Gluing pseudo functors via n -fold categories

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Abstract Gluing of two pseudo functors has been studied by Deligne, Ayoub, and others in the construction of extraordinary direct image functors in étale cohomology, stable homotopy, and mixed motives of schemes. In this article, we study more generally the gluing of finitely many pseudo functors. Given pseudo functors $F_i: \mathcal{A}_i \rightarrow \mathcal{D}$ defined on sub-2-categories $\mathcal{A}_i$ of a 2-category $\mathcal{C}$, we are concerned with the problem of finding pseudo functors $\mathcal{C} \rightarrow \mathcal{D}$ extending F_i up to pseudo natural equivalences. With the help of n -fold categories, we organize gluing data for n pseudo functors into 2-categories. We establish general criteria for equivalence between such 2-categories for n pseudo functors and for $n - 1$ pseudo functors, which can be applied inductively to the gluing problem. Results of this article are used in Zheng (Sci China Math 58(3):565–632, 2015) to construct extraordinary direct image functors in étale cohomology of Deligne-Mumford stacks.

Keywords Pseudo functor · Gluing data · 2-Category · n -Fold category · Hypercube

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1 Introduction

The extraordinary direct image functor $Rf_!$, one of Grothendieck's six operations, between derived categories of étale sheaves, was constructed in SGA 4 XVII [6].

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For a morphism of schemes f in a suitable category of schemes $\mathbf{C}$, $Rf_!$ is a functor between derived categories and functoriality of the construction $f \mapsto Rf_!$ is encoded by a pseudo functor $F: \mathbf{C} \rightarrow \mathcal{D}$, where $\mathcal{D} = \mathcal{Cat}$ is the 2-category of categories. There are obvious candidates for the restrictions $F_{\mathbf{A}}: \mathbf{A} \rightarrow \mathcal{D}$ and $F_{\mathbf{B}}: \mathbf{B} \rightarrow \mathcal{D}$ of F to the subcategories $\mathbf{A}$ and $\mathbf{B}$ consisting of open immersions and proper morphisms, respectively. The problem is thus to glue the pseudo functors $F_{\mathbf{A}}$ and $F_{\mathbf{B}}$, namely to find a pseudo functor F extending $F_{\mathbf{A}}$ and $F_{\mathbf{B}}$ up to pseudo natural equivalences. Deligne developed a theory for gluing two pseudo functors with the same target 2-category $\mathcal{D}$ and defined on subcategories $\mathbf{A}$ and $\mathbf{B}$ of any category $\mathbf{C}$ [6, Section 3].¹ Throughout this article, 2-categories and 2-functors are assumed to be strict. Here $\mathbf{A}$ and $\mathbf{B}$ are assumed to contain all objects of $\mathbf{C}$ and $F_{\mathbf{A}}$ and $F_{\mathbf{B}}$ are assumed to be identical on objects. Deligne assumes that every morphism f of $\mathbf{C}$ is compactifiable in the sense that it can be decomposed (noncanonically) into $f = pj$, where j is in $\mathbf{A}$ and p is in $\mathbf{B}$. One may then define $F(f)$ to be $F_{\mathbf{B}}(p)F_{\mathbf{A}}(j)$, but the latter depends on the noncanonical decomposition $f = pj$ and is not a priori functorial. A necessary condition for $F_{\mathbf{A}}$ and $F_{\mathbf{B}}$ to glue is that for every commutative square D

$$\begin{array}{ccc} X & \xrightarrow{j} & Y \\ q \downarrow & & \downarrow p \\ Z & \xrightarrow{i} & W \end{array}$$

where i, j are in $\mathbf{A}$ and p, q are in $\mathbf{B}$, we have an invertible 2-cell $G_D: F_{\mathbf{A}}(i)F_{\mathbf{B}}(q) \Rightarrow F_{\mathbf{A}}(p)F_{\mathbf{B}}(j)$, compatible with compositions and units. The triple $(F_{\mathbf{A}}, F_{\mathbf{B}}, (G_D))$ is called a *gluing datum*. Deligne shows that under mild assumptions on $\mathbf{B}$ and $\mathbf{C}$, there exists an (essentially unique) pseudo functor F associated to every given gluing datum. Deligne's theory has been applied to other contexts, such as the construction of $Rf_!$ between triangulated categories of mixed motives [5, Proposition 2.2.7]. Ayoub developed a variant of Deligne's theory [1, Théorème 1.3.1] in order to construct $Rf_!$ relative to stable homotopical pseudo functors.

In this article, we study more generally the problem of gluing finitely many pseudo functors with the same target. The case of three pseudo functors is necessary for the construction of $Rf_!$ and base change morphisms in [24] for a morphism f of Deligne-Mumford stacks, because compactification of coarse spaces only allows us to decompose a morphism into a sequence of three morphisms.

Let $\mathcal{C}$ be a $(2, 1)$ -category, namely a 2-category whose 2-cells are all invertible. Let $\mathcal{D}$ be a 2-category. Let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be locally full sub-2-categories of $\mathcal{C}$ each of which contains all objects of $\mathcal{C}$. A *gluing datum* is a pair $((F_i), (G_{ij}))$, where each $F_i: \mathcal{A}_i \rightarrow \mathcal{D}$ is a pseudo functor, and each $G_{ij} = (G_{ijD})$ is a collection of invertible 2-cells such that (F_i, F_j, G_{ij}) is a gluing data for two pseudo functors as above, subject to one extra condition for cubes. We organize gluing data into a 2-category $\mathcal{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$ (see Remark 5.1 for an explicit description in the case $n = 2$ and

¹ Deligne's original formulation focuses on the case $\mathcal{D} = \mathcal{Cat}$ and uses the equivalent language of cofibered categories [16, Section 8], but the argument extends to any target 2-category $\mathcal{D}$.

Remarks 6.1, 6.2 for explicit descriptions in the general case). Our first main result is for the gluing of two pseudo functors.

Theorem 1.1 *Let $\mathcal{C}$ be a $(2, 1)$ -category and let $\mathcal{A}$ and $\mathcal{B}$ be locally full sub-2-categories of $\mathcal{C}$, each containing all objects of $\mathcal{C}$. Assume the following:*

- (1) *For every morphism $f: X \rightarrow Y$ of $\mathcal{C}$, there exist a morphism $j: X \rightarrow Z$ of $\mathcal{A}$, a morphism $p: Z \rightarrow Y$ of $\mathcal{B}$, and a 2-cell $\alpha: pj \Rightarrow f$ of $\mathcal{C}$.*
- (2) *Every diagram $X \rightarrow Y \leftarrow Z$ in $\mathcal{B}$ can be completed into a commutative square with 2-cell in $\mathcal{B}$, Cartesian in $\mathcal{C}$.*

Then the descent 2-functor (as described in Remark 5.2)

$$Q_{\mathcal{D}}^2: \mathcal{PsFun}(\mathcal{C}, \mathcal{D}) \rightarrow \mathcal{GD}_{\mathcal{A}, \mathcal{B}}(\mathcal{C}, \mathcal{D})$$

is a 2-equivalence for every 2-category $\mathcal{D}$.

This is a common generalization of the results of Deligne and Ayoub, as our assumptions are less restrictive. Our result is also more precise in the sense that we establish a 2-equivalence of 2-categories, whereas previous results only dealt with *objects* of $\mathcal{GD}_{\mathcal{A}_1, \mathcal{A}_2}(\mathcal{C}, \mathcal{D})$. This precision is useful for the construction of pseudo natural transformations. We refer the reader to Remark 5.6 for more details on this comparison of results.

Our main result for gluing more than two pseudo functors is the following.

Theorem 1.2 *Let $\mathcal{C}$ and $\mathcal{A}_1, \dots, \mathcal{A}_n$ be as above with $n \geq 3$. Let $\mathcal{B}$ be a locally full sub-2-category of $\mathcal{C}$ containing $\mathcal{A}_1$ and $\mathcal{A}_2$. Under suitable assumptions on $\mathcal{A}_1, \dots, \mathcal{A}_n, \mathcal{B}, \mathcal{C}$, for every 2-category $\mathcal{D}$, the canonical 2-functor (as described in Remark 6.3)*

$$Q_{\mathcal{D}}: \mathcal{GD}_{\mathcal{B}, \mathcal{A}_3, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D}) \rightarrow \mathcal{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$$

is a 2-equivalence.

We refer the reader to Theorem 6.7 for a more precise statement.

Theorem 1.2 reduces the gluing of n pseudo functors to the gluing of $n - 1$ pseudo functors and can be applied recursively. Combining with Theorem 1.1, we obtain sufficient conditions for the descent 2-functor $Q_{\mathcal{D}}^n: \mathcal{PsFun}(\mathcal{C}, \mathcal{D}) \rightarrow \mathcal{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$ to be a 2-equivalence. We illustrate this in the case $n = 3$.

Corollary 1.3 *Let $\mathcal{C}$ be a $(2, 1)$ -category. Let $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ be locally full sub-2-categories of $\mathcal{C}$ each of which contains all objects of $\mathcal{C}$. Assume that $\mathcal{C}$ admits pseudo fiber products, and there exists a locally full sub-2-category $\mathcal{B}$ of $\mathcal{C}$, containing $\mathcal{A}_1, \mathcal{A}_2$, and satisfying the following conditions:*

- (1) *For every morphism f of $\mathcal{C}$, there exist a morphism j of $\mathcal{B}$, a morphism p of $\mathcal{A}_3$, and a 2-cell $pj \Rightarrow f$ of $\mathcal{C}$.*
- (2) *For every morphism f of $\mathcal{B}$, there exist a morphism j of $\mathcal{A}_1$, a morphism p of $\mathcal{A}_2$, and a 2-cell $pj \Rightarrow f$ of $\mathcal{C}$.*

- (3) For every morphism f of $\mathcal{B} \cap \mathcal{A}_3$, there exist a morphism j of $\mathcal{A}_1 \cap \mathcal{A}_3$, a morphism p of $\mathcal{A}_2 \cap \mathcal{A}_3$, and a 2-cell $pj \Rightarrow f$ of $\mathcal{C}$.
- (4) The sub-2-categories $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{B}$ are stable under base change in $\mathcal{C}$. The sub-2-categories $\mathcal{A}_3$ and $\mathcal{B}$ are stable under taking diagonals in $\mathcal{C}$.

Then the descent 2-functor

$$Q_{\mathcal{D}}^3: \mathcal{P}sfun(\mathcal{C}, \mathcal{D}) \rightarrow \mathcal{GD}_{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3}(\mathcal{C}, \mathcal{D})$$

is a 2-equivalence.

Proof Indeed, $Q_{\mathcal{D}}^3$ can be decomposed as

$$\mathcal{P}sfun(\mathcal{C}, \mathcal{D}) \xrightarrow{Q_{\mathcal{D}}^2} \mathcal{GD}_{\mathcal{B}, \mathcal{A}_3}(\mathcal{C}, \mathcal{D}) \xrightarrow{Q_{\mathcal{D}}} \mathcal{GD}_{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3}(\mathcal{C}, \mathcal{D}),$$

where both are 2-equivalences by Theorems 1.1 and 6.7. $\square$

There are other ways to combine the above theorems. In [24, Proposition 1.5], we deduce from the above theorems and Proposition 6.4 a case where $Q_{\mathcal{D}}^2$ is a 2-equivalence but for which condition (1) of Theorem 1.1 is not satisfied.

Note that when $Q_{\mathcal{D}}^n$ is a 2-equivalence, given any gluing datum $((F_i), (G_{ij}))$, there exists a pseudo functor F extending (F_i) up to pseudo natural isomorphisms.

We study the gluing of pseudo functors in the framework of (strict) n -fold categories. An n -fold category structure is an extended categorical structure consisting of hypercubes up to dimension n , endowed with composition laws in each of the n directions. This often encodes more information than a higher category structure of the same dimension, such as an n -category. There is, however, a rich interplay between extended categories and higher categories, of which the most relevant part to our study is the relation between n -fold categories and 2-categories. Given a 2-category $\mathcal{C}$, we construct an n -fold category $\mathbb{Q}^n \mathcal{C}$ of hypercubes in $\mathcal{C}$, extending the double category of up-squares for $n = 2$ due to Bastiani and Ehresmann [2, 2.C.1, p. 272]. We also consider the n -fold subcategory $\mathbb{Q}_{\mathcal{A}_1, \dots, \mathcal{A}_n} \mathcal{C}$ spanned by hypercubes whose edges in direction i are in $\mathcal{A}_i$. The construction $\mathbb{Q}^n$ admits a left adjoint, carrying an n -fold category $\mathbb{C}$ to its reduced 2-category of paths $\mathcal{T}_n^{\text{red}} \mathbb{C}$, related to Gray's tensor product of 2-categories. We further construct a $(2, 1)$ -category $\mathcal{LT}_n \mathbb{C}$, variant of $\mathcal{T}_n^{\text{red}} \mathbb{C}$. The 2-category of gluing data $\mathcal{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$ can be identified with the 2-category $2Fun^{\text{ps}}(\mathcal{E}, \mathcal{D})$ of 2-functors $\mathcal{E} = \mathcal{LT}_n \mathbb{Q}_{\mathcal{A}_1, \dots, \mathcal{A}_n} \mathcal{C} \rightarrow \mathcal{D}$, pseudo natural transformations, and modifications. Properties of the 2-category of gluing data are established by studying the $(2, 1)$ -category $\mathcal{E}$.

The article is organized as follows. In Sect. 2, we fix some conventions and prove some preliminary results on 2-categories. We introduce the 2-category of paths $\mathcal{TC}$ in a 2-category $\mathcal{C}$, which allows one to straighten pseudo functors to 2-functors. In Sect. 3, after recalling the definition of an n -fold category, we investigate the relation between 2-categories and n -fold categories. We construct the n -fold category $\mathbb{Q}^n \mathcal{C}$ of hypercubes and the reduced 2-categories of paths $\mathcal{T}_n^{\text{red}} \mathbb{C}$, and we establish the aforementioned adjunction. A variant of the adjunction is used to define the descent

2-functor $Q^n : \mathcal{P} sFun(\mathcal{C}, \mathcal{D}) \rightarrow \mathcal{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$. In Sect. 4, we study functorial properties of these constructions with respect to the index set $\{1, \dots, n\}$, and construct 2-functors between various 2-categories of gluing data. In Sects. 5 and 6, we apply these constructions to study the gluing of pseudo functors. In Sect. 5, we study the case of two pseudo functors and prove Theorem 1.1, extending results of Deligne and Ayoub. We deduce the theorem from a general criterion involving the 2-category of compactifications. In Sect. 6, we study the case of finitely many pseudo functors and prove Theorem 1.2. In Sects. 7 through 9, we develop several tools for the application of the main theorems. In Sects. 7 and 8, we introduce Cartesian gluing data, first for two pseudo functors, and then for finitely many pseudo functors. Cartesian gluing data are an alternative set of gluing data that only makes use of Cartesian squares instead of commutative squares with 2-cells, and are easier to construct in applications. We show that the 2-category of Cartesian gluing data is isomorphic to the 2-category of gluing data under mild conditions. In Sect. 9, we check the axioms for gluing data in the case when the data are constructed from base change maps via adjunctions. Finally, we include the proof of a preliminary result in Sect. 10 for completeness.

In joint work with Yifeng Liu [20], we establish analogues of some results of this article in the ∞ -categorical setting, which are used in [19] to construct Grothendieck's six operations on Artin stacks. We remark that specific features of 2-categories have been exploited in this article and the full generality of our results cannot be deduced from [20].

2 Preliminaries on 2-categories

In this section, we fix some conventions and notation on 2-categories and record some preliminary results. In particular, we introduce the 2-category of paths $\mathcal{TC}$ in a 2-category $\mathcal{C}$ (Definition 2.13), which allows one to straighten pseudo functors to 2-functors.

Throughout the article, we reserve the symbol $\cong$ for isomorphisms. The symbol will not be used for equivalences or bi-equivalences, which are instead stated verbally. We denote categories by bold letters ($\mathbf{C}$, $\mathbf{D}$, etc.), 2-categories by script letters ($\mathcal{C}$, $\mathcal{D}$, etc.), n -fold categories by blackboard bold letters ($\mathbb{C}$, $\mathbb{D}$, etc.).

The notion of 2-category was introduced by Ehresmann [9, note bibliographique, p. 324] and Bénabou [3, p. 3824]. As in [4, Chapter 7], 2-categories, 2-functors, and 2-natural transformations are assumed to be strict. Pseudo functor preserves composition and unit up to coherent invertible 2-cells. A 2-equivalence is a 2-functor $F : \mathcal{C} \rightarrow \mathcal{D}$ such that there exist a 2-functor $G : \mathcal{D} \rightarrow \mathcal{C}$ and 2-natural isomorphisms $\mathrm{id}_{\mathcal{C}} \cong GF$ and $FG \cong \mathrm{id}_{\mathcal{D}}$. In this case we say that F and G are 2-quasi-inverses of each other. A bi-equivalence is a pseudo functor $F : \mathcal{C} \rightarrow \mathcal{D}$ such that there exist a pseudo functor $G : \mathcal{D} \rightarrow \mathcal{C}$ and pseudo natural equivalences $\mathrm{id}_{\mathcal{C}} \Rightarrow GF$ and $FG \Rightarrow \mathrm{id}_{\mathcal{D}}$ (some authors refer to such pseudo functor as “2-equivalence” [12, Corollary 1.5.26(i)]). In this case we say that F and G are pseudo inverses of each other. We use the term *morphism* for 1-cell of a 2-category.

Let us recall the notion of pseudo fiber product in a 2-category $\mathcal{C}$, which is a type of “2-limit” in the terminology of [12, Definition 1.6.1(i)] or “pseudo-bilimit” in the terminology of [4, Chapter 7]. Let $Z \xrightarrow{i} W \xleftarrow{p} Y$ be morphisms in $\mathcal{C}$. For any object X of $\mathcal{C}$, consider the category $\mathcal{C}(X, Z) \times_{\mathcal{C}(X, W)} \mathcal{C}(X, Y)$ of triples (q, j, α) as shown by the square

$$\begin{array}{ccc} X & \xrightarrow{j} & Y \\ q \downarrow & \alpha \nearrow & \downarrow p \\ Z & \xrightarrow{i} & W, \end{array}$$

where α is an invertible 2-cell. A *pseudo fiber product* of $Z \xrightarrow{i} W \xleftarrow{p} Y$ is an object X of $\mathcal{C}$ equipped with an object (q, j, α) of $\mathcal{C}(X, Z) \times_{\mathcal{C}(X, W)} \mathcal{C}(X, Y)$, such that for every object X' of $\mathcal{C}$, the functor $\mathcal{C}(X', X) \rightarrow \mathcal{C}(X', Z) \times_{\mathcal{C}(X', W)} \mathcal{C}(X', Y)$ induced by composition with (q, j, α) is an equivalence of categories. In this case the square is called *Cartesian*. Given a Cartesian square the morphism q is called a *base change* of p by i . By the diagonal of a morphism $f: X \rightarrow Y$ in $\mathcal{C}$, we mean the morphism $X \rightarrow X \times_Y X$, unique up to equivalence, where $X \times_Y X$ is the pseudo fiber product. We will also consider *strict* fiber products of categories and 2-categories.

Convention 2.1 Unless otherwise stated, all categories and 2-categories are assumed to be small (and strict). Big categories and big 2-categories are occasionally used as a linguistic tool to simplify the narrative. We let **Cat** denote the big category of categories and functors and **2Cat** denote the big category of 2-categories and 2-functors.

Definition 2.2 and Notation 2.3 below are standard.

Definition 2.2 A $(2, 1)$ -category is a 2-category whose 2-cells are invertible.

Notation 2.3 Let $\mathcal{C}$ be a 2-category. We denote by $\mathcal{C}^{\text{coop}}$ (resp. $\mathcal{C}^{\text{co}}$) the 2-category obtained from $\mathcal{C}$ by reversing the morphisms and 2-cells (resp. 2-cells only). In other words $\text{Ob}(\mathcal{C}^{\text{coop}}) = \text{Ob}(\mathcal{C}^{\text{co}}) = \text{Ob}(\mathcal{C})$, and, for any pair of objects X and Y of $\mathcal{C}$, we have $\mathcal{C}^{\text{coop}}(Y, X) = \mathcal{C}^{\text{co}}(X, Y) = \mathcal{C}(X, Y)^{\text{op}}$.

If $\mathcal{C}$ is a $(2, 1)$ -category, inversion of the 2-cells defines an isomorphism $\mathcal{C} \cong \mathcal{C}^{\text{co}}$.

Notation 2.4 (Functor categories) Let **C** and **D** be categories. We let $\text{Fun}(\mathbf{C}, \mathbf{D})$ denote the set of functors $\mathbf{C} \rightarrow \mathbf{D}$. We let **Fun**(**C**, **D**) denote the category of functors $\mathbf{C} \rightarrow \mathbf{D}$ and natural transformations.

Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories. We denote by $2\text{Fun}(\mathcal{C}, \mathcal{D})$ the set of 2-functors $\mathcal{C} \rightarrow \mathcal{D}$. We denote by **2Fun**($\mathcal{C}, \mathcal{D}$) the category of 2-functors $\mathcal{C} \rightarrow \mathcal{D}$ and 2-natural transformations. We denote by $2\text{Fun}(\mathcal{C}, \mathcal{D})$ the 2-category of 2-functors $\mathcal{C} \rightarrow \mathcal{D}$, 2-natural transformations, and modifications. We denote by $2\text{Fun}^{\text{ps}}(\mathcal{C}, \mathcal{D})$ the 2-category of 2-functors $\mathcal{C} \rightarrow \mathcal{D}$, *pseudo* natural transformations, and modifications. We denote by $\mathcal{UPsFun}(\mathcal{C}, \mathcal{D})$ the 2-category of strictly unital pseudo functors $\mathcal{C} \rightarrow \mathcal{D}$, pseudo natural transformations, and modifications. We denote by $\mathcal{PsFun}(\mathcal{C}, \mathcal{D})$ the 2-category of pseudo functors $\mathcal{C} \rightarrow \mathcal{D}$, pseudo natural transformations, and modifications.

For 2-categories $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$, we have

$$2\mathcal{F}un(\mathcal{C}, 2\mathcal{F}un(\mathcal{D}, \mathcal{E})) \cong 2\mathcal{F}un(\mathcal{C} \times \mathcal{D}, \mathcal{E}).$$

For any pseudo functor $G: \mathcal{D} \rightarrow \mathcal{E}$, composition with G provides a pseudo functor $G \circ -: \mathcal{P}sfun(\mathcal{C}, \mathcal{D}) \rightarrow \mathcal{P}sfun(\mathcal{C}, \mathcal{E})$, which is a 2-functor if G is a 2-functor. By contrast, for any pseudo functor $F: \mathcal{C} \rightarrow \mathcal{D}$, composition with F provides a 2-functor

$$- \circ F: \mathcal{P}sfun(\mathcal{D}, \mathcal{E}) \rightarrow \mathcal{P}sfun(\mathcal{C}, \mathcal{E}).$$

Let $\epsilon: F \Rightarrow F'$ be a pseudo natural transformation. Then ϵ induces only a *pseudo* natural transformation $- \circ F \Rightarrow - \circ F'$ in general. However, if $\epsilon(X)$ is an identity for every object X of $\mathcal{C}$, then ϵ induces a 2-natural transformation between the 2-functors $\mathcal{P}sfun(\mathcal{D}, \mathcal{E}) \rightarrow \mathcal{P}sfun(\mathcal{C}, \mathcal{E})$ induced by F and F' .

We will need to work over a base 2-category as follows.

Definition 2.5 (*$\mathcal{D}$ -2-Category*) Let $\mathcal{D}$ be a 2-category. A $\mathcal{D}$ -2-category is a pair $(\mathcal{C}, F)$ consisting of a 2-category $\mathcal{C}$ and a 2-functor $F: \mathcal{C} \rightarrow \mathcal{D}$. If $(\mathcal{B}, E)$ and $(\mathcal{C}, F)$ are $\mathcal{D}$ -categories, we define the category of $\mathcal{D}$ -2-functors $\mathbf{2Fun}_{\mathcal{D}}((\mathcal{B}, E), (\mathcal{C}, F))$ to be the strict fiber at E of the functor $\mathbf{2Fun}(\mathcal{B}, \mathcal{C}) \rightarrow \mathbf{2Fun}(\mathcal{B}, \mathcal{D})$ induced by F . Objects and morphisms of this category are called *$\mathcal{D}$ -2-functors* and *$\mathcal{D}$ -2-natural transformations*, respectively. Thus a $\mathcal{D}$ -2-functor $(\mathcal{B}, E) \rightarrow (\mathcal{C}, F)$ is a 2-functor $G: \mathcal{B} \rightarrow \mathcal{C}$ such that $E = FG$. If $G, H: (\mathcal{B}, E) \rightarrow (\mathcal{C}, F)$ are $\mathcal{D}$ -2-functors, a $\mathcal{D}$ -2-natural transformation is a 2-natural transformation $\alpha: G \Rightarrow H$ such that $F * \alpha: FG \Rightarrow FH$ is id_E . We say that a $\mathcal{D}$ -2-functor $G: (\mathcal{B}, E) \rightarrow (\mathcal{C}, F)$ is a *$\mathcal{D}$ -2-equivalence* if there exist a $\mathcal{D}$ -2-functor $H: (\mathcal{C}, F) \rightarrow (\mathcal{B}, E)$ and $\mathcal{D}$ -2-natural isomorphisms $\text{id}_{\mathcal{C}} \Rightarrow GH$ and $HG \Rightarrow \text{id}_{\mathcal{B}}$. In this case we say that G and H are *$\mathcal{D}$ -2-quasi-inverses* of each other.

We do not consider $\mathcal{D}$ -modifications because we will only be interested in $\mathcal{D}$ -2-categories whose strict fibers are categories. A $\mathcal{D}$ -2-equivalence $\mathcal{B} \rightarrow \mathcal{C}$ induces a 2-equivalence between the strict fiber 2-categories $\mathcal{B}_X \rightarrow \mathcal{C}_X$ for every object X of $\mathcal{D}$.

Let us introduce some terminology on faithfulness.

Definition 2.6 (*Faithfulness, fullness*) Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories and let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a pseudo functor. Given a property (P) on functors, We say that F is *locally (P)* if for every pair of objects X and Y of $\mathcal{C}$, the functor

$$F_{XY}: \mathcal{C}(X, Y) \rightarrow \mathcal{D}(FX, FY)$$

is (P) . In particular, we say that F is a *local equivalence* if F_{XY} is an equivalence of categories for all X and Y (this property is called “fully faithful” in [12, Definition 1.5.11(ii)]).

We say that F is *2-faithful* if F is locally fully faithful and the underlying functor of F is faithful in the 1-categorical sense. We say that F is *2-fully faithful* if F is locally fully faithful and the underlying functor of F is fully faithful in the 1-categorical sense. In other words, F is 2-fully faithful if and only if F_{XY} is an isomorphism of

categories for all X and Y (this property is called “strongly faithful” in [12, Definition 1.5.11(ii)]).

We say that F is *pseudo surjective* if for every object Y of $\mathcal{D}$, there exists an object X of $\mathcal{C}$ and an equivalence $FX \rightarrow Y$ in $\mathcal{D}$.

We say that a sub-2-category $\mathcal{C}$ of $\mathcal{D}$ is *locally full* if for all morphisms f, g in $\mathcal{C}$, the collection of 2-cells between f and g in $\mathcal{C}$ is the same as in $\mathcal{D}$. We say that a sub-2-category $\mathcal{C}$ of $\mathcal{D}$ is *2-full* if for all objects X and Y in $\mathcal{C}$, the category $\mathcal{C}(X, Y)$ equals the category $\mathcal{D}(X, Y)$, or equivalently, if the inclusion 2-functor $\mathcal{C} \rightarrow \mathcal{D}$ is 2-fully faithful.

Example 2.7 The sub-2-categories $2\mathcal{F}un^{\text{ps}}(\mathcal{C}, \mathcal{D}) \subseteq \mathcal{U}Ps\mathcal{F}un(\mathcal{C}, \mathcal{D}) \subseteq \mathcal{P}s\mathcal{F}un(\mathcal{C}, \mathcal{D})$ are 2-full.

Recall that a pseudo functor F is a bi-equivalence if and only if F is a pseudo surjective local equivalence [12, Definition 1.5.11 (iii), Corollary 1.5.26 (i)]. A 2-functor F is a 2-equivalence if and only if F is 2-fully faithful and essentially surjective in the 1-categorical sense. This extends to $\mathcal{D}$ -2-functors as follows.

Lemma 2.8 *A $\mathcal{D}$ -2-functor $G: \mathcal{B} \rightarrow \mathcal{C}$ is a $\mathcal{D}$ -2-equivalence if and only if G is 2-fully faithful and for every object Y of $\mathcal{C}$, there exists an object X of $\mathcal{B}$ and an isomorphism $\epsilon(Y): GX \rightarrow Y$ in $\mathcal{C}$ whose image in $\mathcal{D}$ is an identity.*

Proof The “only if” part is clear. For the “if” part, we construct a $\mathcal{D}$ -2-quasi-inverse $H: \mathcal{C} \rightarrow \mathcal{B}$ carrying Y to X as follows. For objects Y and Y' of $\mathcal{C}$, we take $H_{Y, Y'}: \mathcal{C}(Y, Y') \rightarrow \mathcal{C}(HY, HY')$ to be the unique functor such that $G_{HY, HY'} H_{Y, Y'}$ is given by the formula $f \mapsto \epsilon(Y')^{-1} f \epsilon(Y)$. Then $\epsilon: GH \Rightarrow \text{id}_{\mathcal{C}}$ is a $\mathcal{D}$ -2-natural isomorphism. For each object X of $\mathcal{B}$, we let $\eta(X): HGX \rightarrow X$ be the unique morphism such that $G(\eta(X)) = \epsilon(GX)$. Then $\eta: HG \Rightarrow \text{id}_{\mathcal{B}}$ is a $\mathcal{D}$ -2-natural isomorphism. $\square$

Notation 2.9

- (1) Let S be a set and let $\mathcal{D}$ be a 2-category. We view S as a discrete 2-category and denote by $\mathcal{D}^S$ the 2-category of 2-functors $S \rightarrow \mathcal{D}$. An object of $\mathcal{D}^S$ is a map $S \rightarrow \text{Ob}(\mathcal{D})$. A morphism $\alpha: F \rightarrow F'$ of $\mathcal{D}^S$ is a family

$$(\alpha(X): FX \rightarrow F'X)_{X \in S}$$

of morphisms of $\mathcal{D}$. A 2-cell $\Xi: \alpha \Rightarrow \beta$ of $\mathcal{D}^S$ is a family

$$(\Xi(X): \alpha(X) \Rightarrow \beta(X))_{X \in S}$$

of 2-cells of $\mathcal{D}$.

- (2) Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories. We view $\mathcal{P}s\mathcal{F}un(\mathcal{C}, \mathcal{D})$ as a $\mathcal{D}^{\text{Ob}(\mathcal{C})}$ -2-category via the forgetful 2-functor

$$|-|: \mathcal{P}s\mathcal{F}un(\mathcal{C}, \mathcal{D}) \rightarrow \mathcal{D}^{\text{Ob}(\mathcal{C})}$$

induced by the inclusion $\text{Ob}(\mathcal{C}) \rightarrow \mathcal{C}$.

