$ to the subspace $\Omega B \times F$. Let

$$\zeta \colon \hat{E} = E \otimes \mathring{E} \longrightarrow \mathcal{E}$$

be the morphism of spectral sequences induced by this diagram.

Since $d^s(z \otimes 1) = 0 \in E^s_{*,*}$ for $2 \leq s < r$ and $d^r(z \otimes 1) = \sum_i x_i \otimes v_i$, then for any $b \in \mathring{E}^s$

$$\hat{d}^s((z \otimes 1) \otimes b) = d^s(z \otimes 1) \otimes b = 0$$
$$\hat{d}^r((z \otimes 1) \otimes b) = d^r(z \otimes 1) \otimes b = \sum_i (x_i \otimes v_i) \otimes b,$$

which we use to obtain

$$\begin{aligned}\delta^r(z \otimes y) &= \delta^r(\zeta^r((z \otimes 1) \otimes (1 \otimes y)))\\ &= \zeta^r(\hat{d}^r((z \otimes 1) \otimes (1 \otimes y)))\\ &= \zeta^r\left(\sum_i (x_i \otimes v_i) \otimes (1 \otimes y)\right)\\ &= \sum_i x_i \otimes \theta_*(v_i \otimes y),\end{aligned}$$

and similarly, $\delta^s(z \otimes y) = 0$ for $2 \leq s < r$.

In a similarly manner, we see $\hat{d}^s((\sum_i x_i \otimes v_i) \otimes b) = 0$ for $2 \leq s < r$ and (in turn) $\delta^s(\sum_i x_i \otimes \theta_*(v_i \otimes y)) = 0$ using the fact that $d^s(\sum_i x_i \otimes v_i) = 0$ (so the above equations make sense). □

We now turn our attention towards the free loop space fibration sequence

$$\Omega B \xrightarrow{\vartheta} \mathcal{L}B \xrightarrow{ev_1} B. \tag{4}$$

The map ϑ is the canonical inclusion $\Omega B \subseteq \mathcal{L}B$, and ev_1 is the evaluation map $ev_1(\omega) = \omega(1)$. The homology Serre spectral sequence for this fibration sequence will be denoted by

$$\mathcal{E} = \{\mathcal{E}^r, \delta^r\},$$

and as before $E = \{E^r, d^r\}$ is the homology Serre spectral sequence for the path-loop fibration of B. The path-loop fibration is principal, so E has a left $H_*(\Omega B)$-module as described before which the differentials d respect.

Some basic properties of the free loop space fibration are as follows. The map $\mathcal{L}B \xrightarrow{ev_1} B$ has a section $B \xrightarrow{s} \mathcal{L}B$ defined by mapping a point $b \in B$ to the constant loop at b, which implies the connecting map ϱ for the induced principal homotopy fibration $\Omega B \xrightarrow{\varrho} \Omega B \xrightarrow{\vartheta} \mathcal{L}B$ is null homotopic. The associated left action

$$\theta : \Omega B \times \Omega B \longrightarrow \Omega B$$

is given by

$$\theta(\omega, \lambda) = \omega \cdot \lambda \cdot \omega^{-1}$$

for any $\omega, \lambda \in \Omega B$. If $v \in H_*(\Omega B)$ is primitive, then for any $y \in H_*(\Omega B)$ one has the formula

$$\theta_*(v \otimes y) = (-1)^{|v||y|} yv - vy = -[v, y],$$

where the multiplication on $H_*(\Omega B)$ is the Pontrjagin multiplication induced by loop composition on ΩB. The proof of these can be found in [19] for example. Combining these properties with Proposition 2.1 gives the following description of the differentials in the spectral sequence $\mathcal{E}$.

Proposition 2.2 *Suppose $H_*(B)$ and $H_*(\Omega B)$ are torsion free, and B is 1-connected. Given $z \in H_*(B)$, and $\sum_i x_i \otimes v_i \in E^2_{*,*}$ with v_i primitive in $H_*(\Omega B)$, suppose that $d^s(z \otimes 1) = 0$ and $d^s(\sum_i x_i \otimes v_i) = 0$ in $E^s_{*,*}$ for $2 \leq s < r$, and*

$$d^r(z \otimes 1) = \sum_i x_i \otimes v_i.$$

Then given $z \otimes y \in \mathcal{E}^2_{*,*}$ for any $y \in H_*(\Omega B)$, for each $2 \leq s < r$ we have

$$\delta^s(z \otimes y) = \delta^s\left(\sum_i x_i \otimes [v_i, y]\right) = 0$$

and

$$\delta^r(z \otimes y) = -\sum_i x_i \otimes [v_i, y].$$

□

There are instances where this formula fails to give us enough information to determine some of the higher differentials. For example, if we found ourselves in the situation where $\delta^s(z \otimes y) = 0$ for $s \leq r$ and $d^r(z \otimes y) \neq 0$, then $z \otimes y \in \mathcal{E}^r_{*,*}$ survives to the $\mathcal{E}^{r+1}$ page, while $z \otimes y$ is not an element in $E^{r+1}_{*,*}$. In such case $\delta^s(z \otimes y)$ remains mysterious when $s > r$. An example where this situation happens in practice is the case of 4-manifolds omitted from Theorem 1.1.

3 Based loop space homology

Returning to our $2n$-manifold M in the introduction, we consider the Hopf algebra $H_*(\Omega M)$. This is the last piece in the puzzle required to prove Theorem 1.1. By

Poincaré duality the only nonzero reduced homology groups of M are in degrees n and $2n$. This implies M has a cell decomposition given by attaching an n-cell to an m-fold wedge of n-spheres $\bigvee_m S^n \simeq M - *$.

