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Galois descent for real spectra

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Abstract We prove analogs of faithfully flat descent and Galois descent for categories of modules over E_∞-ring spectra using the ∞-categorical Barr-Beck theorem proved by Lurie. In particular, faithful G-Galois extensions are shown to be of effective descent for modules. Using this we study the category of $ER(n)$-modules, where $ER(n)$ is the $\mathbb{Z}/2$-fixed points under complex conjugation of a generalized Johnson-Wilson spectrum $E(n)$. In particular, we show that $ER(n)$-modules is equivalent to $\mathbb{Z}/2$-equivariant $E(n)$-modules as stable ∞-categories.

Keywords Real-oriented cohomology · Galois extensions of ring spectra · Effective descent for modules · Barr-Beck-Lurie comonadicity

Mathematics Subject Classification 13B05 · 18C15 · 55N20 · 55N91 · 55P42 · 55P43

Contents

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1 Introduction

Let $K \subset L$ be a finite Galois extension of fields with Galois group $\mathrm{Gal}(L|K) = G$. Then the map $K \to L$ is of effective descent for modules. This means any K-module is equivalent to an L-module along with some descent data. More precisely the map

$$K\text{- vector spaces} \xrightarrow{-\otimes_K L} G - (L\text{- vector spaces})$$

is an equivalence of categories. Here $G-(L\text{- vector spaces})$ is the category of L- vector spaces N with a G-action which is semilinear in the sense that $\sigma(cx) = \sigma(c)\sigma(x)$ for all $\sigma \in G$, $c \in L$ and $x \in N$. This is classical Galois descent for fields. A similar descent formalism exists for Galois extensions of commutative rings as well.

Rognes (in [18]) has defined Galois extensions in the category of E_∞-ring spectra. In the first part of this paper we will prove Galois descent for *faithful* Galois extensions of ring spectra. In particular we show that if $A \to B$ is a faithful G-Galois extension of E_∞-ring spectra and G is a finite group then the map

$$A\text{-mod} \xrightarrow{-\wedge_A B} G - (B\text{-mod})$$

is an equivalence of stable ∞-categories. Here $G - (B\text{-mod})$ is the category of G-equivariant B-modules (Definition 2.18). The main tool that we use is the comonadic formulation for descent in ∞-categories and the Barr-Beck-Lurie criteria for comonadicity. Along the way we also prove an analog of faithfully flat descent for E_∞-rings.

Some examples of global faithful Galois extensions of E_∞-rings are the following:

- $HR \to HS$ where $R \to S$ is a Galois extension of commutative rings. The Galois group is $\mathrm{Gal}(S|R)$ ([18, Lemma 4.2.5]).
- $KO \to KU$ with Galois group $\mathbb{Z}/2$ ([18, Prop. 5.3.1])
- $TMF[1/n] \to TMF(n)$ with Galois group $GL_2(\mathbb{Z}/n\mathbb{Z})$ ([17]).

In the second part of this paper we show that Real Johnson-Wilson theories (at the prime 2) produce examples of faithful $\mathbb{Z}/2$-Galois extensions. Complex conjugation acts on the complex K-theory spectrum KU and the homotopy fixed points of this action is *KO*. The complex orientation $MU \to KU$ is equivariant with respect to this $\mathbb{Z}/2$-action. Hu and Kriz ([8]) have realized this as a map of genuine $\mathbb{Z}/2$-equivariant spectra $MU_{\mathbb{R}} \to K_{\mathbb{R}}$ where $K_{\mathbb{R}}$ is Atiyah's *real* K-theory ([1]). More generally, let X be any $2(2^n-1)$-periodic *generalized* Johnson-Wilson theory (see Sect. 3.3) with coefficient ring $\mathbb{Z}_{(2)}[u_1,\ldots,u_n^{\pm 1}]$ with $|u_i| = 2(2^i-1)$ and orientation $MU \to X$ with kernel $\langle u_i, i > n\rangle$. Then Hu and Kriz constructs an equivariant refinement of this orientation to a map

$$MU_{\mathbb{R}} \to X_{\mathbb{R}}.$$

We shall call $X_{\mathbb{R}}$ a *Real generalized Johnson-Wilson spectrum* and its $\mathbb{Z}/2$ homotopy fixed points XR a *real generalized Johnson-Wilson spectrum.*

We show the existence of a fibration (Theorem 3.2)

$$\Sigma^{\lambda(n)} XR \xrightarrow{x(n)} XR \longrightarrow X \tag{1}$$

where the element $x(n)$ has degree $\lambda(n) = 2^{2n+1} - 2^{n+2} + 1$ and is nilpotent with $x(n)^{2^{n+1}-1} = 0$. This is analogous to the Kitchloo-Wilson fibration [10, Thm. 1.7] relating the classical Johnson-Wilson theories $E(n)$ to their real counterparts. Using this fibration we can add to our list of faithful Galois extensions:

- $XR \to X$ with Galois group $\mathbb{Z}/2$ (Theorem 3.3) when X is a generalized Johnson-Wilson theory admitting a E_∞-ring structure:

The main results of this paper can be summarized as follows:

Theorem 1.1 1. *Let $f : A \to B$ be a map of E_∞-rings so that B is faithful and dualizable over A. Then the map f is of effective descent for modules (Sect. 2.6, Theorem 2.6). In particular, there is an equivalence of stable ∞-categories*

$$A\text{-}mod \simeq \varprojlim(B^{\wedge_A^{n+1}}\text{-}mod).$$

2. *Any faithful G-Galois extension $A \to B$ of E_∞-rings (G finite) is of effective descent for modules. In particular, there is an equivalence of stable ∞-categories (Sect. 2.8, Theorem 2.8)*

$$A\text{-}mod \simeq (B\text{-}mod)^{hG}.$$

Theorem 1.2 *Let X be a generalized Johnson-Wilson spectrum admitting an E_∞-ring structure and $X_{\mathbb{R}}$ the Real spectrum associated with it. Let XR denote the homotopy fixed points spectrum. Then the canonical map $XR \to X$ is a faithful $\mathbb{Z}/2$-Galois extension (Theorem 3.3). As a consequence, there is an equivalence of stable ∞-categories*

$$XR\text{-}mod \simeq (X\text{-}mod)^{h\mathbb{Z}/2}.$$

Remark 1.1 A Galois descent result similar to Theorem 1.1(2) has appeared in Meier's thesis ([16, Prop. 6.2.6]). The proof makes use of the main result of [19].

The only examples of generalized Johnson-Wilson spectra ($p = 2$) admitting E_∞ structures that are known of are $KU_{(2)}$ for $n = 1$ and $TMF_1(3)_{(2)}$ for $n = 2$ ([11, Thm.

1.1 and Prop. 8.3]). However, the $K(n)$ localized Johnson-Wilson spectra $E(n)^{\wedge}_{I_n} = L_{K(n)}E(n)$ all admit essentially unique E_∞-ring structures, by work of Baker-Richter ([3]).

1.1 Organization

The paper is organized into two parts.

In Sect. 2.1 we review the classical descent theory for a map of commutative rings. In Sect. 2.2 we recall the essential properties of small stable ∞-categories and in 2.3 we define what it means for a map of E_∞-rings to be of effective descent for modules, using the language of ∞-categories. In Sect. 2.4 we provide definitions of ∞-comonads and their ∞-category of comodules and state the Barr-Beck-Lurie criteria for comonadicity. In Sect. 2.5 we show that the ∞-category of descent data associated to a map of E_∞-rings is equivalent to a category of comodules over an ∞-comonad (Theorem 2.4). In Sect. 2.6 we apply the Barr-Beck-Lurie criteria to show that a map $A \to B$ of E_∞-rings is of effective descent for modules if the extension is faithful and B is dualizable over A. In Sect. 2.8 we show that if furthermore the map is Galois then category of descent data comes from a G-equivariant structure (Theorem 2.8).

In Sects. 3.1 and 3.2 we recall some facts about $\mathbb{Z}/2$-equivariant spectra and the construction of real-oriented spectra. In Sect. 3.3 we define real versions of generalized Johnson-Wilson spectra. In Sect. 3.4 we construct the fibration (1) and finally we show (in Theorem 3.3) that $XR \to X$ is a faithful Galois extension when X is a generalized Johnson-Wilson theory admitting a E_∞-ring structure. The proof depends on the computation of homotopy of $X_{\mathbb{R}}$ using the Borel spectral sequence. This is done in the Appendix.

2 Descent for modules

2.1 Classical descent theory

Let $f : A \to B$ be a map of commutative rings. The question of descent is the following: given a B-module N, what data on N determines an A-module M, together with an isomorphism of B-modules $M \otimes_A B \simeq N$?