Note that the forgetful 2-functor $|-|$ is locally faithful, and the strict fibers are categories.

The following simple technique for modifying a pseudo functor will be of use.

Lemma 2.10 *Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories, let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a pseudo functor, let H be an object of $\mathcal{D}^{\text{Ob}(\mathcal{C})}$, and let $\eta: |F| \rightarrow H$ be an equivalence in $\mathcal{D}^{\text{Ob}(\mathcal{C})}$. Assume that to every morphism $f: X \rightarrow Y$ of $\mathcal{C}$ is associated a square in $\mathcal{D}$*

$$\begin{array}{ccc} FX & \xrightarrow{Ff} & FY \\ \eta(X) \downarrow & \alpha_f \nearrow & \downarrow \eta(Y) \\ HX & \xrightarrow{g_f} & HY \end{array}$$

where α_f is an invertible 2-cell. Then there exists a unique pair (G, ϵ) , where $G: \mathcal{C} \rightarrow \mathcal{D}$ is a pseudo functor and $\epsilon: F \rightarrow G$ is a pseudo natural equivalence, such that $|G| = H$, $|\epsilon| = \eta$, $G(f) = g_f$ and $\epsilon(f) = \alpha_f$ for every morphism f of $\mathcal{C}$.

Proof Indeed, it remains to define the constraints of G , which are uniquely determined by the constraints of F . $\square$

Applying the lemma to the unital constraints of any pseudo functor $F: \mathcal{C} \rightarrow \mathcal{D}$, we obtain a strictly unital pseudo functor $G: \mathcal{C} \rightarrow \mathcal{D}$ and a pseudo natural transformation $\epsilon: F \Rightarrow G$ with $\epsilon(X) = \text{id}_X$ for every object X of $\mathcal{C}$. By Lemma 2.8, we obtain the following.

Proposition 2.11 *Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories. Then the inclusion $\mathcal{UPsFun}(\mathcal{C}, \mathcal{D}) \subseteq \mathcal{PsFun}(\mathcal{C}, \mathcal{D})$ is a $\mathcal{D}^{\text{Ob}(\mathcal{C})}$ -2-equivalence.*

The proof of the following proposition is straightforward. For completeness we give the proof in Sect. 10.

Proposition 2.12 *Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a pseudo functor such that $\text{Ob}(F): \text{Ob}(\mathcal{C}) \rightarrow \text{Ob}(\mathcal{D})$ is a bijection. For every 2-category $\mathcal{E}$, we consider the $\mathcal{E}^{\text{Ob}(\mathcal{C})}$ -2-functor*

$$\Phi_{\mathcal{E}}: \mathcal{PsFun}(\mathcal{D}, \mathcal{E}) \rightarrow \mathcal{PsFun}(\mathcal{C}, \mathcal{E})$$

induced by F .

- (1) *If for every morphism g of $\mathcal{D}$, the identity 2-cell id_g can be decomposed as $g \Rightarrow Ff \Rightarrow g$, then $\Phi_{\mathcal{E}}$ is 2-faithful for every 2-category $\mathcal{E}$.*
- (2) *The following conditions are equivalent:*
 - (a) *The pseudo functor F is a bi-equivalence.*
 - (b) *There exists a pseudo functor $G: \mathcal{D} \rightarrow \mathcal{C}$ such that $\text{Ob}(G)$ is the inverse of $\text{Ob}(F)$ and there are pseudo natural isomorphisms $\eta: \text{id}_{\mathcal{C}} \Rightarrow GF$ and $\epsilon: FG \Rightarrow \text{id}_{\mathcal{D}}$ such that $\eta(X) = \text{id}_X$ and $\epsilon(Y) = \text{id}_Y$ for all objects X of $\mathcal{C}$ and Y of $\mathcal{D}$.*
 - (c) *The 2-functor $\Phi_{\mathcal{E}}$ is an $\mathcal{E}^{\text{Ob}(\mathcal{C})}$ -2-equivalence for every 2-category $\mathcal{E}$.*

- (d) The 2-functor $\Phi_{\mathcal{C}}$ is pseudo surjective and the 2-functor $\Phi_{\mathcal{D}}$ is a local equivalence.

Note that $\Phi_{\mathcal{C}}$ is locally faithful. If F is locally essentially surjective, then the condition in (1) is satisfied. Moreover, the condition in (1) is equivalent to the local essential surjectivity of F if $\mathcal{D}$ is a $(2, 1)$ -category.

Pseudo functors can be straightened to 2-functors via the 2-category of paths as follows.

Definition 2.13 (2-Category of paths $\mathcal{TC}$ in a 2-category $\mathcal{C}$) Let $\mathcal{C}$ be a 2-category. We define the 2-category $\mathcal{TC}$ of paths in $\mathcal{C}$ as follows. The objects of $\mathcal{TC}$ are the objects of $\mathcal{C}$. A morphism $X \rightarrow Y$ of $\mathcal{TC}$ is a path

$$X = X_0 \xrightarrow{f_1} X_1 \rightarrow \cdots \rightarrow X_{m-1} \xrightarrow{f_m} X_m = Y, \quad m \geq 0$$

where each f_i is a morphism of $\mathcal{C}$ for $1 \leq i \leq m$. The identity morphism $\text{id}_X: X \rightarrow X$ in $\mathcal{TC}$ is the path of length 0. Composition of morphisms is given by concatenation of paths and is denoted by $*$. An atomic 2-cell between two paths sharing a source and a target is one of the following:

- (1) (creation of unit) $\iota_{g,f}: g * f \Rightarrow g * \text{id}_Y^{\mathcal{C}} * f$ or (deletion of unit) $\theta_{g,f}: g * \text{id}_Y^{\mathcal{C}} * f \Rightarrow g * f$, where $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a sequence of paths, and $\text{id}_Y^{\mathcal{C}}$ is the identity morphism on Y in $\mathcal{C}$ viewed as a path of length 1.
- (2) (composition of morphisms) $\gamma_{g,h',h,f}: g * h' * h * f \Rightarrow g * (h'h) * f$ or (decomposition of morphism) $\delta_{g,h',h,f}: g * (h'h) * f \Rightarrow g * h' * h * f$, where $X \xrightarrow{f} Y \xrightarrow{h} Y' \xrightarrow{h'} Y'' \xrightarrow{g} Z$ is a sequence of paths, with h and h' of length 1 and $h'h$ being their composition in $\mathcal{C}$.
- (3) (2-cell of $\mathcal{C}$) $\sigma_{g,\alpha,f}: g * h * f \Rightarrow g * h' * f$, where $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are paths, and $h, h': Y \rightarrow Z$ are morphisms of $\mathcal{C}$, and $\alpha: h \Rightarrow h'$ is a 2-cell of $\mathcal{C}$.

A pre-2-cell $f \Rightarrow g$ is a sequence of atomic 2-cells $f = f_0 \Rightarrow f_1 \Rightarrow \cdots \Rightarrow f_{n-1} \Rightarrow f_n = g$. Vertical composition of pre-2-cells, denoted by $\circ$, is given by concatenation. Horizontal composition (or whiskering) of pre-2-cells with morphisms is also given by concatenation: If $f, g: X \rightarrow Y, h: W \rightarrow X, h': Y \rightarrow Z$ are paths and $\alpha: f \Rightarrow g$ is a pre-2-cell, then one has the pre-2-cell $h' * \alpha * h: h' * f * h \Rightarrow h' * g * h$. Consider systems which associate to every pair of paths (f, g) sharing a source and a target, an equivalence relation on the set of pre-2-cells $f \Rightarrow g$. We say that one system $\sim$ is finer than another system $\sim'$ if $\alpha \sim \beta$ implies $\alpha \sim' \beta$. There is a finest system $\sim$ satisfying the following conditions:

- (1) (Stability under vertical and horizontal composition) If $f', f, g, g': X \rightarrow Y, h: W \rightarrow X, h': Y \rightarrow Z$ are paths, $\alpha \sim \beta: f \Rightarrow g, \eta: f' \Rightarrow f, \eta': g \Rightarrow g'$ are pre-2-cells, then $\eta' \circ \alpha \circ \eta \sim \eta' \circ \beta \circ \eta$ and $h' * \alpha * h \sim h' * \beta * h$.
- (2) (Interchange law) If $f, f': X \rightarrow Y, g, g': Y \rightarrow Z$ are paths, $\alpha: f \Rightarrow f'$ and $\beta: g \Rightarrow g'$ are pre-2-cells, then $(f' * \beta) \circ (\alpha * g) \sim (\alpha * g') \circ (f * \beta)$.
- (3) (Creation and deletion of unit) For every object X of $\mathcal{C}$,

$$\theta_{\text{id}_X, \text{id}_X} \circ \iota_{\text{id}_X, \text{id}_X} \sim \text{id}_{\text{id}_X}, \quad \iota_{\text{id}_X, \text{id}_X} \circ \theta_{\text{id}_X, \text{id}_X} \sim \text{id}_{\text{id}_X}.$$

(4) (Unit and composition of morphisms) For every morphism $f: X \rightarrow Y$ of $\mathcal{C}$,

$$\iota_{f, \text{id}_X} \sim \delta_{\text{id}_Y, f, \text{id}_X^{\mathcal{C}}, \text{id}_X}, \quad \iota_{\text{id}_Y, f} \sim \delta_{\text{id}_Y, \text{id}_Y^{\mathcal{C}}, f, \text{id}_X},$$

(5) (Composition and decomposition of morphisms) For every sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ of morphisms in $\mathcal{C}$,

$$\delta_{\text{id}_Z, g, f, \text{id}_X} \circ \gamma_{\text{id}_Z, g, f, \text{id}_X} \sim \text{id}_{g \circ f}, \quad \gamma_{\text{id}_Z, g, f, \text{id}_X} \circ \delta_{\text{id}_Z, g, f, \text{id}_X} \sim \text{id}_{gf}.$$

(6) (Associativity of composition of morphisms) For every sequence $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ of morphisms in $\mathcal{C}$,

$$\gamma_{\text{id}_W, h, gf, \text{id}_X} \circ \gamma_{h, g, f, \text{id}_X} \sim \gamma_{\text{id}_W, hg, f, \text{id}_X} \circ \gamma_{\text{id}_W, h, g, f}.$$

(7) (Unit 2-cell) For every morphism $f: X \rightarrow Y$ of $\mathcal{C}$, we have

$$\sigma_{\text{id}_Y, \text{id}_f^{\mathcal{C}}, \text{id}_X} \sim \text{id}_f,$$

where $\text{id}_f^{\mathcal{C}}$ is the identity 2-cell on f in $\mathcal{C}$.

(8) (Vertical composition of 2-cells) For a sequence of 2-cells $g \xRightarrow{\alpha} g' \xRightarrow{\beta} g''$ in $\mathcal{C}$, where $g, g', g'': X \rightarrow Y$, we have

$$\sigma_{\text{id}_Y, \beta, \text{id}_X} \circ \sigma_{\text{id}_Y, \alpha, \text{id}_X} \sim \sigma_{\text{id}_Y, \beta \circ \alpha, \text{id}_X}.$$

(9) (Horizontal composition of 2-cells with morphisms) For morphisms $f: X \rightarrow Y$ and $h: Z \rightarrow W$ of $\mathcal{C}$, and a 2-cell $\alpha: g \Rightarrow g'$ of $\mathcal{C}$, where $g, g': Y \rightarrow Z$, we have

$$\begin{aligned} \sigma_{\text{id}_Z, \alpha, f} &\sim \delta_{\text{id}_Z, g', f, \text{id}_X} \circ \sigma_{\text{id}_Z, \alpha f, \text{id}_X} \circ \gamma_{\text{id}_Z, g, f, \text{id}_X}, \\ \sigma_{h, \alpha, \text{id}_Y} &\sim \delta_{\text{id}_W, h, g', \text{id}_Y} \circ \sigma_{\text{id}_W, h \alpha, \text{id}_Y} \circ \gamma_{\text{id}_W, h, g, \text{id}_Y}. \end{aligned}$$

In the terminology of Street [23, page 547], we have given a presentation of the 2-category $\mathcal{TC}$ consisting of the underlying graph of $\mathcal{C}$, the derivation scheme given by the atomic 2-cells with f and g being identities, and the relation on 2-cells given by conditions (3) through (9). The 2-category of objects, morphisms, and pre-2-cells in the above definition is the free 2-category on the computed consisting of the graph and the derivation scheme.

It follows from conditions (3), (5), (7), and (8) that $\mathcal{TC}$ is a $(2, 1)$ -category whenever $\mathcal{C}$ is a $(2, 1)$ -category.

We define the *reduction 2-functor* $R: \mathcal{TC} \rightarrow \mathcal{C}$ as follows. We take the identity on objects. To a path $f_m * \dots * f_1$ (where each f_i is a morphism of $\mathcal{C}$) we associate its composition $f_m \dots f_1$ in $\mathcal{C}$. To all atomic 2-cells except σ we associate identities, and to $\sigma_{g, \alpha, f}$ we associate the horizontal composition $g * \alpha * f$ in $\mathcal{C}$. We define a pseudo

functor $G: \mathcal{C} \rightarrow \mathcal{TC}$ as follows. We take the identity on objects. To a morphism f we associate f , considered as a path of length 1. The coherence constraint is given by ι and γ . To a 2-cell $\alpha: f \Rightarrow g$, where $f, g: X \rightarrow Y$, we associate $\sigma_{\text{id}_Y, \alpha, \text{id}_X}$. We have $RG = \text{id}_{\mathcal{C}}$.

Proposition 2.14 *The reduction 2-functor R is a bi-equivalence, with G being a pseudo inverse.*

Proof We construct a pseudo natural isomorphism $\eta: \text{id}_{\mathcal{TC}} \Rightarrow GR$ carrying objects to identities as follows. To a morphism $f_m * \dots * f_1$ of $\mathcal{TC}$, we associate the 2-cell $f_m * \dots * f_1 \Rightarrow f_m \dots f_1$ given by ι (creation of unit) for $m = 0$, identity for $m = 1$, and γ (composition) for $m \geq 2$. $\square$

The following result says that pseudo functors with source $\mathcal{C}$ can be straightened to 2-functors with source $\mathcal{TC}$.

Proposition 2.15 *Composition with G induces an isomorphism $2\text{Fun}^{\text{ps}}(\mathcal{TC}, \mathcal{D}) \cong \mathcal{PsFun}(\mathcal{C}, \mathcal{D})$ of 2-categories.*

Proof The inverse is easily constructed, as follows. A pseudo functor $H: \mathcal{C} \rightarrow \mathcal{D}$ induces a 2-functor $\mathcal{TC} \rightarrow \mathcal{D}$ carrying a path $f_m * \dots * f_1$ to $H(f_m) \dots H(f_1)$ and carrying the classes of ι and γ to the 2-cells of the coherence constraint. $\square$

By the above propositions, the inclusion $2\text{Fun}^{\text{ps}}(\mathcal{TC}, \mathcal{D}) \subseteq \mathcal{PsFun}(\mathcal{TC}, \mathcal{D})$ is a $\mathcal{D}^{\text{Ob}(\mathcal{C})}$ -2-equivalence. This has the following generalization, which will be used later.

Lemma 2.16 *Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories. Assume that the underlying category of $\mathcal{C}$ is a free category on a graph $S \rightrightarrows \text{Ob}(\mathcal{C})$. Then the inclusion $2\text{Fun}^{\text{ps}}(\mathcal{C}, \mathcal{D}) \subseteq \mathcal{PsFun}(\mathcal{C}, \mathcal{D})$ is a $\mathcal{D}^{\text{Ob}(\mathcal{C})}$ -2-equivalence.*

The assumption means that there exists a class S of morphisms of $\mathcal{C}$ such that every morphism of $\mathcal{C}$ is a composition in a unique way of morphisms in S .

Proof By Lemma 2.8, it suffices to show that for every strictly unital pseudo functor $H: \mathcal{C} \rightarrow \mathcal{D}$, there exists a 2-functor $H': \mathcal{C} \rightarrow \mathcal{D}$ and a pseudo natural isomorphism $\epsilon: H \Rightarrow H'$ carrying objects to identities. For any morphism $f = f_m \dots f_1$ of $\mathcal{C}$ with $f_i \in S$, we take $H'(f) = H(f_m) \dots H(f_1)$ and take $\epsilon(f)$ to be the 2-cell $H(f_m) \dots H(f_1) \Rightarrow H(f_m \dots f_1)$ provided by the unital constraint for $m = 0$, identity for $m = 1$, and composition constraint for $m \geq 2$. For any 2-cell $\alpha: f \Rightarrow g$, the 2-cell $H'(\alpha)$ is given by $H(\alpha)$ and the unital and composition constraints. $\square$

We finish this section with a couple of constructions that will be used later.

Construction 2.17 (Adjoint of the inclusion functor $\mathbf{Cat} \rightarrow \mathbf{2Cat}$) For a 2-category $\mathcal{C}$, we define a category $\mathbf{OC}$ with the same objects as $\mathcal{C}$ by $(\mathbf{OC})(X, Y) = \pi_0(\mathcal{C}(X, Y))$ for objects X and Y of $\mathcal{C}$. Here π_0 is the set of connected components. We obtain a functor $\mathbf{O}: \mathbf{2Cat} \rightarrow \mathbf{Cat}$. The 2-functor $\mathcal{C} \rightarrow \mathbf{OC}$ carrying a morphism to its connected component exhibits $\mathbf{O}$ as a left adjoint to the inclusion functor $\mathbf{Cat} \rightarrow \mathbf{2Cat}$. The latter also admits a right adjoint $\mathbf{2Cat} \rightarrow \mathbf{Cat}$ sending a 2-category to its underlying category and a 2-functor to its underlying functor.

A *pseudo* functor $F: \mathcal{C} \rightarrow \mathcal{D}$ also induces a functor $\mathbf{O}F: \mathbf{O}\mathcal{C} \rightarrow \mathbf{O}\mathcal{D}$. A pseudo natural transformation $\alpha: F \rightarrow G$ induces a natural transformation $\mathbf{O}\alpha: \mathbf{O}F \rightarrow \mathbf{O}G$. If there exists a modification $\alpha \rightarrow \beta$, then $\mathbf{O}\alpha = \mathbf{O}\beta$. In particular, $\mathbf{O}$ carries bi-equivalences to equivalences and natural equivalences to natural isomorphisms.

Construction 2.18 (Adjoint of the inclusion functor $(\mathbf{2}, \mathbf{1})\mathbf{Cat} \rightarrow \mathbf{2Cat}$) Let $\mathcal{C}$ be a 2-category. We define two $(2, 1)$ -categories, $\mathcal{LC}$ and $\mathcal{RC}$, with the same objects as $\mathcal{C}$, as follows. For objects X and Y of $\mathcal{C}$, $(\mathcal{RC})(X, Y)$ is the greatest subgroupoid of $\mathcal{C}(X, Y)$ and $(\mathcal{LC})(X, Y)$ is the fundamental groupoid of $\mathcal{C}(X, Y)$, namely the category obtained from $\mathcal{C}(X, Y)$ by inverting all morphisms of $\mathcal{C}(X, Y)$ [13, Section I.1] (there is no set-theoretic issue by Convention 2.1). Thus $\mathcal{RC}$ is the maximal sub- $(2, 1)$ -category of $\mathcal{C}$. If we let $(\mathbf{2}, \mathbf{1})\mathbf{Cat}$ denote the big category of $(2, 1)$ -categories and 2-functors, we obtain functors $\mathcal{L}, \mathcal{R}: \mathbf{2Cat} \rightarrow (\mathbf{2}, \mathbf{1})\mathbf{Cat}$. The “localization” 2-functor $\mathcal{C} \rightarrow \mathcal{LC}$ and the inclusion 2-functor $\mathcal{RC} \rightarrow \mathcal{C}$ exhibit $\mathcal{L}$ and $\mathcal{R}$ as left and right adjoints of the inclusion 2-functor $(\mathbf{2}, \mathbf{1})\mathbf{Cat} \rightarrow \mathbf{2Cat}$, respectively.

The operation $\mathcal{L}$ also acts on pseudo functors, pseudo natural transformations, and modifications, so that $\mathcal{L}$ preserves bi-equivalences and pseudo natural equivalences. The same holds for $\mathcal{R}$ except that $\mathcal{R}$ does not act on modifications in general, but only on invertible modifications. Moreover, the “localization” 2-functor $\mathcal{C} \rightarrow \mathcal{LC}$ and the inclusion 2-functor $\mathcal{RC} \rightarrow \mathcal{C}$ induce isomorphisms of 2-categories

$$\mathcal{PsFun}(\mathcal{LC}, \mathcal{D}) \rightarrow \mathcal{PsFun}(\mathcal{C}, \mathcal{D}), \quad \mathcal{PsFun}(\mathcal{D}, \mathcal{RC}) \rightarrow \mathcal{RPsFun}(\mathcal{D}, \mathcal{C}),$$

for every $(2, 1)$ -category $\mathcal{D}$. For any 2-category $\mathcal{D}$, the 2-functor $\mathcal{PsFun}(\mathcal{LC}, \mathcal{D}) \rightarrow \mathcal{PsFun}(\mathcal{C}, \mathcal{D})$ identifies $\mathcal{PsFun}(\mathcal{LC}, \mathcal{D})$ with the 2-full sub-2-category of $\mathcal{PsFun}(\mathcal{C}, \mathcal{D})$ spanned by pseudo functors that factor through $\mathcal{RD}$.

Our general discussion on categories and 2-categories ends here. In the next section we add n -fold categories to the picture.

3 2-Categories and n -fold categories

In this section, after recalling the definitions of n -fold categories and n -fold functors in Definition 3.2, we investigate the relation between 2-categories and n -fold categories. The functor $\mathbb{K}^n: \mathbf{Cat} \rightarrow n\mathbf{FoldCat}$ carrying a category to its n -fold category of hypercubes [Example 3.8 (2)] admits a left adjoint $\mathbf{T}_n: n\mathbf{FoldCat} \rightarrow \mathbf{Cat}$ carrying an n -fold category $\mathbb{C}$ to its category of paths (Remark 3.11). One goal of this section is to establish an analogue of this adjunction with $\mathbf{Cat}$ replaced by $\mathbf{2Cat}$ (Proposition 3.21). To do this, we extend $\mathbb{K}^n$ to a functor $\mathbb{Q}^n: \mathbf{2Cat} \rightarrow n\mathbf{FoldCat}$ carrying a 2-category $\mathcal{D}$ to its n -fold category of hypercubes $\mathbb{Q}^n\mathcal{D}$ (Definition 3.18, extending the double category of up-squares [2, 2.C.1, p. 272] in the case $n = 2$). We also construct a left adjoint $\mathcal{T}_n^{\text{red}}: n\mathbf{FoldCat} \rightarrow \mathbf{2Cat}$ (Definition 3.13) carrying $\mathbb{C}$ to its reduced 2-category of paths $\mathcal{T}_n^{\text{red}}\mathbb{C}$, which is a refinement of $\mathbf{T}_n\mathbb{C}$. This adjunction is related to Gray’s tensor product of 2-categories. The functors fit into the following diagram:

$$\begin{array}{ccccc}
 & & \mathbb{K}^n & & \\
 & \swarrow & & \searrow & \\
 \text{Cat} & \xleftrightarrow{\quad} & \mathbf{2Cat} & \xleftrightarrow{\quad} & n\text{FoldCat.} \\
 & \searrow & & \swarrow & \\
 & & \mathbb{T}_n & &
 \end{array}$$

$\mathbf{O}$ (between Cat and 2Cat), $\mathcal{T}_n^{\text{red}}$ (between 2Cat and nFoldCat)

We also construct a variant $\mathcal{T}_n$ of $\mathcal{T}_n^{\text{red}}$, closely related to pseudo functors. We use these constructions to define the 2-category of gluing data and the descent 2-functor $Q^n: \mathcal{Psfun}(\mathcal{C}, \mathcal{D}) \rightarrow \mathcal{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$, which are the main objects of study of this article.

The notion of n -fold categories was introduced by Ehresmann [7, Définition 15, p. 396]. His original definition proceeds by induction on n . For our purpose, it is more convenient to adopt a direct combinatorial definition: an n -fold category is a collection $(\mathbb{C}_J)_J$ of sets, where J runs through subsets of $\{1, \dots, n\}$ and $\mathbb{C}_J$ can be visualized as a set of hypercubes of dimension $\#J$, endowed with various sources, targets, units, and compositions. To specify the compatibility between these data, it is convenient to introduce the following notation. Note that the map carrying J to its characteristic function χ_J is a bijection from the set of subsets of $\{1, \dots, n\}$ onto $\{0, 1\}^n$.

Notation 3.1 For $m \geq 0$, we let $[m]$ denote the totally ordered set $\{0, 1, \dots, m\}$. We let $\mathbf{I}$ denote the category whose objects are $[0]$ and $[1]$ and whose morphisms are nondecreasing maps. In other words, the morphisms of $\mathbf{I}$ are the identity maps, the face maps $d_0^1, d_1^1: [0] \rightarrow [1]$ which skip 0 and 1 respectively, and the degeneracy map $s: [1] \rightarrow [0]$. We denote by $\mathbf{Set}$ the big category of sets. A category $\mathbf{C}$ can be viewed as a functor $\mathbf{C}: \mathbf{I}^{\text{op}} \rightarrow \mathbf{Set}$ with $\text{Ob}(\mathbf{C}) = \mathbf{C}([0])$ and $\text{Mor}(\mathbf{C}) = \mathbf{C}([1])$ endowed with a composition map

$$\mathbf{C}([1]) \times_{\mathbf{C}(d_1^1), \mathbf{C}([0]), \mathbf{C}(d_0^1)} \mathbf{C}([1]) \rightarrow \mathbf{C}([1]).$$

We omit the maps $\mathbf{C}(d_1^1)$ and $\mathbf{C}(d_0^1)$ when no confusion arises.

For $n \geq 0$, we identify objects of $\mathbf{I}^n$ with elements of $\{0, 1\}^n$ and let ϵ_i denote the element such that $\epsilon_i(i) = 1$ and $\epsilon_i(j) = 0$ for $j \neq i$. For an object of α , we let $\Sigma\alpha$ denote the sum $0 \leq \alpha_1 + \dots + \alpha_n \leq n$.

The following definition is similar to [11, Definition 2.2].²

Definition 3.2 An n -fold category is a functor $\mathbb{C}: (\mathbf{I}^n)^{\text{op}} \rightarrow \mathbf{Set}$ endowed with a composition map

$$\circ^i: \mathbb{C}(\alpha') \times_{\mathbb{C}(\alpha)} \mathbb{C}(\alpha') \rightarrow \mathbb{C}(\alpha')$$

for every $1 \leq i \leq n$ and every pair (α, α') of objects of $\mathbf{I}^n$ satisfying $\alpha' = \alpha + \epsilon_i$, such that the following axioms hold:

- (1) For all i and (α, α') as above, the composite $\mathbf{I}^{\text{op}} \xrightarrow{\iota} (\mathbf{I}^n)^{\text{op}} \xrightarrow{\mathbb{C}} \mathbf{Set}$, where ι is the functor carrying $[0]$ to α and $[1]$ to α' , is a category.

² Our axiom (2) appears to be missing in [11, Definition 2.2].

- (2) (functoriality) For every $1 \leq i \leq n$ and every morphism $\beta \rightarrow \alpha$ in $\mathbf{I}^n$ satisfying $\alpha_i = \beta_i = 0$, the diagram

$$\begin{array}{ccc} \mathbb{C}(\alpha') \times_{\mathbb{C}(\alpha)} \mathbb{C}(\alpha') & \xrightarrow{\circ^i} & \mathbb{C}(\alpha') \\ \downarrow & & \downarrow \\ \mathbb{C}(\beta') \times_{\mathbb{C}(\beta)} \mathbb{C}(\beta') & \xrightarrow{\circ^i} & \mathbb{C}(\beta'), \end{array}$$

where $\alpha' = \alpha + \epsilon_i$, $\beta' = \beta + \epsilon_i$, commutes.

- (3) (interchange law) For all $1 \leq i < j \leq n$ and every object α of $\mathbf{I}^n$ satisfying $\alpha_i = \alpha_j = 0$, the diagram

$$\begin{array}{ccc} X & \xrightarrow{\circ^i \times \circ^i} & \mathbb{C}(\bar{\alpha}') \times_{\mathbb{C}(\alpha')} \mathbb{C}(\bar{\alpha}') \\ \downarrow \circ^j \times \circ^j & & \downarrow \circ^j \\ \mathbb{C}(\bar{\alpha}') \times_{\mathbb{C}(\bar{\alpha})} \mathbb{C}(\bar{\alpha}') & \xrightarrow{\circ^i} & \mathbb{C}(\bar{\alpha}') \end{array}$$

commutes. Here $\alpha' = \alpha + \epsilon_i$, $\bar{\alpha} = \alpha + \epsilon_j$, $\bar{\alpha}' = \alpha + \epsilon_i + \epsilon_j$, and X is the limit of the diagram

$$\begin{array}{ccccc} \mathbb{C}(\bar{\alpha}') & \longrightarrow & \mathbb{C}(\alpha') & \longleftarrow & \mathbb{C}(\bar{\alpha}') \\ \downarrow & & & & \downarrow \\ \mathbb{C}(\bar{\alpha}) & & & & \mathbb{C}(\bar{\alpha}) \\ \uparrow & & & & \uparrow \\ \mathbb{C}(\bar{\alpha}') & \longrightarrow & \mathbb{C}(\alpha') & \longleftarrow & \mathbb{C}(\bar{\alpha}'). \end{array}$$

Elements of the set $\text{Ob}(\mathbb{C}) := \mathbb{C}(\mathbf{0})$, where $\mathbf{0} = (0, \dots, 0)$, are called *objects* of $\mathbb{C}$. Elements of $\mathbb{C}(\epsilon_j)$ are called *morphisms in direction j* . Elements of $\mathbb{C}(\alpha)$ are called α -*hypercubes* (or *J-hypercubes* for $\alpha = \chi_J$) of $\mathbb{C}$.

An n -fold functor $\mathbb{C} \rightarrow \mathbb{D}$ between n -fold categories $\mathbb{C}$ and $\mathbb{D}$ is a natural transformation compatible with the composition maps. We let $n\text{FoldFun}(\mathbb{C}, \mathbb{D})$ denote the set of n -fold functors from $\mathbb{C}$ to $\mathbb{D}$.

We let $n\text{FoldCat}$ denote the big category of n -fold categories and n -fold functors.

We have an isomorphism $1\text{FoldCat} \cong \text{Cat}$.