Generally, if a space Y is formed by attaching a k-cell to a space X via an attaching map $S^{k-1} \xrightarrow{\alpha} X$, and α' is its adjoint, the composite with the looped inclusion $S^{k-2} \xrightarrow{\alpha'} \Omega X \xrightarrow{\Omega i} \Omega Y$ is nullhomotopic, so one obtains a factorization of Hopf algebras through Hopf algebra maps

$$\begin{array}{ccc} & & H_*(\Omega X; R)/I \\ & \nearrow & \downarrow \theta \\ H_*(\Omega X; R) & \xrightarrow{(\Omega i)_*} & H_*(\Omega Y; R), \end{array} \tag{5}$$

where I is the two-sided ideal generated by $\alpha'([S^{k-2}]) \in H_{k-2}(\Omega X; R)$. The problem of determining the conditions under which θ is a Hopf algebra isomorphism is part of what is known as the *cell-attachment problem*. One of these conditions—the *inert* condition—states somewhat suprisingly that θ is a Hopf algebra isomorphism when R is a field if and only if $(\Omega i)_*$ is a surjection [11,15,18]. Here we select $k = 2n$, $Y \simeq M$, and $X \simeq M - *$, and use the inert condition to prove the following:

Proposition 3.1 *Suppose $n \geq 2$, $n \neq 2, 4, 8$, and $m \geq 1$.*

(i) *There is an isomorphism of Hopf algebras (free as R-modules)*

$$H_*(\Omega M) \cong \frac{T(V)}{I}$$

where $V = R\{u_1, \ldots, u_m\}$, $|u_i| = n - 1$.

(ii) *The element $\alpha'_*([S^{2n-2}])$ generating the two-sided ideal I is given by*

$$\alpha'_*([S^{2n-2}]) = \sum_{i<j} c_{ij}[u_j, u_i] + \sum_i c_{ii} u_i^2.$$

Proof of part (i) In [4], ΩM is shown to be a homotopy retract of $\Omega(M - *)$ when $n \neq 2, 4, 8$. Therefore $(\Omega i)_*$ is a split epimorphism, so we obtain $H_*(\Omega M; F) \cong H_*(\Omega(M - *); F)/I$ for any field F. Moreover, since $M - *$ is homotopy equivalent to $\bigvee_m S^n$, the $\mathbb{Z}$-module $H_*(\Omega(M - *); \mathbb{Z}) \cong T(V)$ is torsion-free. Therefore $H_*(\Omega M; \mathbb{Z})$ is torsion-free, and the Hopf algebra isomorphism holds for $R = \mathbb{Z}$ as well. □

Proof of part (ii) We will write $u_j = (\Omega i)_*(u_j) \in H_{n-1}(\Omega M)$, and take u_j to be the transgression of $a_j \in H_n(M)$.

Since the elements $u_1, \ldots, u_m$ in $H_{n-1}(\Omega(M - *))$ are primitive, and there are no monomials of length greater than 2 in degree $2n - 2$, the elements u_i^2 and $[u_j, u_i]$ form a basis for the primitives in $H_{2n-2}(\Omega(M - *))$. Now $\alpha'_*([S^{2n-2}])$ is primitive since $[S^{2n-2}]$ is primitive, so we can set

$$(\alpha')_*([S^{2n-2}]) = \sum_{i<j} c''_{ij}[u_i, u_j] + \sum_i c''_{ii} u_i^2$$

for some integers c''_{ij}.

Consider the homology Serre spectral sequence $E = (E^r, d^r)$ for the (principal) path-loop fibration sequence M, with

$$E^2_{*,*} = H_*(M) \otimes H_*(\Omega M).$$

On the dual cohomology spectral sequence we have the formula

$$\begin{aligned} d_n(a_j \otimes u_i) &= d_n(a_j \otimes 1)(1 \otimes u_i) + (-1)^n (a_j \otimes 1) d_n(1 \otimes u_i) \\ &= (-1)^n (a_j \otimes 1)(a_i \otimes 1) = c_{ij}([M]^* \otimes 1), \end{aligned}$$

so dualizing back to the homology spectral sequence gives us

$$d^n([M] \otimes 1) = \sum_{i,j} c_{ij}(a_j \otimes u_i). \tag{6}$$

Take $\bar{E} = (\bar{E}^r, \bar{d}^r)$ to be the homology Serre spectral sequence for the path-loop fibration of $M - *$. The inclusion $(M - *) \longrightarrow M$ induces an inclusion of the corresponding path-loop fibrations of $(M - *)$ and M, and in turn a morphism of spectral sequences $\gamma \colon \bar{E} \longrightarrow E$. On the second page of spectral sequences γ_2 maps $1 \otimes u_i$ to $1 \otimes u_i$ and $a_i \otimes 1$ to $a_i \otimes 1$, and $\bar{E}^r_{n,n-1} \xrightarrow{\gamma_r} E^r_{n,n-1}$ is an isomorphism for $2 \leq r \leq n$.

By part (i) of the theorem (and preceeding discussion), $(\alpha')_*([S^{2n-2}])$ generates the kernel of $(\Omega i)_* \colon H_{2n-2}(\Omega(M - *)) \longrightarrow H_{2n-2}(\Omega M)$, so $1 \otimes (\alpha)_*([S^{2n-2}])$ generates the kernel of $\gamma_2 \colon E^2_{0,2n-2} \longrightarrow E^2_{0,2n-2}$. Since $\gamma_r \colon \bar{E}^r_{i,j} \longrightarrow E^r_{i,j}$ is an isomorphism for $i < n$, $j < 2n - 2$, and all r, then in fact $1 \otimes (\alpha')_*([S^{2n-2}])$ generates the kernel of the map $\bar{E}^r_{0,2n-2} \xrightarrow{\gamma_r} E^r_{0,2n-2}$ for $2 \leq r \leq n$.