Given a B-module N the descent data for N is necessary gluing data for the module to descend to a module over A. The descent data can be defined precisely in the following way. Let us begin with the following truncated semi-cosimplicial diagram.

$$B \underset{\phi_2}{\overset{\phi_1}{\rightrightarrows}} B \otimes_A B \overset{\phi_{23}, \phi_{12}, \phi_{13}}{\rightrightarrows\!\!\!\to} B \otimes_A B \otimes_A B \tag{2}$$

The maps are $\phi_1 = B \otimes 1$, $\phi_2 = 1 \otimes B$, $\phi_{12} = B \otimes B \otimes 1$, $\phi_{23} = 1 \otimes B \otimes B$ and $\phi_{13} = B \otimes 1 \otimes B$. The diagram is semi-cosimplicial in the sense the following equalities hold:

$$\begin{aligned} \phi_{12}\phi_1 &= \phi_{13}\phi_1 \\ \phi_{12}\phi_2 &= \phi_{23}\phi_1 \\ \phi_{13}\phi_2 &= \phi_{23}\phi_2 \end{aligned} \tag{3}$$

A descent datum for a B module N is the following:

- an isomorphism $\phi : \phi_1^* N \simeq \phi_2^* N$ of $B \otimes_A B$-modules,
- satisfying a cocycle condition: $\phi_{13}^* \phi = \phi_{23}^* \phi \circ \phi_{12}^* \phi$ in the category of $B \otimes_A B \otimes_A B$-modules.

The B-modules together with descent data form a category $Desc(f)$. If M is an A-module and $N = M \otimes_A B$ then there is an obvious descent datum from the isomorphisms $\phi : \phi_1^* N = B \otimes_A N \simeq B \otimes_A B \otimes_A M \simeq N \otimes_A B = \phi_2^* N$. There is a map

$$\mathrm{Mod}(A) \to Desc(f)$$

given by $M \mapsto (M \otimes_A B, \phi)$. The map $f : A \to B$ is said to have *effective descent* for modules when this is an equivalence of categories.

The category of descent data can be described as the limit of a truncated semi-cosimplicial category [induced by (2)] in the 2-category of categories.

$$Desc(f) = \varprojlim \left(\mathrm{Mod}(B) \rightrightarrows \mathrm{Mod}(B \otimes_A B) \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} \mathrm{Mod}(B \otimes_A B \otimes_A B) \right)$$

There is another description of the category of descent data as the category of comodules over a *comonad* on $\mathrm{Mod}(B)$. Recall (see [14]) that a comonad K on a category $\mathcal{D}$ is a coalgebra object in the functor category $\mathrm{End}(\mathcal{C})$ with respect to the composition monoidal structure. Let $\mathcal{D}_K$ denote the category of K-comodules in $\mathcal{D}$. The Eilenberg-Moore adjunction $(U_K \dashv F_K) : \mathcal{D}_K \to \mathcal{D}$ is the final representation of K; for any other representation of K via a pair of adjoint functors $(F \dashv G) : \mathcal{C} \rightleftarrows \mathcal{D}$ (so that $K = F \circ G$), there is a canonical map $\mathcal{C} \to \mathcal{D}_K$. The functor F is said to be *comonadic* if this is an equivalence of categories. The Barr-Beck theorem gives necessary and sufficient conditions for the comonadicity of F ([14, VI. 7]).

The map $f : A \to B$ induces a pair of adjoint functors

$$(f^* \dashv f_*) : \mathrm{Mod}(A) \rightleftarrows \mathrm{Mod}(B).$$

Here $f_* = - \otimes_A B$ and f_* is the forgetful function. Let $\mathrm{Mod}(B)_K$ be the Eilenberg-Moore category of comodules over the comonad $K = f_* f^*$ on $\mathrm{Mod}(B)$. There is a forgetful functor $U : Desc(f) \to \mathrm{Mod}(B)$ which forgets the descent data. The functor U admits a *right* adjoint F, $F(M) = (f_* f^*(M), \phi)$ and $F \circ U = K$. So

$$(U \dashv F) : Desc(f) \rightleftarrows \mathrm{Mod}(B)$$

is a presentation of the comonad K. Therefore by the universal property of the Eilenberg-Moore representation, there is a canonical map $Desc(f) \to \mathrm{Mod}(B)_K$.

Proposition 2.1 *The canonical map $Desc(f) \to Mod(B)_K$ is an equivalence of categories, i.e. U is comonadic.*

Proof From the square below

$$\begin{array}{ccc} \mathrm{Mod}(A) & \underset{f_*}{\overset{f^*}{\rightleftarrows}} & \mathrm{Mod}(B) \\ {\scriptstyle f_*}\uparrow\downarrow{\scriptstyle f^*} & & {\scriptstyle \phi_{1*},\phi_{2*}}\uparrow\downarrow{\scriptstyle \phi_1^*,\phi_2^*} \\ \mathrm{Mod}(B) & \underset{\phi_{1*},\phi_{2*}}{\overset{\phi_1^*,\phi_2^*}{\rightleftarrows}} & \mathrm{Mod}(B \otimes_A B) \end{array}$$

we have the following isomorphisms of comonads on Mod(B):

$$K = f_* f^* \simeq \phi_{1*}\phi_2^* \simeq \phi_{2*}\phi_1^* \tag{4}$$

A K-comodule structure on N is a given by a B-map $N \to N \otimes_A B = f_* f^* N \simeq \phi_{1*}\phi_2^* N \simeq \phi_{2*}\phi_1^* N$. So by adjunction, the comodule structure map gives the descent datum $\phi : \phi_1^* N \to \phi_2^* N$. □

The question of effective descent therefore is equivalent to the question of whether the map

$$f^* : \mathrm{Mod}(A) \to \mathrm{Mod}(B)$$

is comonadic, which may be verified by applying the theorem of Barr and Beck.

Theorem 2.1 (Grothendieck) *Let $f : A \to B$ be faithfully flat, then f^* is comonadic, hence f is of effective descent for modules.*

Our aim is to obtain a homotopical version of this theorem, where we can replace the abelian categories of modules with their derived categories. However it is known that one cannot glue objects in the derived category of a cover to obtain objects in the derived category of the base. What one can do instead is consider the more enriched stable ∞-category or dg-category versions of the derived categories. In this paper we will work with stable ∞-categories as they contain important examples from stable homotopy theory that we want our results to apply to.

2.2 Stable ∞-categories

By an ∞-category we shall mean an $(\infty, 1)$-category modelled using a *weak-Kan complex*. This theory was first developed by Boardman-Vogt and Joyal and later in great detail by Lurie in [12,13]. These will be our main references.

Definition 2.1 [12, *Def. 1.1.2.4*] A simplicial set K is an ∞-*category* if it satisfies the following condition: for any $0 < i < n$, any map $f_0 : \Lambda_i^n \to K$ admits (possibly non-unique) extension $f : \Delta^n \to K$. Here $\Lambda_i^n \subseteq \Delta$ denotes the ith horn, obtained from the simplex Δ^n by deleting the face opposite the ith vertex.

Let K be a simplicial set underlying an ∞-category $\mathcal{C}$. The objects of $\mathcal{C}$ are the elements of K_0, the 1-morphisms of $\mathcal{C}$ are the elements of K_1. The hom set $\mathrm{Maps}_{\mathcal{C}}(x, y)$ is a Kan complex. So every ∞-category has an underlying simplicial category.

A *functor* between ∞-categories is a map of simplicial sets. The functors between ∞-categories $\mathcal{C}$ and $\mathcal{D}$ assemble in an ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. We say a functor is an *equivalence of ∞-categories* when the map of the underlying simplicial categories is a Dwyer-Kan equivalence. The *homotopy category* of $\mathcal{C}$ is the homotopy category of the underlying simplicial category.

Definition 2.2 [12, *Chapter 3*] Let $\mathcal{C}\mathrm{at}_\infty^\Delta$ be the simplicial category whose objects are small ∞-categories. Given two ∞-categories $\mathcal{C}$ and $\mathcal{D}$ define the mapping space $\mathrm{Maps}_{\mathcal{C}\mathrm{at}_\infty^\Delta}(\mathcal{C}, \mathcal{D})$ to be the maximal Kan complex contained in the ∞-category of functors $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. The ∞-category $\mathcal{C}\mathrm{at}_\infty$ is defined to be the simplicial nerve $N(\mathcal{C}\mathrm{at}_\infty^\Delta)$.

The ∞-category $\mathcal{C}\mathrm{at}_\infty$ admits small limits ([12, Section 3.3.3]).

Definition 2.3 [13, *Definition 1.1.1.9*] An ∞-category $\mathcal{C}$ is *stable* if

1. There is a zero object $0 \in \mathcal{C}$
2. Every morphism in $\mathcal{C}$ has a fiber and a cofiber
3. A triangle in $\mathcal{C}$ is a fiber sequence if and only it is a cofiber sequence.

The homotopy category of a stable ∞-category is canonically triangulated ([13, 1.1.2]). So stable ∞-categories can be thought of as natural enrichments of triangulated categories.