Example 3.3 A 2-fold category is called a *double category* [7, Définition 10, p. 389]. A double category $\mathbb{C}$ consists of a set $\text{Ob}(\mathbb{C}) = \mathbb{C}(0, 0)$ of objects, a set $\text{Hor}(\mathbb{C}) = \mathbb{C}(1, 0)$ of horizontal morphisms, a set $\text{Ver}(\mathbb{C}) = \mathbb{C}(0, 1)$ of vertical morphisms, and a set $\text{Sq}(\mathbb{C}) = \mathbb{C}(1, 1)$ of squares, equipped with various sources, targets, and associative and unital compositions. We will sometimes enumerate the elements of

the set of directions $\{1, 2\}$ using the notation $h = 1$ and $v = 2$. Functoriality implies that given two squares that are horizontally composable as shown in the diagram

$$\begin{array}{ccccc} X_1 & \xrightarrow{j} & X_2 & \xrightarrow{j'} & X_3 \\ \downarrow & & \downarrow & & \downarrow \\ Y_1 & \xrightarrow{i} & Y_2 & \xrightarrow{i'} & Y_3, \end{array}$$

the upper arrow of the horizontal composition $D' \circ^h D$ is $j' \circ j$, and the lower arrow is $i' \circ i$. The same holds for vertical composition. The interchange law means that given four squares as shown in the diagram

$$\begin{array}{ccccc} X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 \\ \downarrow & & \downarrow & & \downarrow \\ Y_1 & \longrightarrow & Y_2 & \longrightarrow & Y_3 \\ \downarrow & & \downarrow & & \downarrow \\ Z_1 & \longrightarrow & Z_2 & \longrightarrow & Z_3, \end{array}$$

the two ways of composing them produce the same square:

$$(E' \circ^h E) \circ^v (D' \circ^h D) = (E' \circ^v D') \circ^h (E \circ^v D).$$

Remark 3.4 Fiore and Paoli defined the n -fold nerve functor, which is a fully faithful functor from the big category $n\mathbf{FoldCat}$ to the big category of n -fold simplicial sets [11, Definition 2.14, Proposition 2.17]. As in the case $n = 1$ (see for example [21, Proposition 1.1.2.2]; see also [17, Proposition VI.2.2.3]), the essential image of the functor consists of n -fold simplicial sets satisfying the unique right lifting property with respect to the inclusions

$$\Lambda_k^{m_i} \boxtimes \left(\bigboxtimes_{\substack{1 \leq j \leq n \\ j \neq i}} \Delta^{m_j} \right) \subseteq \Delta^{m_1} \boxtimes \dots \boxtimes \Delta^{m_n},$$

$1 \leq i \leq n$, $0 < k < m_i$, and $m_1, \dots, m_n \geq 0$. Here $\Lambda_k^{m_i} \subseteq \Delta^{m_i}$ denotes the k th horn in the m_i -simplex. Multisimplicial sets satisfying suitable lifting properties play an essential role in the theory of gluing functors between ∞ -categories developed in [20].

Notation 3.5 (External product $\mathbb{C} \boxtimes \mathbb{D}$ of an m -fold category $\mathbb{C}$ and an n -fold category $\mathbb{D}$) Let $\mathbb{C}$ be an m -fold category and let $\mathbb{D}$ be an n -fold category. We let $\mathbb{C} \boxtimes \mathbb{D}$ denote their *external product*, which is the $(m+n)$ -fold category $\mathbb{E}$ given by $\mathbb{E}(\alpha) = \mathbb{C}(\beta) \times \mathbb{D}(\gamma)$ for $\alpha = (\beta_1, \dots, \beta_m, \gamma_1, \dots, \gamma_n)$, with $\circ_{\mathbb{E}}^i$ for $1 \leq i \leq m$ given by the $\circ_{\mathbb{C}}^i$ and $\circ_{\mathbb{E}}^{m+i}$ for $1 \leq i \leq n$ given by $\circ_{\mathbb{D}}^i$. We get a functor

$$\boxtimes: m\mathbf{FoldCat} \times n\mathbf{FoldCat} \rightarrow (m+n)\mathbf{FoldCat}.$$

In particular, given categories $\mathbf{C}_1, \dots, \mathbf{C}_n$, the external product $\mathbb{C} = \mathbf{C}_1 \boxtimes \dots \boxtimes \mathbf{C}_n$ is the n -fold category defined in [11, Definition 2.9]. We have $\mathbb{C}(\alpha) = \prod_{1 \leq i \leq n} \mathbf{C}_i(\alpha_i)$ with $\circ^i$ given by the composition in $\mathbf{C}_i$.

For any n -fold category $\mathbb{D}$ and any object α of $\mathbf{I}^n$, the set $\mathbb{D}(\alpha)$ of α -hypercubes can be identified with $n\mathbf{FoldFun}(\alpha_1 \boxtimes \dots \boxtimes \alpha_n, \mathbb{D})$.

Remark 3.6 Let $\mathbb{C}$ and $\mathbb{D}$ be n -fold categories. The n -fold functors $\mathbb{C} \rightarrow \mathbb{D}$ can be organized into an n -fold category $n\mathbf{FoldFun}(\mathbb{C}, \mathbb{D})$ as follows. For any object α of $\mathbf{I}^n$, we define $n\mathbf{FoldFun}(\alpha) := n\mathbf{FoldFun}((\alpha_1 \boxtimes \dots \boxtimes \alpha_n) \times \mathbb{C}, \mathbb{D})$. If $\mathbb{E}$ is another n -fold category, we have

$$n\mathbf{FoldFun}(\mathbb{C}, n\mathbf{FoldFun}(\mathbb{D}, \mathbb{E})) \cong n\mathbf{FoldFun}(\mathbb{C} \times \mathbb{D}, \mathbb{E}).$$

We now define the pullback functor ϕ^* for a map ϕ between two sets of directions.

Definition 3.7 (*The functor ϕ^**) Let $\phi: \{1, \dots, m\} \rightarrow \{1, \dots, n\}$ be a map and let $\mathbb{C}$ be an n -fold category. We define an m -fold category $\phi^*\mathbb{C}$ by

$$(\phi^*\mathbb{C})(\alpha) := n\mathbf{FoldFun}\left(\boxtimes_{j=1}^n \prod_{\phi(i)=j} \alpha_i, \mathbb{C}\right) \quad (3.1)$$

for every object α of $\mathbf{I}^m$. In particular, we have $(\phi^*\mathbb{C})(\epsilon_i) = \mathbb{C}(\epsilon_{\phi(i)})$. For $1 \leq i \leq m$, the composition $\circ^i$ in $\phi^*\mathbb{C}$ is given by $\circ^{\phi(i)}$ in $\mathbb{C}$. More formally, for objects α and α' of $\mathbf{I}^m$ satisfying $\alpha' = \alpha + \epsilon_i$, the composition $\circ^i$ is given by the map

$$(\phi^*\mathbb{C})(\alpha') \times_{(\phi^*\mathbb{C})(\alpha)} (\phi^*\mathbb{C})(\alpha') \cong n\mathbf{FoldFun}\left(\boxtimes_{j=1}^n \prod_{\phi(k)=j} \tilde{\alpha}_k, \mathbb{C}\right) \rightarrow (\phi^*\mathbb{C})(\alpha')$$

induced by d_1^2 , where

$$\tilde{\alpha}_k = \begin{cases} [2] & \text{if } k = i, \\ \alpha_k & \text{if } k \neq i. \end{cases}$$

We obtain a functor $\phi^*: n\mathbf{FoldCat} \rightarrow m\mathbf{FoldCat}$.

For a sequence of maps $\{1, \dots, l\} \xrightarrow{\psi} \{1, \dots, m\} \xrightarrow{\phi} \{1, \dots, n\}$, we have $(\phi\psi)^* \cong \psi^*\phi^*$.

Example 3.8 (Special cases of the functor ϕ^*)

- (1) Let $\iota_i: \{1\} \rightarrow \{1, \dots, n\}$ be the map with image $\{i\}$. For an n -fold category $\mathbb{C}$, we have $\text{Ob}(\iota_i^* \mathbb{C}) \cong \mathbb{C}(0, \dots, 0)$ and $\text{Mor}(\iota_i^* \mathbb{C}) \cong \mathbb{C}(\epsilon_i)$. More generally, the functor ϕ^* for ϕ strictly increasing has been studied by Fiore and Paoli [11, Notation 2.12].
- (2) Let $\text{pr}_n: \{1, \dots, n\} \rightarrow \{1\}$. We denote the functor pr_n^* by $\mathbb{K}^n$. For a category $\mathbb{C}$, we have $(\mathbb{K}^n \mathbb{C})(\alpha) = \text{Fun}(\alpha_1 \times \dots \times \alpha_n, \mathbb{C})$. In other words, α -hypercubes of $\mathbb{K}^n \mathbb{C}$ are the commutative hypercubes of dimension $\Sigma \alpha$ in $\mathbb{C}$. In particular, morphisms in each direction are the morphisms of $\mathbb{C}$. We call $\mathbb{K}^n \mathbb{C}$ the *n -fold category of hypercubes in $\mathbb{C}$* . This was defined by Ehresmann [7, end of p. 398] and, in the case $n = 2$, was called the double category of quartets (*quatuors* in French) [7, Proposition 12, p. 394].
- (3) Let $t: \{1, 2\} \rightarrow \{1, 2\}$ be the map swapping 1 and 2. Then $\mathbb{C}^t := t^* \mathbb{C}$ is the transpose of $\mathbb{C}$ in the sense that $\text{Ob}(\mathbb{C}^t) = \text{Ob}(\mathbb{C})$, $\text{Hor}(\mathbb{C}^t) = \text{Ver}(\mathbb{C})$, $\text{Ver}(\mathbb{C}^t) = \text{Hor}(\mathbb{C})$, and $\text{Sq}(\mathbb{C}^t)$ is obtained from $\text{Sq}(\mathbb{C})$ by transposing the squares.

Remark 3.9 (The functor $\phi_!$) For any map $\phi: \{1, \dots, m\} \rightarrow \{1, \dots, n\}$, the functor ϕ^* admits a left adjoint $\phi_!: m\mathbf{FoldCat} \rightarrow n\mathbf{FoldCat}$. Let $\mathbb{C}$ be an m -fold category. In the case where ϕ is injective, $\phi_! \mathbb{C}$ can be easily described by the formula $(\phi_! \mathbb{C})(\alpha) = \mathbb{C}(\alpha \circ \phi)$ for every object α of $\mathbf{I}^n$, where $\alpha \circ \phi$ is the object of $\mathbf{I}^m$ given by $(\alpha \circ \phi)_i = \alpha_{\phi(i)}$, and composition in direction i of $\mathbb{C}$ provides composition in direction $\phi(i)$ of $\phi_! \mathbb{C}$. In general, the assignment $\alpha \mapsto \mathbb{C}(\alpha \circ \phi)$ does not provide an n -fold category, since composition in direction j is not well-defined whenever $\#\phi^{-1}(j) > 1$. In general, $\phi_! \mathbb{C}$ is the fundamental n -fold category [11, Proposition 2.18] of the n -fold simplicial set $(\alpha_1, \dots, \alpha_n) \mapsto n\text{FoldFun}(\alpha_{\phi(1)} \boxtimes \dots \boxtimes \alpha_{\phi(m)}, \mathbb{C})$, where each α_i is a (combinatorial) simplex.

For an m -fold category $\mathbb{C}$ and an n -fold category $\mathbb{D}$, we have $\mathbb{C} \boxtimes \mathbb{D} \cong (\phi_! \mathbb{C}) \times (\psi_! \mathbb{D})$, where $\phi: \{1, \dots, m\} \rightarrow \{1, \dots, m+n\}$ is the inclusion and $\psi: \{1, \dots, n\} \rightarrow \{1, \dots, m+n\}$ is the map given by $\psi(i) = m+i$. In (3.1), we have $\boxtimes_{j=1}^n \prod_{\phi(k)=j} \alpha_k \cong \phi_!(\alpha_1 \boxtimes \dots \boxtimes \alpha_m)$.

For the map $\text{pr}_n: \{1, \dots, n\} \rightarrow \{1\}$, the category $(\text{pr}_n)_! \mathbb{C}$ associated to an n -fold category $\mathbb{C}$ is the fundamental category of the simplicial set $[m] \mapsto n\text{FoldFun}([m]^{\boxtimes n}, \mathbb{C})$. A priori this construction involves $\mathbb{C}(1, \dots, 1)$ hence hypercubes of $\mathbb{C}$ of all dimensions. However, we have the following simpler description.

Definition 3.10 (Category of paths $\mathbf{T}_n \mathbb{C}$ in an n -fold category $\mathbb{C}$) Let $\mathbb{C}$ be an n -fold category. We define the *category $\mathbf{T}_n \mathbb{C}$ of paths in $\mathbb{C}$* as follows. The objects of $\mathbf{T}_n \mathbb{C}$ are the objects of $\mathbb{C}$. A *morphism* $X \rightarrow Y$ of $\mathbf{T}_n \mathbb{C}$ is an equivalence class of paths

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \dots \xrightarrow{f_{m-1}} X_{m-1} \xrightarrow{f_m} X_m = Y, \quad m \geq 0$$

where $f_i \in \coprod_{1 \leq k \leq n} \mathbb{C}(\epsilon_k)$, $1 \leq i \leq m$ under the equivalence relation $\sim$ generated by the following conditions:

- (1) (unit) $g * f \sim g * \text{id}_Y^k * f$, where $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a sequence of paths, $\text{id}_Y^k \in \mathbb{C}(\epsilon_k)$ is identity morphism of Y .

- (2) (composition) $g * h' * h * f \sim g * (h'h) * f$, where $X \xrightarrow{f} Y \xrightarrow{h} Y' \xrightarrow{h'} Y'' \xrightarrow{g} Z$ is a sequence of paths, h and h' belong to the same $\mathbb{C}(\epsilon_k)$ and $h'h$ is their composition in $\mathbb{C}$.
- (3) (square) $g * i * q * f \sim g * p * j * f$, where $i, j \in \mathbb{C}(\epsilon_k)$, $p, q \in \mathbb{C}(\epsilon_{k'})$, $k < k'$, $D \in \mathbb{C}(\epsilon_k + \epsilon_{k'})$ is of the form

$$\begin{array}{ccc} X & \xrightarrow{j} & Y \\ q \downarrow & D & \downarrow p \\ Z & \xrightarrow{i} & W, \end{array} \quad (3.2)$$

$g: W \rightarrow W'$ and $f: X' \rightarrow X$ are paths.

Here $*$ denotes concatenation of paths. The identity morphism $\text{id}_X: X \rightarrow X$ in $\mathbf{T}_n\mathbb{C}$ is given by the path of length 0. Composition of morphisms is given by concatenation of paths.

We obtain a functor $\mathbf{T}_n: n\mathbf{FoldCat} \rightarrow \mathbf{Cat}$.

In other words, $\mathbf{T}_n\mathbb{C}$ is the quotient of the free category on the graph $\coprod_{1 \leq k \leq n} \mathbb{C}(\epsilon_k) \Rightarrow \text{Ob}(\mathbb{C})$ by the relations (1) through (3) above.

Note that $\mathbf{T}_n\mathbb{C}$ does not depend on hypercubes of $\mathbb{C}$ of dimension > 2 . We have an isomorphism $\mathbf{T}_n\mathbb{C} \cong (\text{pr}_n)_! \mathbb{C}$ that is the identity on objects and carries the class of $f \in \mathbb{C}(\epsilon_i)$ to the class of the image of f under the degeneracy map $\mathbb{C}(\epsilon_i) \rightarrow \mathbb{C}([1], \dots, [1])$. The inverse carries the class of a hypercube $C \in \mathbb{C}(\alpha)$ to the class of any path from the initial vertex to the final vertex of the hypercube, the class of the path being independent of the choice of the path.

For n -fold categories $\mathbb{C}$ and $\mathbb{D}$, we have an isomorphism $\mathbf{T}_n(\mathbb{C} \times \mathbb{D}) \cong (\mathbf{T}_n\mathbb{C}) \times (\mathbf{T}_n\mathbb{D})$ that is the identity on objects and carries the class of $(f, g) \in (\mathbb{C} \times \mathbb{D})(\epsilon_k)$ to $([f], [g])$, where $[f]$ and $[g]$ denote the classes of f and g , respectively. The inverse carries $([f_m * \dots * f_1], [g_l * \dots * g_1])$ to the path $(f_m, \text{id}) * \dots * (f_1, \text{id}) * (\text{id}, g_l) * \dots * (\text{id}, g_1)$, which is equivalent to the path $(\text{id}, g_l) * \dots * (\text{id}, g_1) * (f_m, \text{id}) * \dots * (f_1, \text{id})$ by conditions (2) and (3) above.

For an m -fold category $\mathbb{C}$ and an n -fold category $\mathbb{D}$, we have $\mathbf{T}_{m+n}(\mathbb{C} \boxtimes \mathbb{D}) \cong (\mathbf{T}_m\mathbb{C}) \times (\mathbf{T}_n\mathbb{D})$. In particular, for categories $\mathbf{C}_1, \dots, \mathbf{C}_n$, we have $\mathbf{T}_n(\mathbf{C}_1 \boxtimes \dots \boxtimes \mathbf{C}_n) \cong \mathbf{C}_1 \times \dots \times \mathbf{C}_n$.

Remark 3.11 (Adjunction between $\mathbf{T}_n$ and $\mathbb{K}^n$) Let $\mathbb{K}^n = \text{pr}_n^*: \mathbf{Cat} \rightarrow n\mathbf{FoldCat}$ be the functor carrying a category to its n -fold category of hypercubes as in Example 3.8 (2). Let us describe the adjunction between $\mathbf{T}_n$ and $\mathbb{K}^n$ more explicitly. For every category $\mathbf{D}$, we have an isomorphism of categories $\mathbf{T}_n\mathbb{K}^n\mathbf{D} \cong \mathbf{D}$ which is the identity on objects and carries the morphism represented by a path $f_m * \dots * f_1$, where $f_i \in \coprod_{1 \leq k \leq n} (\mathbb{K}^n\mathbf{D})(\epsilon_k)$ to the composite $f_m \dots f_1$. For every n -fold category $\mathbb{C}$, we have an n -fold functor $\mathbb{C} \rightarrow \mathbb{K}^n\mathbf{T}_n\mathbb{C}$ carrying a hypercube $C \in \mathbb{C}(\alpha)$, corresponding to an n -fold functor $\alpha_1 \boxtimes \dots \boxtimes \alpha_n \rightarrow \mathbb{C}$, to the functor $\alpha_1 \times \dots \times \alpha_n \cong \mathbf{T}_n(\alpha_1 \boxtimes \dots \boxtimes \alpha_n) \rightarrow \mathbf{T}_n\mathbb{C}$. These constructions exhibit $\mathbf{T}_n$ as a left adjoint of $\mathbb{K}^n$. Furthermore, we have an isomorphism of n -fold categories

$$\mathbb{K}^n \mathbf{Fun}(\mathbf{T}_n \mathbb{C}, \mathbf{D}) \cong n\mathbf{FoldFun}(\mathbb{C}, \mathbb{K}^n \mathbf{D}), \quad (3.3)$$

functorial in $\mathbb{C}$ and $\mathbf{D}$. Indeed, we have

$$\begin{aligned} \mathbf{Fun}(\alpha_1 \times \cdots \times \alpha_n, \mathbf{Fun}(\mathbf{T}_n \mathbb{C}, \mathbf{D})) &\cong \mathbf{Fun}(\mathbf{T}_n((\alpha_1 \boxtimes \cdots \boxtimes \alpha_n) \times \mathbb{C}), \mathbf{D}) \\ &\cong n\mathbf{FoldFun}((\alpha_1 \boxtimes \cdots \boxtimes \alpha_n) \times \mathbb{C}, \mathbb{K}^n \mathbf{D}). \end{aligned}$$

The remainder of this section is devoted to 2-categorical analogues of the above. For the inclusion map $\iota_i: \{1\} \rightarrow \{1, \dots, n\}$ with image i , the functor $\iota_i^*: n\mathbf{FoldCat} \rightarrow \mathbf{Cat}$ has the following 2-categorical refinement.

Definition 3.12 To any double category $\mathbb{C}$, one associates the *underlying horizontal 2-category* $\mathcal{H}\mathbb{C}$ and the *underlying vertical 2-category* $\mathcal{V}\mathbb{C}$. The underlying category of $\mathcal{H}\mathbb{C}$ is $\iota_1^* \mathbb{C} = (\mathrm{Ob}(\mathbb{C}), \mathrm{Hor}(\mathbb{C}))$ and the underlying category of $\mathcal{V}\mathbb{C}$ is $\iota_2^* \mathbb{C} = (\mathrm{Ob}(\mathbb{C}), \mathrm{Ver}(\mathbb{C}))$. A 2-cell $\alpha: f \Rightarrow g$ in $\mathcal{H}\mathbb{C}$ is a square in $\mathbb{C}$ of the boundary

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ \parallel & & \parallel \\ X & \xrightarrow{f} & Y. \end{array} \quad (3.4)$$

A 2-cell $\alpha: f \Rightarrow g$ in $\mathcal{V}\mathbb{C}$ is a square in $\mathbb{C}$ of the boundary

$$\begin{array}{ccc} X & \xlongequal{\quad} & X \\ f \downarrow & & \downarrow g \\ Y & \xlongequal{\quad} & Y. \end{array} \quad (3.5)$$

We have isomorphisms $\mathcal{H}(\mathbb{C}') \cong (\mathcal{V}\mathbb{C})^{\mathrm{co}}$, $\mathcal{V}(\mathbb{C}') \cong (\mathcal{H}\mathbb{C})^{\mathrm{co}}$ (see Notation 2.3).

More generally, to an n -fold category $\mathbb{C}$, and $1 \leq i, j \leq n$, $i \neq j$, one associates the 2-category

$$\mathcal{H}_{i,j}\mathbb{C} = \begin{cases} \mathcal{H}(\iota_{i,j}^* \mathbb{C}) & \text{if } i < j, \\ \mathcal{V}(\iota_{j,i}^* \mathbb{C}) & \text{if } i > j, \end{cases}$$

where $\iota_{i,j}: \{1, 2\} \rightarrow \{1, \dots, n\}$ is the map sending 1 to i and 2 to j . We obtain a functor $\mathcal{H}_{i,j}: n\mathbf{FoldCat} \rightarrow 2\mathbf{Cat}$.

We now proceed to give a 2-categorical analogue of the adjunction between $\mathbf{T}_n$ and $\mathbb{K}^n$. We start by 2-categorical refinements of $\mathbf{T}_n$.

Definition 3.13 (2-Categories of paths $\mathcal{T}_n \mathbb{C}$ and $\mathcal{T}_n^{\mathrm{red}} \mathbb{C}$ in an n -fold category $\mathbb{C}$) Let $\mathbb{C}$ be an n -fold category. We define the 2-category $\mathcal{T}_n \mathbb{C}$ of paths in $\mathbb{C}$ as follows. The objects of $\mathcal{T}_n \mathbb{C}$ are the object of $\mathbb{C}$. A morphism $X \rightarrow Y$ of $\mathcal{T}_n \mathbb{C}$ is a path

$$X = X_0 \xrightarrow{f_1} X_1 \rightarrow \cdots \rightarrow X_{m-1} \xrightarrow{f_m} X_m = Y, \quad m \geq 0$$

where $f_i \in \coprod_{1 \leq k \leq n} \mathbb{C}(\epsilon_k)$, $1 \leq i \leq m$. The identity morphism $\text{id}_X: X \rightarrow X$ in $\mathcal{T}_n\mathbb{C}$ is the path of length 0. Composition of morphisms is given by concatenation of paths and is denoted by $*$. An *atomic 2-cell* between two paths sharing a source and a target is one of the following:

- (1) (creation of unit) $\iota_{g,f}^k: g * f \Rightarrow g * \text{id}_Y^k * f$ or (deletion of unit) $\theta_{g,f}^k: g * \text{id}_Y^k * f \Rightarrow g * f$, where $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a sequence of paths, $\text{id}_Y^k \in \mathbb{C}(\epsilon_k)$ is identity morphism of Y .
- (2) (composition) $\gamma_{g,h',h,f}: g * h' * h * f \Rightarrow g * (h'h) * f$ or (decomposition) $\delta_{g,h',h,f}: g * (h'h) * f \Rightarrow g * h' * h * f$, where $X \xrightarrow{f} Y \xrightarrow{h} Y' \xrightarrow{h'} Y'' \xrightarrow{g} Z$ is a sequence of paths, h and h' belong to the same $\mathbb{C}(\epsilon_k)$ and $h'h$ is their composition in $\mathbb{C}$.
- (3) (square) $\sigma_{g,D,f}: g * i * q * f \Rightarrow g * p * j * f$, where $i, j \in \mathbb{C}(\epsilon_k)$, $p, q \in \mathbb{C}(\epsilon_{k'})$, $k < k'$, $D \in \mathbb{C}(\epsilon_k + \epsilon_{k'})$ is of the form (3.2), $g: W \rightarrow W'$ and $f: X' \rightarrow X$ are paths.

A *pre-2-cell* $f \Rightarrow g$ is a sequence of atomic 2-cells $f = f_0 \Rightarrow f_1 \Rightarrow \cdots \Rightarrow f_{n-1} \Rightarrow f_n = g$. Vertical composition of pre-2-cells, denoted by $\circ$, is given by concatenation. Horizontal composition of pre-2-cells with morphisms is also given by concatenation: If $f, g: X \rightarrow Y$, $h: W \rightarrow X$, $h': Y \rightarrow Z$ are paths and $\alpha: f \Rightarrow g$ is a pre-2-cell, then one has the pre-2-cell $h' * \alpha * h: h' * f * h \Rightarrow h' * g * h$. Consider systems which associate to every pair of paths (f, g) sharing a source and a target, an equivalence relation on the set of pre-2-cells $f \Rightarrow g$. We say that one system $\sim$ is finer than another system $\sim'$ if $\alpha \sim \beta$ implies $\alpha \sim' \beta$. There is a finest system $\sim$ satisfying the following conditions:

- (1) (Stability under horizontal and vertical composition) If $f, g: X \rightarrow Y$, $h: W \rightarrow X$, $h': Y \rightarrow Z$ are paths, $\alpha \sim \beta: f \Rightarrow g$, $\eta: f' \Rightarrow f$, $\eta': g \Rightarrow g'$ are pre-2-cells, then $\eta' \circ \alpha \circ \eta \sim \eta' \circ \beta \circ \eta$ and $h' * \alpha * h \sim h' * \beta * h$.
- (2) (Interchange law) If $f, f': X \rightarrow Y$, $g, g': Y \rightarrow Z$ are paths, $\alpha: f \Rightarrow f'$ and $\beta: g \Rightarrow g'$ are pre-2-cells, then $(f' * \beta) \circ (\alpha * g) \sim (\alpha * g') \circ (f * \beta)$.
- (3) (Creation and deletion of unit) For $1 \leq k \leq n$ and every object X of $\mathbb{C}$,

$$\theta_{\text{id}_X, \text{id}_X}^k \circ \iota_{\text{id}_X, \text{id}_X}^k \sim \text{id}_{\text{id}_X}, \quad \iota_{\text{id}_X, \text{id}_X}^k \circ \theta_{\text{id}_X, \text{id}_X}^k \sim \text{id}_{\text{id}_X}.$$

- (4) (Unit and composition) For $1 \leq k \leq n$ and $f: X \rightarrow Y$ belonging to $\mathbb{C}(\epsilon_k)$,

$$\iota_{f, \text{id}_X}^k \sim \delta_{\text{id}_Y, f, \text{id}_X, \text{id}_X}^k, \quad \iota_{\text{id}_Y, f}^k \sim \delta_{\text{id}_Y, \text{id}_Y^k, f, \text{id}_X}^k,$$

- (5) (Composition and decomposition) For $1 \leq k \leq n$ and every sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ of morphisms in $\mathbb{C}(\epsilon_k)$,

$$\delta_{\text{id}_Z, g, f, \text{id}_X} \circ \gamma_{\text{id}_Z, g, f, \text{id}_X} \sim \text{id}_{g * f}, \quad \gamma_{\text{id}_Z, g, f, \text{id}_X} \circ \delta_{\text{id}_Z, g, f, \text{id}_X} \sim \text{id}_{g * f}.$$

- (6) (Associativity of composition) For $1 \leq k \leq n$ and every sequence $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ of morphisms in $\mathbb{C}(\epsilon_k)$,

$$\gamma_{\text{id}_W, h, g, f, \text{id}_X} \circ \gamma_{h, g, f, \text{id}_X} \sim \gamma_{\text{id}_W, h, g, f, \text{id}_X} \circ \gamma_{\text{id}_W, h, g, f}.$$

(7) (Unit square) For $1 \leq k, k' \leq n$ ($k \neq k'$) and $f: X \rightarrow Y$ belonging to $\mathbb{C}(\epsilon_k)$, we have

$$\begin{cases} \sigma_{\text{id}_Y, D, \text{id}_X} \circ \iota_{f, \text{id}_X}^{k'} \sim \iota_{\text{id}_Y, f}^{k'} & \text{if } k < k', \\ \sigma_{\text{id}_Y, D, \text{id}_X} \circ \iota_{\text{id}_Y, f}^{k'} \sim \iota_{f, \text{id}_X}^{k'} & \text{if } k' < k, \end{cases}$$

where $D = s(f) \in \mathbb{C}(\epsilon_k + \epsilon_{k'})$ is the identity square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \parallel & & \parallel \\ X & \xrightarrow{f} & Y. \end{array} \quad (3.6)$$

(8) (Composition of squares) For $1 \leq k, k' \leq n$ ($k \neq k'$) and $D'' = D' \circ^k D$ in $\mathbb{C}(\epsilon_k + \epsilon_{k'})$ of the form

$$\begin{array}{ccccc} X_1 & \xrightarrow{j} & X_2 & \xrightarrow{j'} & X_3 \\ p_1 \downarrow & & D & \downarrow p_2 & D' \downarrow p_3 \\ Y_1 & \xrightarrow{i} & Y_2 & \xrightarrow{i'} & Y_3, \end{array} \quad (3.7)$$

we have

$$\begin{cases} \gamma_{p_3, j', j, \text{id}_{X_1}} \circ \sigma_{\text{id}_{Y_3}, D', j} \circ \sigma_{i', D, \text{id}_{X_1}} \sim \sigma_{\text{id}_{Y_3}, D'', \text{id}_{X_1}} \circ \gamma_{\text{id}_{Y_3}, i', i, p_1} & \text{if } k < k', \\ \gamma_{\text{id}_{Y_3}, i', i, p_1} \circ \sigma_{i', D, \text{id}_{X_1}} \circ \sigma_{\text{id}_{Y_3}, D', j} \sim \sigma_{\text{id}_{Y_3}, D'', \text{id}_{X_1}} \circ \gamma_{p_3, j', j, \text{id}_{X_1}} & \text{if } k' < k. \end{cases}$$

(9) (Commutativity of cubes) For $1 \leq k < k' < k'' \leq n$ and $C \in \mathbb{C}(\epsilon_k + \epsilon_{k'} + \epsilon_{k'')}$ of the form

$$\begin{array}{ccccc} X' & \xrightarrow{b'} & Y' & & \\ \downarrow x & \searrow q' & \downarrow y & \searrow p' & \\ & Z' & \xrightarrow{a'} & W' & \\ & \downarrow z & \downarrow & \downarrow w & \\ X & \xrightarrow{b} & Y & \xrightarrow{p} & W \\ & \searrow q & \downarrow & \searrow a & \\ & Z & \xrightarrow{a} & W & \end{array}$$

where $a, b, a', b' \in \mathbb{C}(\epsilon_k)$, $p, q, p', q' \in \mathbb{C}(\epsilon_{k'})$, $x, y, z, w \in \mathbb{C}(\epsilon_{k''})$, we have

$$\sigma_{\text{id}_W, I, b'} \circ \sigma_{p', J', \text{id}_{X'}} \circ \sigma_{\text{id}_W, K, x} \sim \sigma_{w, K', \text{id}_{X'}} \circ \sigma_{\text{id}_W, J, q'} \circ \sigma_{a, I', \text{id}_{X'}},$$

where I, I', J, J', K, K' are respectively the right, left, front, back, bottom, top faces of the cube.