Take the element

$$\zeta'' = \sum_{i \leq j} c''_{ij}(a_j \otimes u_i - a_i \otimes u_j)$$

in $\bar{E}^r_{n,n-1}$, for $2 \leq r \leq n$. Then

$$\gamma_n(\zeta'') = \sum_{i \leq j} c''_{ij}(a_j \otimes u_i - a_i \otimes u_j), \tag{7}$$

and in $\bar{E}^n_{0,2n-2}$ we have

$$1 \otimes (\alpha')_*([S^{2n-2}]) = \sum_{i \leq j} c''_{ij}(1 \otimes [u_i, u_j]) = \bar{d}^n(\zeta'').$$

Since $\bar{E}^r_{i,j} = \{0\}$ for $i > n$ and $\bar{E}^\infty_{*,*} = \{0\}$, the differential $\bar{E}^n_{n,n-1} \xrightarrow{\bar{d}^n} \bar{E}^n_{0,2n-2}$ is an isomorphism, and since $\bar{E}^n_{n,n-1} \xrightarrow{\gamma_n} E^n_{n,n-1}$ is an isomorphism and $1 \otimes (\alpha')_*([S^{2n-2}])$ generates the kernel of $\bar{E}^n_{0,2n-2} \xrightarrow{\gamma_n} E^n_{0,2n-2}$, by naturality we see that the kernel of the differential $E^n_{n,n-1} \xrightarrow{d^n} E^n_{0,2n-2}$ is generated by $\gamma_n(\zeta'')$. In particular, we may project $\gamma_n(\zeta'')$ down to $E^\infty_{*,*}$.

Let

$$\mathcal{I} = \operatorname{Im} d^n \colon E^n_{2n,0} \longrightarrow E^n_{n,n-1}$$
$$\mathcal{K} = \ker d^n \colon E^n_{n,n-1} \longrightarrow E^n_{0,2n-2}.$$

As we saw above, $\mathcal{I}$ is generated by $d^n([M] \otimes 1)$, and $\gamma_n(\zeta'')$ generates $\mathcal{K}$. But the short exact sequence

$$0 \longrightarrow E^n_{2n,0} \xrightarrow{d^n} E^n_{n,n-1} \xrightarrow{d^n} E^n_{0,n-2} \longrightarrow 0$$

implies $\mathcal{I} \subseteq \mathcal{K}$. Therefore $d^n([M] \otimes 1) = \pm\gamma_n(\zeta'')$. Now comparing coefficients in Eqs. (6) and (7), the result follows. □

4 Proof of Theorem 1.1

We now have everything required to prove Theorem 1.1 via a routine Serre spetral sequence argument. Let $\mathcal{E} = \{\mathcal{E}^r, \delta^r\}$ be the homology Serre spectral sequence for the free loop space fibration sequence

$$\Omega M \xrightarrow{\vartheta} \mathcal{L}M \xrightarrow{ev_1} M.$$

By Proposition 3.1 we have an isomorphism $H_*(\Omega M) \cong U = T(V)/I$ of Hopf algebras, which are free as R-modules. So we start with an isomorphism of free R-modules

$$\mathcal{E}^2_{*,*} \cong R\{1, a_1, \ldots, a_m, [M]\} \otimes U.$$

By Proposition 2.2

$$\delta^n(a_i \otimes y) = -1 \otimes [u_i, y]$$

where u_i is the transgression of a_i, and using (6),

$$\delta^n([M] \otimes y) = -\sum_{i,j} c_{ij}(a_j \otimes [u_i, y]).$$

Therefore $\mathcal{E}^{2n}_{0,*} \cong \mathcal{Q}$, $\mathcal{E}^\infty_{n,*} \cong \mathcal{E}^{2n}_{n,*} \cong \mathcal{W}$, and $\mathcal{E}^{2n}_{2n,*} \cong \mathcal{Z}$, while all other entries in the spectral sequence are zero. Here, the only possible nonzero differentials are

$\delta^{2n}\colon \mathcal{E}^{2n}_{2n,*} \longrightarrow \mathcal{E}^{2n}_{0,*+2n-1}$. But since the nonzero elements in $\mathcal{Z}$ and $\mathcal{Q}$ are concentrated in total degrees $2n+k(n-1)$ and $k(n-1)$ respectively, one can check the differentials δ^{2n} are zero for degree placement reasons whenever $n > 2$. Thus these isomorphisms carry over to the infinity page, that is,

$$\mathcal{E}^{\infty}_{*,*} \cong \mathcal{E}^{\infty}_{0,*} \oplus \mathcal{E}^{\infty}_{n,*} \oplus \mathcal{E}^{\infty}_{2n,*} \cong \mathcal{Q} \oplus \mathcal{W} \oplus \mathcal{Z}.$$

Generally, one has torsion here when $R = \mathbb{Z}$ (or at least in $\mathcal{Q}$, and possibly $\mathcal{W}$), so we must consider a potential extension problem. Once again placement reasons allow us to skirt around the issue.

From the construction of the homology Serre spectral sequence there are increasing filtrations $\mathcal{F}_{i,j} = \mathcal{F}_i H_j(\mathcal{L}M) \subseteq H_j(\mathcal{L}M)$ such that $\mathcal{F}_{k,k} = H_k(\mathcal{L}M)$, $\mathcal{F}_{i,j} = 0$ for $i < 0$, and

$$\mathcal{E}^{\infty}_{i,j} \cong \frac{\mathcal{F}_{i,i+j}}{\mathcal{F}_{i-1,i+j}}.$$

Since the nonzero elements in $\mathcal{Q}$, $\mathcal{W}$, and $\mathcal{Z}$ are in degrees $k(n-1)$, $n+k(n-1)$, and $2n+k(n-1)$, $\mathcal{Q}$, $\mathcal{W}$, and $\mathcal{Z}$ pairwise have no nonzero elements in the same degrees when $n > 3$. Since $\mathcal{F}_{n-1,*} = \mathcal{F}_{0,*} = \mathcal{Q}$, we have $\mathcal{F}_{n-1,n+k(n-1)} = \{0\}$, and we see that $\mathcal{F}_{n,*} \cong \mathcal{F}_{0,*} \oplus \mathcal{E}^{\infty}_{n,*} \cong \mathcal{Q} \oplus \mathcal{W}$. Then $\mathcal{F}_{2n-1,2n+k(n-1)} = \mathcal{F}_{n,2n+k(n-1)} = \{0\}$, so $\mathcal{F}_{2n,*} \cong \mathcal{F}_{n,*} \oplus \mathcal{E}^{\infty}_{2n,*}$, and we have

$$\mathcal{E}^{\infty}_{2n,*} \cong \mathcal{F}_{2n,*} = H_*(\mathcal{L}M)$$

whenever $n > 3$.