Some examples of stable ∞-categories:

- For R an ordinary commutative ring, the ∞-category of unbounded chain complexes of modules over R is stable [13, 1.3.5], called the derived category $\mathcal{D}(R)$. Its homotopy category is the derived category $D(R)$ of the ring R.
- For X a scheme or an algebraic stack one can assign a stable ∞-category $QC(X)$ which has the usual unbounded quasicoherent derived category $D_{qc}(X)$ as the homotopy category (see [5]).
- For a E_∞-ring A, the ∞-category of A-modules A-mod is stable [13, 1.4]. When $A = S^0$, the homotopy category is the classical stable homotopy category. When $A = HR$ for a discrete ring R, the homotopy category is the derived category $D(R)$ [6, Thm. 4.2.4].

There is a good notion of homotopy limits in the ∞-category of stable ∞-categories. This is important for the descent formalism we develop later. We make precise statements here.

Given two stable ∞-categories $\mathcal{C}$ and $\mathcal{D}$, an exact functor between them is an ∞ functor that preserves 0 and fiber sequences. The identity functor is exact and composition of exacts functors is exact. This gives us the following definition.

Definition 2.4 The ∞-category $\mathcal{C}\mathrm{at}_\infty^{Ex}$ is the subcategory of $\mathcal{C}\mathrm{at}_\infty$ whose objects are small stable ∞-categories and morphisms are exact functors.

Theorem 2.2 ([13, Thm. 1.1.4.4]) *The ∞-category $\mathcal{C}at_\infty^{Ex}$ admits small limits and $\mathcal{C}at_\infty^{Ex} \subseteq \mathcal{C}at_\infty$ preserves small limits.*

2.3 Higher descent theory

Let (Sp, $\wedge$) denote the symmetric monoidal stable ∞-category of spectra. Denote by CAlg(Sp) the category of E_∞-rings. If A is an E_∞ ring, the category A-mod is symmetric monoidal by the relative smash product $\wedge_A$. Denote by CAlg(A-mod) the category of commutative A-algebras.

Given a map of $f : A \to B$ of E_∞-rings there is a map of stable ∞-categories

$$f^* : A\text{-mod} \to B\text{-mod}$$

which is defined on the 0-simplices as $f^*(M) = M \wedge_A B$. This extends (see [13]) to an ∞-functor

$$\text{QC} : \text{CAlg(Sp)} \to \mathcal{C}\text{at}_\infty^{Ex}.$$

Let $A^\bullet : N(\Delta) \to \text{CAlg(Sp)}$ be a cosimplicial object in E_∞-rings. The functor QC applied to $A^\bullet$ levelwise produces a cosimplicial stable ∞-category QC($A^\bullet$). Since the category $\mathcal{C}\text{at}_\infty^{Ex}$ closed under small limits, the totalization Tot(QC($A^\bullet$)) is a stable ∞-category.

Definition 2.5 [18, *Definition 8.2.1*] Let $f : A \to B$ be a map of E_∞-rings. The *Amitsur complex* associated with f is a cosimplicial commutative A-algebra,

$$C^\bullet(B/A) : N(\Delta) \to \text{CAlg}(A\text{-mod})$$

with $C^q(B/A) = B^{\wedge_A^{q+1}}$, coaugmented by $A \to B = C^0(B/A)$. The ith coface map, denoted by $\phi_{1\ldots\hat{i}\ldots q}$, is induced by smashing with $A \to B$ after the first i-copies of B and the jth codegeneracy map is induced by smashing with $B \wedge_A B \to B$ after the first j copies of B.

Definition 2.6 Given a map $f : A \to B$ of E_∞-rings, the ∞-category Tot($QC(C^\bullet$ $(B/A))$) is the *category of descent data* for f.

Remark 2.1 The following observation justifies Definition 2.6. The objects of Tot($QC(C^\bullet(B/A))$) are in one-one correspondence with commutative diagrams of the following form:

$$\begin{array}{ccccccc}
\Delta^0 & \xrightarrow{d^0,d^1} & \Delta^1 & \xrightarrow{d^0,d^1,d^2} & \Delta^2 & & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
B\text{-mod} & \xrightarrow{\phi_1^*,\phi_2^*} & B^{\wedge_A^2}\text{-mod} & \xrightarrow{\phi_{12}^*,\phi_{23}^*,\phi_{13}^*} & B^{\wedge_A^3}\text{-mod} & & \cdots
\end{array}$$

We can think of an object of Tot($QC(C^\bullet(B/A))$) informally as the following data:

- A B-module N
- An equivalence $\phi : \phi_1^* N \simeq \phi_2^* N$ in $B \wedge_A B$-mod

- A 2-simplex $\phi_{13}^*\phi \to \phi_{23}^*\phi \circ \phi_{12}^*\phi$ in $B \wedge_A B \wedge_A B$-mod
- ...

The k-simplices of $\text{Tot}(QC(C^\bullet(B/A)))$ are in one-one correspondence with commutative diagrams

$$\begin{array}{ccccccc} \Delta^0 \times \Delta^k & \xrightarrow{d^0,d^1} & \Delta^1 \times \Delta^k & \xrightarrow{d^0,d^1,d^2} & \Delta^2 \times \Delta^k & & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ B\text{-mod} & \xrightarrow{\phi_1^*,\phi_2^*} & B^{\wedge_A^2}\text{-mod} & \xrightarrow{\phi_{12}^*,\phi_{23}^*,\phi_{13}^*} & B^{\wedge_A^3}\text{-mod} & & \cdots \end{array}$$

Definition 2.7 A map $f : A \to B$ of E_∞-rings is of *effective descent for modules* when the map

$$\theta_f : A\text{-mod} \to \text{Tot}(QC(C^\bullet(B/A)))$$

induced from the coaugmentation $A \to B = C^0(B/A)$ is an equivalence of stable ∞-categories.

2.4 Comonads in ∞-categories

In this section we discuss the theory of comonads in the ∞-categorical setting and the ∞-categorical analog of the Barr-Beck theorem as developed by Lurie. The main reference for this is [13, section 4], in particular [13, 4.7]. Lurie develops the theory of monads, we need the dual version of comonads here. We provide definitions of ∞-comonads and their ∞-categories of comodules.

Definition 2.8 (*see* [13, *Definition 2.1.3.1, Definition 4.2.1.3*])

- Given a monoidal ∞-category $\mathcal{C}$, there is an ∞-*category of (associative) algebra objects in* $\mathcal{C}$ denoted by $\text{Alg}(\mathcal{C})$.
- Let $\mathcal{M}$ be an ∞-category left tensored over a monoidal ∞-category $\mathcal{C}$. Then there exists an ∞-*category of left module objects of* $\mathcal{M}$ denoted by $\text{LMod}(\mathcal{M})$ and a map

$$\text{LMod}(\mathcal{M}) \to \text{Alg}(\mathcal{C}).$$

  If $A \in \text{Alg}(\mathcal{C})$, then we let $\text{LMod}_A(\mathcal{M})$ denote the fiber $\text{LMod}(\mathcal{M}) \times_{\text{Alg}(\mathcal{C})} \{A\}$. We refer to $\text{LMod}_A(\mathcal{M})$ as the ∞-*category of left* A-*modules in* $\mathcal{M}$.

Definition 2.9 *(Coalgebras and comodules)*

- Let $\mathcal{C}$ be a monoidal ∞-category. Define $\text{CoAlg}(\mathcal{C})$ to be $\text{Alg}(\mathcal{C}^{op})^{op}$. we refer to this as the ∞-*category of (coassociative) coalgebra objects in* $\mathcal{C}$.

- Let $\mathcal{M}$ be an ∞-category left tensored over a monoidal ∞-category $\mathcal{C}$. Define $\mathrm{LComod}(\mathcal{M})$ to be $\mathrm{LMod}(\mathcal{M}^{op})^{op}$. We refer to this as the *∞-category of (left) comodule objects of $\mathcal{M}$*. There is a map of ∞-categories

$$\mathrm{LComod}(\mathcal{M}) \to \mathrm{CoAlg}(\mathcal{C}).$$

If $H \in \mathrm{CoAlg}(\mathcal{C})$, then we let $\mathrm{LComod}_H(\mathcal{M})$ denote the fiber $\mathrm{LComod}(\mathcal{M}) \times_{\mathrm{CoAlg}(\mathcal{C})} \{H\}$. We refer to $\mathrm{LComod}_H(\mathcal{M})$ as the *∞-category of left H-comodules in $\mathcal{M}$*. Alternately, $\mathrm{LComod}_H(\mathcal{M}) \simeq \mathrm{LMod}_H(\mathcal{M}^{op})^{op}$.

Definition 2.10 *(Comonads and comodules)* Given an ∞-category $\mathcal{D}$, the ∞-category of functors $\mathrm{Fun}(\mathcal{D}, \mathcal{D})$ is monoidal and $\mathcal{D}$ is left tensored over $\mathrm{Fun}(\mathcal{D}, \mathcal{D})$.