A 2-cell $f \Rightarrow g$ of $\mathcal{T}_n\mathbb{C}$ is an equivalence class of pre-2-cells under this system of equivalence relations. Composition of 2-cells is given by composition of pre-2-cells.

We define the *reduced 2-category* $\mathcal{T}_n^{\text{red}}\mathbb{C}$ of paths in $\mathbb{C}$ as follows. Objects of $\mathcal{T}_n^{\text{red}}\mathbb{C}$ are objects of $\mathbb{C}$. A *morphism* $X \rightarrow Y$ of $\mathcal{T}_n^{\text{red}}\mathbb{C}$ is an equivalence class of paths under the equivalence condition generated by conditions (1) and (2) (unit, composition) of Definition 3.10. The definition of 2-cells in $\mathcal{T}_n^{\text{red}}\mathbb{C}$ is similar to the above, with only atomic 2-cells of type (3) (square). The atomic 2-cells of types (1) and (2) (creation and deletion of unit, composition and decomposition) appearing in conditions (3) through (8) are replaced by identity pre-2-cells. In particular, conditions (3) through (6) become tautological.

We obtain functors $\mathcal{T}_n, \mathcal{T}_n^{\text{red}}: n\text{FoldCat} \rightarrow 2\text{Cat}$.

As the (last) referee pointed out, the definition of $\mathcal{T}_n\mathbb{C}$ above is a presentation of the 2-category in the terminology of Street [23, page 547]. The presentation consists of the graph $\coprod_{1 \leq k \leq n} \mathbb{C}(\epsilon_k) \rightrightarrows \text{Ob}(\mathbb{C})$, the derivation scheme given by the atomic 2-cells with f and g being identities, and the relation on 2-cells given by conditions (3) through (9). Note that the underlying category of $\mathcal{T}_n\mathbb{C}$ is a free category, so that Lemma 2.16 applies.

For a 1-fold category $\mathbb{C}$, the 2-category $\mathcal{T}_1\mathbb{C}$ is $\mathcal{TC}$ in Definition 3.10 and $\mathcal{T}_1^{\text{red}}\mathbb{C}$ can be identified with $\mathbb{C}$. Note that $\mathcal{T}_n\mathbb{C}$ and $\mathcal{T}_n^{\text{red}}\mathbb{C}$ do not depend on hypercubes of $\mathbb{C}$ of dimension > 3 . This is justified by a theorem of Gray that we will recall soon.

Remark 3.14 The functors $\mathcal{T}_n$ and $\mathcal{T}_n^{\text{red}}$ are refinements of $\mathbf{T}_n$, in the sense that we have isomorphisms $\mathbf{T}_n \cong \mathbf{OT}_n \cong \mathbf{OT}_n^{\text{red}}$, where $\mathbf{O}: 2\text{Cat} \rightarrow \text{Cat}$ is the functor in Construction 2.17.

We define the *reduction 2-functor* $R: \mathcal{T}_n\mathbb{C} \rightarrow \mathcal{T}_n^{\text{red}}\mathbb{C}$ as follows. On objects we take the identity. To a morphism, we associate its equivalence class. To the class of $\sigma_{g,D,f}$, we associate the class of $\sigma_{g,D,f}$. To the class of atomic 2-cells of types (1) and (2), we associate identities. We have the following analogue of Proposition 2.14.

Proposition 3.15 *The reduction 2-functor $R: \mathcal{T}_n\mathbb{C} \rightarrow \mathcal{T}_n^{\text{red}}\mathbb{C}$ is a bi-equivalence.*

Despite the proposition, we need both $\mathcal{T}_n$ and $\mathcal{T}_n^{\text{red}}$, the latter for the 2-categorical refinement of the adjunction in Remark 3.11, and the former for applications to pseudo functors.

Proof We construct a strictly unital pseudo functor $G: \mathcal{T}_n^{\text{red}}\mathbb{C} \rightarrow \mathcal{T}_n\mathbb{C}$ satisfying $RG = \text{id}_{\mathcal{T}_n^{\text{red}}\mathbb{C}}$ and a pseudo natural isomorphism $\eta: \text{id}_{\mathcal{T}_n\mathbb{C}} \rightarrow GF$ carrying X to id_X as follows. Note that the length of the target of each atomic 2-cell of type θ (deletion of unit) or γ (composition) is one less than the source. Thus, for any path f in $\mathbb{C}$, there exists a sequence of atomic 2-cells $f \Rightarrow \dots \Rightarrow f^{\text{red}}$ of types θ and γ such that f^{red} is not the source of any θ or γ . We denote the composite 2-cell in $\mathcal{T}_n\mathbb{C}$ by $\eta(f)$, which does not depend on choices by conditions (4) (unit and composition) and (6) (associativity) of Definition 3.13. For an atomic 2-cell $\alpha: f_1 \Rightarrow f_2$ of type θ or γ , we have $\eta(f_1) = \eta(f_2)\alpha$. In particular, $f_1^{\text{red}} = f_2^{\text{red}}$. Thus f^{red} depends only on the equivalence class of f , and can be characterized as the unique path of minimal length in the class of f . For any morphism g of $\mathcal{T}_n^{\text{red}}\mathbb{C}$, we take $G(g)$ to be the unique path

of minimal length representing g . For a sequence of morphisms $X \xrightarrow{g} Y \xrightarrow{h} Z$ in $\mathcal{T}_n^{\text{red}}\mathcal{C}$, the composition constraint of G is given by $\eta(G(h)G(g))$. An atomic 2-cell $\beta: g_1 \Rightarrow g_2$ in $\mathcal{T}_n^{\text{red}}\mathcal{C}$ can be lifted to an atomic 2-cell $\alpha: f_1 \Rightarrow f_2$ in $\mathcal{T}_n\mathcal{C}$. We take $G(\beta)$ to be the composite $G(f_1) \xrightarrow{\eta(f_1)^{-1}} g_1 \xRightarrow{\alpha} g_2 \xrightarrow{\eta(f_2)} G(f_2)$. $\square$

Our next goal is to extend the functor $\mathbb{K}^n: \mathbf{Cat} \rightarrow n\mathbf{FoldCat}$ [Example 3.8 (2)] to a functor $\mathbb{Q}^n: \mathbf{2Cat} \rightarrow n\mathbf{FoldCat}$. In the case $n = 2$, for a 2-category $\mathcal{C}$, $\mathbb{Q}^2\mathcal{C}$ is the *double category of up-squares* of $\mathcal{C}$ defined by Bastiani and Ehresmann [2, 2.C.1, p. 272]. In the case $\mathcal{C} = \mathbf{Cat}$, the double category $\mathbb{Q}^2\mathbf{Cat}$ was studied earlier by Ehresmann and called the double category of quintets [8]. Objects of $\mathbb{Q}^2\mathcal{C}$ are objects of $\mathcal{C}$, horizontal morphisms are morphisms of $\mathcal{C}$, vertical morphisms are morphisms of $\mathcal{C}$, and squares are up-squares in $\mathcal{C}$, namely, diagrams in $\mathcal{C}$ of the form

$$\begin{array}{ccc} X & \xrightarrow{j} & Y \\ q \downarrow & \alpha \nearrow & \downarrow p \\ Z & \xrightarrow{i} & W. \end{array} \quad (3.8)$$

In general, for α in $\mathbf{I}^n$, we define α -hypercubes of $\mathbb{Q}^n\mathcal{C}$ to be the set of commutative hypercubes of dimension $\Sigma\alpha$ with 2-cells in $\mathcal{C}$. The directions of the 2-cells are determined by the natural order on the set $\{1, \dots, n\}$.

By a theorem of Gray [15], a commutative hypercube of dimension n with 2-cells is encoded by the 2-category $\gamma_n = \mathcal{T}_n^{\text{red}}([1]^{\boxtimes n})$, despite the fact that no hypercubes of dimension > 3 appear in the definition of $\mathcal{T}_n^{\text{red}}$. More precisely, the theorem says that γ_n is locally a partially ordered set, and hence commutative in the sense that, given any pair (f, g) of morphisms, there exists at most one 2-cell $f \Rightarrow g$.

Notation 3.16 (The α -hypercube γ_α) Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be an object of $\mathbf{I}^n$ (each α_i being either $[0]$ or $[1]$). We define the (commutative) α -hypercube γ_α with 2-cells to be $\mathcal{T}_n^{\text{red}}(\alpha_1 \boxtimes \dots \boxtimes \alpha_n)$.

Of course for the 2-category γ_α is isomorphic to $\gamma_{\Sigma\alpha}$, but the above notation makes it clear that γ_α is functorial with respect to α .

Let us recall in our notation the explicit description of the hypercube by Street [23, Section 6]. We have $\text{Ob}(\gamma_\alpha) = \alpha_1 \times \dots \times \alpha_n$. Consider a path $f_1 * \dots * f_{\#J}$ from the initial vertex to the final vertex of a J -hypercube, each f_k in direction $i_k \in J$. The path is uniquely determined by the total order $i_1 <_f \dots <_f i_{\#J}$ on J (or equivalently, by the permutation of J that carries the natural order to f). The partial order on the set of such paths is precisely the weak order on permutations. Let us give a description in terms of total orders, which is more natural. For objects a and b of γ_α , the category of morphisms $\gamma_\alpha(a, b)$ is given by the partially ordered set

$$\begin{cases} \text{set of total orders on } J_{a,b} = \{1 \leq i \leq n \mid a_i \neq b_i\} & \text{if } a \leq b, \\ \emptyset & \text{otherwise.} \end{cases}$$

For total orders f and g on $J_{a,b}$, there exists a 2-cell $f \Rightarrow g$ if and only if $\text{Inv}(f) \subseteq \text{Inv}(g)$. In this case there exists a unique 2-cell. Here $\text{Inv}(f)$ denote the set of pairs $(i, j) \in J_{a,b}^2$ for which $i < j$ but $i >_f j$ (i greater than j under the order f). For $f: a \rightarrow b$ and $g: b \rightarrow c$, the composition $g \circ f$ is the unique order on $J_{a,c} = J_{b,c} \amalg J_{a,b}$ extending f and g such that $i <_{g \circ f} j$ for all $i \in J_{b,c}$ and $j \in J_{a,b}$.

Remark 3.17 Note that γ_α is also the locally full sub-2-category of $\mathcal{T}_n(\alpha_1 \boxtimes \cdots \boxtimes \alpha_n)$ spanned by paths that do not contain id^k . Let S be the set of edges of the hypercube γ_α , namely the set of morphisms $f: a \rightarrow b$ of γ_α such that $b = a + \epsilon_i$ for some i . Then the underlying category of γ_α is free on the graph $S \rightrightarrows \text{Ob}(\gamma_\alpha)$, so that Lemma 2.16 applies.

Definition 3.18 (The n -fold category $\mathbb{Q}^n \mathcal{C}$ of hypercubes in a 2-category $\mathcal{C}$) Let $\mathcal{C}$ be a 2-category. We define an n -fold category $\mathbb{Q}^n \mathcal{C}$ by $(\mathbb{Q}^n \mathcal{C})(\alpha) = 2\text{Fun}(\gamma_\alpha, \mathcal{C})$. For $\alpha, \alpha' \in \mathbf{I}^n$ satisfying $\alpha' = \alpha + \epsilon_i$, the composition $\circ^i$ in direction i is given by the map

$$(\mathbb{Q}^n \mathcal{C})(\alpha') \times_{(\mathbb{Q}^n \mathcal{C})(\alpha)} (\mathbb{Q}^n \mathcal{C})(\alpha') \cong 2\text{Fun}(\mathcal{B}, \mathcal{C}) \rightarrow (\mathbb{Q}^n \mathcal{C})(\alpha')$$

induced by d_1^2 , where $\mathcal{B} = \mathcal{T}_n^{\text{red}}(\tilde{\alpha}_1 \boxtimes \cdots \boxtimes \tilde{\alpha}_n)$ ($\tilde{\alpha}_j$ as in Definition 3.7).

We obtain a functor $\mathbb{Q}^n: \mathbf{2Cat} \rightarrow n\mathbf{FoldCat}$.

Now let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be locally full sub-2-categories of $\mathcal{C}$, each containing all objects of $\mathcal{C}$. We denote by $\mathbb{Q}_{\mathcal{A}_1, \dots, \mathcal{A}_n} \mathcal{C}$ the largest n -fold subcategory of $\mathbb{Q}^n \mathcal{C}$ such that $(\mathbb{Q}_{\mathcal{A}_1, \dots, \mathcal{A}_n} \mathcal{C})(\epsilon_i) = 2\text{Fun}(\gamma_{\epsilon_i}, \mathcal{A}_i)$. In other words, α -hypercubes of $\mathbb{Q}_{\mathcal{A}_1, \dots, \mathcal{A}_n} \mathcal{C}$ are commutative hypercubes of dimension $\Sigma \alpha$ with 2-cells in $\mathcal{C}$, in the directions i satisfying $\alpha_i = 1$, such that every edge in direction i is in $\mathcal{A}_i$.

For a category $\mathbf{C}$, we have $\mathbb{Q}^n \mathbf{C} \cong \mathbb{K}^n \mathbf{C}$. For $n = 1$, the 1-fold category $\mathbb{Q}^1 \mathcal{C}$ is the underlying category of $\mathcal{C}$. By definition, $\mathbb{Q}^n \mathcal{C} = \mathbb{Q}_{\mathcal{C}, \dots, \mathcal{C}} \mathcal{C}$. For the map $\iota_i: \{1\} \rightarrow \{1, \dots, n\}$ with image of i , the category $\iota_i^* \mathbb{Q}_{\mathcal{A}_1, \dots, \mathcal{A}_n} \mathcal{C}$ is isomorphic to the underlying category of $\mathcal{A}_i$. Moreover, for $i \neq j$, we have an isomorphism of 2-categories $\mathcal{H}_{i,j}(\mathbb{Q}_{\mathcal{A}_1, \dots, \mathcal{A}_n} \mathcal{C}) \cong \mathcal{A}_i$.

Example 3.19 In the case $n = 2$, let $\mathcal{A}$ and $\mathcal{B}$ be locally full sub-2-categories of $\mathcal{C}$, each containing all objects of $\mathcal{C}$. The objects of the double category $\mathbb{Q}_{\mathcal{A}, \mathcal{B}} \mathcal{C}$ are the objects of $\mathcal{C}$. Horizontal morphisms are morphisms of $\mathcal{A}$, vertical morphisms are morphisms of $\mathcal{B}$, and squares are $(\mathcal{A}, \mathcal{B})$ -squares in $\mathcal{C}$, namely, diagrams in $\mathcal{C}$ of the form (3.8) where the horizontal morphisms i, j are in $\mathcal{A}$ and the vertical morphisms p, q are in $\mathcal{B}$. We have isomorphisms $\mathcal{H}(\mathbb{Q}_{\mathcal{A}, \mathcal{B}} \mathcal{C}) \cong \mathcal{A}$ and $\mathcal{V}(\mathbb{Q}_{\mathcal{A}, \mathcal{B}} \mathcal{C}) \cong \mathcal{B}$.

Construction 3.20 (Adjunction between $\mathcal{T}_n^{\text{red}}$ and $\mathbb{Q}^n$) Let $\mathbb{C}$ be an n -fold category. We construct an n -fold functor

$$\mathbb{C} \rightarrow \mathbb{Q}^n \mathcal{T}_n^{\text{red}} \mathbb{C} \quad (3.9)$$

as follows. To a hypercube $C \in \mathbb{C}(\alpha)$, corresponding to an n -fold functor $\alpha_1 \boxtimes \cdots \boxtimes \alpha_n \rightarrow \mathbb{C}$, we associate the 2-functor $\gamma_\alpha \rightarrow \mathcal{T}_n^{\text{red}} \mathbb{C}$ obtained by applying $\mathcal{T}_n^{\text{red}}$.

Let $\mathcal{D}$ be a 2-category. We construct a 2-functor

$$E_n^{\text{red}}: \mathcal{T}_n^{\text{red}} \mathbb{Q}^n \mathcal{D} \rightarrow \mathcal{D} \quad (3.10)$$

as follows. On objects we take the identity. To a path $f_m * \cdots * f_1$, where $f_i \in \coprod_{1 \leq k \leq n} (\mathbb{Q}^n \mathcal{D})(\epsilon_k)$, we associate the composite $f_m \cdots f_1$. We take $F(\sigma_{g,D}, f)$ to be the 2-cell induced by D .

As in Remark 3.11, it is easy to check the following.

Proposition 3.21 *The above construction exhibits $\mathcal{T}_n^{\text{red}}: n\mathbf{FoldCat} \rightarrow \mathbf{2Cat}$ as a left adjoint of $\mathbb{Q}^n: \mathbf{2Cat} \rightarrow n\mathbf{FoldCat}$. In other words, we get a bijection*

$$2\text{Fun}(\mathcal{T}_n^{\text{red}} \mathbb{C}, \mathcal{D}) \cong n\text{FoldFun}(\mathbb{C}, \mathbb{Q}^n \mathcal{D}),$$

functorial in $\mathbb{C}$ and $\mathcal{D}$.

For $n = 1$, this adjunction reduces to the one mentioned at the end of Construction 2.17.

Remark 3.22 In the case where $\mathbb{C} = \mathbb{C}_1 \boxtimes \cdots \boxtimes \mathbb{C}_n$ is the external product of categories $\mathbb{C}_1, \dots, \mathbb{C}_n$, this adjunction was studied by Gray [14, Section I.4]. In this case, the 2-category $\mathcal{T}_n^{\text{red}}(\mathbb{C}_1 \boxtimes \cdots \boxtimes \mathbb{C}_n)$ is Gray's tensor product $\mathbb{C}_1 \otimes \cdots \otimes \mathbb{C}_n$,³ and n -fold functors $\mathbb{C}_1 \boxtimes \cdots \boxtimes \mathbb{C}_n \rightarrow \mathbb{Q}^n \mathcal{D}$ are precisely Gray's "quasi-functors" of several variables $\mathbb{C}_1 \times \cdots \times \mathbb{C}_n \rightarrow \mathcal{D}$. We emphasize that this is only a special case of Gray's theory, which defines more generally tensor products of 2-categories in relation to the 2-category of 2-functors, lax natural transformations, and modifications.

The functor $\mathbb{K}^n$ is fully faithful and we have $\mathbf{T}_n \mathbb{K}^n \mathbf{D} \cong \mathbf{D}$. For $n \geq 2$, the 2-functor E_n^{red} is a bijection on objects and a surjection on morphisms and 2-cells, so that the functor $\mathbb{Q}^n$ is faithful. On the other hand, E_n^{red} is seldom an isomorphism and $\mathbb{Q}^n$ is never fully faithful. Nonetheless, we have the following 2-categorical analogue.

Proposition 3.23 *Let $\mathcal{D}$ be a 2-category. For $n \geq 2$, the 2-functor $E_n^{\text{red}}: \mathcal{T}_n^{\text{red}} \mathbb{Q}^n \mathcal{D} \rightarrow \mathcal{D}$ is a bi-equivalence. More generally, if $\mathcal{A}_1, \dots, \mathcal{A}_n$ are locally full sub-2-categories of $\mathcal{D}$ each of which contains all objects of $\mathcal{D}$ and $\mathcal{A}_i = \mathcal{D}$ for some $1 \leq i \leq n$, then the 2-functor $\mathcal{T}_n^{\text{red}} \mathbb{Q}_{\mathcal{A}_1, \dots, \mathcal{A}_n} \mathcal{D} \rightarrow \mathcal{D}$ induced by E_n^{red} is a bi-equivalence.*

For $n = 1$, E_1^{red} is the inclusion of the underlying category of $\mathcal{D}$ into $\mathcal{D}$, which is seldom a bi-equivalence. The functor $\mathbb{Q}^1$ is not faithful.

Proof We begin with a construction for an n -fold category $\mathbb{C}$. For $1 \leq i, j \leq n$ and $i \neq j$, we define a 2-functor

$$\mathcal{H}_{i,j}: \mathbb{C} \rightarrow \mathcal{T}_n^{\text{red}} \mathbb{C} \quad (3.11)$$

³ The identification with Gray's tensor product follows from the definition for $n = 2$ and from Proposition 3.21 for $n \geq 3$.

as follows. On objects, we take the identity. To $f \in \mathbb{C}(\epsilon_i)$, we associate f . If $i < j$ (resp. $i > j$), to any 2-cell $\alpha: f \Rightarrow g$ in $\mathcal{H}_{i,j}\mathbb{C}$, where $f, g: X \rightarrow Y$, we associate the 2-cell

$$f = f * \text{id}_X^j \xrightarrow{\sigma_{\text{id}_Y, D, \text{id}_X}} \text{id}_Y^j * g = g \quad (\text{resp. } f = \text{id}_Y^j * f \xrightarrow{\sigma_{\text{id}_Y, D, \text{id}_X}} g * \text{id}_X^j = g), \quad (3.12)$$

where $D \in \mathbb{C}(\epsilon_i + \epsilon_j)$ is the square induced by α of the boundary (3.4) [resp. (3.5)]. Taking $\mathbb{C} = \mathbb{Q}^n\mathcal{D}$, we get a 2-functor

$$G_i^{\text{red}}: \mathcal{D} \cong \mathcal{H}_{i,j}\mathbb{Q}^n\mathcal{D} \rightarrow \mathcal{T}_n^{\text{red}}\mathbb{Q}^n\mathcal{D}. \quad (3.13)$$

Note that in this case the 2-cell (3.12) does not depend on the choice of j by conditions (7) (unit square) and (9) (cube) of Definition 3.13.

We still use E_n^{red} and G_i^{red} to denote the induced 2-functors between $\mathcal{T}_n^{\text{red}}\mathbb{Q}_{\mathcal{A}_1, \dots, \mathcal{A}_n}\mathcal{D}$ and $\mathcal{D}$. We have $E^{\text{red}}G_i^{\text{red}} = \text{id}_{\mathcal{D}}$. We define a pseudo natural isomorphism $\epsilon: G_i^{\text{red}}E^{\text{red}} \Rightarrow \text{id}_{\mathcal{T}_n\mathbb{Q}_{\mathcal{A}_1, \dots, \mathcal{A}_n}\mathcal{D}}$ as follows. For any object X of $\mathcal{D}$, we take $\epsilon(X) = \text{id}_X$. For any morphism $f: X \rightarrow Y$ of $\mathcal{D}$, we denote by $f^k \in (\mathbb{Q}^n\mathcal{D})(\epsilon_k)$ the image of f . We take $\epsilon(f^i) = \text{id}_{f^i}$, and take $\epsilon(f^k), i < k$ (resp. $i > k$) to be the composition

$$\begin{aligned} f^i &= f^i * \text{id}_Y^k \xrightarrow{\sigma_{\text{id}_Y, D, \text{id}_X}} f^k * \text{id}_X^i = f^k \\ (\text{resp. } f^i &= \text{id}_Y^k * f^i \xrightarrow{\sigma_{\text{id}_Y, E, \text{id}_X}} \text{id}_X^i * f^k = f^k), \end{aligned}$$

where $D, E \in (\mathbb{Q}^n\mathcal{D})(\epsilon_i + \epsilon_k)$ are the squares

$$\begin{array}{ccc} X & \xlongequal{\quad} & X \\ \parallel & \begin{array}{c} D \\ \downarrow f \end{array} & \downarrow f \\ X & \xrightarrow{\quad f \quad} & Y \end{array} \quad \begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ \downarrow f & \begin{array}{c} E \\ \parallel \end{array} & \downarrow \\ Y & \xlongequal{\quad} & Y \end{array}$$

induced by id_f . It is straightforward to check that this definition is compatible with 2-cells of $\mathcal{T}_n^{\text{red}}\mathbb{Q}_{\mathcal{A}_1, \dots, \mathcal{A}_n}\mathcal{D}$. $\square$

Since our goal is to study pseudo functors between 2-categories, it would be nice to have a version of the above adjunction for pseudo functors. Between double categories, a notion of double pseudo functors, weakly compatible with compositions in both directions, was defined by Shulman [22, Definition 6.1].⁴ For $n \geq 3$, however, to define n -fold pseudo functors between n -fold categories, one would need, for each direction i , to choose an auxiliary direction $j \neq i$ that provides the pseudoness. Fortunately, we will only need n -fold pseudo functors from an n -fold category $\mathbb{C}$ to $\mathbb{Q}^n\mathcal{D}$, where $\mathcal{D}$ is a 2-category. In this case, no choice of directions is necessary. Taking our clue from Propositions 2.15 and 3.21, we regard $2\mathcal{F}un^{\text{ps}}(\mathcal{T}_n\mathbb{C}, \mathcal{D})$ as the 2-category of n -fold pseudo functors from $\mathbb{C}$ to $\mathbb{Q}^n\mathcal{D}$.

⁴ This is more general than the notion of pseudo double functors by Fiore [10, Definition 6.4], which are weakly compatible with horizontal composition and strictly compatible with vertical composition.

Remark 3.24 As in Proposition 3.21, it is possible to describe $2\mathcal{F}un^{\text{ps}}(\mathcal{T}_n\mathbb{C}, \mathcal{D})$ in terms of $\mathbb{C}$ and $\mathbb{Q}^n\mathcal{D}$, as follows. *Objects* are pairs $(F, (F_i)_{1 \leq i \leq n})$ consisting of a natural transformation $F: \mathbb{C} \rightarrow \mathbb{Q}^n\mathcal{D}$, where $\mathbb{C}$ and $\mathbb{Q}^n\mathcal{D}$ are viewed as functors $(\mathbf{I}_+^{\text{op}})^n \rightarrow \mathbf{Set}$, and pseudo functors $F_i: \iota_i^*\mathbb{C} \rightarrow \mathcal{D}$ extending F_0 and F_{ϵ_i} . Here $\mathbf{I}_+$ is the subcategory of $\mathbf{I}$ (Notation 3.1) spanned by the strictly increasing maps. For all $1 \leq k, k' \leq n$ ($k \neq k'$), and $D \in \mathbb{C}(\epsilon_k + \epsilon_{k'})$, we denote by G_D the 2-cell in $F_{\epsilon_k + \epsilon_{k'}}(D)$. The pair $(F, (F_i))$ is subject to the following conditions:

- (a) For $f: X \rightarrow Y$ belonging to $\mathbb{C}(\epsilon_k)$, the following triangle (resp. with the horizontal arrow reversed)

$$\begin{array}{ccc} F_k(f) & & \\ \Downarrow & \searrow & \\ F_k(f)F_{k'}(\text{id}_X^{k'}) & \xRightarrow{G_D} & F_{k'}(\text{id}_Y^{k'})F_k(f) \end{array}$$

commutes if $k < k'$ (resp. $k' < k$), where $D = s(f) \in \mathbb{C}(\epsilon_k + \epsilon_{k'})$ is the identity square (3.6).

- (b) For $D'' = D' \circ^k D$ in $\mathbb{C}(\epsilon_k + \epsilon_{k'})$ of the form (3.7), the following pentagon (resp. with the horizontal arrows reversed)

$$\begin{array}{ccccc} F_k(i')F_k(i)F_{k'}(p_1) & \xRightarrow{G_D} & F_k(i')F_{k'}(p_2)F_k(j) & \xRightarrow{G_{D'}} & F_{k'}(p_3)F_k(j')F_k(j) \\ \Downarrow & & & & \Downarrow \\ F_k(i'i)F_{k'}(p_1) & \xRightarrow{G_{D''}} & & & F_{k'}(p_3)F_k(j'j) \end{array}$$

commutes if $k < k'$ (resp. $k' < k$).

A *morphism* $\mu: F \rightarrow F'$ is a collection $(\mu_i)_{1 \leq i \leq n}$ of morphisms (i.e. pseudo natural transformations) $\mu_i: F_i \rightarrow F'_i$ of $\mathcal{P}sfun(\iota_i^*\mathbb{C}, \mathcal{D})$ satisfying $|\mu_1| = \cdots = |\mu_n|$ (Notation 2.9 (2)) and such that for every square (3.2) in $\mathbb{C}$, the cube

$$\begin{array}{ccccc} FX & \xrightarrow{F_k(j)} & FY & & \\ \alpha X \downarrow & \searrow F_{k'}(q) & \downarrow \alpha Y & \searrow F_{k'}(p) & \\ & FZ & \xrightarrow{F_k(i)} & FW & \\ & \downarrow \alpha Z & & \downarrow \alpha W & \\ F'X & \xrightarrow{F'_k(j)} & F'Y & & \\ & \searrow F'_{k'}(q) & \downarrow & \searrow F'_{k'}(p) & \\ & F'Z & \xrightarrow{F'_k(i)} & F'W & \end{array}$$

is commutative. Here the top, bottom, front, back, right, and left faces are respectively given by $F_{\epsilon_k + \epsilon_{k'}}(D)$, $F'_{\epsilon_k + \epsilon_{k'}}(D)$, $\mu_k(i)$, $\mu_k(j)$, $\mu_{k'}(p)$, and $\mu_{k'}(q)$.

A 2-cell $\Xi: \mu \Rightarrow \nu$ consists of a collection $(\Xi_i)_{1 \leq i \leq n}$ of 2-cells (i.e. modifications) $\Xi_i: \mu_i \Rightarrow \nu_i$ of $\mathcal{P}Fun(\iota_i^* \mathbb{C}, \mathcal{D})$ such that $|\Xi_1| = \dots = |\Xi_n|$. In other words, a 2-cell $\mu \Rightarrow \nu$ of $2Fun^{ps}(\mathcal{T}_n \mathbb{C}, \mathcal{D})$ is a function from $Ob(\mathbb{C})$ to the set of 2-cells of $\mathcal{D}$ which is a modification $\mu_i \Rightarrow \nu_i$ for all $1 \leq i \leq n$ at the same time.

Composition is given by composition in $\mathcal{D}$.

The 2-category $2Fun^{ps}(\mathcal{L}\mathcal{T}_n \mathbb{C}, \mathcal{D})$ can be identified with the 2-full sub-2-category spanned by objects for which all the G_D are invertible, where $\mathcal{L}: 2Cat \rightarrow (2, 1)Cat$ is the functor in Construction 2.18.