When $n = 3$, the common nonzero degrees shared between any pair of these three modules are of the form $2(k+3)$, and these are only between $\mathcal{Q}$ and $\mathcal{Z}$. But since $\mathcal{Z}$ is torsion-free and $\mathcal{Q} = \mathcal{F}_{0,*}$ is at the bottom of the filtration, there are no extension issues here either.

5 Eilenberg–Maclane spaces and the BV-operator

We will need some information about the action of the BV-operator on products of Eilenberg–Maclane spaces before getting into the proof Theorem 1.2. The approach we take here is similar to the one taken by Hepworth in [17] to compute the BV-operator for Lie groups. We begin this section by recalling it. Fix R to be a principal ideal domain, and X (homotopy type of a CW-complex) a path-connected topological group with multiplication $X \times X \xrightarrow{\mu} X$. This makes $\mathcal{L}X$ into topological group with multiplication $\mathcal{L}X \times \mathcal{L}X \xrightarrow{\mathcal{L}\mu} \mathcal{L}X$ defined by point-wise multiplication of loops $(\omega \cdot \gamma)(t) = \omega(t) \cdot \gamma(t)$. There is a well-known homeomorphism

$$h\colon X \times \Omega X \longrightarrow \mathcal{L}X$$
$$h(x, \omega) = x \cdot \omega$$

with inverse $h^{-1}\colon \mathcal{L}X \longrightarrow X \times \Omega X$ given by $h^{-1}(\omega) = (\omega(0), \omega(0)^{-1} \cdot \omega)$, where $x \cdot \omega$ is the loop defined at each point by $(x \cdot \omega)(t) = x \cdot \omega(t)$. These homeomorphisms are equivariant with respect to our action $S^1 \times \mathcal{L}X \xrightarrow{\nu} \mathcal{L}X$, and the action

$$\bar{\nu}\colon S^1 \times X \times \Omega X \longrightarrow X \times \Omega X$$

defined by the formula

$$\begin{aligned}\bar{\nu}(t, x, \omega) &= h^{-1} \circ \nu(t, x \cdot \omega) = (x \cdot \omega_t(0), (x \cdot \omega_t(0))^{-1} \cdot x \cdot \omega_t)\\ &= (x \cdot \omega_t(0), \omega_t(0)^{-1} \cdot \omega_t)\end{aligned}$$

where $\omega_t(s) = \nu(t, \omega)(s) = \omega(s + t)$. On homology we have a commutative square

$$\begin{array}{ccc} H_*(X \times \Omega X; R) & \xrightarrow[\cong]{h_*} & H_*(\mathcal{L}X; R) \\ \bar{\Delta}\downarrow & & \downarrow\Delta \\ H_{*+1}(X \times \Omega X; R) & \xrightarrow[h_*]{\cong} & H_{*+1}(\mathcal{L}X; R) \end{array}$$

where $\bar{\Delta}(e) = \bar{\nu}_*([S^1] \otimes e)$. Clearly, after transposing X and S^1, $\bar{\nu}$ is the composite

$$X \times (S^1 \times \Omega X) \xrightarrow{\mathbb{1}_X \times \Delta} X \times (S^1 \times \Omega X) \times (S^1 \times \Omega X) \xrightarrow{\mathbb{1}_X \times ev \times \phi} (X \times X) \times \Omega X \xrightarrow{\mu \times \mathbb{1}} X \times \Omega X,$$

with $ev\colon S^1 \times \Omega X \longrightarrow X$ the evaluation map $ev(t, \omega) = \omega(t) = \omega_t(0)$, and $\phi\colon S^1 \times \Omega X \longrightarrow \Omega X$ defined by $\phi(t, \omega) = \omega_t(0)^{-1} \cdot \omega_t$. Thus, if $H_*(\Omega X; R)$ is a free R-module, so that (for simplicity) the cross product $H_*(X; R) \otimes H_*(\Omega X; R) \xrightarrow{\times} H_*(X \times \Omega X; R)$ is an isomorphism, and the coproduct on an element $b \in H_*(\Omega X; R)$ has the form $\triangle_*(b) = \sum_i d_i \otimes e_i$, then $\bar{\Delta}$ satisfies

$$\begin{aligned}(-1)^{|a|}\bar{\Delta}(a \otimes b) &= \sum_i (-1)^{|d_i|}(a(ev_*(1 \otimes d_i)) \otimes \phi_*([S^1] \otimes e_i))\\ &\quad + \sum_i (a(ev_*([S^1] \otimes d_i)) \otimes \phi_*(1 \otimes e_i))\\ &= \sum_i (-1)^{|d_i|}(a\epsilon(d_i) \otimes \phi_*([S^1] \otimes e_i))\\ &\quad + \sum_i (a(ev_*([S^1] \otimes d_i)) \otimes e_i) \qquad (8)\end{aligned}$$