- A functor $K \in \mathrm{Fun}(\mathcal{D}, \mathcal{D})$ is a *comonad* if $K \in \mathrm{CoAlg}(\mathrm{Fun}(\mathcal{D}, \mathcal{D}))$.
- There is an ∞-category $\mathrm{LComod}_K(\mathcal{D})$ of comodules over a comonad K in $\mathcal{D}$.

There is a natural forgetful map $U_K : \mathrm{LComod}_K(\mathcal{D}) \to \mathcal{D}$.

Remark 2.2 Informally, a comonad K on an ∞-category $\mathcal{D}$ is an endofunctor $K : \mathcal{D} \to \mathcal{D}$ equipped with maps $K \to 1$ and $K \to K \circ K$ which satisfies the usual counit and co-associativity conditions up to coherent homotopy. A comodule over the comonad K is an object $x \in \mathcal{D}$ equipped with a structure map $\eta : x \to K(x)$ which is compatible with the coalgebra structure on K, again up to coherent homotopy. The forgetful map takes a comodule to the underlying object in $\mathcal{D}$.

Proposition 2.2 (see [13, Prop. 4.7.4.3]) *Given a functor $F : \mathcal{C} \to \mathcal{D}$ of ∞-categories which admits a right adjoint G. Then the composition $K = F \circ G \in Fun(\mathcal{D}, \mathcal{D})$ is a comonad on $\mathcal{D}$.*

There is a canonial map $F' : \mathcal{C} \to LComod_K(\mathcal{D})$ so that $F' \circ U_K \simeq F \in Fun(\mathcal{C}, \mathcal{D})$.

Remark 2.3 In ordinary categorical setting it is easy to check that the composition K is a comonad on $\mathcal{D}$. However, as Lurie notes in [13, Remark 4.7.0.4], this a not so straightforward in the ∞-categorical setting. In order to give a coalgebra structure on the composition $K = F \circ G \in \mathrm{Fun}(\mathcal{D}, \mathcal{D})$ it is not enough to give a produce a single natural transformation $K \to K \circ K$ but an infinite system of coherence data, which is not easy to describe explicitly.

Definition 2.11 Let $F : \mathcal{C} \to \mathcal{D}$ be a map of ∞-categories that admits a right adjoint G and let $K = F \circ G$ be the composition comonad on $\mathcal{D}$. Then F is said to be *comonadic* if the comparison map $F' : \mathcal{C} \to \mathrm{LComod}_K(\mathcal{D})$ is an equivalence of ∞-categories.

Theorem 2.3 (see [13, Thm. 4.7.4.5]) (∞-categorical Barr-Beck theorem) *Let $F : \mathcal{C} \to \mathcal{D}$ be an ∞-functor that admits a right adjoint G. Then F is comonadic if and only if F satisfies the following two conditions:*

(a) *F reflects equivalences*
(b) *Let U be a cosimplicial object in $\mathcal{C}$ which is F-split then U admits a limit in $\mathcal{C}$ and the limit is preserved by F*

2.5 Higher descent data as comodules over an ∞-comonad

Our aim is to give an ∞-categorical version of Proposition 2.1. In other words, given an E_∞-ring map $f : A \to B$ we want to express the category of descent data in Definition 2.6 as the category of comodules over a comonad on B-mod. (See [4] for a different perspective on this.)

Given a map $f : A \to B$ of E_∞ rings, the map $f^* : A\text{-mod} \to B\text{-mod}$ of stable ∞-categories admits a right adjoint f_*. The composition $K = f^* \circ f_*$, by Proposition 2.2, is a comonad on B-mod.

Theorem 2.4 *There is an equivalence of stable ∞-categories,*

$$Tot(QC(C^\bullet(B/A))) \simeq LComod_K(B\text{-}mod).$$

First we need a few results from Lurie ([13, Section 4.7.6]).

Proposition 2.3 (see [13, Prop. 4.7.6.1]) *Let $\mathcal{C}^\bullet$ be a cosimplicial ∞-category and $F : \varprojlim \mathcal{C}^\bullet \to \mathcal{C}^0$ be the natural projection map. If F admits a right adjoint then F satisfies the conditions of Theorem 2.3. i.e. there is a comonad K on $\mathcal{C}^0$ so that there is an equivalence $\varprojlim \mathcal{C}^\bullet \simeq LComod_K(\mathcal{C}^0)$.*

Definition 2.12 (*Right-adjointability*) ([13, *Definition 4.7.5.13*]) Given a diagram of ∞-categories σ:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{G} & \mathcal{D} \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle V} \\ \mathcal{C}' & \xrightarrow{G'} & \mathcal{D}' \end{array}$$

and a specified equivalence $\alpha : G' \circ U \simeq V \circ G$. We say σ is *right adjointable* if G and G' admit right adjoints H and H', and the composition transformation

$$U \circ H \to H \circ G' \circ U \circ H \to^{\alpha} H' \circ V \circ G \circ H \to H' \circ V$$

is an equivalence.

Theorem 2.5 ([13, Thm. 4.7.6.2]) *Let $\mathcal{C}^\bullet$ be a cosimplicial ∞-category. If for every $[m] \to [n]$ in Δ the diagram*

$$\begin{array}{ccc} \mathcal{C}^m & \xrightarrow{d^0} & \mathcal{C}^{m+1} \\ \downarrow & & \downarrow \\ \mathcal{C}^n & \xrightarrow{d^0} & \mathcal{C}^{n+1} \end{array}$$

is right adjointable (in particular, $d^0 : \mathcal{C}^n \to \mathcal{C}^{n+1}$ admits a right adjoint $H(0)$), then

1. *the forgetful functor $\varprojlim \mathcal{C}^\bullet \to \mathcal{C}^0$ admits a right adjoint*

2. *The square*

$$\begin{array}{ccc} \varprojlim(\mathcal{C}^\bullet) & \xrightarrow{U} & \mathcal{C}^0 \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle d^1} \\ \mathcal{C}^0 & \xrightarrow{d^0} & \mathcal{C}^1 \end{array}$$

is right adjointable and there is an equivalence $U \circ H \simeq H(0) \circ d^1 \in Fun(\mathcal{C}^0, \mathcal{C}^0)$.

Remark 2.4 The the right adjointabily of the square in consequence (2) of Theorem 2.5 above condition is similar to (4).

Proof (of Theorem 2.4) We need to show the projection map $p : \mathrm{Tot}(QC(C^\bullet(B/A))) \to B$-mod admits a right adjoint $p_!$ so that $p \circ p_! \simeq K$. Then the theorem can be proved by applying Proposition 2.3.

By Theorem 2.5, p admits a right adjoint if the following diagrams are right adjointable:

$$\begin{array}{ccc} B^{\wedge_A^{m}}\text{-mod} & \xrightarrow{d^0} & B^{\wedge_A^{m+1}}\text{-mod} \\ \downarrow & & \downarrow \\ B^{\wedge_A^{n}}\text{-mod} & \xrightarrow{d^0} & B^{\wedge_A^{n+1}}\text{-mod} \end{array}$$

where coface maps d^0 are basechange along the map $1 \wedge \cdots \wedge B : B^{\wedge_A^{q}} \to B^{\wedge_A^{q}}$. The maps d^0 admits right adjoints; forgetting the $B^{\wedge_A^{q}}$-module structure, and the right adjointability condition is satisfied.

Therefore, by Theorem 2.5(1), the map p has a right adjoint, and by Theorem 2.5(2), if $p_!$ is the right adjoint then $p \circ p_! \simeq H(0) \circ d^1$ as comonads over B-mod. It only remains to check that $H(0) \circ d_1 \simeq K = f^* \circ f_*$. This can be verified on objects, since for any B-module N, $(H(0) \circ d_1)(N) \simeq N \wedge_A B$ as a B-module. □

2.6 Faithfully dualizable descent

Definition 2.13 Let A and B be E_∞-rings.

1. An extension $A \to B$ is *faithful* if for any A-module M, $B \wedge_A M \simeq *$ implies $M \simeq *$.
2. Let M be a A-module and $D_A M = F_A(M, A)$ the functional dual of M. Then M is *dualizable* over A if the canonical map $\nu : D_A M \wedge_A M \to F_A(M, M)$ is an equivalence.

Lemma 2.1 ([18, Lemma3.3.2])

1. *If M is a dualizable over A and N any A-module then the canonical map $D_A M \wedge_A N \to F_A(M, N)$ is an equivalence.*
2. *If M is dualizable over A, then $D_A M$ is also dualizable and the canonical map $M \to D_A D_A M$ is an equivalence.*

Proposition 2.4 *Let $f : A \to B$ be a faithful map of E_∞-rings and B be dualizable over A then the functor*

$$f^* : A\text{-}mod \to B\text{-}mod$$

is comonadic.