Construction 3.25 (The 2-functor $E_n: \mathcal{T}_n \mathbb{Q}^n \mathbb{C} \rightarrow \mathcal{TC}$) Let $\mathcal{C}$ be a 2-category. We construct a variant

$$E_n: \mathcal{T}_n \mathbb{Q}^n \mathbb{C} \rightarrow \mathcal{TC} \quad (3.14)$$

of the adjunction 2-functor (3.10) as follows. On objects we take the identity. To a path $f_m * \dots * f_1$ in $\mathbb{Q}^n \mathbb{C}$, we associate the path $f_m * \dots * f_1$ in $\mathcal{C}$. To the 2-cells ι^k and θ^k (creation and deletion of unit), we associate ι and θ . To the 2-cells γ and δ (composition and decomposition), we associate γ and δ . Finally, to the 2-cell $\sigma_{g,D,f}$ (square), we associate the composite

$$\begin{aligned} (E_n g) * i * q * (E_n f) &\xrightarrow{\gamma_{E_n g, i, q, E_n f}} (E_n g) * i q * (E_n f) \xrightarrow{\sigma_{E_n g, \alpha, E_n f}} E_n g * p j * E_n f \\ &\xrightarrow{\delta_{E_n g, p, j, E_n f}} E_n g * p * j * E_n f, \end{aligned}$$

where $\alpha: i q \Rightarrow p j$ is the 2-cell in $\mathcal{D}$. We have a commutative square

$$\begin{array}{ccc} \mathcal{T}_n \mathbb{Q}^n \mathbb{C} & \xrightarrow{E_n} & \mathcal{TC} \\ R \downarrow & & \downarrow R_C \\ \mathcal{T}_n^{\text{red}} \mathbb{Q}^n \mathbb{C} & \xrightarrow{E_n^{\text{red}}} & \mathcal{C}, \end{array}$$

where the vertical arrows are the reduction 2-functors and are bi-equivalences by Propositions 2.14 and 3.15. For $n \geq 2$, the lower horizontal arrow is a bi-equivalence by Proposition 3.23, so that $E_n: \mathcal{T}_n \mathbb{Q}^n \mathbb{C} \rightarrow \mathcal{TC}$ is a bi-equivalence as well.

This construction is related to pseudo functors as follows. For any 2-category $\mathcal{D}$, the 2-functor E_n induces a $\mathcal{D}^{Ob(\mathcal{C})}$ -2-functor

$$\mathcal{P}Fun(\mathcal{C}, \mathcal{D}) \cong 2Fun^{ps}(\mathcal{TC}, \mathcal{D}) \rightarrow 2Fun^{ps}(\mathcal{T}_n \mathbb{Q}^n \mathbb{C}, \mathcal{D}),$$

which is a $\mathcal{D}^{Ob(\mathcal{C})}$ -2-equivalence for $n \geq 2$ by Proposition 2.12 and Lemma 2.16.

For reference in the next section, for $n \geq 2$, we define 2-functors

$$G_i: \mathcal{TC} \rightarrow \mathcal{T}_n \mathbb{Q}^n \mathbb{C}, \quad (3.15)$$

pseudo inverses of E_n , as follows. The definition is similar to (3.13). On objects we take the identity. To a path $f_m * \dots * f_1$ in $\mathcal{C}$, we associate the path $f_m^i * \dots * f_1^i$ in $\mathbb{Q}^n \mathbb{C}$, where $f_j^i \in (\mathbb{Q}^n \mathbb{C})(\epsilon_i)$ is f_j considered as a morphism in direction i . We take

$G_i(\iota_{g,f}) = \iota_{G_i g, G_i f}^i$ and $G_i(\gamma_{g,h',h,f}) = \gamma_{G_i g, h^i, h^i, G_i f}$. Choose $1 \leq j \leq n$ with $j \neq i$. We take $G_i(\sigma_{g,\alpha,f})$ to be the 2-cell

$$\begin{aligned} G_i(g) * h^i * G_i(f) &\xRightarrow{\iota^j} G_i(g) * h^i * \text{id}_Y^j * G_i(f) \\ &\xRightarrow{\sigma_{G_i g, D, G_i f}} G_i(g) * \text{id}_Z^j * h^i * G_i(f) \xRightarrow{\theta^j} G_i(g) * h^i * G_i(f) \end{aligned}$$

for $i < j$ and the 2-cell

$$\begin{aligned} G_i(g) * h^i * G_i(f) &\xRightarrow{\iota^j} G_i(g) * \text{id}_Y^j * h^i * G_i(f) \\ &\xRightarrow{\sigma_{G_i g, D, G_i f}} G_i(g) * h^i * \text{id}_Z^j * G_i(f) \xRightarrow{\theta^j} G_i(g) * h^i * G_i(f) \end{aligned}$$

for $i > j$. Here D denotes the square induced by α . The 2-cell does not depend on the choice of j .

Construction 3.26 (The descent 2-functor $Q_{\mathcal{D}}^n$) Let $\mathcal{C}$ be a $(2, 1)$ -category. Consider the 2-functor

$$\mathcal{L}E_n: \mathcal{L}T_n \mathbb{Q}^n \mathcal{C} \rightarrow \mathcal{L}TC \cong TC$$

obtained from (3.14) by applying the functor $\mathcal{L}$ (Construction 2.18). This induces, for every 2-category $\mathcal{D}$, a $\mathcal{D}^{\text{Ob}(\mathcal{C})}$ -2-functor

$$\mathcal{P} sFun(\mathcal{C}, \mathcal{D}) \cong 2Fun^{\text{ps}}(TC, \mathcal{D}) \rightarrow 2Fun^{\text{ps}}(\mathcal{L}T_n \mathbb{Q}^n \mathcal{C}, \mathcal{D}), \quad (3.16)$$

For $n \geq 2$, the 2-functor $\mathcal{L}E_n$ is a bi-equivalence and (3.16) is a $\mathcal{D}^{\text{Ob}(\mathcal{C})}$ -2-equivalence.

One main goal of this article is to study more generally, for locally full sub-2-categories $\mathcal{A}_1, \dots, \mathcal{A}_n \subseteq \mathcal{C}$ each of which contains all objects of $\mathcal{C}$, the *descent 2-functor*

$$Q^n = Q_{\mathcal{D}}^n: \mathcal{P} sFun(\mathcal{C}, \mathcal{D}) \cong 2Fun^{\text{ps}}(TC, \mathcal{D}) \rightarrow 2Fun^{\text{ps}}(\mathcal{L}T_n \mathbb{Q}_{\mathcal{A}_1, \dots, \mathcal{A}_n} \mathcal{C}, \mathcal{D}) \quad (3.17)$$

induced by the 2-functor

$$\mathcal{L}E_n: \mathcal{L}T_n \mathbb{Q}_{\mathcal{A}_1, \dots, \mathcal{A}_n} \mathcal{C} \rightarrow TC$$

obtained by restriction via the inclusion $\mathbb{Q}_{\mathcal{A}_1, \dots, \mathcal{A}_n} \mathcal{C} \subseteq \mathbb{Q}^n \mathcal{C}$.

Definition 3.27 (2-Category of gluing data) For $n \geq 2$, we call

$$\mathcal{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D}) := 2Fun^{\text{ps}}(\mathcal{L}T_n \mathbb{Q}_{\mathcal{A}_1, \dots, \mathcal{A}_n} \mathcal{C}, \mathcal{D})$$

the 2-category of *gluing data* from $\mathcal{C}$ to $\mathcal{D}$, relative to $\mathcal{A}_1, \dots, \mathcal{A}_n$. For $n = 1$, we put $\mathcal{GD}_{\mathcal{A}_1}(\mathcal{C}, \mathcal{D}) := \mathcal{P} sFun(\mathcal{A}_1, \mathcal{D})$.

We will give more explicit descriptions of the 2-category of gluing data later (see Remark 5.1 for the case $n = 2$ and Remarks 6.1, 6.2 for the general case), which also justifies our convention for the case $n = 1$. In order to treat the cases $n = 1$ and $n \geq 2$ uniformly, we adopt the following notation for $n \geq 2$:

$$\begin{aligned}\overline{T_n \mathbb{Q}_{\mathcal{A}_1, \dots, \mathcal{A}_n} \mathcal{C}} &:= T_n \mathbb{Q}_{\mathcal{A}_1, \dots, \mathcal{A}_n} \mathcal{C}, & \overline{T_1 \mathbb{Q}_{\mathcal{A}_1} \mathcal{C}} &:= T \mathcal{A}_1, \\ \overline{T_n^{\text{red}} \mathbb{Q}_{\mathcal{A}_1, \dots, \mathcal{A}_n} \mathcal{C}} &:= T_n^{\text{red}} \mathbb{Q}_{\mathcal{A}_1, \dots, \mathcal{A}_n} \mathcal{C}, & \overline{T_1^{\text{red}} \mathbb{Q}_{\mathcal{A}_1} \mathcal{C}} &:= \mathcal{A}_1.\end{aligned}$$

By Proposition 2.12 and Lemma 2.16, we have the following criterion.

Proposition 3.28 *Let $\mathcal{L}E_n^{\text{red}}: \mathcal{L}T_n^{\text{red}} \mathbb{Q}_{\mathcal{A}_1, \dots, \mathcal{A}_n} \mathcal{C} \rightarrow \mathcal{C}$ be the 2-functor induced by (3.10). Consider the descent 2-functor $Q_{\mathcal{D}}^n$ (3.17).*

- (1) *If $\mathcal{L}E_n^{\text{red}}$ is locally essentially surjective, then $Q_{\mathcal{D}}^n$ is 2-faithful for every 2-category $\mathcal{D}$.*
- (2) *The following conditions are equivalent:*
 - (a) *The 2-functor $\mathcal{L}E_n^{\text{red}}$ is a bi-equivalence.*
 - (b) *The 2-functor $Q_{\mathcal{D}}^n$ is a bi-equivalence for every 2-category $\mathcal{D}$.*
 - (c) *The 2-functor $Q_{\mathcal{D}}^n$ is a $\mathcal{D}^{\text{Ob}(\mathcal{C})}$ -2-equivalence for every 2-category $\mathcal{D}$.*

Combining with Proposition 3.23, we obtain the following.

Corollary 3.29 *Assume $n \geq 2$ and $\mathcal{A}_i = \mathcal{C}$ for some $1 \leq i \leq n$. Then the descent 2-functor*

$$Q_{\mathcal{D}}^n: \mathcal{P} \mathcal{S} \mathcal{F} \mathcal{u} \mathcal{n}(\mathcal{C}, \mathcal{D}) \rightarrow \mathcal{G} \mathcal{D}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$$

is a $\mathcal{D}^{\text{Ob}(\mathcal{C})}$ -2-equivalence for every 2-category $\mathcal{D}$.

The corollary does not produce *new* pseudo functors from $\mathcal{C}$ to $\mathcal{D}$, since a pseudo functor from $\mathcal{C}$ to $\mathcal{D}$ is part of the gluing datum by the assumption $\mathcal{A}_i = \mathcal{C}$. In Section 5, we will give a more substantial application of Proposition 3.28 for the gluing of two pseudo functors (case $n = 2$) defined on *proper* sub-2-categories. For the gluing of more than two functors (case $n > 2$) defined on proper sub-2-categories, we will do so indirectly, by relating the gluing problem for n pseudo functors to the gluing problem for $n - 1$ pseudo functors via the constructions of the following section.

4 Functoriality of gluing data with respect to the index set $\{1, \dots, n\}$

In this section, we record some functorial properties, with respect to the index set $\{1, \dots, n\}$, of gluing data and of the operations T_n and $\mathbb{Q}^n$ defining them. Such properties will play an essential role in relating gluing problems for different numbers of pseudo functors in Sect. 6. Section 5 does not depend on the construction of this section.

We start with the case where the map $\phi: \{1, \dots, m\} \rightarrow \{1, \dots, n\}$ between index sets is nondecreasing.

Construction 4.1 (Functoriality of $\mathcal{T}_n$ for ϕ nondecreasing) Let $\phi: \{1, \dots, m\} \rightarrow \{1, \dots, n\}$ be a nondecreasing map and let $\mathbb{C}$ be an n -fold category. We define 2-functors

$$\mathcal{T}_m \phi^* \mathbb{C} \rightarrow \mathcal{T}_n \mathbb{C}, \quad \mathcal{T}_m^{\text{red}} \phi^* \mathbb{C} \rightarrow \mathcal{T}_n^{\text{red}} \mathbb{C} \quad (4.1)$$

as follows. To an object X , we associate X . To a morphism f^k , where $f \in (\phi^* \mathbb{C})(\epsilon_k) = \mathbb{C}(\epsilon_{\phi(k)})$, we associate $f^{\phi(k)}$. To an atomic 2-cell in $\mathcal{T}_m \phi^* \mathbb{C}$ or $\mathcal{T}_m^{\text{red}} \phi^* \mathbb{C}$ of type ι , θ (unit), γ , δ ((de)composition), or $\sigma_{g,D,f}$ with $D \in (\phi^* \mathbb{C})(\epsilon_k + \epsilon_{k'}) = \mathbb{C}(\epsilon_{\phi(k)} + \epsilon_{\phi(k')})$ satisfying $\phi(k) \neq \phi(k')$, we associate the corresponding atomic 2-cell in $\mathcal{T}_n \mathbb{C}$ or $\mathcal{T}_n^{\text{red}} \mathbb{C}$. An element $D \in \mathbb{C}(\epsilon_k + \epsilon_{k'})$ with $k < k'$ and $\phi(k) = \phi(k') = l$ corresponds to a commutative square in $\iota_l^* \mathbb{C}$, and we take $\mathcal{T}_n(\sigma_{g,D,f})$ to be the composite

$$g * i * q * f \xrightarrow{\gamma_{g,i,q,f}} g * i q * f = g * p j * f \xrightarrow{\delta_{g,p,j,f}} g * p * j * f$$

and take $\mathcal{T}_n^{\text{red}}(\sigma_{g,D,f})$ to be the identity. The two 2-functors are compatible with reduction $\mathcal{T}_n \rightarrow \mathcal{T}_n^{\text{red}}$.

For a sequence of nondecreasing maps

$$\{1, \dots, l\} \xrightarrow{\psi} \{1, \dots, m\} \xrightarrow{\phi} \{1, \dots, n\}, \quad (4.2)$$

the composition

$$\mathcal{T}_l(\phi\psi)^* \mathbb{C} \cong \mathcal{T}_l \psi^* \phi^* \mathbb{C} \rightarrow \mathcal{T}_m \phi^* \mathbb{C} \rightarrow \mathcal{T}_n \mathbb{C}$$

equals the 2-functor induced by $\phi\psi$. The same holds for $\mathcal{T}_n^{\text{red}}$.

Consider the natural transformation induced by (4.1)

$$\mathcal{T}_m^{\text{red}} \rightarrow \mathcal{T}_m^{\text{red}} \phi^* \phi! \rightarrow \mathcal{T}_n^{\text{red}} \phi!, \quad (4.3)$$

which is a natural isomorphism when ϕ is injective (hence strictly increasing).

Construction 4.2 (Functoriality of $\mathbb{Q}^n$ for ϕ nondecreasing) By adjunction (4.3) induces a natural transformation

$$\phi^* \mathbb{Q}^n \mathcal{C} \rightarrow \mathbb{Q}^m \mathcal{C}. \quad (4.4)$$

When ϕ is injective (hence strictly increasing), (4.4) is a natural isomorphism, and we have

$$\phi^* \mathbb{Q}_{\mathcal{A}_1, \dots, \mathcal{A}_n} \mathcal{C} \cong \mathbb{Q}_{\mathcal{A}_{\phi(1)}, \dots, \mathcal{A}_{\phi(m)}} \mathcal{C}$$

for any 2-category $\mathcal{C}$ and locally full sub-2-categories $\mathcal{A}_1, \dots, \mathcal{A}_n$ of $\mathcal{C}$ each containing all objects of $\mathcal{C}$.

For a sequence of nondecreasing maps (4.2), the composition

$$(\phi\psi)^* \mathbb{Q}^n \cong \psi^* \phi^* \mathbb{Q}^n \rightarrow \psi^* \mathbb{Q}^m \rightarrow \mathbb{Q}^l$$

equals the natural transformation induced by $\phi\psi$.

Construction 4.3 (Functoriality of $\overline{T_n \mathbb{Q}^n}$ for ϕ nondecreasing) Let $\phi: \{1, \dots, m\} \rightarrow \{1, \dots, n\}$ be a nondecreasing map and let $\mathcal{C}$ be a 2-category. We define 2-functors

$$E_\phi: \overline{T_m \mathbb{Q}^m} \mathcal{C} \rightarrow \overline{T_n \mathbb{Q}^n} \mathcal{C}, \quad E_\phi^{\text{red}}: \overline{T_m^{\text{red}} \mathbb{Q}^m} \mathcal{C} \rightarrow \overline{T_n^{\text{red}} \mathbb{Q}^n} \mathcal{C},$$

compatible with reduction, as follows.

For $m = n = 1$, we take $E_\phi: \mathcal{TC} \rightarrow \mathcal{TC}$ and $E_\phi^{\text{red}}: \mathcal{C} \rightarrow \mathcal{C}$ to be the identities. For $m = 1$ and $n \geq 2$, we take

$$E_\phi: \overline{T_1 \mathbb{Q}^1} \mathcal{C} = \mathcal{TC} \rightarrow \overline{T_n \mathbb{Q}^n} \mathcal{C}, \quad E_\phi^{\text{red}}: \overline{T_1^{\text{red}} \mathbb{Q}^1} \mathcal{C} = \mathcal{C} \rightarrow \overline{T_n^{\text{red}} \mathbb{Q}^n} \mathcal{C}$$

to be the 2-functors $G_{\phi(1)}$ (3.15) and $G_{\phi(1)}^{\text{red}}$ (3.13). For $m \geq 2$ and $n = 1$, we take

$$E_\phi = E_m: \overline{T_m \mathbb{Q}^m} \mathcal{C} \rightarrow \mathcal{TC}, \quad E_\phi^{\text{red}} = E_m^{\text{red}}: \overline{T_m^{\text{red}} \mathbb{Q}^m} \mathcal{C} \rightarrow \mathcal{C}.$$

For $m, n \geq 2$, we construct E_ϕ as follows. To an object X , we associate X . To a morphism $f_a * \dots * f_1$, where $f_i \in \coprod_{1 \leq k \leq m} (\mathbb{Q}^m \mathcal{C})(\epsilon_k)$ for $1 \leq i \leq a$, we associate $f_a * \dots * f_1$. To the atomic 2-cell $\sigma_{g,D,f}$ of $\overline{T_m \mathbb{Q}^m} \mathcal{C}$, where $D \in \mathbb{Q}^m \mathcal{C}(\epsilon_k + \epsilon_{k'})$ with $\phi(k) = \phi(k') = l, k < k'$, we associate the composition

$$g * i * q * f \xrightarrow{\gamma_{g,i,q,f}} g * i q * f \xrightarrow{g * \alpha * f} g * p j * f \xrightarrow{\delta_{g,p,j,f}} g * p * j * f,$$

where $\alpha: i q \Rightarrow p j$ is the image of the 2-cell in D under the 2-functor $G_l: \mathcal{TC} \rightarrow \overline{T_n \mathbb{Q}^n} \mathcal{C}$ (the assumption $n \geq 2$ is used in the definition of G_l). To other atomic 2-cells in $\overline{T_m \mathbb{Q}^m} \mathcal{C}$, we associate the corresponding atomic 2-cells in $\overline{T_n \mathbb{Q}^n} \mathcal{C}$. The 2-functor E_ϕ^{red} is defined similarly, with G_l replaced by G_l^{red} .

If $\mathcal{A}_1, \dots, \mathcal{A}_m, \mathcal{B}_1, \dots, \mathcal{B}_n$ are locally full sub-2-categories of $\mathcal{C}$, each containing all objects of $\mathcal{C}$, such that $\mathcal{A}_i \subseteq \mathcal{B}_{\phi(i)}$ for all $1 \leq i \leq m$, then E_ϕ restricts to a 2-functor

$$\overline{T_m \mathbb{Q}}_{\mathcal{A}_1, \dots, \mathcal{A}_m} \mathcal{C} \rightarrow \overline{T_n \mathbb{Q}}_{\mathcal{B}_1, \dots, \mathcal{B}_n} \mathcal{C}$$

and similarly for E_ϕ^{red} .

For a sequence of nondecreasing maps $\{1, \dots, l\} \xrightarrow{\psi} \{1, \dots, m\} \xrightarrow{\phi} \{1, \dots, n\}$, the composition

$$\overline{T_l \mathbb{Q}^l} \mathcal{C} \xrightarrow{E_\psi} \overline{T_m \mathbb{Q}^m} \mathcal{C} \xrightarrow{E_\phi} \overline{T_n \mathbb{Q}^n} \mathcal{C}$$

equals $E_{\phi\psi}$. Similarly $E_{\phi\psi}^{\text{red}} = E_\phi^{\text{red}} E_\psi^{\text{red}}$.

The following is a generalization of Proposition 3.23.

Proposition 4.4 Let $\phi: \{1, \dots, m\} \rightarrow \{1, \dots, n\}$ be a nondecreasing map with $n \geq 2$ and let ψ be a section of ϕ . Let $\mathcal{A}_1, \dots, \mathcal{A}_m$ be locally full sub-2-categories of $\mathcal{C}$, each containing all objects of $\mathcal{C}$, such that $\mathcal{A}_i \subseteq \mathcal{A}_{\psi\phi(i)}$ for all $1 \leq i \leq m$. Then

$$\begin{aligned} E_\phi: \overline{T_m} \overline{\mathbb{Q}}_{\mathcal{A}_1, \dots, \mathcal{A}_m} \mathcal{C} &\rightarrow \overline{T_n} \overline{\mathbb{Q}}_{\mathcal{A}_{\psi(1)}, \dots, \mathcal{A}_{\psi(n)}} \mathcal{C}, \\ E_\psi: \overline{T_n} \overline{\mathbb{Q}}_{\mathcal{A}_{\psi(1)}, \dots, \mathcal{A}_{\psi(n)}} \mathcal{C} &\rightarrow \overline{T_m} \overline{\mathbb{Q}}_{\mathcal{A}_1, \dots, \mathcal{A}_m} \mathcal{C} \end{aligned}$$

are pseudo inverses of each other. The same holds for E_ϕ^{red} and E_ψ^{red} .

Proof We have $E_\phi E_\psi = E_{\text{id}} = \text{id}$. Furthermore, one can construct a pseudo natural isomorphism $E_\psi E_\phi \Rightarrow \text{id}$ as in the proof of Proposition 3.23. $\square$

Remark 4.5 Let $t: \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ be the map sending i to $n + 1 - i$. For an n -fold category $\mathbb{C}$ and 2-categories $\mathcal{C}$ and $\mathcal{D}$, we have isomorphisms

$$\mathcal{T}_n(t^* \mathbb{C}) \cong (\mathcal{T}_n \mathbb{C})^{\text{co}}, \quad \mathcal{T}_n^{\text{red}}(t^* \mathbb{C}) \cong (\mathcal{T}_n^{\text{red}} \mathbb{C})^{\text{co}}, \quad t^*(\mathbb{Q}_{\mathcal{A}_1, \dots, \mathcal{A}_n} \mathcal{C}) \cong \mathbb{Q}_{\mathcal{A}_1^{\text{co}}, \dots, \mathcal{A}_n^{\text{co}}} \mathcal{C}^{\text{co}}.$$

When ϕ is not nondecreasing, the functoriality involves inverting 2-cells.

Construction 4.6 (Functoriality of $\mathcal{LT}$ and $\mathbb{Q}^n$ for ϕ arbitrary) Let $\phi: \{1, \dots, m\} \rightarrow \{1, \dots, n\}$ be an arbitrary map. We have the following analogues of Constructions 4.1 and 4.2. For any n -fold category $\mathbb{C}$, we have 2-functors

$$\mathcal{LT}_m \phi^* \mathbb{C} \rightarrow \mathcal{LT}_n \mathbb{C}, \quad \mathcal{LT}_m^{\text{red}} \phi^* \mathbb{C} \rightarrow \mathcal{LT}_n^{\text{red}} \mathbb{C} \quad (4.5)$$

sending $\sigma_{g, D, f}$, with $D \in \mathbb{C}(\epsilon_i + \epsilon_j)$, $i < j$, $\phi(i) > \phi(j)$, to $\sigma_{g, D, f}^{-1}$. Consider the composite natural transformation

$$\mathcal{LT}_m \rightarrow \mathcal{LT}_m \phi^* \phi! \rightarrow \mathcal{LT}_n \phi!.$$

By adjunction, we obtain, for any $(2, 1)$ -category $\mathcal{C}$, an m -fold functor

$$\phi^* \mathbb{Q}^n \mathcal{C} \rightarrow \mathbb{Q}^m \mathcal{C},$$

which is an isomorphism when ϕ is injective.

Construction 4.7 (Functoriality of $\mathcal{LT}_n \overline{\mathbb{Q}^n}$ and of gluing data) Let $\phi: \{1, \dots, m\} \rightarrow \{1, \dots, n\}$ be an arbitrary map and let $\mathcal{C}$ be a $(2, 1)$ -category. Similarly to Construction 4.3, we have 2-functors

$$\mathcal{LE}_\phi: \mathcal{LT}_m \overline{\mathbb{Q}^m} \mathcal{C} \rightarrow \mathcal{LT}_n \overline{\mathbb{Q}^n} \mathcal{C}, \quad \mathcal{LE}_\phi^{\text{red}}: \mathcal{LT}_m^{\text{red}} \overline{\mathbb{Q}^m} \mathcal{C} \rightarrow \mathcal{LT}_n^{\text{red}} \overline{\mathbb{Q}^n} \mathcal{C} \quad (4.6)$$

sending $\sigma_{g, D, f}$ to $\sigma_{g, D^*, f}^{-1}$, where $D \in (\mathbb{Q}^m \mathcal{C})(\epsilon_i + \epsilon_j)$, $i < j$, $\phi(i) > \phi(j)$, and D^* is the transpose of D obtained by inverting the 2-cell.

Now let $\mathcal{A}_1, \dots, \mathcal{A}_n, \mathcal{B}_1, \dots, \mathcal{B}_m$ be locally full sub-2-categories of $\mathcal{C}$, each containing all objects of $\mathcal{C}$, satisfying $\mathcal{B}_i \subseteq \mathcal{A}_{\phi(i)}$ for all $1 \leq i \leq m$. By restriction we obtain 2-functors

$$\mathcal{LE}_\phi: \mathcal{LT}_m \overline{\mathbb{Q}}_{\mathcal{B}_1, \dots, \mathcal{B}_m} \mathcal{C} \rightarrow \mathcal{LT}_n \overline{\mathbb{Q}}_{\mathcal{A}_1, \dots, \mathcal{A}_n} \mathcal{C}, \quad \mathcal{LE}_\phi^{\text{red}}: \mathcal{LT}_m^{\text{red}} \overline{\mathbb{Q}}_{\mathcal{B}_1, \dots, \mathcal{B}_m} \mathcal{C} \rightarrow \mathcal{LT}_n^{\text{red}} \overline{\mathbb{Q}}_{\mathcal{A}_1, \dots, \mathcal{A}_n} \mathcal{C} \quad (4.7)$$

Our extended goal is to study the 2-functor

$$Q^\phi = Q_D^\phi: \mathcal{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D}) \rightarrow \mathcal{GD}_{\mathcal{B}_1, \dots, \mathcal{B}_m}(\mathcal{C}, \mathcal{D}) \quad (4.8)$$

induced by $\mathcal{LE}_\phi$. For $n = 1$ and $m \geq 2$, the 2-functor Q^ϕ is the descent 2-functor Q^m (3.17).

For a sequence of arbitrary maps $\{1, \dots, l\} \xrightarrow{\psi} \{1, \dots, m\} \xrightarrow{\phi} \{1, \dots, n\}$ and locally full sub-2-categories $\mathcal{C}_1, \dots, \mathcal{C}_l$ of $\mathcal{C}$, each containing all objects of $\mathcal{C}$, the composite

$$\mathcal{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D}) \xrightarrow{Q^\phi} \mathcal{GD}_{\mathcal{B}_1, \dots, \mathcal{B}_m}(\mathcal{C}, \mathcal{D}) \xrightarrow{Q^\psi} \mathcal{GD}_{\mathcal{C}_1, \dots, \mathcal{C}_l}(\mathcal{C}, \mathcal{D})$$

equals $Q^{\phi\psi}$. In particular, for $m, n \geq 2$, the composite

$$\mathcal{PsFun}(\mathcal{C}, \mathcal{D}) \xrightarrow{Q^n} \mathcal{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D}) \xrightarrow{Q^\phi} \mathcal{GD}_{\mathcal{B}_1, \dots, \mathcal{B}_m}(\mathcal{C}, \mathcal{D})$$

equals Q^m . If ϕ is a bijection and $\mathcal{B}_i = \mathcal{A}_{\phi(i)}$ for all $1 \leq i \leq m$, then Q^ϕ is a bijection.

By Proposition 2.12 and Lemma 2.16, we have the following criterion.

Proposition 4.8 *Let $\mathcal{LE}_\phi^{\text{red}}: \mathcal{LT}_\phi^{\text{red}}\mathbb{Q}_{\mathcal{A}_1, \dots, \mathcal{A}_n}\mathcal{C} \rightarrow \mathcal{C}$ be the 2-functor (4.7).*

- (1) *If $\mathcal{LE}_\phi^{\text{red}}$ is locally essentially surjective, then Q_D^ϕ is 2-faithful for every 2-category $\mathcal{D}$.*
- (2) *The following conditions are equivalent:*
 - (a) *The 2-functor $\mathcal{LE}_\phi^{\text{red}}$ is a bi-equivalence.*
 - (b) *The 2-functor Q_D^ϕ is a bi-equivalence for every 2-category $\mathcal{D}$.*
 - (c) *The 2-functor Q_D^ϕ is a $\mathcal{D}^{\text{Ob}(\mathcal{C})}$ -2-equivalence for every 2-category $\mathcal{D}$.*

Later we will give an explicit (and uniform) description of the 2-functor Q^ϕ (see Remark 5.2). However, the above construction via $\mathcal{LE}_\phi$ will be essential in proving Theorem 1.2.

5 Gluing two pseudo functors

In this section, we study the gluing of two pseudo functors and prove Theorem 1.1. Throughout this section, $\mathcal{C}$ is a $(2, 1)$ -category, and $\mathcal{A}$ and $\mathcal{B}$ are locally full sub-2-categories that each contain all objects of $\mathcal{C}$. In particular, we have $\text{Ob}(\mathcal{A}) = \text{Ob}(\mathcal{B}) = \text{Ob}(\mathcal{C})$. The theorem gives sufficient conditions for the descent 2-functor (3.17)

$$Q_{\mathcal{D}} = Q_{\mathcal{D}}^2: \mathcal{PsFun}(\mathcal{C}, \mathcal{D}) \rightarrow \mathcal{GD}_{\mathcal{A}, \mathcal{B}}(\mathcal{C}, \mathcal{D}),$$

where the target is the 2-category of gluing data, to be a 2-equivalence. It is possible to state the theorem in concrete terms, without reference to the constructions of Sect. 3.

In fact, we will give an explicit description of the 2-category of gluing data (Remark 5.1) and the descent 2-functor (Remark 5.2).