where $\epsilon\colon H_*(\Omega X; R) \longrightarrow R$ is the augmentation. To complete this formula one needs to determine the maps ϕ_* and ev_*. This latter map defines the *homology suspension* $\sigma\colon H_*(\Omega X; R) \longrightarrow H_{*+1}(X; R)$, $\sigma(a) = ev_*([S^1] \otimes a)$, which satisfies the formula

$$\sigma(ab) = \sigma(a)\epsilon(b) + \epsilon(a)\sigma(b) \qquad (9)$$

for any product $ab \in H_*(\Omega X; R)$ induced by the loop composition multiplication on ΩX. In particular, σ is zero on decomposable elements. If X is an H-space, one can derive this formula by observing that the following diagram commutes

$$\begin{array}{ccccc} (S^1 \times S^1) \times (\Omega X \times \Omega X) & \xrightarrow{\mathbb{1}\times T\times\mathbb{1}} & (S^1 \times \Omega X) \times (S^1 \times \Omega X) & \xrightarrow{ev\times ev} & X \times X \\ \uparrow{\scriptstyle \Delta\times\mathbb{1}\times\mathbb{1}} & & & & \downarrow{\scriptstyle \mu} \\ S^1 \times (\Omega X \times \Omega X) & \xrightarrow{\mathbb{1}\times\Omega\mu} & S^1 \times \Omega X & \xrightarrow{ev} & X, \end{array}$$

and that point-wise multiplication of based loops $\Omega\mu$ on ΩX is homotopy commutative and homotopic to the loop composition multiplication on ΩX (this is a mapping space analogue of Theorem 5.21, Chapter III in [26]). Alternatively, it is a consequence of the homology suspension theorem [26, Chapter VIII]. The map $\kappa(a) = \phi_*([S^1] \otimes a)$ is a bit more mysterious. At the very least, when μ is commutative one obtains an analogous commutative diagram for ϕ together with a derivation formula $\kappa(ab) = \kappa(a)b + a\kappa(b)$, while for the case of compact Lie groups, κ is trivial since $H_*(\Omega X)$ is concentrated even degrees. We consider the case where X is an Eilenberg–Maclane space $K(R, n)$. These can be taken to be commutative topological groups, and we may write $K(G, n-1) = \Omega K(G, n)$ with commutative multiplication induced by the one on $K(R, n)$, which by the way is homotopic to the loop composition multiplication.

Proposition 5.1 *Let J be the image of the cross product $H_*(K(R, n-1); R) \otimes H_*(K(R, n); R) \xrightarrow{\times} H_*(K(R, n-1) \times K(R, n); R)$ (which is injective by the Künneth formula). Suppose the coproduct on $a \in H_*(K(R, n-1); R)$ is in the image of the cross product, that is, it is of the form $\triangle_*(b) = \sum_i d_i \times e_i$. Then with respect to the isomorphism h_*, the BV-operator is given on $a \times b \in J \subseteq H_*(\mathcal{L}K(R, n); R)$ by the formula*

$$\Delta(a \times b) = (-1)^{|a|} \sum_i (a(\rho_*([S^1] \otimes d_i)) \times e_i),$$

where $\Sigma K(R, n-1) \xrightarrow{\rho} K(R, n)$ is a classifying map for $[S^1] \otimes \iota_{n-1} \in H^*(\Sigma K(R, n-1); R) \cong \bar{H}^*(S^1; R) \otimes \bar{H}^*(K(R, n-1); R)$, and $\iota_{n-1} \in \bar{H}^{n-1}(K(R, n-1); R)$ is the fundamental class.

Proof Since our map $S^1 \times K(R, n-1) \xrightarrow{\phi} K(R, n-1)$ restricts to the identity on the right factor, $\phi^*(\iota_{n-1}) = 1 \otimes \iota_{n-1}$, or in other words, ϕ is a classifying map of the cohomology class $1 \otimes \iota_{n-1} \in \bar{H}^{n-1}(S^1 \times K(R, n-1); R)$. The projection map onto the right factor $S^1 \times K(R, n-1) \xrightarrow{*\times\mathbb{1}} K(R, n-1)$ is also a classifying map for $1 \otimes \iota_{n-1}$. Since cohomology classes are in one-to-one correspondance with the homotopy classes of the classifying maps representing them, ϕ must be homotopic to $* \times \mathbb{1}$. Therefore $\phi_*([S^1] \otimes d) = 0$ for any d.

Next, recall the suspension isomorphism $H_{n-1}(K(R,n-1); R) \xrightarrow{\cong} H_n(\Sigma K(R, n-1); R)$, sending $a \mapsto [S^1] \otimes a$, factors as the composite

$$\begin{aligned} H_{n-1}(K(R,n-1); R) &\xrightarrow{\cong} [K(R,n-1), K(R,n-1)] \\ &\xrightarrow{\cong} [\Sigma K(R,n-1), K(R,n)] \end{aligned}$$

where the last map is the adjoint isomorphism. Since the evaluation map $S^1 \times K(R, n-1) \xrightarrow{ev} K(R,n)$ restricts to the constant map on both the left and right factors, it factors as the composite

$$ev\colon S^1 \times K(R,n-1) \xrightarrow{quotient} \Sigma K(R,n-1) \xrightarrow{ev'} K(R,n),$$

where the last map ev' (also known as the evaluation map in the literature) is the adjoint of the identity map $K(R,n-1) \xrightarrow{\mathbb{1}} K(R,n-1)$. Since the identity is a classifying map of ι_{n-1}, by the above factorization of the suspension, its adjoint ev' is a classifying map of $[S^1] \otimes \iota_{n-1}$. The proposition now follows using Eq. (8). □