Proof Condition (i) of Theorem 2.3 is satisfied since $A \to B$ is faithful. As for condition (ii), suppose $U^\bullet$ is a cosimplicial object in A-mod, then $\mathrm{Tot}\, U^\bullet \in A$-mod. To show that f^* preserves totalization we need B is dualizable over A. To see this we note the following.

$$\begin{aligned} f^*\mathrm{Tot}(U^\bullet) &= B \wedge_A \mathrm{Tot}\, U^\bullet \\ &\simeq F_A(D_A B, \mathrm{Tot}\, U^\bullet) \\ &\simeq \mathrm{Tot}\, F_A(D_A B, U^\bullet) \\ &= \mathrm{Tot}(B \wedge_A U^\bullet) \end{aligned}$$

The equivalences follow from Lemma 2.1. □

As a corollary we get faithfully dualizable descent for modules:

Theorem 2.6 *Let $f : A \to B$ be a faithful map of E_∞-rings and B be dualizable over A. Then $A \to B$ is of effective descent for modules.*

Proof Proposition 2.4 and Theorem 2.4. □

2.7 Galois extensions of E_∞-ring spectra

Definition 2.14 Let $\mathcal{C}$ be an ∞-category and let X an object in $\mathcal{C}$ with an action by a finite group G, given by a map $X : BG \to \mathcal{C}$ of ∞-categories. Then the associated *group cobar complex* is a cosimplicial object

$$C^\bullet(G; X) : N(\Delta) \to \mathcal{C}$$

where $C^q(G; X) = \Pi_{G^q} X$ and the coface and codegeneracies are induced by the structure maps of BG. The limit $\mathrm{Tot}(C^\bullet(G; X)) = X^{hG}$.

Remark 2.5 If B is a commutative A-algebra for an E_∞-ring A and G acts on B by A-algebra maps, then the cobar complex of Definition 2.14 is a cosimplicial object

$$C^\bullet(G; B) : N(\Delta) \to \mathrm{CAlg}(A\text{-mod})$$

and is coaugmented by $A \to B = C^0(B; G)$. This induces a map $A \to \mathrm{Tot}(C^\bullet(B; G)) = B^{hG}$.

Definition 2.15 (*Rognes* [18]) Let $f : A \to B$ be a map of E_∞-rings and G a finite group acting on B through A-algebra maps. Then f is a G-Galois extension if the canonical maps

(i) $i : A \to B^{hG}$
(ii) $h : B \wedge_A B \to \Pi_G B$ (informally given by $(b_1 \wedge b_2) \mapsto \{b_1 \wedge g(b_2)\}_{g \in G}$)

are equivalences.

Definition 2.16 Let $A \to B$ be a map of E_∞-rings and let G act on B by A-algebra maps. Then there is a map of cosimplicial A-algebras

$$h^\bullet : C^\bullet(B/A) \to C^\bullet(G; B)$$

given in codegree q by the map $h^q : B^{\wedge_A^{q+1}} \to \Pi_{G^q} B$ given symbolically by

$$b_0 \wedge \cdots \wedge b_q \mapsto \{b_0 \wedge g_1(b_1) \wedge \cdots \wedge g_q(b_q)\}_{(g_1,\ldots,g_q) \in G^q}.$$

Lemma 2.2 ([18, Lemma 8.2.7]) *If $A \to B$ is a G-Galois extension then the map $h^\bullet$ is a codegreewise equivalence.*

Proposition 2.5 ([18, 6.2.1]) *If $f : A \to B$ is a Galois extension then B is dualizable over A.*

2.8 Higher Galois descent

Let $f : A \to B$ be a Galois extension of ordinary commutative rings. The truncated semi-cosimplicial ring of (2) is isomorphic to the following truncated semi-cosimplicial ring,

$$B \underset{\phi_2}{\overset{\phi_1}{\rightrightarrows}} \Pi_G B \overset{\phi_{23},\ \phi_{12},\ \phi_{13}}{\rightrightarrows\!\!\rightarrow} \Pi_{G\times G} B$$

where $\phi_1(x) = (g \mapsto gx)_{g\in G}$ and $\phi_2(x) = (g \mapsto x)_{g\in G}$. The category of descent data

$$Desc(f) = \varprojlim \left(\mathrm{Mod}(B) \underset{\phi_2^*}{\overset{\phi_1^*}{\rightrightarrows}} \Pi_G \mathrm{Mod}(B) \overset{\phi_{23}^*,\ \phi_{12}^*,\ \phi_{13}^*}{\rightrightarrows\!\!\rightarrow} \Pi_{G\times G} \mathrm{Mod}(B) \right)$$

can be denoted by $\mathrm{Mod}(B)^G$. Given a B module N, a descent datum is equivalent to a semi-linear G-action on N. In this setting, classical Galois descent can be stated as follows.

Theorem 2.7 (Galois descent) *Let $A \to B$ be a G-Galois extension of commutative rings, then there is an equivalence of categories*

$$Mod(A) \simeq Mod(B)^G.$$

Definition 2.17 Let $\mathcal{C}$ be an ∞-category with an action of a group G, given by a map $BG \to \mathcal{C}\mathrm{at}_\infty$. The group cobar complex is a cosimplicial ∞-category $C^\bullet(G; \mathcal{C})$. We can consider the limit

$$\mathcal{C}^{hG} = \mathrm{Tot}(C^\bullet(G; \mathcal{C})).$$

An object X of $\mathcal{C}$ will be called a *G-equivariant object of $\mathcal{C}$* if X is an object of $\mathcal{C}^{hG}$.

Remark 2.6 Informally, the objects of $\mathcal{C}^{hG}$ consist of the following data:

- An object $X \in \mathcal{C}$
- An equivalence $\phi_{g,X} : g.X \to X$ for all $g \in G$
- A 2-simplex $$\begin{array}{ccc} & X & \\ {}^{\phi_{g_1,X}}\nearrow & & \nwarrow^{\phi_{g_1 g_2,X}} \\ g_1.X & \xleftarrow[g_2, g_1.X]{} & g_2 g_1.X \end{array}$$ for all $(g_1, g_2) \in G^2$
- ...

Definition 2.18 Let B be an E_∞-ring with an action of a group G. Then the ∞-category B-mod has an action of G. On objects it is given as follows. Let N be a B-module with structure map $\alpha : B \wedge N \to N$. Then define $g.N$ to be the spectrum N with a B-module structure by the map $B \wedge N \xrightarrow{g \wedge 1} B \wedge N \xrightarrow{\alpha} N$. We then refer to objects of the stable ∞-category $(B\text{-mod})^{hG}$ as *G-equivariant B-modules.*

Proposition 2.6 *Let $f : A \to B$ be G-Galois extension of E_∞-rings. Then there is an equivalence of stable ∞-categories*

$$Desc(f) \simeq (B\text{-}mod)^{hG}.$$

Proof There is an equivalence

$$\mathrm{Tot}(QC(C^\bullet(G; B)) \simeq \mathrm{Tot}(C^\bullet(G; B\text{-mod}))$$

since there is a codegreewise equivalance $(\Pi_{G^q} B)$-mod $\simeq \Pi_{G^q}(B\text{-mod})$. Also, by Lemma 2.2 the map

$$\mathrm{Tot}(QC(h^\bullet)) : \mathrm{Tot}(QC(C^\bullet(B/A)) \to \mathrm{Tot}(QC(C^\bullet(G; B)))$$

induced by $h^\bullet$ is an equivalence of stable ∞-categories. □

We can state the Galois descent theorem for a faithful Galois extension of E_∞-rings.

Theorem 2.8 (Galois descent for E_∞-rings) *Let $f : A \to B$ be a faithful G-Galois extension of E_∞-rings. Then the map*

$$A\text{-}mod \to (B\text{-}mod)^{hG}$$

given by $N \mapsto Tot(QC(h^\bullet))(\theta_f(N))$ is an equivalence of stable ∞-categories of the category of A-modules and the category of G-equivariant B-modules.

Proof Apply Theorem 2.6, Propositions 2.5 and 2.6. □

3 Generalized Real Johnson-Wilson theories

3.1 $\mathbb{Z}/2$-equivariant spectra

Let α denote the one-dimensional sign representation of $\mathbb{Z}/2$ and $\mathcal{U} = \mathbb{R}^\infty \oplus \mathbb{R}^{\infty\alpha}$ be a complete $\mathbb{Z}/2$-universe. An indexing space is a finite-dimensional subrepresentation of $\mathcal{U}$, therefore of the form $V = m + n\alpha$. If $V \subset W$, let $W - V$ be the orthogonal complement of V in W.

A $\mathbb{Z}/2$-spectrum $X_{\mathbb{Z}/2}$ is a collection of pointed $\mathbb{Z}/2$-spaces $X_{\mathbb{Z}/2}(V)$ and a system of $\mathbb{Z}/2$-homeomorphisms

$$X_{\mathbb{Z}/2}(V) \simeq \Omega^{W-V} X_{\mathbb{Z}/2}(V).$$

Since any indexing space is contained in $n(1+\alpha)$ for a large enough n, we have the following simple description of a $\mathbb{Z}/2$-spectrum.