In Sect. 3, the descent 2-functor was defined more conceptually using the operations $\mathcal{T}_2$ (2-category of paths) and $\mathbb{Q}^2$ (double category of squares). Up to 2-equivalences, the descent 2-functor is induced by the 2-functor

$$E = \mathcal{L}E_2^{\text{red}}: \mathcal{L}\mathcal{T}_2^{\text{red}}\mathbb{Q}_{\mathcal{A},\mathcal{B}}\mathcal{C} \rightarrow \mathcal{C}. \quad (5.1)$$

This conceptual interpretation is an essential ingredient in the proof of Theorem 1.1. Indeed, we will deduce Theorem 1.1 from a criterion involving fundamental groups in the 2-category $\mathcal{L}\mathcal{T}_2^{\text{red}}\mathbb{Q}_{\mathcal{A},\mathcal{B}}\mathcal{C}$ (Theorem 5.9).

Remark 5.1 (Explicit description of gluing data) Let us explicitly describe the $\mathcal{D}^{\text{Ob}(\mathcal{C})}$ -2-category $\mathcal{GD}_{\mathcal{A},\mathcal{B}}(\mathcal{C}, \mathcal{D})$. An *object* of it is a triple $(F_{\mathcal{A}}, F_{\mathcal{B}}, (G_D))$ consisting of an object $F_{\mathcal{A}}$ of $\mathcal{PsFun}(\mathcal{A}, \mathcal{D})$, an object $F_{\mathcal{B}}$ of $\mathcal{PsFun}(\mathcal{B}, \mathcal{D})$ satisfying $|F_{\mathcal{A}}| = |F_{\mathcal{B}}|$ (recall from Notation 2.9 that $|-|$ denotes the restriction to $\text{Ob}(\mathcal{C})$), and a family of invertible 2-cells of $\mathcal{D}$

$$G_D: F_{\mathcal{A}}(i)F_{\mathcal{B}}(q) \Rightarrow F_{\mathcal{B}}(p)F_{\mathcal{A}}(j),$$

D running over $(\mathcal{A}, \mathcal{B})$ -squares in $\mathcal{C}$ of the form (3.8). The triple is subject to the following conditions:

- (a) For any square D of the form

$$\begin{array}{ccc} X & \xrightarrow{j} & Y \\ \parallel & \nearrow \alpha & \parallel \\ X & \xrightarrow{i} & Y, \end{array}$$

the following square commutes

$$\begin{array}{ccc} F_{\mathcal{A}}(i) & \xRightarrow{F_{\mathcal{A}}(\alpha)} & F_{\mathcal{A}}(j) \\ \Downarrow & & \Downarrow \\ F_{\mathcal{A}}(i)F_{\mathcal{B}}(\text{id}_X) & \xRightarrow{G_D} & F_{\mathcal{B}}(\text{id}_Y)F_{\mathcal{A}}(j). \end{array}$$

- (a') For any square D of the form

$$\begin{array}{ccc} X & \xlongequal{\quad} & X \\ q \downarrow & \nearrow \alpha & \downarrow p \\ Y & \xlongequal{\quad} & Y, \end{array}$$

the following square commutes

$$\begin{array}{ccc} F_{\mathcal{B}}(q) & \xrightarrow{F_{\mathcal{B}}(\alpha)} & F_{\mathcal{B}}(p) \\ \Downarrow & & \Downarrow \\ F_{\mathcal{A}}(\text{id}_Y)F_{\mathcal{B}}(q) & \xrightarrow{G_D} & F_{\mathcal{B}}(p)F_{\mathcal{A}}(\text{id}_X). \end{array}$$

(b) If D, D', D'' are respectively the upper, lower and outer squares of the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{i_1} & Y_1 \\ q \downarrow & \alpha \nearrow & \downarrow p \\ X_2 & \xrightarrow{i_2} & Y_2 \\ q' \downarrow & \alpha' \nearrow & \downarrow p' \\ X_3 & \xrightarrow{i_3} & Y_3, \end{array}$$

then the following pentagon commutes

$$\begin{array}{ccc} F_{\mathcal{A}}(i_3)F_{\mathcal{B}}(q')F_{\mathcal{B}}(q) & \xrightarrow{G_{D'}} & F_{\mathcal{B}}(p')F_{\mathcal{A}}(i_2)F_{\mathcal{B}}(q) \xrightarrow{G_D} F_{\mathcal{B}}(p')F_{\mathcal{B}}(p)F_{\mathcal{A}}(i_1) \\ \Downarrow & & \Downarrow \\ F_{\mathcal{A}}(i_3)F_{\mathcal{B}}(q'q) & \xrightarrow{G_{D''}} & F_{\mathcal{B}}(p'p)F_{\mathcal{A}}(i_1). \end{array}$$

(b') If D, D', D'' are respectively the left, right and outer squares of the diagram

$$\begin{array}{ccccc} X_1 & \xrightarrow{j} & X_2 & \xrightarrow{j'} & X_3 \\ p_1 \downarrow & \alpha \nearrow & \downarrow p_2 & \alpha' \nearrow & \downarrow p_3 \\ Y_1 & \xrightarrow{i} & Y_2 & \xrightarrow{i'} & Y_3, \end{array}$$

then the following pentagon commutes

$$\begin{array}{ccc} F_{\mathcal{A}}(i')F_{\mathcal{A}}(i)F_{\mathcal{B}}(p_1) & \xrightarrow{G_D} & F_{\mathcal{A}}(i')F_{\mathcal{B}}(p_2)F_{\mathcal{A}}(j) \xrightarrow{G_{D'}} F_{\mathcal{B}}(p_3)F_{\mathcal{A}}(j')F_{\mathcal{A}}(j) \\ \Downarrow & & \Downarrow \\ F_{\mathcal{A}}(i'i)F_{\mathcal{B}}(p_1) & \xrightarrow{G_{D''}} & F_{\mathcal{B}}(p_3)F_{\mathcal{A}}(j'j). \end{array}$$

A morphism $(F_{\mathcal{A}}, F_{\mathcal{B}}, G) \rightarrow (F'_{\mathcal{A}}, F'_{\mathcal{B}}, G')$ of $\mathcal{GD}_{\mathcal{A}, \mathcal{B}}(\mathcal{C}, \mathcal{D})$ is a pair $(\alpha_{\mathcal{A}}, \alpha_{\mathcal{B}})$ consisting of a morphism $\alpha_{\mathcal{A}}: F_{\mathcal{A}} \rightarrow F'_{\mathcal{A}}$ of $\mathcal{PsFun}(\mathcal{A}, \mathcal{D})$ and a morphism $\alpha_{\mathcal{B}}: F_{\mathcal{B}} \rightarrow F'_{\mathcal{B}}$ of $\mathcal{PsFun}(\mathcal{B}, \mathcal{D})$ such that $|\alpha_{\mathcal{A}}| = |\alpha_{\mathcal{B}}|$ and satisfying the following condition

(m) For any $(\mathcal{A}, \mathcal{B})$ -square D (3.8), the following hexagon commutes

$$\begin{array}{ccccc} \alpha_0(W)F_{\mathcal{A}}(i)F_{\mathcal{B}}(q) & \xrightarrow{\alpha_{\mathcal{A}}(i)} & F'_{\mathcal{A}}(i)\alpha_0(Z)F_{\mathcal{B}}(q) & \xrightarrow{\alpha_{\mathcal{B}}(q)} & F'_{\mathcal{A}}(i)F'_{\mathcal{B}}(q)\alpha_0(X) \\ \downarrow G_D & & & & \downarrow G'_D \\ \alpha_0(W)F_{\mathcal{B}}(p)F_{\mathcal{A}}(j) & \xrightarrow{\alpha_{\mathcal{B}}(p)} & F'_{\mathcal{B}}(p)\alpha_0(Y)F_{\mathcal{A}}(j) & \xrightarrow{\alpha_{\mathcal{A}}(j)} & F'_{\mathcal{B}}(p)F'_{\mathcal{A}}(j)\alpha_0(X). \end{array}$$

Here $\alpha_0 = |\alpha_{\mathcal{A}}| = |\alpha_{\mathcal{B}}|$.

A 2-cell of $\mathcal{GD}_{\mathcal{A},\mathcal{B}}(\mathcal{C}, \mathcal{D})$ is a pair $(\Xi_{\mathcal{A}}, \Xi_{\mathcal{B}}): (\alpha_{\mathcal{A}}, \alpha_{\mathcal{B}}) \Rightarrow (\alpha'_{\mathcal{A}}, \alpha'_{\mathcal{B}})$ consisting of a 2-cell $\Xi_{\mathcal{A}}: \alpha_{\mathcal{A}} \Rightarrow \alpha'_{\mathcal{A}}$ of $\mathcal{PsFun}(\mathcal{A}, \mathcal{D})$ and a 2-cell $\Xi_{\mathcal{B}}: \alpha_{\mathcal{B}} \Rightarrow \alpha'_{\mathcal{B}}$ of $\mathcal{PsFun}(\mathcal{B}, \mathcal{D})$ such that $|\Xi_{\mathcal{A}}| = |\Xi_{\mathcal{B}}|$.

The $\mathcal{D}^{\text{Ob}(\mathcal{C})}$ -2-category structure of $\mathcal{GD}_{\mathcal{A},\mathcal{B}}(\mathcal{C}, \mathcal{D})$ is given by the 2-functor defined by

$$\begin{aligned} (F_{\mathcal{A}}, F_{\mathcal{B}}, G) &\mapsto |F_{\mathcal{A}}| = |F_{\mathcal{B}}|, \quad (\alpha_{\mathcal{A}}, \alpha_{\mathcal{B}}) \mapsto |\alpha_{\mathcal{A}}| = |\alpha_{\mathcal{B}}|, \\ (\Xi_{\mathcal{A}}, \Xi_{\mathcal{B}}) &\mapsto |\Xi_{\mathcal{A}}| = |\Xi_{\mathcal{B}}|. \end{aligned}$$

In an object $(F_{\mathcal{A}}, F_{\mathcal{B}}, G)$ of $\mathcal{GD}_{\mathcal{A},\mathcal{B}}(\mathcal{C}, \mathcal{D})$, the family G of 2-cells expresses compatibility between the pseudo functors $F_{\mathcal{A}}$ and $F_{\mathcal{B}}$. The necessity of such compatibility for gluing $F_{\mathcal{A}}$ and $F_{\mathcal{B}}$ into a pseudo functor $\mathcal{C} \rightarrow \mathcal{D}$ is also clear from the following remark.

Remark 5.2 (Explicit description of the descent 2-functor) Let us explicitly describe the descent 2-functor

$$Q_{\mathcal{D}}: \mathcal{PsFun}(\mathcal{C}, \mathcal{D}) \rightarrow \mathcal{GD}_{\mathcal{A},\mathcal{B}}(\mathcal{C}, \mathcal{D}).$$

For an object F of $\mathcal{PsFun}(\mathcal{C}, \mathcal{D})$, we have $Q_{\mathcal{D}}(F) = (F|_{\mathcal{A}}, F|_{\mathcal{B}}, G)$. Here, $F|_{\mathcal{A}}$ and $F|_{\mathcal{B}}$ denote restrictions of F . For an $(\mathcal{A}, \mathcal{B})$ -square D (3.8), G_D is the composition

$$F(i)F(q) \Longrightarrow F(iq) \xrightarrow{F(\alpha)} F(pj) \Longrightarrow F(p)F(j).$$

For a morphism $\alpha: F \rightarrow F'$ of $\mathcal{PsFun}(\mathcal{C}, \mathcal{D})$, $Q_{\mathcal{D}}(\alpha)$ is

$$(\alpha|_{\mathcal{A}}, \alpha|_{\mathcal{B}}): (F|_{\mathcal{A}}, F|_{\mathcal{B}}, G) \rightarrow (F'|_{\mathcal{A}}, F'|_{\mathcal{B}}, G').$$

For a 2-cell $\Xi: \alpha \Rightarrow \alpha'$ of $\mathcal{PsFun}(\mathcal{A}, \mathcal{B})$, $Q_{\mathcal{D}}(\Xi)$ is

$$(\Xi|_{\mathcal{A}}, \Xi|_{\mathcal{B}}): (\alpha|_{\mathcal{A}}, \alpha|_{\mathcal{B}}) \Rightarrow (\alpha'|_{\mathcal{A}}, \alpha'|_{\mathcal{B}}).$$

Remark 5.3 The isomorphism of $\mathcal{D}^{\text{Ob}(\mathcal{C})}$ -2-categories (4.8)

$$Q^!: \mathcal{GD}_{\mathcal{A},\mathcal{B}}(\mathcal{C}, \mathcal{D}) \cong \mathcal{GD}_{\mathcal{B},\mathcal{A}}(\mathcal{C}, \mathcal{D})$$

associated to the map $t: \{1, 2\} \rightarrow \{1, 2\}$ swapping 1 and 2 can be described as follows. To an object $(F_{\mathcal{A}}, F_{\mathcal{B}}, G)$ of $\mathcal{GD}_{\mathcal{A}, \mathcal{B}}(\mathcal{C}, \mathcal{D})$, we associate $(F_{\mathcal{B}}, F_{\mathcal{A}}, G^*)$. Here, for every $(\mathcal{A}, \mathcal{B})$ -square D (3.8), $G_{D^*}^* = G_D^{-1}$, where D^* is the square obtained from D by inverting α . To a morphism $(\alpha_{\mathcal{A}}, \alpha_{\mathcal{B}}): (F_{\mathcal{A}}, F_{\mathcal{B}}, G) \rightarrow (F'_{\mathcal{A}}, F'_{\mathcal{B}}, G')$ of $\mathcal{GD}_{\mathcal{A}, \mathcal{B}}(\mathcal{C}, \mathcal{D})$, we associate

$$(\alpha_{\mathcal{B}}, \alpha_{\mathcal{A}}): (F_{\mathcal{B}}, F_{\mathcal{A}}, G^*) \rightarrow (F'_{\mathcal{B}}, F'_{\mathcal{A}}, G'^*).$$

To a 2-cell $(\Xi_{\mathcal{A}}, \Xi_{\mathcal{B}}): (\alpha_{\mathcal{A}}, \alpha_{\mathcal{B}}) \Rightarrow (\alpha'_{\mathcal{A}}, \alpha'_{\mathcal{B}})$ of $\mathcal{GD}_{\mathcal{A}, \mathcal{B}}(\mathcal{C}, \mathcal{D})$, we associate

$$(\Xi_{\mathcal{B}}, \Xi_{\mathcal{A}}): (\alpha_{\mathcal{B}}, \alpha_{\mathcal{A}}) \Rightarrow (\alpha'_{\mathcal{B}}, \alpha'_{\mathcal{A}}).$$

Definition 5.4 We say that $\mathcal{A}$ and $\mathcal{B}$ generate $\mathcal{C}$ if for every morphism f of $\mathcal{C}$, there exist morphisms $i_1, \dots, i_n$ of $\mathcal{A}$ and $p_1, \dots, p_n$ of $\mathcal{B}$ and a 2-cell $p_1 i_1 \dots p_n i_n \Rightarrow f$ of $\mathcal{C}$.

Remark 5.5 We have $\mathcal{A}$ and $\mathcal{B}$ generate $\mathcal{C}$ if and only if $E: \mathcal{LT}_2^{\text{red}} \mathbb{Q}_{\mathcal{A}, \mathcal{B}} \mathcal{C} \rightarrow \mathcal{C}$ is locally essentially surjective. In particular, if $\mathcal{A}$ and $\mathcal{B}$ generate $\mathcal{C}$, then $\mathcal{Q}_{\mathcal{D}}$ is 2-faithful by Proposition 3.28 (1).

Remark 5.6 Let us recall the results of Deligne [6, Proposition 3.3.2] and Ayoub [1, Theorem 1.3.1]. Both assume $\mathcal{C} = \mathbf{C}$ to be an (ordinary) category. Let $\mathcal{A} = \mathbf{A}$ and $\mathcal{B} = \mathbf{B}$ be subcategories of $\mathbf{C}$, each containing all objects of $\mathbf{C}$. Consider the conditions (1) and (2) of Theorem 1.1, as well as the following conditions:

- (2a) Fiber products exist in $\mathbf{B}$ and are fiber products in $\mathbf{C}$.
- (2b) $\mathbf{C}$ admits fiber products and morphisms of $\mathbf{A}$ and $\mathbf{B}$ are stable under base change by morphisms of $\mathbf{C}$. (In particular, all isomorphisms in $\mathbf{C}$ are in $\mathbf{A} \cap \mathbf{B}$.) Moreover, the diagonal of every morphism in $\mathbf{C}$ is in $\mathbf{A}$.

Deligne assumes (1) and (2a), while Ayoub assumes (1) and (2b). In our language, their conclusions can be stated as saying that $\mathcal{Q}_{\mathcal{D}}$ induces a bijection between equivalence classes of $\mathcal{PSFun}(\mathbf{C}, \mathcal{D})$ and $\mathcal{GD}_{\mathbf{A}, \mathbf{B}}(\mathbf{C}, \mathcal{D})$. There are no implications between (2a) and (2b). Moreover, each of (2a) and (2b) is stronger than (2). Thus Theorem 1.1 is a common generalization of the results of Deligne and Ayoub. Even under the assumptions of Deligne and Ayoub, the conclusion of Theorem 1.1 is more precise in the sense that it establishes a 2-equivalence of 2-categories. As we have remarked in the Introduction, this precision is useful in the construction of pseudo natural transformations.

By Proposition 3.28 (2), the conclusion of Theorem 1.1 is equivalent to saying that $E: \mathcal{LT}_2^{\text{red}} \mathbb{Q}_{\mathcal{A}, \mathcal{B}} \mathcal{C} \rightarrow \mathcal{C}$ is a bi-equivalence. We will construct a pseudo inverse F as follows. The quadruple (Z, i, p, α) in Theorem 1.1 (1) corresponding to the diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & Z \\ & \searrow f & \downarrow p \\ & & Y \end{array} \quad \begin{array}{c} \swarrow \alpha \\ \swarrow \alpha \end{array}$$

is called a *compactification* of f (relative to $\mathcal{A}, \mathcal{B}$). Given a choice of compactification, $p * i$ is a candidate for Ff . To express the dependence on choices, we need to organize the set of compactifications $\text{Comp}_{\mathcal{A}, \mathcal{B}}(f)$ into 2-categories.

Definition 5.7 (2-Category $\text{Comp}_{\mathcal{A}, \mathcal{B}}^{\mathcal{P}}(f)$ of compactifications of a morphism f) Let $f: X \rightarrow Y$ be a morphism of $\mathcal{C}$. Let $\mathcal{P}$ be a locally full sub-2-category of $\mathcal{C}$ containing all objects of $\mathcal{C}$. We define the 2-category of compactifications of f (relative to $\mathcal{A}, \mathcal{B}$, and $\mathcal{P}$), denoted $\text{Comp}^{\mathcal{P}}(f) = \text{Comp}_{\mathcal{A}, \mathcal{B}}^{\mathcal{P}}(f)$, as follows. The *objects* are compactifications of f (relative to $\mathcal{A}, \mathcal{B}$, independent of $\mathcal{P}$). A *morphism* $(Z, i, p, \alpha) \rightarrow (W, j, q, \beta)$ is a triple (r, γ, δ) consisting of a morphism $r: Z \rightarrow W$ of $\mathcal{P}$, and 2-cells $\gamma: ri \Rightarrow j$ and $\delta: qr \Rightarrow p$ of $\mathcal{C}$, fitting in the diagram

$$\begin{array}{ccccc} & & Z & & \\ & i \nearrow & \downarrow r & \nwarrow p & \\ X & \xrightarrow{j} & W & \xleftarrow{\delta^{-1}} & \\ & f \searrow & \downarrow q & & \\ & & Y & & \end{array}$$

$\Downarrow \gamma$ $\Downarrow \beta$

where the outer triangle is α . A 2-cell $(r, \gamma, \delta) \Rightarrow (r', \gamma', \delta')$ is a 2-cell $\epsilon: r \Rightarrow r'$ of $\mathcal{P}$ such that $\gamma = \gamma' \circ (\epsilon i)$ and $\delta = \delta' \circ (q \epsilon)$. We omit $\mathcal{A}$ and $\mathcal{B}$ from the notation when no confusion arises.

The 2-category $\text{Comp}^{\mathcal{P}}(f)$ is a $(2, 1)$ -category, and is a locally full sub-2-category of $\text{Comp}^{\mathcal{C}}(f)$ containing all objects of $\text{Comp}^{\mathcal{C}}(f)$. We will mainly work with the category $\mathbf{OComp}^{\mathcal{P}}(f)$ (Construction 2.17). A compactification (Z, i, p, α) of f gives a morphism $p * i: X \rightarrow Y$ of $\mathcal{T}_2^{\text{red}} \mathbb{Q}_{\mathcal{A}, \mathcal{B}} \mathcal{C}$. A morphism (r, γ, δ) of $\text{Comp}^{\mathcal{B}}(f)$ as above gives a 2-cell

$$q * j = q * j * \text{id}_X^v \xrightarrow{\sigma_{q, D, \text{id}_X}} q * r * i = qr * i \Rightarrow p * i$$

of $\mathcal{T}_2^{\text{red}} \mathbb{Q}_{\mathcal{A}, \mathcal{B}} \mathcal{C}$, where the last 2-cell is given by the image of δ under the pseudo functor $\mathcal{B} \rightarrow \mathcal{T}_2^{\text{red}} \mathbb{Q}_{\mathcal{A}, \mathcal{B}} \mathcal{C}$ induced by G_v (3.13), and D is the square

$$\begin{array}{ccc} X & \xrightarrow{i} & Z \\ \parallel & & \downarrow r \\ X & \xrightarrow{j} & W \end{array}$$

given by γ^{-1} . This defines a functor

$$\mathbf{OComp}^{\mathcal{B}}(f) \rightarrow (\mathcal{T}_2^{\text{red}} \mathbb{Q}_{\mathcal{A}, \mathcal{B}} \mathcal{C})(X, Y)^{\text{op}}.$$

Composing with the functor inverting 2-cells in $\mathcal{E} = \mathcal{L}\mathcal{T}_2^{\text{red}} \mathbb{Q}_{\mathcal{A}, \mathcal{B}} \mathcal{C}$, we get a functor

$$G_f: \mathbf{OComp}^{\mathcal{B}}(f) \rightarrow \mathcal{E}(X, Y). \quad (5.2)$$

To investigate the compatibility of F with composition, we also need to consider compactifications of sequences of morphisms. Recall that, for $n \geq 0$, $[n]$ denotes the totally ordered set $\{0, \dots, n\}$. We define the 2-category of n -simplices of $\mathcal{C}$ to be the 2-category $\mathcal{UPsFun}([n], \mathcal{C})$ of strictly unital pseudo functors $[n] \rightarrow \mathcal{C}$ (Notation 2.4). A sequence of morphisms $X_0 \xrightarrow{f_1} \dots \xrightarrow{f_n} X_n$ defines a 2-functor $[n] \rightarrow \mathcal{C}$, which we denote by $(f_n, \dots, f_1)$. Let ∇_n be the partially ordered set

$$\{(k, l) \in [n] \times [n] \mid k \geq l\}.$$

The diagonal embedding $[n] \rightarrow \nabla_n$ induces a 2-functor $\mathcal{UPsFun}(\nabla_n, \mathcal{C}) \rightarrow \mathcal{UPsFun}([n], \mathcal{C})$. The following generalization of Definition 5.7 slightly generalizes [6, Définition 3.2.5] and [1, Section 1.3.1].

Definition 5.8 (2-Category $\text{Comp}_{\mathcal{A}, \mathcal{B}}^{\mathcal{P}}(\sigma)$ of compactifications of an n -simplex σ) Let σ be an n -simplex of $\mathcal{C}$ and let $\mathcal{P}$ be a locally full sub-2-category of $\mathcal{C}$ containing all objects of $\mathcal{C}$. The 2-category of compactifications of σ (relative to $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{P}$), denoted $\text{Comp}^{\mathcal{P}}(\sigma) = \text{Comp}_{\mathcal{A}, \mathcal{B}}^{\mathcal{P}}(\sigma)$, is the locally full sub-2-category of the strict fiber product

$$\mathcal{UPsFun}(\nabla_n, \mathcal{C}) \times_{\mathcal{UPsFun}([n], \mathcal{C})} \{\sigma\}$$

spanned by pseudo functors F such that $F(k \rightarrow k', l)$ is a morphism of $\mathcal{A}$ and $F(k', l \rightarrow l')$ is a morphism of $\mathcal{B}$, for all elements (k, l) and (k', l') of ∇_n , and pseudo natural transformations α such that $\alpha(k, l)$ is a morphism in $\mathcal{P}$ for every element (k, l) of ∇_n . We denote the set of objects by $\text{Comp}_{\mathcal{A}, \mathcal{B}}(\sigma)$, which does not depend on $\mathcal{P}$.

The 2-category $\text{Comp}^{\mathcal{P}}(\sigma)$ is a $(2, 1)$ -category. A compactification of $(f_n, \dots, f_1)$ can be represented by a diagram of the form

with horizontal arrows in $\mathcal{A}$, vertical arrows in $\mathcal{B}$, and suitable 2-cells.

Let $d: [m] \rightarrow [n]$ be a nondecreasing map and let σ be an n -simplex. We have a 2-functor

$$\mathcal{C}omp^{\mathcal{P}}(\sigma) \rightarrow \mathcal{C}omp^{\mathcal{P}}(\sigma \circ d) \quad (5.3)$$

sending $F: \nabla_n \rightarrow \mathcal{C}$ to the composition $dF: \nabla_m \xrightarrow{d \times d} \nabla_n \xrightarrow{F} \mathcal{C}$.

We will deduce Theorem 1.1 from the following criterion.

Theorem 5.9 *Let $\mathcal{C}$ be a $(2, 1)$ -category and let $\mathcal{A}$ and $\mathcal{B}$ be locally full sub-2-categories of $\mathcal{C}$, each containing all objects of $\mathcal{C}$. Let $\mathcal{E} = \mathcal{L}T_2^{\text{red}}\mathbb{Q}_{\mathcal{A}, \mathcal{B}}\mathcal{C}$. Assume the following:*

- (1) *For any morphism $f: X \rightarrow Y$ of $\mathcal{C}$, there exists a compactification $\kappa = (Z, i, p, \alpha)$ of f relative to $\mathcal{A}, \mathcal{B}$ such that the homomorphism of fundamental groups*

$$\pi_1(\mathbf{O}Comp^{\mathcal{B}}(f), \kappa) \rightarrow \pi_1(\mathcal{E}(X, Y), p * i)$$

induced by G_f (5.2) has trivial image.

- (2) *For any sequence of two morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, $Comp^{\mathcal{B}}(g, f)$ is connected. Then the 2-functor $E: \mathcal{E} \rightarrow \mathcal{C}$ (5.1) is a bi-equivalence.*

Note that (2) implies that $Comp^{\mathcal{B}}(f)$ is connected, so that (1) holds for every compactification κ of f .

Remark 5.10 The proof of Theorem 5.9 makes use of the following generalization of (5.2). There exists a unique way to associate, for every $n \geq 0$ and every n -simplex σ of $\mathcal{C}$, a functor

$$G_{\sigma}: \mathbf{O}Comp^{\mathcal{B}}(\sigma) \rightarrow \mathcal{UPsFun}([n], \mathcal{E}), \quad (5.4)$$

satisfying the following conditions:

- (0) For every 0-simplex (i.e. object) X of $\mathcal{C}$, the image of G_X is X .
- (1) For every 1-simplex (i.e. morphism) f of $\mathcal{C}$, the functor G_f is (5.2).
- (2) For every 2-simplex σ of $\mathcal{C}$, the functor G_{σ} sends a compactification as partly shown by the diagram

$$\begin{array}{ccccc}
 & & k & & \\
 & & \curvearrowright & & \\
 X & \xrightarrow{i} & X' & \xrightarrow{\ell} & X'' \\
 & & \downarrow p & \searrow D & \downarrow s \\
 & & Y & \xrightarrow{j} & Y' \xrightarrow{\beta} Y'' \\
 & & & & \downarrow q \\
 & & & & Z
 \end{array}$$

(A curved arrow labeled r goes from X'' to Z .)

to the 2-simplex of $\mathcal{E}$

$$\begin{array}{ccc}
 & r * k & \\
 & \curvearrowright & \\
 X & \xrightarrow{p * i} & Y & \xrightarrow{q * j} & Z
 \end{array}$$

where the 2-cell is the composition

$$q * j * p * i \xrightarrow{\sigma_{q,D,i}} q * s * \ell * i = (qs) * (\ell i) \xrightarrow{\beta * \alpha} r * k.$$

(3) For any non-decreasing map $d: [m] \rightarrow [n]$, the diagram

$$\begin{array}{ccc} \mathbf{OComp}^{\mathcal{B}}(\sigma) & \xrightarrow{G_\sigma} & \mathcal{UPsFun}([n], \mathcal{E}) \\ (5.3) \downarrow & & \downarrow \mathcal{UPsFun}(d, \mathcal{E}) \\ \mathbf{OComp}^{\mathcal{B}}(\sigma \circ d) & \xrightarrow{G_{\sigma \circ d}} & \mathcal{UPsFun}([m], \mathcal{E}) \end{array}$$

commutes.

It follows from (0) and (3) that the image of G_σ lies in the category $\mathcal{UPsFun}([n], \mathcal{E}) \times_{\mathcal{E}^{\text{Ob}([n])}} \{[\sigma]\}$.

Lemma 5.11 *Let $\mathcal{C}$ be a $(2, 1)$ -category and let $\mathcal{A}$ and $\mathcal{B}$ be locally full sub-2-categories of $\mathcal{C}$, each containing all objects of $\mathcal{C}$, such that $\text{Comp}_{\mathcal{A}, \mathcal{B}}(f)$ is nonempty for every morphism f of $\mathcal{C}$. Then $\text{Comp}_{\mathcal{A}, \mathcal{B}}(\sigma)$ is nonempty for every n -simplex σ .*

Recall that $\text{Comp}_{\mathcal{A}, \mathcal{B}}^{\mathcal{P}}(\sigma)$ denotes the set of objects of $\text{Comp}_{\mathcal{A}, \mathcal{B}}^{\mathcal{P}}(\sigma)$, which is independent of $\mathcal{P}$.