The BV-operator has a very clean form on decomposable elements when we take our multiplication on $H_*(\mathcal{L}X)$ to be the one induced by point-wise multiplication of loops $\mathcal{L}\mu$ (instead of the multiplication $(\Omega\mu \times \mu) \circ (\mathbb{1} \times T \times \mathbb{1})$ based on each coordinate of $\Omega X \times X \cong \mathcal{L}X$). Tamanoi [24] gave a derivation formula with respect to this product

$$\Delta(ab) = \Delta(a)b + (-1)^{|a|} a\Delta(b),$$

which is a straightforward consequence of the following commutative diagram

$$\begin{array}{ccccc} (S^1 \times S^1) \times (\mathcal{L}X \times \mathcal{L}X) & \xrightarrow{\mathbb{1}\times T\times\mathbb{1}} & (S^1 \times \mathcal{L}X) \times (S^1 \times \mathcal{L}X) & \xrightarrow{\nu\times\nu} & \mathcal{L}X \times \mathcal{L}X \\ \uparrow{\scriptstyle \Delta\times\mathbb{1}\times\mathbb{1}} & & & & \downarrow{\scriptstyle \mathcal{L}\mu} \\ S^1 \times (\mathcal{L}X \times \mathcal{L}X) & \xrightarrow{\mathbb{1}\times\mathcal{L}\mu} & S^1 \times \mathcal{L}X & \xrightarrow{\nu} & \mathcal{L}X. \end{array}$$

Both multiplications on $\mathcal{L}X$ are equal when the multiplication on X is commutative. Since this is the case for $K(R,n)$, our formula in Proposition 5.1 satisfies

$$\begin{aligned} (-1)^{|b||c|}\Delta(ac \times bd) &= \Delta((a \times b)(c \times d)) \\ &= \Delta(a \times b)(c \times d) + (-1)^{|a|+|b|}(a \times b)\Delta(c \times d). \quad (10) \end{aligned}$$

The derivation formula can also be used to determine how the BV-operator interacts with the cross-product, as we see in the following:

Proposition 5.2 *Let $X = X_1 \times \cdots \times X_k$ be a product of topological groups (X_i, μ_i). Then the BV-operator for $\mathcal{L}X \cong \mathcal{L}X_1 \times \cdots \times \mathcal{L}X_k$ satisfies*

$$\Delta(a_1 \times \cdots \times a_k) = \sum_i (-1)^{k_i} (a_1 \times \cdots \times \Delta(a_i) \times \cdots \times a_k)$$

for $a_i \in H_*(\mathcal{L}X_i)$, where $k_i = \sum_{j=1}^{i-1} |a_j|$ and $k_1 = 0$.

Proof It suffices to prove the statement for length-2 products $X = X_1 \times X_2$. One can then iterate to obtain the general formula. Since the inclusion of the left factor $\mathcal{L}X_1 \xrightarrow{\mathbb{1}\times *} \mathcal{L}X_1 \times \mathcal{L}X_2$ induces the map on homology sending $a \mapsto a \times 1$ for any a, by naturality of the BV-operator we have $\Delta(a_1 \times 1) = (\mathbb{1} \times *)_*(\Delta(a_1)) = \Delta(a_1) \times 1$. Similarly, $\Delta(1 \times a_2) = 1 \times \Delta(a_2)$. Since X is a topological group with multiplication μ defined by the composite $X \times X \xrightarrow{\mathbb{1}\times T\times\mathbb{1}} (X_1 \times X_1) \times (X_2 \times X_2) \xrightarrow{\mu_1\times\mu_2} X$, the point-wise loop multiplication $\mathcal{L}\mu$ is the composite

$$\mathcal{L}X \times \mathcal{L}X \xrightarrow{\cong} (\mathcal{L}X_1 \times \mathcal{L}X_2) \times (\mathcal{L}X_1 \times \mathcal{L}X_2) \xrightarrow{\mathbb{1}\times T\times\mathbb{1}} (\mathcal{L}X_1 \times \mathcal{L}X_1) \times (\mathcal{L}X_2 \times \mathcal{L}X_2) \xrightarrow{\mathcal{L}\mu_1\times\mathcal{L}\mu_2} \mathcal{L}X.$$

Therefore $(a_1 \times 1)(1 \times a_2) = a_1 \times a_2$ with respect to this induced product, and by the derivation formula we have

$$\begin{aligned}\Delta(a_1 \times a_2) &= \Delta(a_1 \times 1)(1 \times a_2) + (-1)^{|a_1|}(a_1 \times 1)\Delta(a_2 \times 1)\\ &= \Delta(a_1) \times a_2 + (-1)^{|a_1|} a_1 \times \Delta(a_2).\end{aligned}$$

□

We have, for the sake of simplicity, been restricting X to be a topological group. Some of the material above however extends (up-to-homotopy) to where X is a homotopy associative H-space. In this scenario h is a homotopy equivalence since it defines is a weak equivalence between the free loop fibration and the trivial fibration. If X has an inverse $-\mathbb{1}: X \longrightarrow X, x \mapsto x^{-1}$, the null homotopy $H: X \times X \times I \longrightarrow X$, with $H_0 = *$ and $H_1 = \mathbb{1} \times -\mathbb{1}$, allows us to define the homotopy inverse h^{-1} just as before, except this time composing the loop $\omega(0)^{-1} \cdot \omega$ with the based path given by $H_t(\omega(0)^{-1}, \omega(0))$, and the action $\bar{\nu}$ will have a similar form.

In the case of rational coefficients, a simply connected H-space X has a rational decomposition $X_{\mathbb{Q}} \simeq \prod_i K(\mathbb{Q}, n_i)$, and the classifying maps $\Sigma K(\mathbb{Q}, n_i - 1) \longrightarrow K(\mathbb{Q}, n_i)$ can be identified with the Freudenthal suspension $S^{n_i}_{\mathbb{Q}} \longrightarrow \Omega\Sigma S^{n_i}_{\mathbb{Q}}$ in the n_i even case, and evaluation $\Sigma\Omega S^{n_i}_{\mathbb{Q}} \longrightarrow S^{n_i}_{\mathbb{Q}}$ in the odd case. We see then that the action of Δ on $H_*(\mathcal{L}X; \mathbb{Q})$ with respect to the algebra structure induced by the group multiplication on $\prod_i K(\mathbb{Q}, n_i)$ can be determined by applying Propositions 5.1 and 5.2.