Lemma 3.1 *A $\mathbb{Z}/2$-spectrum $X_{\mathbb{Z}/2}$ is a collection of pointed $\mathbb{Z}/2$-equivariant spaces X_n with equivariant structure maps,*

$$S^{1+\alpha} \wedge X_n \to X_{n+1}.$$

Definition 3.1 ([9]) The Borel, co-Borel, geometric and Tate spectra associated with $X_{\mathbb{Z}/2}$ are defined respectively as follows:

$$\begin{aligned} c(X_{\mathbb{Z}/2}) &= F(E\mathbb{Z}/2_+, X_{\mathbb{Z}/2}) \\ f(X_{\mathbb{Z}/2}) &= E\mathbb{Z}/2_+ \wedge X_{\mathbb{Z}/2} \\ g(X_{\mathbb{Z}/2}) &= X_{\mathbb{Z}/2} \wedge \tilde{E}\mathbb{Z}/2 \\ t(X_{\mathbb{Z}/2}) &= F(E\mathbb{Z}/2_+, X_{\mathbb{Z}/2}) \wedge \tilde{E}\mathbb{Z}/2 \end{aligned}$$

where $\tilde{E}\mathbb{Z}/2$ is the unreduced suspension of $E\mathbb{Z}/2$.

The Tate diagram is a commutative diagram of $\mathbb{Z}/2$-spectra.

$$\begin{array}{ccccc} f(X_{\mathbb{Z}/2}) & \longrightarrow & X_{\mathbb{Z}/2} & \longrightarrow & g(X_{\mathbb{Z}/2}) \\ & \searrow & \downarrow & & \downarrow \\ & & c(X_{\mathbb{Z}/2}) & \longrightarrow & t(X_{\mathbb{Z}/2}) \end{array} \tag{5}$$

The top and bottom rows are fibrations of $\mathbb{Z}/2$-spectra.

There are spectral sequences (Borel, co-Borel and Tate resp.)

$$\begin{aligned} H^p(\mathbb{Z}/2; X_{\mathbb{Z}/2}{}^q) &\Rightarrow c(X_{\mathbb{Z}/2})_\star \\ H_p(\mathbb{Z}/2; X_{\mathbb{Z}/2}{}^q) &\Rightarrow f(X_{\mathbb{Z}/2})_\star \\ \widehat{H}^p(\mathbb{Z}/2; X_{\mathbb{Z}/2}{}^q) &\Rightarrow t(X_{\mathbb{Z}/2})_\star \end{aligned} \tag{6}$$

where $q \in RO(\mathbb{Z}/2)$ and $p \geq 0$, $p \geq 0$ and $p \in \mathbb{Z}$ respectively.

The *homotopy fixed points* $X^{h\mathbb{Z}/2}$ of $X_{\mathbb{Z}/2}$ is the ordinary fixed points of the Borel spectrum. The standard inclusion map $\mathbb{Z}/2_+ \subset E\mathbb{Z}/2_+$ induces a map of non-equivariant spectra.

$$X^{h\mathbb{Z}/2} \to X = F(\mathbb{Z}/2_+, X_{\mathbb{Z}/2})^{\mathbb{Z}/2}$$

3.2 Real-oriented cohomology theories

In this section we recall the construction of real-oriented spectra from [8]. Let $MU(n)$ denote the Thom space of the universal bundle γ_n over $BU(n)$. Complex conjugation induces an action of $\mathbb{Z}/2$ on $MU(n)$. The canonical Real bundle γ_n of dimension n over $BU(n)$ gives equivariant maps between Thom spaces, $\Sigma^{1+\alpha} BU(n)^{\gamma_n} \to BU(n+1)^{\gamma_{n+1}}$. The resulting genuine $\mathbb{Z}/2$-spectrum is denoted $MU_\mathbb{R}$. The spectrum $MU_\mathbb{R}$ is an E_∞ ring spectrum. The underlying non-equivariant spectrum of $MU_\mathbb{R}$ is MU.

Definition 3.2 Let $B\mathbb{S}^1$ be the classifying space of $\mathbb{S}^1$ considered as $\mathbb{Z}/2$ equivariant space via the inclusion $\mathbb{S}^1 \subset \mathbb{C}^*$. We have $\Omega B\mathbb{S}^1 \simeq \mathbb{S}^1$ in the category of based $\mathbb{Z}/2$-spaces. Therefore by adjunction we have a canonical equivariant based map $\eta : S^{1+\alpha} \to B\mathbb{S}^1$. Let $X_{\mathbb{Z}/2}$ be a homotopy commutative and associative $\mathbb{Z}/2$-ring spectrum. A *Real orientation* of $X_{\mathbb{Z}/2}$ is a cohomology class $u : B\mathbb{S}^1 \to \Sigma^{1+\alpha} X_{\mathbb{Z}/2}$ such that $\eta^* u = 1$ in $\pi_0 X_{\mathbb{Z}/2}$.

The spectrum $MU_\mathbb{R}$ is real-oriented, hence it supports a formal group law. The forgetful map $MU_{\mathbb{R}\star} \to MU_\star$ is split by the map of rings $MU_* \to MU_{\mathbb{R}\star}$ classifying this formal group law, where the image of $x_i \in MU_{2i}$ is in degree $i(1+\alpha)$. Here $\star$ denotes the bigraded coefficient ring of any $\mathbb{Z}/2$-spectrum.

3.3 $BP\langle n, \mathbf{u}\rangle_{\mathbb{R}}$ and $E(n; \mathbf{u})_{\mathbb{R}}$

Working 2-locally there is a Real analogue of the Quillen idempotent which produces a $\mathbb{Z}/2$-equivariant spectrum $BP_{\mathbb{R}}$ such that $(MU_{\mathbb{R}})_{(2)}$ splits as a wedge of suspensions of $BP_{\mathbb{R}}$ ([8, Thm. 2.33]).

Using the map $MU_* \to MU_{\mathbb{R}\star}$ we may identify classes $v_n \in \pi_{(2^n-1)(1+\alpha)}MU_{\mathbb{R}}$. Using these elements some constructions of complex-oriented spectra can be mimicked to give real versions. Consider the MU-module spectra $E(n)$ and $BP\langle n\rangle$ with coefficient rings:

$$BP\langle n\rangle_* = \mathbb{Z}_{(2)}[v_1, \ldots, v_n]$$
$$E(n)_* = \mathbb{Z}_{(2)}[v_1, \ldots, v_{n-1}, v_n^{\pm 1}]$$

We can mod out by the lifts $v_{n+1}, v_{n+2}, \ldots \in \pi_\star BP_{\mathbb{R}}$ to construct Real truncated BP theory $BP\langle n\rangle_{\mathbb{R}}$. We can also invert v_n to construct Real Johnson Wilson theory $E(n)_{\mathbb{R}}$ (see [8, Sec. 3]).

Definition 3.3 Let $\mathbf{u} = u_0, u_1, \ldots, u_n, \ldots$ be a regular sequence in BP_* such that $|u_i| = 2(2^i - 1)$ and $(2, u_1, u_2, \ldots, u_{n-1}) = I_n$. There are commutative ring spectra (see [2]) $BP\langle n; \mathbf{u}\rangle$ and $E(n; \mathbf{u})$ such that

$$BP\langle n; \mathbf{u}\rangle_* = BP_*/(u_i : i \geq n+1)$$

and

$$E(n; \mathbf{u})_* = BP_*/(u_i : i \geq n+1)[u_n^{-1}].$$

We refer to $E(n; \mathbf{u})$ as a *generalized Johnson-Wilson spectrum.*

Remark 3.1 If we set $\mathbf{u} = \mathbf{v}$, the Hazewinkel generators of BP_*, then we recover the standard Johnson-Wilson spectra $BP\langle n\rangle$ and $E(n)$. The rings $\pi_* E(n; \mathbf{u})$ are isomorphic for different choices of $\mathbf{u}$, but support different formal group laws and therefore are non-equivalent as complex-oriented spectra.

Definition 3.4 We can identify classes $u_i \in \pi_{(2^i-1)(1+\alpha)}BP_{\mathbb{R}}$ which are lifts of $u_i \in BP_{(2^i-1)2}$. Going modulo the appropriate classes in $\pi_\star BP_{\mathbb{R}}$ we can define real-oriented spectra $BP\langle n; \mathbf{u}\rangle_{\mathbb{R}}$ and $E(n; \mathbf{u})_{\mathbb{R}}$. We refer to $E(n; \mathbf{u})_{\mathbb{R}}$ as a *generalized Real Johnson-Wilson spectrum.*

3.4 Fibrations related to $E(n; \mathbf{u})_{\mathbb{R}}$

Theorem 3.1 *Let X be a generalized Johnson-Wilson spectrum $E(n; \mathbf{u})$ and $X_{\mathbb{R}} = E(n; \mathbf{u})_{\mathbb{R}}$ the associated real-oriented spectrum.*

The following are true.