Proof Consider the subset

$$\nabla_{n,j} = \{(k, l) \in [n] \times [n] \mid 0 \leq k - l \leq j\} \subseteq \nabla_n$$

for $0 \leq j \leq n$. We show by induction on j that there exists a strictly unital pseudo functor $F: \nabla_{n,j} \rightarrow \mathcal{C}$ such that F composed with the diagonal embedding $[n] \rightarrow \nabla_{n,j}$ equals σ and that $F(k \rightarrow k', \ell)$ is in $\mathcal{A}$, $F(k', \ell \rightarrow \ell')$ is in $\mathcal{B}$. The case $j = 0$ is trivial as the diagonal embedding $[n] \rightarrow \nabla_{n,0}$ is an isomorphism. For $j \geq 1$, assume that $F: \nabla_{n,j-1} \rightarrow \mathcal{C}$ has been constructed. Applying the hypothesis, we get a decomposition $F((k + j - 1, k) \rightarrow (k + j, k + 1)) \cong pi$ for $0 \leq k \leq n - j$. We take $F(k + j - 1 \rightarrow k + j, k) = i$ and $F(k + j, k \rightarrow k + 1) = p$. In general, for $P \leq Q$ in $\nabla_{n,j}$ with $P \neq Q$, we take

$$F(P \rightarrow Q) = F(Q' \rightarrow Q)F(P' \rightarrow Q')F(P \rightarrow P')$$

where $P', Q' \in \nabla_{n,j-1}$ are given by

$$P' = \begin{cases} P & \text{if } P \in \nabla_{n,j-1}, \\ P + (0, 1) & \text{if } P \notin \nabla_{n,j-1}, \end{cases} \quad Q' = \begin{cases} Q & \text{if } Q \in \nabla_{n,j-1}, \\ Q - (1, 0) & \text{if } Q \notin \nabla_{n,j-1}. \end{cases}$$

The coherence constraint of F is given by the obvious 2-cells. For $j = n$, we have $\nabla_{n,n} = \nabla_n$ so that $\text{Comp}_{\mathcal{A}, \mathcal{B}}(\sigma)$ is nonempty. $\square$

Proof of Theorem 5.9 For any set S , we denote by $S^\sim$ the contractible groupoid of underlying set S . In other words, we have $\text{Ob}(S^\sim) = S$ and for all elements X and Y of S , there exists a unique morphism in $S^\sim$ from X to Y . By assumption, for any morphism $f: X \rightarrow Y$ of $\mathcal{C}$, G_f (5.2) factorizes through a functor $G_f^\sim: \text{Comp}(f)^\sim \rightarrow \mathcal{E}(X, Y)$. For every f , choose an element κ_f of $\text{Comp}(f)$ such that $\kappa_{\text{id}_X} = (X, \text{id}_X, \text{id}_X, \text{id}_{\text{id}_X})$ for every object X of $\mathcal{C}$. We denote by $d_i^n: [n-1] \rightarrow [n]$ and $s_i^n: [n+1] \rightarrow [n]$, $0 \leq i \leq n$, the face and degeneracy maps, respectively.

We construct a strictly unital pseudo functor $F: \mathcal{C} \rightarrow \mathcal{E}$ as follows. On objects we let F be the identity. For any morphism f of $\mathcal{C}$, we take $Ff = G_f(\kappa_f)$. In particular, $F(\text{id}_X) = \text{id}_X$. A 2-cell $\alpha: f \Rightarrow g$ of $\mathcal{C}$ induces an isomorphism of 2-categories $H_\alpha: \text{Comp}^{\mathcal{B}}(f) \rightarrow \text{Comp}^{\mathcal{B}}(g)$ such that $G_g H_\alpha = G_f$. We take $F\alpha = G_g^\sim(H_\alpha(\kappa_f) \rightarrow \kappa_g)$.

We construct the coherence constraint of F as follows. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a sequence of morphisms of $\mathcal{C}$. We consider the 2-simplex (g, f) of $\mathcal{C}$ and the 2-functor $G_{g,f} := G_{(g,f)}$ (5.4). For any object λ of $\text{Comp}^{\mathcal{B}}(g, f)$, the image $G_{g,f}(\lambda)$ is a 2-simplex of $\mathcal{E}$ with edges $G_f(d_2^2\lambda)$, $G_{gf}(d_1^2\lambda)$, $G_g(d_0^2\lambda)$. Applying Lemma 2.10 to $G_{g,f}(\lambda)$, we get a unique pair $(F_\lambda, \phi_\lambda)$, where F_λ is a 2-simplex of $\mathcal{E}$ with edges $F(f)$, $F(gf)$, $F(g)$, and $\phi_\lambda: G_{g,f}(\lambda) \rightarrow F_\lambda$ is a morphism of $\mathcal{UPsFun}([2], \mathcal{E}) \times_{\mathcal{E}^{\text{Ob}([2])}} \{(g, f)\}$ satisfying

$$\begin{aligned}\phi_\lambda(0 \rightarrow 1) &= G_f^\sim(d_2^2\lambda \rightarrow \kappa_f), \quad \phi_\lambda(0 \rightarrow 2) = G_{gf}^\sim(d_1^2\lambda \rightarrow \kappa_{gf}), \\ \phi_\lambda(1 \rightarrow 2) &= G_g^\sim(d_0^2\lambda \rightarrow \kappa_g).\end{aligned}$$

By the uniqueness of the pair, for any morphism $\psi: \lambda \rightarrow \lambda'$ of $\text{Comp}^{\mathcal{B}}(g, f)$, we have $F_\lambda = F_{\lambda'}$ and $\phi_\lambda = \phi_{\lambda'} \circ G_{g,f}(\psi)$. It then follows from the connectedness of $\text{Comp}^{\mathcal{B}}(g, f)$ that F_λ does not depend on the choice of λ and we denote the corresponding 2-cell of $\mathcal{E}$ by

$$F_{g,f}: F(g)F(f) \Rightarrow F(gf).$$

A 2-cell $\alpha: f \Rightarrow f'$ of $\mathcal{C}$ induces an isomorphism of 2-categories $H_{g,\alpha}: \text{Comp}^{\mathcal{B}}(g, f) \rightarrow \text{Comp}^{\mathcal{B}}(g, f')$, compatible with H_α and $H_{g\alpha}$ and such that $G_{g,f'}H_{g,\alpha} = G_{g,f}$. Thus $F_{g,f}$ is functorial in f . Similarly $F_{g,f}$ is functorial in g . By construction, $F_{\text{id}_X, f}$ is given by $G_{\text{id}_X, f}(s_0^1\kappa_f) = s_0^1G_f(\kappa_f)$, so that $F_{\text{id}_X, f} = \text{id}_{F(f)}$. Similarly, $F_{f, \text{id}_Y} = \text{id}_{F(f)}$ since it is given by $G_{f, \text{id}_Y}(s_1^1\kappa_f)$.

Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ be a sequence of morphisms of $\mathcal{C}$. We consider the 3-simplex $\sigma = (h, g, f)$ of $\mathcal{C}$ and the 2-functor G_σ (5.4). For any object λ of $\text{Comp}^{\mathcal{B}}(\sigma)$, the image $G_\sigma(\lambda)$ is a 3-simplex of $\mathcal{E}$ with faces $G_{h,g}(d_0^3\lambda)$, $G_{h,gf}(d_1^3\lambda)$, $G_{hg,f}(d_2^3\lambda)$, $G_{g,f}(d_3^3\lambda)$. Applying Lemma 2.10 to $G_\sigma(\lambda)$, we get a pair $(F_\lambda, \phi_\lambda)$, where F_λ is a 3-simplex of $\mathcal{E}$ of edges $\kappa_f, \kappa_g, \kappa_h, \kappa_{gf}, \kappa_{hg}, \kappa_{hgf}$, and $\phi_\lambda: G_\sigma(\lambda) \rightarrow F_\lambda$ is a morphism of $\mathcal{UPsFun}([3], \mathcal{E}) \times_{\mathcal{E}^{\text{Ob}([3])}} \{(h, g, f)\}$ such that

$$\phi_\lambda(e) = G_{\sigma(e)}^\sim(d_e\lambda \rightarrow \kappa_{\sigma(e)})$$

for all edges e of $[3]$. Here $d_e: [1] \rightarrow [3]$ denotes the map determined by e . By construction, $d_i^3 F_\lambda = F_{d_i^3\lambda}$ for $0 \leq i \leq 3$. Thus F_λ implies that the diagram

$$\begin{array}{ccc}
 F(h)F(g)F(f) & \xRightarrow{F_{g,f}} & F(h)F(gf) \\
 \downarrow F_{h,g} & & \downarrow F_{h,gf} \\
 F(hg)F(f) & \xRightarrow{F_{hg,f}} & F(hgf)
 \end{array}$$

commutes, which proves the composition axiom. This finishes the construction of F .

Let $\tilde{F}: \mathcal{TC} \rightarrow \mathcal{E}$ be the 2-functor induced by F . Consider the commutative square

$$\begin{array}{ccc}
 \mathcal{LT}_2\mathbb{Q}_{\mathcal{A},\mathcal{B}}\mathcal{C} & \xrightarrow{\tilde{E}} & \mathcal{TC} \\
 R_{\mathcal{E}} \downarrow & & \downarrow R_{\mathcal{C}} \\
 \mathcal{E} & \xrightarrow{E} & \mathcal{C},
 \end{array}$$

where $\tilde{E} = \mathcal{L}E_2$, and the vertical arrows $R_{\mathcal{C}}$ and $R_{\mathcal{E}}$ are the reduction 2-functors. Since $R_{\mathcal{C}}$ and $R_{\mathcal{E}}$ are bi-equivalences, it suffices to construct pseudo natural equivalences $E\tilde{F} \Rightarrow R_{\mathcal{C}}$ and $R_{\mathcal{E}} \Rightarrow \tilde{F}\tilde{E}$.

We define a pseudo natural isomorphism $\epsilon: E\tilde{F} \Rightarrow R_{\mathcal{C}}$ sending X to id_X as follows. To every morphism f of $\mathcal{C}$, we associate the 2-cell $\alpha: pi \Rightarrow f$ in $\kappa_f = (Z, i, p, \alpha)$.

We define a pseudo natural isomorphism $\eta: R_{\mathcal{E}} \Rightarrow \tilde{F}\tilde{E}$ sending X to id_X as follows. To every morphism f of $\mathcal{A}$ (resp. $\mathcal{B}$), we associate the 2-cell

$$f = f * \text{id}_X^v \xRightarrow{\sigma_{\text{id}_Y, D, \text{id}_X}} p * i \quad (\text{resp. } f = \text{id}_Y^h * f \xRightarrow{\sigma_{\text{id}_Y, D', \text{id}_X}} p * i), \quad (5.5)$$

where D (resp. D') is the square

$$\begin{array}{ccc}
 X & \xrightarrow{i} & Z \\
 \parallel & D & \downarrow p \\
 X & \xrightarrow{f} & Y
 \end{array}
 \quad (\text{resp. }
 \begin{array}{ccc}
 X & \xrightarrow{i} & Z \\
 f \downarrow & D' & \downarrow p \\
 Y & \xlongequal{\quad} & Y
 \end{array}
)$$

induced by κ_f . □

Remark 5.12 Similarly to (5.2), we have a functor

$$G'_f: \mathbf{OComp}^{\mathcal{A}}(f) \rightarrow (\mathcal{T}_2^{\text{red}}\mathbb{Q}_{\mathcal{A},\mathcal{B}}\mathcal{C})(X, Y) \rightarrow \mathcal{E}(X, Y). \quad (5.6)$$

It sends every morphism $(r, \gamma, \delta): (Z, i, p, \alpha) \rightarrow (W, j, q, \beta)$ of $\mathbf{Comp}^{\mathcal{A}}(f)$ to the composition

$$p * i = \text{id}_Y^h * p * i \xRightarrow{\sigma_{\text{id}_Y, D, i}} q * r * i = q * ri \xRightarrow{\gamma} q * j,$$

where D is the square

$$\begin{array}{ccc}
 Z & \xrightarrow{r} & W \\
 p \downarrow & & \downarrow q \\
 Y & \xlongequal{\quad} & Y
 \end{array}$$

given by δ^{-1} . The analogue of Theorem 5.9 holds with $\text{Comp}^{\mathcal{B}}$ replaced by $\text{Comp}^{\mathcal{A}}$.

Moreover, (5.2) and (5.6) extend to a functor

$$\mathbf{T}_2\text{Comp}(f) \rightarrow \mathcal{E}(X, Y), \quad (5.7)$$

where

$$\text{Comp}(f) = \mathbb{Q}_{\mathbf{OComp}^{\mathcal{A}}(f), \mathbf{OComp}^{\mathcal{B}}(f)} \mathbf{OComp}^{\mathcal{C}}(f)$$

is the double subcategory of $\mathbb{K}^2(\mathbf{OComp}^{\mathcal{C}}(f))$.

By Proposition 3.28 (2), Theorem 1.1 follows from the following.

Proposition 5.13 *Under the assumptions of Theorem 1.1, for every n -simplex σ of $\mathcal{C}$, the 2-category $\text{Comp}^{\mathcal{B}}(\sigma)^{\text{coop}}$ is directed. Moreover, the conditions of Theorem 5.9 are satisfied.*

Here we say that a 2-category $\mathcal{D}$ is *directed* if its underlying category is directed, namely, if it is nonempty and if for every pair of objects X and Y of $\mathcal{D}$, there exist an object Z of $\mathcal{D}$ and morphisms $X \rightarrow Z$ and $Y \rightarrow Z$ of $\mathcal{D}$. A directed 2-category is connected.

Proof By Lemma 5.11, assumption (1) of Theorem (1) implies that $\text{Comp}^{\mathcal{B}}(\sigma)$ is nonempty. Next we show the following:

- (*) Every morphism of n -simplices $\sigma \rightarrow \tau$ in $\mathcal{C}$, where τ is an n -simplex of $\mathcal{B}$, is isomorphic in $\mathcal{UPsFun}([n], \mathcal{C})$ to the composition $\sigma \xrightarrow{i} \tau' \xrightarrow{p} \tau$, where p is a morphism of n -simplices of $\mathcal{B}$ and $i(j)$ is a morphism of $\mathcal{A}$ for all $0 \leq j \leq n$.

We proceed by induction on n . The case $n = 0$ is assumption (1) of Theorem 1.1. For $n \geq 0$, induction hypothesis provides the restrictions $p \mid \{1, \dots, n\}$ and $i \mid \{1, \dots, n\}$. Applying assumption (2) of Theorem 1.1, we obtain a commutative diagram with 2-cells in $\mathcal{C}$

$$\begin{array}{ccccc} \sigma(0) & \xrightarrow{f} & X & \longrightarrow & \tau(0) \\ \downarrow & & \downarrow & & \downarrow \\ \sigma(1) & \xrightarrow{i(1)} & \tau'(1) & \xrightarrow{p(1)} & \tau(1) \end{array}$$

where the square on the right is in $\mathcal{B}$. Applying assumption (1) of Theorem 1.1 to f , we get $f \cong qh$. Replacing f by h , we may assume f is a morphism of $\mathcal{A}$. Then it suffices to take $\tau'(0) = X$ with the restrictions $p \mid \{0, 1\}$ and $i \mid \{0, 1\}$ given by the diagram.

Now let F and F' be two objects of $\text{Comp}_{\mathcal{A}, \mathcal{B}}^{\mathcal{B}}(\sigma)$. Applying assumption (2) of Theorem 1.1, we obtain morphisms $F'' \rightarrow F$ and $F'' \rightarrow F'$ of $\text{Comp}_{\mathcal{C}, \mathcal{B}}^{\mathcal{B}}(\sigma)$, where $F''(k, l)$ is a pseudo fiber product $F(k, l) \times_{\sigma(k)} F'(k, l)$ in $\mathcal{C}$. To show that $\text{Comp}_{\mathcal{A}, \mathcal{B}}^{\mathcal{B}}(\sigma)$

is directed, it then suffices to show that, for every object F'' of $\text{Comp}_{\mathcal{C}, \mathcal{B}}^{\mathcal{B}}(\sigma)$, there exists a morphism $F''' \rightarrow F''$ of $\text{Comp}_{\mathcal{C}, \mathcal{B}}^{\mathcal{B}}(\sigma)$ such that F''' is an object of $\text{Comp}_{\mathcal{A}, \mathcal{B}}^{\mathcal{B}}(\sigma)$. We will construct a sequence

$$F''' = F_n \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 = F''$$

of morphisms of $\text{Comp}_{\mathcal{C}, \mathcal{B}}^{\mathcal{B}}(\sigma)$ such that $F_j \mid \nabla_j$ is an object of $\text{Comp}_{\mathcal{A}, \mathcal{B}}^{\mathcal{B}}(\sigma \mid [j])$, $0 \leq j \leq n$. For $j \geq 1$, assume F_{j-1} constructed. Applying $(*)$, we get a decomposition $F_{j-1} \mid \{j-1\} \times [j-1] \rightarrow \tau \rightarrow F_{j-1} \mid \{j\} \times [j-1]$. We take $F_j \mid \nabla_{j-1} = F_{j-1} \mid \nabla_{j-1}$, $F_j \mid \{j\} \times [j-1] = \tau$, and $F_j \mid (\nabla_n - \nabla_j) = F_{j-1} \mid (\nabla_n - \nabla_j)$. This finishes the proof of the fact that $\text{Comp}_{\mathcal{A}, \mathcal{B}}^{\mathcal{B}}(\sigma)^{\text{coop}}$ is directed. In particular, we have verified condition (2) of Theorem 5.9.

The verification of condition (1) of Theorem 5.9 is reminiscent of the proof of [1, Lemme 1.3.8]. Let $f: X \rightarrow Y$ be a morphism of $\mathcal{C}$. Endow $S = \text{Comp}(f)$ with the following preorder: $\kappa \leq \lambda$ if and only if there exists a morphism $\lambda \rightarrow \kappa$ in $\text{Comp}^{\mathcal{B}}(f)$. Since S is directed, it is simply connected. Thus it suffices to show that for every pair of morphisms $(r_1, r_2): \lambda = (W, j, q, \tau) \rightrightarrows \kappa = (Z, i, p, \sigma)$ of $\text{Comp}^{\mathcal{B}}(f)$, we have $G_f(r_1) = G_f(r_2)$, so that G_f factorizes through S . Applying assumption (2) of Theorem 1.1, we obtain a diagram in $\mathcal{B}$

$$\begin{array}{ccccc} W \times_Y W & \longrightarrow & Z \times_Y W & \longrightarrow & W \\ \downarrow & & \downarrow & & \downarrow r_2 \\ W \times_Y Z & \longrightarrow & Z \times_Y Z & \xrightarrow{p_2} & Z \\ \downarrow & & \downarrow p_1 & & \downarrow p \\ W & \xrightarrow{r_1} & Z & \xrightarrow{p} & Y \end{array}$$

where all the squares are Cartesian in $\mathcal{C}$. Let $q_\alpha: W \times_Y W \rightarrow W$, $\alpha = 1, 2$ be the projections. Applying $(*)$, we obtain a decomposition

$$\begin{array}{ccccc} X & \longrightarrow & W' & \xrightarrow{t} & W \times_Y W \\ \parallel & & \downarrow u & & \downarrow r_1 \times_Y r_2 \\ X & \longrightarrow & Z' & \xrightarrow{s} & Z \times_Y Z, \end{array}$$

where s, t and u are in $\mathcal{B}$ and the horizontal arrows on the left are in $\mathcal{A}$. This induces commutative squares with 2-cells

$$\begin{array}{ccc} W' & \xrightarrow{q_\alpha t} & W \\ u \downarrow & & \downarrow r_\alpha \\ Z' & \xrightarrow{p_\alpha s} & Z \end{array}$$

in $\text{Comp}^{\mathcal{B}}(f)$. Since $\mathcal{E}(X, Y)$ is a groupoid, it suffices to show $G_f(p_1 s) = G_f(p_2 s)$ (and $G_f(q_1 t) = G_f(q_2 t)$). Applying $(*)$ again, we obtain a decomposition

$$\begin{array}{ccccc}
 X & \longrightarrow & Z''' & \xrightarrow{v} & Z' \\
 \downarrow i & & \downarrow v' & & \downarrow s \\
 Z & \xrightarrow{h} & Z'' & \xrightarrow{s'} & Z \times_Y Z,
 \end{array} \quad (5.8)$$

where s', v, v' are in $\mathcal{B}$, the left horizontal arrows are in $\mathcal{A}$, and the composition hs' of the second line is isomorphic to the diagonal. The diagram (5.8) induces commutative squares with 2-cells

$$\begin{array}{ccc}
 Z''' & \xrightarrow{v} & Z' \\
 \downarrow v' & & \downarrow p_\alpha s \\
 Z'' & \xrightarrow{p_\alpha s'} & Z
 \end{array}$$

in $\text{Comp}^{\mathcal{B}}(f)$. Thus it suffices to show $G_f(p_1 s') = G_f(p_2 s')$. Consider the 2-functor (5.7). Note that h induces a morphism in $\text{Comp}^{\mathcal{A}}(f)$. Since $p_\alpha s' * h = \text{id}_{(Z, i, p, \sigma)}$ in $\mathbf{T}_2\text{Comp}(f)$, its image under (5.7) is the identity, so that $G_f(p_1 s') = G_f(p_2 s')$. One can also check more directly that $G_f(p_\alpha s')^{-1} = G'_f(h)$, where G'_f is defined in (5.6). Indeed, both $G_f(p_\alpha s')^{-1}$ and $G'_f(h)$ are equal to the morphism $p * i \Rightarrow q * j$ induced by the diagram

$$\begin{array}{ccccc}
 X & \xrightarrow{i} & Z & \xrightarrow{h} & Z'' \\
 \parallel & & \parallel & & \downarrow p_\alpha s' \\
 X & \xrightarrow{i} & Z & \xlongequal{\quad} & Z \\
 & & \downarrow p & & \downarrow p \\
 & & Y & \xlongequal{\quad} & Y.
 \end{array}$$

□

This finishes the proof of Theorem 1.1, which relates pseudo functors to gluing data for two sub-2-categories. In the next section we relate gluing data for two sub-2-categories to gluing data for more sub-2-categories.

6 Gluing finitely many pseudo functors

In this section, we study the gluing of finitely many pseudo functors in general. The main result of this section is Theorem 6.7, which is the precise form of Theorem 1.2. Although the constructions of Sections 3 and 4 can be avoided in the statement of the theorem (see Remarks 6.1, 6.2, and 6.3), they allow us to give a more conceptual interpretation and are used in the proof. We deduce Theorem 6.7 from a criterion (Theorem 6.10) involving 2-functors of type (5.1) studied in the previous section.

Throughout this section, $\mathcal{C}$ is a $(2, 1)$ -category, $\mathcal{A}_1, \dots, \mathcal{A}_n, n \geq 1$, are locally full sub-2-categories of $\mathcal{C}$, each containing all objects of $\mathcal{C}$. Let $\mathcal{D}$ be a 2-category.

Remark 6.1 (Explicit description of gluing data) Let us explicitly describe the $\mathcal{D}^{\text{Ob}(\mathcal{C})}$ -2-category $\mathcal{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$. An *object* is a pair

$$((F_i)_{1 \leq i \leq n}, (G_{ij})_{1 \leq i < j \leq n}),$$

where $F_i: \mathcal{A}_i \rightarrow \mathcal{D}$ is an object of $\mathcal{PSFun}(\mathcal{A}_i, \mathcal{D})$, and (F_i, F_j, G_{ij}) is an object of $\mathcal{GD}_{\mathcal{A}_i, \mathcal{A}_j}(\mathcal{C}, \mathcal{D})$, as described in Remark 5.1, satisfying the following condition:

(D) For $1 \leq i < j < k \leq n$ and any commutative cube with 2-cells of the form

$$\begin{array}{ccccc} X' & \xrightarrow{b'} & Y' & & \\ \downarrow x & \searrow q' & \downarrow y & \searrow p' & \\ & Z' & \xrightarrow{a'} & W' & \\ \downarrow z & & \downarrow & & \downarrow w \\ X & \xrightarrow{b} & Y & & \\ \downarrow q & \searrow & \downarrow p & \searrow & \\ & Z & \xrightarrow{a} & W & \end{array} \quad (6.1)$$

where a, b, a', b' are morphisms of $\mathcal{A}_i$, p, q, p', q' are morphisms of $\mathcal{A}_j$, and x, y, z, w are morphisms of $\mathcal{A}_k$, the following hexagon commutes

$$\begin{array}{ccc} F_i(a)F_j(q)F_k(x) & \xrightarrow{G_{jkI'}} & F_i(a)F_k(z)F_j(q') & \xrightarrow{G_{ikJ}} & F_k(w)F_i(a')F_j(q') \\ \downarrow G_{ijK} & & \downarrow G_{ijK'} & & \downarrow G_{ijK'} \\ F_j(p)F_i(b)F_k(x) & \xrightarrow{G_{ikJ'}} & F_j(p)F_k(y)F_i(b') & \xrightarrow{G_{jkI}} & F_k(w)F_j(p')F_i(b') \end{array}$$

where I, I', J, J', K, K' are respectively the right, left, front, back, bottom, top faces of the cube.

A *morphism* $((F_i), (G_{ij})) \rightarrow ((F'_i), (G'_{ij}))$ of $\mathcal{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$ is a collection $(\alpha_i)_{1 \leq i \leq n}$ of morphisms $\alpha_i: F_i \rightarrow F'_i$ of $\mathcal{PSFun}(\mathcal{A}_i, \mathcal{D})$, such that for $1 \leq i < j \leq n$,

$$(\alpha_i, \alpha_j): (F_i, F_j, G_{ij}) \rightarrow (F'_i, F'_j, G'_{ij})$$

is a morphism of $\mathcal{GD}_{\mathcal{A}_i, \mathcal{A}_j}(\mathcal{C}, \mathcal{D})$.

A *2-cell* of $\mathcal{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$ is a collection $(\Xi_i)_{1 \leq i \leq n}: (\alpha_i)_{1 \leq i \leq n} \Rightarrow (\alpha'_i)_{1 \leq i \leq n}$ of 2-cells $\Xi_i: \alpha_i \Rightarrow \alpha'_i$ of $\mathcal{PSFun}(\mathcal{A}_i, \mathcal{D})$ such that $|\Xi_i| = \dots = |\Xi_n|$.

The $\mathcal{D}^{\text{Ob}(\mathcal{C})}$ -2-category structure of $\mathcal{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$ is given by the 2-functor defined by

$$\begin{aligned} ((F_i), G) &\mapsto |F_1| = \dots = |F_n|, & (\alpha_i) &\mapsto |\alpha_1| = \dots = |\alpha_n|, \\ (\Xi_i) &\mapsto |\Xi_1| = \dots = |\Xi_n|. \end{aligned}$$

Remark 6.2 (Alternative description of gluing data) Let us give an alternative description of the objects and morphisms of $\mathcal{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$. An *object* of it is a pair $((F_i)_{1 \leq i \leq n}, (G_{ijD})_{1 \leq i, j \leq n})$ (here we do *not* assume $i < j$), where $F_i: \mathcal{A}_i \rightarrow \mathcal{D}$ is an object of $\mathcal{PSFun}(\mathcal{A}_i, \mathcal{D})$ such that $|F_1| = \dots = |F_n|$, and

$$G_{ijD}: F_i(a)F_j(q) \Rightarrow F_j(p)F_i(b)$$

is an invertible 2-cell of $\mathcal{D}$, where D runs over $(\mathcal{A}_i, \mathcal{A}_j)$ -squares in $\mathcal{C}$ (Example 3.19) of the form

$$\begin{array}{ccc} X & \xrightarrow{b} & Y \\ q \downarrow & \nearrow \alpha & \downarrow p \\ Z & \xrightarrow{a} & W, \end{array} \quad (6.2)$$

satisfying condition (D) of Remark 6.1 for all $1 \leq i, j, k \leq n$ (again here we do *not* assume $i < j < k$) and the following conditions:

(A) For $1 \leq i, j \leq n$ and any square D of the form

$$\begin{array}{ccc} X & \xrightarrow{b} & Y \\ \parallel & \nearrow \alpha & \parallel \\ X & \xrightarrow{a} & Y \end{array}$$

where a and b are morphisms of $\mathcal{A}_i$, the following square commutes

$$\begin{array}{ccc} F_i(a) & \xRightarrow{F_i(\alpha)} & F_i(b) \\ \Downarrow & & \Downarrow \\ F_i(a)F_j(\text{id}_X) & \xRightarrow{G_{ijD}} & F_j(\text{id}_Y)F_i(b). \end{array}$$

(O) For any square D (6.2) with $i = j$, the following square commutes

$$\begin{array}{ccc} F_i(a)F_i(q) & \xRightarrow{G_{iiD}} & F_i(p)F_i(b) \\ \Downarrow & & \Downarrow \\ F_i(aq) & \xRightarrow{F_i(\alpha)} & F_i(pb). \end{array}$$

In fact, given $(G_{ij})_{1 \leq i < j \leq n}$, it suffices to take (O) as a definition of G_{ii} , and to put $G_{ji} = G_{ij}^*$ (Remark 5.3) for $1 \leq i < j \leq n$. Condition (D) for the triple (i, i, j) follows from axiom (b) of Remark 5.1 for G_{ij} .

A *morphism* $((F_i), (G_{ij})) \rightarrow ((F'_i), (G'_{ij}))$ of $\mathcal{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$ is a collection $(\alpha_i)_{1 \leq i \leq n}$ of morphisms $\alpha_i: F_i \rightarrow F'_i$ of $\mathcal{PSFun}(\mathcal{A}_i, \mathcal{D})$ such that for every $(\mathcal{A}_i, \mathcal{A}_j)$ -square D (6.2), the following hexagon commutes

$$\begin{array}{ccc}
 \alpha_0(W)F_i(a)F_j(q) & \xrightarrow{\alpha_i(a)} & F'_i(a)\alpha_0(Z)F_j(q) \xrightarrow{\alpha_j(q)} F'_i(a)F'_j(q)\alpha_0(X) \\
 \downarrow G_{ijD} & & \downarrow G'_{ijD} \\
 \alpha_0(W)F_j(p)F_i(b) & \xrightarrow{\alpha_j(p)} & F'_j(p)\alpha_0(Y)F_i(b) \xrightarrow{\alpha_i(b)} F'_j(p)F'_i(b)\alpha_0(X).
 \end{array}$$

Here $\alpha_0 = |\alpha_1| = \dots = |\alpha_n|$.

Remark 6.3 (Explicit description of the 2-functor Q^ϕ) Let $\phi: \{1, \dots, n\} \rightarrow \{1, \dots, m\}$ be a map and let $\mathcal{A}_1, \dots, \mathcal{A}_n, \mathcal{B}_1, \dots, \mathcal{B}_m$ be locally full sub-2-categories of $\mathcal{C}$, each containing all objects of $\mathcal{C}$, such that $\mathcal{A}_i \subseteq \mathcal{B}_{\phi(i)}$ for $1 \leq i \leq n$. In Construction 4.7 we defined a $\mathcal{D}^{\text{Ob}(\mathcal{C})}$ -2-functor

$$Q^\phi = Q^\phi_{\mathcal{D}}: \mathcal{GD}_{\mathcal{B}_1, \dots, \mathcal{B}_m}(\mathcal{C}, \mathcal{D}) \rightarrow \mathcal{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$$

via a 2-functor E_ϕ . The definition of E_ϕ in the cases $m = 1$ and $m = 1$ needs special treatment. The description in Remark 6.2 allows us to explicitly and uniformly describe Q^ϕ as

$$((F_i), (G_{ij})) \mapsto ((F_{\phi(i)}), (G_{\phi(i)\phi(j)})), \quad (\alpha_i) \mapsto (\alpha_{\phi(i)}), \quad (\Xi_i) \mapsto (\Xi_{\phi(i)}).$$

It follows immediately from this description that for a sequence of maps $\{1, \dots, n\} \xrightarrow{\phi} \{1, \dots, m\} \xrightarrow{\psi} \{1, \dots, l\}$, we have $Q^{\psi\phi} = Q^\phi Q^\psi$.