This technique can still be used to obtain some useful information for more general coefficients. Suppose $H_*(X; R)$ is free as an R-module, and $a \in H_n(X; R)$ is an indecomposable element in the Hopf algebra $H_*(X; R)$. Then the cohomology dual

$\hat{a} \in H^n(X; R)$ of a is a primitive element in the dual Hopf algebra $H^*(X; R)$, the classifying map $X \stackrel{c}{\longrightarrow} K(R, n)$ of $\hat{a}$ is an H-map, and moreover it is natural with respect to the homeomorphism h. That is, the following squares commute up to homotopy

$$\begin{array}{ccc} X \times X & \xrightarrow{c\times c} & K(R,n) \times K(R,n) \\ \mu \downarrow & & \downarrow mult. \\ X & \xrightarrow{c} & K(R,n) \end{array} \qquad \begin{array}{ccc} X \times \Omega X & \xrightarrow{c\times \Omega c} & K(R,n) \times \Omega K(R,n) \\ h \downarrow \cong & & \cong \downarrow h \\ \mathcal{L}X & \xrightarrow{\mathcal{L}c} & \mathcal{L}K(R,n). \end{array} \tag{11}$$

The proof of commutativity is as follows. For degree reasons, the fundamental class ι_n satisfies $(mult.)^*(\iota_n) = (\iota_n \times 1 + 1 \times \iota_n)$, so we have $(c \times c)^* \circ (mult.)^*(\iota_n) = \hat{a} \otimes 1 + 1 \otimes \hat{a}$. Likewise, since $\hat{a}$ is primitive, $\mu^* \circ c^*(\iota_n) = \mu^*(\hat{a}) = \hat{a} \otimes 1 + 1 \otimes \hat{a}$. Thus both the composites in the first square are classifying maps of $\hat{a} \otimes 1 + 1 \otimes \hat{a}$, meaning they are homotopic. This gives the first square. To obtain the second square, let $H: (X \times X) \times I \longrightarrow K(R, n)$ be a choice of homotopy between the composites in the first square. Define the homotopy $G: (X \times \Omega X) \times I \longrightarrow \mathcal{L}K(R, n)$ by $G(x, \omega, t) = \omega_{x,t}$, where $\omega_{x,t}: S^1 \longrightarrow X$ is the loop given by $\omega_{x,t}(s) = H(x, \omega(s), t)$. Then G defines a homotopy between the two composites in the second square. As a consequence of these diagrams, $\mathcal{L}c_*$ is an algebra map with respect to the algebra structure induced by the isomorphisms h_*, given by $(\mathcal{L}c)_*(v \otimes b) = c_*(v) \times (\Omega c)_*(b)$.

Now suppose n is odd, a is trangressive, and $\tau(a) \in H_{n-1}(\Omega X; R)$ is its trangression. Since c_* maps a to the homology dual $\hat{\iota}_n$ of ι_n, and $\hat{\iota}_n$ is trangressive onto $\tau(\hat{\iota}_n) = \hat{\iota}_{n-1}$, the homology dual of the fundamental class of $\Omega K(R, n) = K(R, n-1)$, we have $(\Omega c)_*(\tau(a)) = \hat{\iota}_{n-1}$. Then $(\mathcal{L}c)_*(\Delta(v \otimes \tau(a))) = \Delta((\mathcal{L}c)_*(v \otimes \tau(a))) = \Delta(c_*(v) \times \hat{\iota}_{n-1}) = (-1)^{|v|}(c_*(v)\hat{\iota}_n) \times 1$ by Proposition 5.1, and applying the derivation formula (10) inductively,

$$(\mathcal{L}c)_*(\Delta(v \otimes \tau(a)^k)) = \Delta(c_*(v) \otimes \hat{\iota}_{n-1}^k) = k(-1)^{|v|}((c_*(v)\hat{\iota}_n) \times \hat{\iota}_{n-1}^{k-1}).$$

Since $(\mathcal{L}c)_*(va \otimes \tau(a)^{k-1}) = (c_*(v)\hat{\iota}_n) \times \hat{\iota}_{n-1}^{k-1}$, if we assume $\tau(a)^{k-1}$ generates $H_{(k-1)(n-1)}(\Omega X; R)$, and va generates $H_{n+|v|}(X; R)$, then

$$\Delta(v \otimes \tau(a)^k) = k(-1)^{|v|}(va \otimes \tau(a)^{k-1}).$$

For example, if we take $R = \mathbb{Z}_p$ for p an odd prime, $X = S^n_{(p)}$ as a p-localized sphere (which is an H-space for n odd [1]), and $a = [S^n]$, then this formula completely determines the action of Δ on $H(\mathcal{L}S^n; \mathbb{Z}_p) \cong H(\mathcal{L}X; \mathbb{Z}_p)$. This is a somewhat different approach for spheres than the one taken by Westerland [25], and Menichi [20].

6 Proof of Theorem 1.2

For degree placement reasons, it is clear that $\Delta(\mathcal{Q}) \subseteq \mathcal{W}$, $\Delta(\mathcal{W}) \subseteq \mathcal{Z}$, and $\Delta(\mathcal{Z}) = \{0\}$ when $n > 3$. Consider the composite

$$f: M \xrightarrow{\Delta} \prod_{i=1}^{m} M \xrightarrow{\prod_i f_i} \prod_{i=1}^{m} K(\mathbb{Q}, n) = P,$$

where f_i is the classifying map of the generator $a_i \in H^n(M; \mathbb{Q})$. Let $\iota_i \in H_n(K(\mathbb{Q}, n); \mathbb{Q})$ denote the homology dual of the fundamental class for the i^{th} factor, and $\bar{\iota}_i \in H_{n-1}(K(\mathbb{Q}, n-1); \mathbb{Q})$ the corresponding trangression. Let $W = \mathbb{Q}\{\iota_1, \ldots, \iota_m\}$ and $\bar{W} = \mathbb{Q}\{\bar{\iota}_1, \ldots, \bar{\iota}_m\}$.