(i) *There is an invertible element $y(n) \in c(X_{\mathbb{R}})_\star$ in degree $\lambda(n) + \alpha$, where $\lambda(n) = 2^{2n+1} - 2^{n+2} + 1$.*

(ii) *The Euler class $a \in c(X_{\mathbb{R}})_\star$ in degree α, coming from $a : S^0 \to S^\alpha$, is nilpotent.*
(iii) *The Tate spectrum $t(X_{\mathbb{R}})$ is trivial.*

Proof Using Cor. 3.1 define the invertible element $y(n) = u_n^{2^n-1}\sigma^{-2^{n+1}(2^{n-1}-1)} \in c(X_{\mathbb{R}})_\star$. Since u_n is invertible in $c(X_{\mathbb{R}})_\star$ it follows from the differential of equation (12) that $a^{2^{n+1}-1} = 0$ in $c(X_{\mathbb{R}})_\star$.

Recall that $t(X_{\mathbb{Z}/2})$ is the localization of $c(X_{\mathbb{Z}/2})$ away from a ([9, 16.3]). In the case of $X_{\mathbb{R}}$ there is a commutative square of $\mathbb{Z}/2$-equivariant ring spectra:

$$\begin{array}{ccccc} X_{\mathbb{R}} & \longrightarrow & g(X_{\mathbb{R}}) & & = X_{\mathbb{R}}[a^{-1}] \\ \downarrow & & \downarrow & & \\ c(X_{\mathbb{R}}) & \longrightarrow & t(X_{\mathbb{R}}) & & = c(X_{\mathbb{R}})[a^{-1}] \end{array}$$

The element $a : S^0 \subset S^\alpha$ acts

(a) nilpotently on $c(X_{\mathbb{R}})_\star$.
(b) invertibly on $g(X_{\mathbb{R}})_\star$ and $t(X_{\mathbb{R}})_\star$.

The Tate spectrum $t(X_{\mathbb{R}})$ is the localization of $c(X_{\mathbb{R}})$ away from a, so on $t(X_{\mathbb{R}})_\star$, a acts invertibly as well as nilpotently. Therefore $t(X_{\mathbb{R}})$ is equivariantly contractible. □

Theorem 3.2 *There is a fibration of XR-algebras,*

$$\Sigma^{\lambda(n)} XR \xrightarrow{x(n)} XR \longrightarrow X \tag{7}$$

where $XR := X_{\mathbb{R}}^{h\mathbb{Z}/2}$ The element $x(n)$ has degree $\lambda(n) = 2^{2n+1} - 2^{n+2} + 1$ and is nilpotent with $x(n)^{2^{n+1}-1} = 0$.

Example 3.1 (i) When $X = KU$, this is the fibration

$$\Sigma^1 KO \xrightarrow{\eta} KO \xrightarrow{c} KU$$

of KO-module spectra. The map c is complexification and the fiber can be identified with ΣKO using the equivalence $KU \simeq KO \wedge C_\eta$ of Reg Wood. Here η is the stable Hopf map $\eta : S^1 \to S^0$ and $\eta^3 = 0$.

(ii) The spectrum of $TMF_1(3)$ of topological modular forms with $\Gamma_1(3)$ level structures is an example of X when $n = 2$. There is a $\mathbb{Z}/2 = \Gamma_0(3)/\Gamma_1(3)$ action on $TMF_1(3)$ coming from the level 3 structures. In a recent preprint ([7]) Hill and Meier has shown that $TMF_1(3)$ with this $\mathbb{Z}/2$-action is real-oriented, so that the $\mathbb{Z}/2$-homotopy fixed points of $TMF_1(3)_{\mathbb{R}}$ is $TMF_0(3)$. The fibration of Theorem 3.2 then is the Maholwald-Rezk fibration ([15, Remark 4.2]),

$$\Sigma^{17} TMF_0(3) \xrightarrow{x} TMF_0(3) \longrightarrow TMF_1(3)$$

of $TMF_0(3)$-module spectra. The element $x \in \Sigma^{17} TMF_0(3)$ is nilpotent with $x^7 = 0$ (see [15, Prop 4.1]).

(iii) More generally, with $X = E(n)$ the nth Johnson-Wilson spectrum, Kitchloo and Wilson ([10]) have produced fibrations

$$\Sigma^{\lambda(n)} ER(n) \to ER(n) \to E(n).$$

Proof From Theorem 3.1 we have the following equivalence as part of the Tate diagram.

$$\begin{array}{ccccc} f(X_{\mathbb{R}}) & \longrightarrow & X_{\mathbb{R}} & \longrightarrow & g(X_{\mathbb{R}}) \\ & \searrow^{\simeq} & \downarrow & & \\ & & c(X_{\mathbb{R}}) & & \end{array}$$

This implies a splitting of $\mathbb{Z}/2$- ring spectra

$$X_{\mathbb{R}} \simeq c(X_{\mathbb{R}}) \vee g(X_{\mathbb{R}}).$$

There is a fibration $\mathbb{Z}/2_+ \longrightarrow S^0 \xrightarrow{a} S^\alpha$ inducing a fibration

$$\begin{array}{ccccc} F(S^\alpha, g(X_{\mathbb{R}})) & \xrightarrow{a^*} & F(S^0, g(X_{\mathbb{R}})) & \longrightarrow & F(\mathbb{Z}/2_+, g(X_{\mathbb{R}})) \\ \downarrow \simeq & & \downarrow \simeq & & \\ \Sigma^{-\alpha} g(X_{\mathbb{R}}) & \underset{\simeq}{\xrightarrow{a}} & g(X_{\mathbb{R}}) & & \end{array}$$

This equivalence induced by a implies that $F(\mathbb{Z}/2_+, g(X_{\mathbb{R}}))$ is trivial. We have the analogous fibration:

$$\begin{array}{ccccc} \Sigma^{-\alpha} c(X_{\mathbb{R}}) & \xrightarrow{a} & c(X_{\mathbb{R}}) & \longrightarrow & F(\mathbb{Z}/2_+, c(X_{\mathbb{R}})) \\ y(n) \uparrow \simeq & \nearrow_{x(n)} & & & \downarrow = \\ \Sigma^{\lambda(n)} c(X_{\mathbb{R}}) & & & & F(\mathbb{Z}/2_+, X_{\mathbb{R}}) \end{array}$$

The desired fibration is obtained by taking the ordinary fixed points of the bottom fibration. □

Theorem 3.3 *Let X be a generalized Johnson-Wilson-spectrum having a E_∞-ring structure. Then the extension $XR \to X$ is a faithful $\mathbb{Z}/2$-Galois extension.*

Proof The proof proceeds along the lines of [18, Prop. 5.3.1]. The main ingredient is the fibration of Proposition 3.2. By definition, $XR = c(X_{\mathbb{R}})^{\mathbb{Z}/2}$ and the group $\mathbb{Z}/2$

acts through XR-algebra maps. Therefore in order to show $XR \to X$ is Galois it only remains to show that the map

$$h : X \wedge_{XR} X \to \Pi_{\mathbb{Z}/2} X$$

is an equivalence. Consider the part of the cofibration (7) induced by fibration

$$S^{\alpha-1} \to \mathbb{Z}/2_+ \to S^0. \tag{8}$$

$$\begin{array}{ccccc}
XR & \longrightarrow & X & \longrightarrow & \Sigma^{\lambda(n)+1} XR \\
\| & & \| & & \| \\
c(X_{\mathbb{R}})^{\mathbb{Z}/2} & \longrightarrow & c(F(\mathbb{Z}/2_+, X_{\mathbb{R}}))^{\mathbb{Z}/2} & \longrightarrow & c(\Sigma^{\alpha-1} X_{\mathbb{R}})^{\mathbb{Z}/2}
\end{array}$$

The homotopy fixed points inclusion

$$c(X_G)^G \to X$$

then produces a commutative diagram.

$$\begin{array}{ccccc}
XR & \longrightarrow & X & \longrightarrow & \Sigma^{\lambda(n)+1} XR \\
\downarrow & & \downarrow & & \downarrow \\
X & \xrightarrow[\Delta]{} & F(\mathbb{Z}/2_+, X) & \xrightarrow[\delta]{} & X
\end{array}$$

This is a map of cofibrations and the bottom row is induced by the non-equivariant version of the cofibration (8) on X. The map Δ is the trivial $\mathbb{Z}/2$-Galois extension over X (Δ is the diagonal inclusion) and the homotopy cofiber δ can be identified with X.

This is a diagram of XR-algebras. Inducing the upper row along $XR \to X$, gives the following map between cofiber sequences. Furthermore, by adjunction, this is a commutative diagram of X-algebras.

$$\begin{array}{ccccc}
X \wedge_{XR} XR & \longrightarrow & X \wedge_{XR} X & \longrightarrow & X \wedge_{XR} \Sigma^{\lambda(n)+1} XR \\
\simeq\downarrow & & \downarrow & & \downarrow\simeq \\
X & \xrightarrow[\Delta]{} & \Pi_{\mathbb{Z}/2} X & \xrightarrow[\delta]{} & X
\end{array}$$

The right hand column is an equivalence since $X \wedge_{XR} \Sigma^{\lambda(n)+1} XR \simeq \Sigma^{\lambda(n)+1} X \simeq X$. This follows from the fact that X is $|v_n| = 2(2^n - 1)$ periodic and $2(2^n - 1)(2^n - 1) = \lambda(n) + 1$. Therefore h is an equivalence.

Finally, $XR \to X$ is faithful. If N is a XR-module then applying to the cofibration (7) we get a cofibration

$$\Sigma^{\lambda(n)} N \xrightarrow{x(n)} N \longrightarrow N \wedge_{XR} X.$$

If we assume $N \wedge_{XR} X \simeq *$, then $x(n) : \Sigma^{\lambda(n)} N \to N$ is an equivalence. But $x(n)$ is nilpotent. Therefore $N \simeq *$. □

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Appendix: Borel spectral sequence for $E(n; \mathbf{u})_{\mathbb{R}}$

In this section we compute the coefficient ring of the Borel spectrum of $E(n; \mathbf{u})_{\mathbb{R}}$. Hu an Kriz have computed the Borel spectral sequence for $BP_{\mathbb{R}}$ and $E(n)_{\mathbb{R}}$. Here we recall their results.

Definition 3.5 Let $\mathbf{u} = u_0, u_1, \ldots, u_n, \ldots$ be a regular sequence in BP_* such that $|u_i| = 2(p^i - 1)$ and $(p, u_1, u_2, \ldots, u_{n-1}) = I_n$. Define the following ring.

$$E^c_\infty((BP; \mathbf{u})_{\mathbb{R}}) = \mathbb{Z}_{(2)}[u_n \sigma^{l2^{n+1}}, \quad a|l \in \mathbb{Z}, n \geq 0]/I(\mathbf{u})$$

The ideal $I(\mathbf{u})$ is generated by the following relations,

$$\begin{aligned} u_0 &= 2 \\ \left(u_n \sigma^{l2^{n+1}}\right) a^{2^{n+1}-1} &= 0 \\ \left(u_m \sigma^{k2^{m+1}}\right)\left(u_n \sigma^{l2^{m-n}2^{n+1}}\right) &= u_n u_m \sigma^{(k+l)2^{m+1}} \end{aligned} \tag{9}$$

where the $u_n \sigma^{l2^{n+1}}$ has bidegree $(2^n - 1)(1 + \alpha) + l2^{n+1}(\alpha - 1)$ and a has bidegree $-\alpha$.

Theorem 3.4 (Hu, Kriz [8]) *The Borel spectral sequence for $BP_{\mathbb{R}}$ is*

$$H^*(\mathbb{Z}/2, BP_*[\sigma^{\pm}]) \Rightarrow \pi_\star c(BP_{\mathbb{R}})$$

where $\sigma \in \pi_{1-\alpha} F(\mathbb{Z}/2_+, BP_{\mathbb{R}})$ coming from the homeomorphism $S^1 \wedge \mathbb{Z}/2_+ \simeq S^\alpha \wedge \mathbb{Z}/2_+$. The differentials are

$$d_{2^{k+1}-1}(\sigma^{-2^k}) = v_k a^{2^{k+1}-1} \tag{10}$$

where $a \in \pi_\alpha BP_{\mathbb{R}}$ comes from the embedding $S^0 \subset S^\alpha$.

The E_∞-page is $E_\infty^c((BP;\mathbf{v})_\mathbb{R})$. Furthermore, there are no extension problems. Therefore,

$$c(BP_\mathbb{R})_\star \simeq E_\infty^c((BP;\mathbf{v})_\mathbb{R}).$$

In the following proposition we show that running the Borel spectral sequence with different generators $\mathbf{u}$ for BP_* gives similar results.

Proposition 3.1 *For any* $\mathbf{u}$ *in Definition 3.5,*

$$c(BP_\mathbb{R})_\star \simeq E_\infty^c((BP;\mathbf{u})_\mathbb{R}).$$

Proof We compute the Borel spectral sequence for $BP_\mathbb{R}$ starting with the generators $\mathbf{u}$ of BP_*.

Since $(2, u_1, \dots, u_k) = (2, v_1, \dots, v_k)$, $u_k = \alpha_k v_k \mod I_n$ where α_k is a unit. In the $E_{2^{k+1}-1}$-page

$$\begin{aligned} u_k a^{2^k-1} &= \left(\alpha_k v_k + \Sigma_0^{k-1} x_i v_i\right) a^{2^k-1} \\ &= \alpha_k v_k a^{2^k-1} \end{aligned} \tag{11}$$

since the other terms are zero. Therefore we can rewrite the differentials as

$$d_{2^{k+1}-1}(\sigma^{-2^k}) = v_k a^{2^k-1} a^{2^{k+1}-2^k} = \alpha_k^{-1} u_k a^{2^{k+1}-1}. \tag{12}$$

Since the differentials don't change under the new generators (up to a unit), the E_∞-page is $E_\infty^c((BP;\mathbf{u})_\mathbb{R})$.

Next, we have to show there are no extension problems. For this it is enough to show that $E_\infty^c((BP;\mathbf{u})_\mathbb{R})$ and $E_\infty^c((BP;\mathbf{v})_\mathbb{R})$ have the same multiplicative structure.

We have to show $I(\mathbf{u}) = I(\mathbf{v})$. We have $u_0 = v_0 = 2$. To show,

$$0 = v_n \sigma^{l2^{n+1}} a^{2^{n+1}-1} = u_n \sigma^{l2^{n+1}} a^{2^{n+1}-1}.$$

Proceed by induction on n. For $n = 1$, $u_1 = v_1 \mod(2)$, let $u_1 = \alpha_1 v_1 + 2x$, then $u_1 \sigma^{l4} a^3 = (\alpha_1 v_1 + 2x)\sigma^{4l} a^3 = \alpha_1 v_1 \sigma^{4l} a^3 + x2\sigma^{4l} a^3$. But $v_0 \sigma^{2l} a = 0 \Rightarrow x2\sigma^{4l} a^3 = 0$.

$u_n = \alpha_n v_n \mod I_n \Rightarrow$

$$\begin{aligned} u_n \sigma^{l2^{n+1}} a^{2^{n+1}-1} &= (\alpha_n v_n + \Sigma_0^{n-1} x_i v_i)\sigma^{l2^{n+1}} a^{2^{n+1}-1} \\ &= \alpha_n v_n \sigma^{l2^{n+1}} a^{2^{n+1}-1} \end{aligned} \tag{13}$$

by induction hypothesis.

To show, $\left(u_m\sigma^{k2^{m+1}}\right)\left(u_n\sigma^{l2^{n+1}}\right) = u_n u_m \sigma^{k2^{m+1}+l2^{n+1}}$. Since $u_n = \alpha_n v_n \mod I_n$ and $u_m = \alpha_m v_m \mod I_m$, the left hand side

$$\left(u_m\sigma^{k2^{m+1}}\right)\left(u_n\sigma^{l2^{n+1}}\right) = \left((\alpha_m v_m + \Sigma_0^{m-1} x_i v_i)\sigma^{k2^{m+1}}\right)\left((\alpha_n v_n + \Sigma_0^{n-1} y_j v_j)\sigma^{l2^{n+1}}\right) \tag{14}$$

$$= \left(\alpha_m v_m \sigma^{k2^{m+1}}\right)\left(\alpha_n v_n \sigma^{l2^{n+1}}\right) + \Sigma_{0,0}^{m-1,n-1}\left(x_i v_i \sigma^{k2^{m+1}}\right)\left(y_j v_j \sigma^{l2^{n+1}}\right) \tag{15}$$

$$= \alpha_m\alpha_n v_m v_n \sigma^{k2^{m+1}+l2^{n+1}} + \Sigma_{0,0}^{m-1,n-1}\left(x_i y_j v_i v_j \sigma^{k2^{m+1}+l2^{n+1}}\right) \tag{16}$$

$$= \left(\alpha_m\alpha_n v_m v_n + \Sigma_{0,0}^{m-1,n-1} x_i y_j v_i v_j\right)\sigma^{k2^{m+1}+l2^{n+1}}$$

$$= u_m u_n \sigma^{k2^{m+1}+l2^{n+1}}. \tag{17}$$

□

As a corollary we obtain the coefficient ring of the Borel spectrum of $E(n; \mathbf{u})_{\mathbb{R}}$.

Corollary 3.1 *The* $\mathbb{Z}/2$*-equivariant homotopy of the generalized Real Johnson-Wilson spectrum is given by,*

$$\pi_\star c(E(n; \mathbf{u})_{\mathbb{R}}) = \mathbb{Z}_{(2)}[u_k\sigma^{l2^{k+1}}, a, u_n^{\pm 1}, \sigma^{\pm 2^{n+1}}]/I(\mathbf{u}), \quad l \in \mathbb{Z}, n, k \geq 0.$$

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