Since n is odd, $H_*(K(\mathbb{Q}, n); \mathbb{Q}) \cong \Lambda_{\mathbb{Q}}[\iota_i]$, $H_*(K(\mathbb{Q}, n-1); \mathbb{Q}) \cong \mathbb{Q}[\bar{\iota}_i]$, f induces the injection $H_*(M; \mathbb{Q}) \cong V \oplus K \longrightarrow \Lambda_{\mathbb{Q}}[W]$, mapping $a_i \mapsto \iota_i$ and $[M] \mapsto \beta = \sum_{i<j}(c_{ij}\iota_i\iota_j)$, and Ωf induces the algebra map $\mathcal{Q} \xrightarrow{\eta_q} \mathbb{Q}[\bar{W}] \cong S(V)$, mapping $u_i \mapsto \bar{\iota}_i$.

Consider the morphism of rational homology Serre spectral sequences $\mathcal{E} \xrightarrow{\phi} E$ induced by the map of free loop space fibrations

$$\begin{array}{ccccc} \Omega M & \longrightarrow & \mathcal{L}M & \xrightarrow{ev_1} & M \\ \downarrow{\scriptstyle \Omega f} & & \downarrow{\scriptstyle \mathcal{L}f} & & \downarrow{\scriptstyle f} \\ \Omega P & \longrightarrow & \mathcal{L}P & \xrightarrow{ev_1} & P. \end{array}$$

The spectral sequence E for the bottom fibration collapses since the total space is a topological group with section. On the infinity page

$$H_*(\mathcal{L}P; \mathbb{Q}) \cong H_*(P; \mathbb{Q}) \otimes H_*(\Omega P; \mathbb{Q}) \cong \bigoplus_{i=0}^{m} E^{\infty}_{ni,*},$$

and ϕ^∞ restricts to the maps $\mathcal{Q} \xrightarrow{\eta_q} \mathbb{Q}[\bar{W}] \cong E^{\infty}_{0,*}$, $\mathcal{W} \xrightarrow{\eta_w} W \otimes \mathbb{Q}[\bar{W}] \cong E^{\infty}_{n,*}$, and $\mathcal{Z} \xrightarrow{\eta_z} \mathbb{Q}\{\beta\} \otimes \mathbb{Q}[\bar{W}] \subseteq E^{\infty}_{2n,*}$ [note $W \cong V$, $\mathbb{Q}\{\beta\} \cong K$, and $\mathbb{Q}[\bar{W}] \cong S(V)$ in the introduction].

Let F be the filtration of $H_*(\mathcal{L}P; \mathbb{Q})$ associated with the spectral sequence E. Notice $E^{\infty}_{n,*} \cong F_{n,n+*}/\mathbb{Q}[\bar{W}]$, and $\mathbb{Q}[\bar{W}]$ is concentrated in degrees $k(n-1)$, while $\mathcal{W}$ is concentrated in degrees $n+k(n-1)$, which are never equal when $n > 3$, so they do not share any nonzero elements in the same degree. Similarly, $E^{\infty}_{2n,*} \cong F_{2n,2n+*}/F_{n,2n+*}$, $F_{n,*} \cong \mathbb{Q}[\bar{W}] \oplus (W \otimes \mathbb{Q}[\bar{W}])$ is concentrated in degrees $k(n-1)$ and $n + k(n-1)$, and $\mathcal{Z}$ is concentrated in degrees $2n + k(n-1)$, which are never equal when $n > 3$. Therefore, with respect to our isomorphism $H_*(\mathcal{L}M; \mathbb{Q}) \cong \mathcal{Q} \oplus \mathcal{W} \oplus \mathcal{Z}$, $(\mathcal{L}f)_*$ restricts to the maps η_q, η_w, and η_z on each summand.

The action of Δ on $H_*(\mathcal{L}K(\mathbb{Q}, n-1); \mathbb{Q})$ is given by $\Delta(1 \otimes \bar{\iota}_i^k) = k(\iota_i \otimes \bar{\iota}_i^{k-1})$ and $\Delta(a \otimes \bar{\iota}_i) = 0$ when $|a| > 0$. This follows from Proposition 5.1, and iterating formula (10). Alternatively, it follows from [20,25]. Now by Proposition 5.2,

$$\Delta(a \otimes \bar{\iota}_1^{k_1} \ldots \bar{\iota}_m^{k_m}) = \sum_{i=1}^{m} k_i(a\iota_i \otimes \bar{\iota}_i^{k_1} \ldots \bar{\iota}_i^{k_i-1} \ldots \bar{\iota}_m^{k_m}) \subseteq W \otimes \mathbb{Q}[\bar{W}] \cong A \otimes S(V)$$

for any integers $k_i \geq 0$. Since for any $q \in \mathcal{Q}$, we have $\Delta(q) \in \mathcal{W}$,

$$\Delta \circ \eta_q(q) = \Delta \circ (\mathcal{L}f)_*(q) = (\mathcal{L}f)_* \circ \Delta(q) = \eta_w \circ \Delta(q),$$

we obtain the formula for the composite $\mathcal{Q} \xrightarrow{\Delta} \mathcal{W} \xrightarrow{\eta_w} A \otimes S(V)$. Similarly we obtain the formula for the composite $\mathcal{W} \xrightarrow{\Delta} \mathcal{Z} \xrightarrow{\eta_z} K \otimes S(V)$.

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