We have the following generalization of Corollary 3.29.

Proposition 6.4 Let $\phi: \{1, \dots, n\} \rightarrow \{1, \dots, m\}$ be a map with a section ψ . Assume $\mathcal{A}_i \subseteq \mathcal{A}_{\psi\phi(i)}$ for all $1 \leq i \leq n$. Then

$$Q^\phi: \mathcal{GD}_{\mathcal{A}_{\psi(1)}, \dots, \mathcal{A}_{\psi(m)}}(\mathcal{C}, \mathcal{D}) \rightarrow \mathcal{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$$

and Q^ψ are $\mathcal{D}^{\text{Ob}(\mathcal{C})}$ -2-quasi-inverses to each other.

Proof This follows from Proposition 4.4. The above description of Q^ϕ and Q^ψ allows us to give a more direct proof as follows. We have $Q^\psi Q^\phi = \text{id}$. We construct a $\mathcal{D}^{\text{Ob}(\mathcal{C})}$ -2-natural isomorphism $\epsilon: Q^\phi Q^\psi \Rightarrow \text{id}$ as follows. For any object $((F_i), (G_{ij}))$ of $\mathcal{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$, we associate the morphism $Q^\phi Q^\psi((F_i), (G_{ij})) \rightarrow ((F_i), (G_{ij}))$ given by $\rho_{i, \psi\phi(i)}: F_{\psi\phi(i)}|_{\mathcal{A}_i} \rightarrow F_i$. See Remark 8.1 for the definition of $\rho_{i, \psi\phi(i)}$. $\square$

Similarly to Corollary 3.29, Proposition 6.4 does not produce essentially new gluing data (see however Remark 6.8 for an application of the proposition). The goal of this section is to prove a more substantial criterion for Q^ϕ to be a 2-equivalence, where $\phi: \{1, \dots, n\} \rightarrow \{1, \dots, m\}$ is a surjection with $m = n - 1$. In the case $n = 2$, such a criterion was given in Theorem 1.1. Here we assume $n \geq 3$. Without loss of generality, we may assume that ϕ is nondecreasing and $\phi(1) = \phi(2)$, so that $\phi(i) = \max\{1, i - 1\}$

for $1 \leq i \leq n$. We assume $\mathcal{B}_i = \mathcal{A}_{i+1}$ for $2 \leq i \leq m$. We simplify notation as follows. We put $\mathcal{B} = \mathcal{B}_1$ and consider

$$\begin{aligned} Q_{\mathcal{D}} &= Q_{\mathcal{D}}^{\phi}: \mathcal{GD}_{\mathcal{B}, \mathcal{A}_3, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D}) \rightarrow \mathcal{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D}), \\ E &= \mathcal{LE}_{\phi}^{\text{red}}: \mathcal{LT}_n^{\text{red}} \mathbb{Q}_{\mathcal{A}_1, \dots, \mathcal{A}_n} \mathcal{C} \rightarrow \mathcal{LT}_{n-1}^{\text{red}} \mathbb{Q}_{\mathcal{B}, \mathcal{A}_3, \dots, \mathcal{A}_n} \mathcal{C}. \end{aligned} \quad (6.3)$$

Remark 6.5 If $\mathcal{A}_1$ and $\mathcal{A}_2$ generate $\mathcal{B}$ (Definition 5.4), then E is locally essentially surjective, so that $Q_{\mathcal{D}}$ is 2-faithful by Proposition 4.8 (1).

To state the main result of this section, we need to introduce some terminology.

Definition 6.6 Let $\mathcal{C}$ be a (2, 1)-category and let $\mathcal{A}, \mathcal{B}, \mathcal{A}_i, \mathcal{A}_j, \mathcal{A}_k$ be locally full sub-2-categories of $\mathcal{C}$, each containing all objects of $\mathcal{C}$.

- (1) We say that $(\mathcal{A}, \mathcal{B})$ is *squarable* in $\mathcal{C}$ if every pair of morphisms $i: Z \rightarrow W$ in $\mathcal{A}$ and $p: Y \rightarrow W$ in $\mathcal{B}$ with the same target can be completed into an $(\mathcal{A}, \mathcal{B})$ -square (3.8), Cartesian in $\mathcal{C}$.
- (2) We say that $(\mathcal{A}, \mathcal{B})$ is *squaring* in $\mathcal{C}$ if every $(\mathcal{A}, \mathcal{B})$ -square (3.8) can be decomposed as

$$\begin{array}{ccccc} X & & & & \\ & \searrow f & & \nearrow j & \\ & & X' & \xrightarrow{k} & Y \\ & \nearrow \gamma & \downarrow r & \nearrow \beta & \downarrow p \\ q & & Z & \xrightarrow{i} & W \end{array} \quad (6.4)$$

where k is a morphism of $\mathcal{A}$, and r is a morphism of $\mathcal{B}$, and f is a morphism of $\mathcal{A} \cap \mathcal{B}$, and the inner square is Cartesian in $\mathcal{C}$.

- (3) We say that $(\mathcal{A}_i, \mathcal{A}_j)$ is $\mathcal{A}_k$ -*squaring* in $\mathcal{C}$ if every cube with 2-cells (6.1) can be decomposed as

$$\begin{array}{ccccccc} X' & & & & & & \\ & \searrow f' & & \nearrow b' & & & \\ & & V' & \xrightarrow{c'} & Y' & & \\ & \nearrow \gamma' & \downarrow v & \nearrow r' & \downarrow y & \nearrow p' & \\ q' & & Z' & \xrightarrow{a'} & W' & & \\ & \searrow f & & \nearrow b & & & \\ & & V & \xrightarrow{c} & Y & & \\ & \nearrow \gamma & \downarrow r & \nearrow \beta & \downarrow p & \nearrow w & \\ q & & Z & \xrightarrow{a} & W & & \end{array} \quad (6.5)$$

where c, c' are morphisms of $\mathcal{A}_i$, and r, r' are morphisms of $\mathcal{A}_j$, and f, f' are morphisms of $\mathcal{A}_i \cap \mathcal{A}_j$, and v is a morphism of $\mathcal{A}_k$, and the bottom face L and the top face L' of the inner cube are Cartesian in $\mathcal{C}$. Note that if the right face I (resp. the front face J) of the inner cube is Cartesian in $\mathcal{C}$, so is the left face I'' (resp. the back face J'') by [18, Corollary 3.11].

In this terminology, condition (2) of Theorem 1.1 says that $(\mathcal{B}, \mathcal{B})$ is squarable. If pseudo fiber products exist in $\mathcal{B}$ and are pseudo fiber products in $\mathcal{C}$ (cf. condition (2a) of Remark 5.6), then $(\mathcal{B}, \mathcal{B})$ is squarable and squaring in $\mathcal{C}$. One sufficient condition for $(\mathcal{A}, \mathcal{B})$ to be squaring in $\mathcal{C}$ is that pseudo fiber products exist in $\mathcal{C}$, and $\mathcal{A}$ and $\mathcal{B}$ are stable under base change in $\mathcal{C}$ and taking diagonals in $\mathcal{C}$.

One sufficient condition for $(\mathcal{A}_i, \mathcal{A}_j)$ to be $\mathcal{A}_k$ -squaring in $\mathcal{C}$ is that $\mathcal{C}$ admits pseudo fiber products, and $\mathcal{A}_i, \mathcal{A}_j, \mathcal{A}_k$ are stable under base change in $\mathcal{C}$ and taking diagonals in $\mathcal{C}$.

The following is the main result of this section.

Theorem 6.7 *Let $\mathcal{C}$ be a $(2, 1)$ -category. Let $n \geq 3$ and let $\mathcal{A}_1, \dots, \mathcal{A}_n, \mathcal{B}$ be locally full sub-2-categories of $\mathcal{C}$, each containing all objects of $\mathcal{C}$. Assume the following:*

- (1) *For every morphism f of $\mathcal{B}$, there exist a morphism a of $\mathcal{A}_1$, a morphism p of $\mathcal{A}_2$, and a 2-cell $pa \Rightarrow f$ of $\mathcal{C}$.*
- (2) *For $3 \leq i \leq n$ and every morphism f of $\mathcal{B} \cap \mathcal{A}_i$, there exist a morphism a of $\mathcal{A}_1 \cap \mathcal{A}_i$, a morphism p of $\mathcal{A}_2 \cap \mathcal{A}_i$, and a 2-cell $pa \Rightarrow f$ of $\mathcal{C}$.*
- (3) *For $3 \leq i < j \leq n$, the sub-2-categories $\mathcal{A}_1 \cap \mathcal{A}_i \cap \mathcal{A}_j$ and $\mathcal{A}_2 \cap \mathcal{A}_i \cap \mathcal{A}_j$ generate $\mathcal{B} \cap \mathcal{A}_i \cap \mathcal{A}_j$ (Definition 5.4).*
- (4) *$(\mathcal{A}_2, \mathcal{A}_2)$ is squarable in $\mathcal{B}$ and every Cartesian $(\mathcal{A}_2, \mathcal{A}_2)$ -square in $\mathcal{B}$ is Cartesian in $\mathcal{C}$. For $3 \leq i \leq n$, $(\mathcal{A}_2 \cap \mathcal{A}_i, \mathcal{A}_2 \cap \mathcal{A}_i)$ is squarable in $\mathcal{B} \cap \mathcal{A}_i$.*
- (5) *For $3 \leq i \leq n$, the sub-2-categories $\mathcal{A}_1 \cap \mathcal{A}_i$ and $\mathcal{A}_2 \cap \mathcal{A}_i$ are stable under base change in $\mathcal{C}$ by morphisms of $\mathcal{A}_1$ whenever such base change exists.*
- (6) *For $3 \leq i \leq n$, the sub-2-category $\mathcal{A}_1$ is stable under base change in $\mathcal{C}$ by morphisms of $\mathcal{A}_i$ whenever such base change exists, $(\mathcal{A}_2, \mathcal{A}_i)$ is squarable in $\mathcal{C}$, and $(\mathcal{B}, \mathcal{A}_i)$ is squaring in $\mathcal{C}$.*
- (7) *For $3 \leq i, j \leq n$ with $i \neq j$, the sub-2-category $\mathcal{A}_1 \cap \mathcal{A}_j$ is stable under base change in $\mathcal{C}$ by morphisms of $\mathcal{A}_i$ whenever such base change exists, $(\mathcal{A}_2 \cap \mathcal{A}_j, \mathcal{A}_i)$ is squarable in $\mathcal{C}$, and $(\mathcal{B} \cap \mathcal{A}_j, \mathcal{A}_i)$ is squaring in $\mathcal{C}$.*
- (8) *For $3 \leq i, j \leq n$ with $i \neq j$, the pair $(\mathcal{B}, \mathcal{A}_i)$ is $\mathcal{A}_j$ -squaring.*

Then

$$Q_{\mathcal{D}}: \mathcal{GD}_{\mathcal{B}, \mathcal{A}_3, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D}) \rightarrow \mathcal{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$$

is a $\mathcal{D}^{\text{Ob}(\mathcal{C})}$ -2-equivalence for every 2-category $\mathcal{D}$.

Note that assumptions (4) through and (8) of Theorem 6.7 are all satisfied if $\mathcal{C}$ admits pseudo fiber products, $\mathcal{A}_1, \dots, \mathcal{A}_n, \mathcal{B}$ are stable under 2-base change in $\mathcal{C}$, and $\mathcal{A}_3, \dots, \mathcal{A}_n, \mathcal{B}$ are stable under taking diagonals in $\mathcal{C}$. For $n = 3$, assumptions (3), (7), and (8) trivially hold.

Remark 6.8 Since every surjection $\phi: \{1, \dots, n\} \rightarrow \{1, \dots, m\}$ (for arbitrary m, n) is a composition of surjections satisfying $m = n - 1$, Theorem 1.1 for $n = 2$ and Theorem 6.7 for $n \geq 3$ produce results for Q^ϕ for every surjection ϕ , and in particular for the descent 2-functor Q^n for $n \geq 2$. In the case $n = 3$, this is illustrated by Corollary 1.3. We note that there are other ways to combine results of this paper can be combined. In fact, we deduce in [24, Proposition 1.5] from Theorem 1.1, Theorem 6.7 (case $n = 3$), and Proposition 6.4, a case where Q^2 is a 2-equivalence but for which condition (1) of Theorem 1.1 is not satisfied.

We will deduce Theorem 6.7 from a general criterion. The idea is to consider not only the gluing of the two sub-2-categories $\mathcal{A}_1$ and $\mathcal{A}_2$ of $\mathcal{B}$, but also similar gluing problems of sub-2-categories with $\mathcal{B}$ replaced by the 2-category $\mathcal{A}r(\mathcal{A}_i; \mathcal{B})$ whose objects are arrows in $\mathcal{A}_i$ and whose morphisms are $(\mathcal{A}_i, \mathcal{B})$ -squares, for all $3 \leq i \leq n$. To state this formally, we introduce the following.

Notation 6.9 Let $\mathcal{S}, \mathcal{S}', \mathcal{R}$ be locally full sub-2-categories of $\mathcal{C}$. We denote by $\mathcal{A}r(\mathcal{S}; \mathcal{R}) \subseteq \mathcal{U}PsFun([1], \mathcal{C})$ the locally full sub-2-category spanned by strictly unital pseudo functors that factor through $\mathcal{S}$ and pseudo natural transformations α such that α_0 and α_1 are both morphisms of $\mathcal{R}$. We denote by $\mathcal{T}r(\mathcal{S}; \mathcal{R}) \subseteq \mathcal{U}PsFun([2], \mathcal{C})$ the locally full sub-2-category spanned by strictly unital pseudo functors that factor through $\mathcal{S}$ and pseudo natural transformations α such that $\alpha_0, \alpha_1, \alpha_2$ are morphisms of $\mathcal{R}$. We denote by $\mathcal{S}_q(\mathcal{S}, \mathcal{S}'; \mathcal{R}) \subseteq \mathcal{U}PsFun([1] \times [1], \mathcal{C})$ the locally full sub-2-category spanned by strictly unital pseudo functors F such that $F(0 \rightarrow 1, 0)$ and $F(0 \rightarrow 1, 1)$ are morphisms of $\mathcal{S}$ and $F(0, 0 \rightarrow 1)$ and $F(1, 0 \rightarrow 1)$ are morphisms of $\mathcal{S}'$ and by pseudo natural transformations α such that $\alpha(i, j)$, $0 \leq i, j \leq 1$ are all morphisms of $\mathcal{T}$.

Note that if $(\mathcal{A}_i, \mathcal{A}_j)$ is $\mathcal{A}_k$ -squaring in $\mathcal{C}$, then $(\mathcal{A}_i, \mathcal{A}_j)$ is squaring in $\mathcal{C}$ and $(\mathcal{A}r(\mathcal{A}_k; \mathcal{A}_i), \mathcal{A}r(\mathcal{A}_k; \mathcal{A}_j))$ is squaring in $\mathcal{A}r(\mathcal{A}_k; \mathcal{C})$.

For $3 \leq i \leq n$, we consider the 2-functor

$$G_i: \mathcal{G}_i := \mathcal{L}\mathcal{T}_2^{\text{red}} \mathbb{Q}_{\mathcal{A}r(\mathcal{A}_i; \mathcal{A}_1), \mathcal{A}r(\mathcal{A}_i; \mathcal{A}_2)} \mathcal{A}r(\mathcal{A}_i, \mathcal{B}) \rightarrow \mathcal{A}r(\mathcal{A}_i; \mathcal{B})$$

induced by (5.1). For a morphism $f: x \rightarrow y$ of $\mathcal{A}r(\mathcal{A}_i, \mathcal{B})$, we let $G_i^{-1}(f)$ denote the groupoid of pairs (g, π) , where $g: x \rightarrow y$ is a morphism of $\mathcal{G}_i$ and $\pi: G_i(g) \Rightarrow f$ is a 2-cell of $\mathcal{A}r(\mathcal{A}_i; \mathcal{B})$.

Theorem 6.10 Let $\mathcal{C}$ be a $(2, 1)$ -category. Let $n \geq 3$ and let $\mathcal{A}_1, \dots, \mathcal{A}_n, \mathcal{B}$ be locally full sub-2-categories of $\mathcal{C}$ such that $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{B}$. Assume the following:

- (1) The 2-functor $G: \mathcal{G} := \mathcal{L}\mathcal{T}_2^{\text{red}} \mathbb{Q}_{\mathcal{A}_1, \mathcal{A}_2} \mathcal{B} \rightarrow \mathcal{B}$ (5.1) is a bi-equivalence.
- (2) For every morphism $f: x \rightarrow y$ of $\mathcal{A}r(\mathcal{A}_i; \mathcal{B})$, $3 \leq i \leq n$, the groupoid $G_i^{-1}(f)$ is connected.
- (3) For $3 \leq i \leq n$, the sub-2-categories $\mathcal{T}r(\mathcal{A}_i; \mathcal{A}_1)$ and $\mathcal{T}r(\mathcal{A}_i; \mathcal{A}_2)$ generate $\mathcal{T}r(\mathcal{A}_i; \mathcal{B})$.

- (4) For $3 \leq i < j \leq n$, $Sq(\mathcal{A}_i, \mathcal{A}_j; \mathcal{A}_1)$ and $Sq(\mathcal{A}_i, \mathcal{A}_j; \mathcal{A}_2)$ generate $Sq(\mathcal{A}_i, \mathcal{A}_j; \mathcal{B})$.

Then the 2-functor $E: \mathcal{LT}_n^{\text{red}} \mathbb{Q}_{\mathcal{A}_1, \dots, \mathcal{A}_n} \mathcal{C} \rightarrow \mathcal{LT}_{n-1}^{\text{red}} \mathbb{Q}_{\mathcal{B}, \mathcal{A}_3, \dots, \mathcal{A}_n} \mathcal{C}$ (6.3) is a bi-equivalence.

For $3 \leq i \leq n$, assumption (3) implies that $\mathcal{A}r(\mathcal{A}_i; \mathcal{A}_1)$ and $\mathcal{A}r(\mathcal{A}_i; \mathcal{A}_2)$ generate $\mathcal{A}r(\mathcal{A}_i; \mathcal{B})$, so that the groupoid $G_i^{-1}(f)$ in (2) is nonempty. Note also that for $n = 3$, assumption (4) trivially holds.

Proof For $3 \leq i \leq n$, let $\mathcal{B}_i = \mathcal{A}r(\mathcal{A}_i; \mathcal{B})$. We have source and target 2-functors $\tau_0, \tau_1: \mathcal{B}_i \rightarrow \mathcal{B}$, sending an object $f: X \rightarrow Y$ of $\mathcal{B}_i$ to X and Y , respectively. These 2-functors induce 2-functors $\mathcal{G}_i \rightarrow \mathcal{G}$, which we still denote by τ_0 and τ_1 . We have $\tau_0 G_i = G \tau_0$ and $\tau_1 G_i = G \tau_1$.

By Proposition 2.12, as G is a bi-equivalence (assumption (1) of the theorem), there exist a pseudo functor $H: \mathcal{B} \rightarrow \mathcal{G}$ and pseudo natural isomorphisms $\eta: \text{id}_{\mathcal{G}} \Rightarrow HG$ and $\epsilon: GH \Rightarrow \text{id}_{\mathcal{B}}$ such that $H(X) = X$, $\eta(X) = \text{id}_X$, and $\epsilon(X) = \text{id}_X$ for every object X of $\mathcal{B}$. We may assume that H is strictly unital.

Consider the commutative square

$$\begin{array}{ccc} \tilde{\mathcal{E}} & \xrightarrow{\tilde{E}} & \tilde{\mathcal{D}} \\ R_{\mathcal{E}} \downarrow & & \downarrow R_{\mathcal{D}} \\ \mathcal{E} & \xrightarrow{E} & \mathcal{D}, \end{array}$$

where $\tilde{E} = \mathcal{L}E_{\phi}: \mathcal{LT}_n \mathbb{Q}_{\mathcal{A}_1, \dots, \mathcal{A}_n} \mathcal{C} \rightarrow \mathcal{LT}_{n-1} \mathbb{Q}_{\mathcal{B}, \mathcal{A}_3, \dots, \mathcal{A}_n} \mathcal{C}$ and the vertical arrows $R_{\mathcal{D}}$ and $R_{\mathcal{E}}$ are the reduction 2-functors.

We construct a 2-functor $F: \tilde{\mathcal{D}} \rightarrow \mathcal{E}$ as follows. For an object X of $\tilde{\mathcal{D}}$, we take $FX = X$. For a morphism of $\tilde{\mathcal{D}}$ of length 1, we take $F(f^k) = f^{k+1}$ for $2 \leq k \leq n-1$ and f in $\mathcal{A}_{k+1}$ and take $F(f^1) = H(f)$ for f in $\mathcal{B}$. We thus obtain a functor from the underlying category of $\tilde{\mathcal{D}}$ to the underlying category of $\mathcal{C}$.

Next we define the effects of F on the pre-2-cells of $\mathcal{T}_{n-1} \mathbb{Q}_{\mathcal{B}, \mathcal{A}_3, \dots, \mathcal{A}_n} \mathcal{C}$ (Definition 3.13). We take $F(\iota_{g,f}^k) = \text{id}$ and $F(\theta_{g,f}^k) = \text{id}$ for all $1 \leq k \leq n-1$. We take $F(\gamma_{g,h^k,h^k,f}) = \text{id}$ for $2 \leq k \leq n-1$ and h', h in $\mathcal{A}_{k+1}$ and take $F(\gamma_{g,h^1,h^1,f}): F(g) * H(h') * H(h) * F(f) \Rightarrow F(g) * H(h'h) * F(f)$ to be the 2-cell induced by the coherence constraint $H(h') * H(h) \Rightarrow H(h'h)$ of H for morphisms h', h of $\mathcal{B}$. In both cases, we take $F(\delta_{g,h^k,h^k,f}) = F(\gamma_{g,h^k,h^k,f})^{-1}$. Now let $D \in (\mathbb{Q}_{\mathcal{B}, \mathcal{A}_3, \dots, \mathcal{A}_n} \mathcal{C})(\epsilon_k + \epsilon_{k'})$, $k < k'$. For $k \geq 2$, we take $F(\sigma_{g,D,f}) = \sigma_{F(g), D, F(f)}$. For $k = 1$, we view D as a morphism $x \rightarrow y$ of $\mathcal{B}_{k'+1} = \mathcal{A}r(\mathcal{A}_{k'+1}; \mathcal{B})$. For an object $\rho = (D', \pi)$ of $G_{k'+1}^{-1}(D)$, where $D': x \rightarrow y$ is a morphism of $\mathcal{G}_{k'+1}$ and $\pi: G_{k'+1}(D') \Rightarrow D$ is a 2-cell of $\mathcal{B}_{k'+1}$, we consider the 2-cell

$$\begin{aligned} \alpha_{\rho}: H \tau_1 D * x &\xrightarrow{\pi^{-1}} HG \tau_1 D' * x \xrightarrow{\eta^{-1}} \tau_1 D' * x \xrightarrow{\sigma_{\text{id}, D', \text{id}}} y * \tau_0 D' \\ &\xrightarrow{\eta} y * HG \tau_0 D' \xrightarrow{\pi} y * H \tau_0 D. \end{aligned}$$

It is straightforward to check that for a morphism $\rho \rightarrow \rho'$ of $G_{k'+1}^{-1}(D)$, we have $\alpha_\rho = \alpha_{\rho'}$. Thus by the connectedness of $G_{k'+1}^{-1}(D)$ (assumption (2) of the theorem), α_ρ does not depend on the choice of ρ , and we take $F(\sigma_{g,D,f})$ to be the 2-cell induced by α_ρ .

To show that F factors through a 2-functor $\tilde{\mathcal{D}} \rightarrow \mathcal{E}$, it suffices to check that the equivalence system defined by $\alpha \sim \beta$ if and only if $F(\alpha) = F(\beta)$ satisfies conditions (1) through (9) of Definition 3.13. Conditions (1) through (5) of Definition 3.13 follow immediately from the construction. For the other conditions, the cases that only concerns the directions $\mathcal{A}_3, \dots, \mathcal{A}_n$ are also trivial. Let us check the nontrivial cases. The case $k = 1$ of condition (6) (associativity of composition) of Definition 3.13 follows from the composition axiom of the coherence constraint of H . For the case $k' = 1$ of condition (7) (unit square in directions k and k' , with units in direction k') of Definition 3.13, it suffices to take ρ to be the object of $G_{k'+1}^{-1}(D)$ given by the identity square (3.6). For the case $k = 1$ of condition (7) of Definition 3.13, it suffices to take ρ to be the object of $G_{k'+1}^{-1}(D)$ given by the identity on an object of the category $G^{-1}(f)$ of pairs (g, π) , where $g: X \rightarrow Y$ is a morphism of $\mathcal{G}$ and $\pi: g \Rightarrow f$ is a 2-cell of $\mathcal{B}$. For the case $k' = 1$ of condition (8) (composition in direction k of squares in directions k and k') of Definition 3.13, it suffices to take, for objects ρ of $G_{k+1}^{-1}(D)$ and ρ' of $G_{k+1}^{-1}(D')$, their composite $\rho' \circ \rho$ in $G_{k+1}^{-1}(D'')$. For the case $k = 1$ of condition (8) of Definition 3.13, the diagram defines a morphism of $Tr(\mathcal{A}_{k'+1}; \mathcal{B})$, and it suffices to take objects of $G_{k'+1}^{-1}(D)$, $G_{k'+1}^{-1}(D')$, and $G_{k'+1}^{-1}(D'')$ given by assumption (3) of the theorem. For the case $k = 1$ of condition (9) (cube in directions k, k' , and k'') of Definition 3.13, the diagram defines a morphism in $Sq(\mathcal{A}_{k'+1}, \mathcal{A}_{k''+1}; \mathcal{B})$, and it suffices to take objects of $G_{k'+1}^{-1}(K)$, $G_{k'+1}^{-1}(K')$, $G_{k'+1}^{-1}(J)$, $G_{k'+1}^{-1}(J')$ given by assumption (4) of the theorem. This finishes the construction of the 2-functor $F: \tilde{\mathcal{D}} \rightarrow \mathcal{E}$.

We define a pseudo natural isomorphism $\tilde{\eta}: R_{\mathcal{E}} \rightarrow F\tilde{E}$ sending X to id_X as follows. To every morphism f of $\mathcal{A}_i$ with $3 \leq i \leq n$, we associate id_f . To every morphism f of $\mathcal{A}_1$ (resp. $\mathcal{A}_2$), we associate $\eta(f)$.

We define a pseudo natural isomorphism $\tilde{\epsilon}: EF \Rightarrow R_{\mathcal{D}}$ sending X to id_X as follows. To every morphism f of $\mathcal{A}_i$ with $3 \leq i \leq n$, we associate id_f . To every morphism f of $\mathcal{B}$ such that $H(f) = f_k * \dots * f_1$, we associate the composition

$$f_k * \dots * f_1 \Rightarrow f_k \circ \dots \circ f_1 \xrightarrow{\epsilon(f)} f,$$

where the first 2-cell is given by γ . □

Proof of Theorem 6.7 By Proposition 4.8 (2), it suffices to show that the assumptions of Theorem 6.7 imply the conditions of Theorem 6.10. Let us first note that by assumption (5) of Theorem 6.7, each of $\mathcal{A}_1, \dots, \mathcal{A}_n$ contains all equivalences in $\mathcal{C}$.

To check condition (1) of Theorem 6.10, it suffices to apply Theorem 1.1 to the sub-2-categories $\mathcal{A}_1$ and $\mathcal{A}_2$ of $\mathcal{B}$. Here we used assumption (1) and the assumption that $(\mathcal{A}_2, \mathcal{A}_2)$ is squarable in $\mathcal{B}$ (assumption (4)) of Theorem 6.7.

Next we check condition (2) of Theorem 6.10. To simplify the notation, we put $\mathcal{B}_i = \mathcal{A}r(\mathcal{A}_i; \mathcal{B})$, $\mathcal{A}_{ki} = \mathcal{A}r(\mathcal{A}_i; \mathcal{A}_k)$, $k = 1, 2$. Let f be a morphism of $\mathcal{B}_i$. Since $(\mathcal{B}, \mathcal{A}_i)$ is squaring in $\mathcal{C}$ (assumption (6)), f can be decomposed as follows

$$\begin{array}{ccccc} X & \longrightarrow & X' & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ Z & \xlongequal{\quad} & Z & \longrightarrow & W, \end{array}$$

where the upper left horizontal arrow is in $\mathcal{B} \cap \mathcal{A}_i$ and the square on the right is a Cartesian $(\mathcal{B}, \mathcal{A}_i)$ -square in $\mathcal{C}$. We fix some notation in order to better analyze this decomposition. We let $\tau_0, \tau_1: \mathcal{B}_i \rightarrow \mathcal{B}$ denote the source and target 2-functors. We let $\mathcal{B}'_i \subseteq \mathcal{A}r(\mathcal{A}_i; \mathcal{B} \cap \mathcal{A}_i)$ denote the locally full sub-2-category spanned by morphisms f' such that $\tau_1 f'$ is an identity. We let $\mathcal{B}''_i \subseteq \mathcal{B}_i$ denote the locally full sub-2-category spanned by morphisms corresponding to Cartesian $(\mathcal{B}, \mathcal{A}_i)$ -squares in $\mathcal{C}$. The decomposition above provides morphisms f' in $\mathcal{B}'_i$ and f'' in $\mathcal{B}''_i$, and a 2-cell $f'' f' \Rightarrow f$ of $\mathcal{B}_i$. We put $\mathcal{A}'_{ki} = \mathcal{B}'_i \cap \mathcal{A}_{ki}$, $\mathcal{A}''_{ki} = \mathcal{B}''_i \cap \mathcal{A}_{ki}$. By Theorem 1.1, the 2-functors

$$G'_i: \mathcal{L}T_2^{\text{red}} \mathbb{Q}_{\mathcal{A}'_{1i}, \mathcal{A}'_{2i}} \mathcal{B}'_i \rightarrow \mathcal{B}'_i, \quad G''_i: \mathcal{L}T_2^{\text{red}} \mathbb{Q}_{\mathcal{A}''_{1i}, \mathcal{A}''_{2i}} \mathcal{B}''_i \rightarrow \mathcal{B}''_i$$

are bi-equivalences. Here for G'_i , we used assumption (2) and the second sentence of assumption (4) (which implies that $(\mathcal{A}'_{2i}, \mathcal{A}'_{2i})$ is squarable in $\mathcal{B}'_i$) of Theorem 6.7. For G''_i , we used the assumption that $(\mathcal{A}_2, \mathcal{A}_i)$ is squarable in $\mathcal{C}$ and that $\mathcal{A}_1$ is stable under base change in $\mathcal{C}$ by $\mathcal{A}_i$ whenever such base change exists (assumption (6) of Theorem 6.7) for condition (1) of Theorem 1.1 and the first sentence of assumption (4) of Theorem 6.7 (as well as the fact that $\mathcal{A}_2$ contains all equivalences in $\mathcal{C}$) for the condition that $(\mathcal{A}''_{2i}, \mathcal{A}''_{2i})$ is squarable in $\mathcal{B}''_i$ [condition (2) of Theorem 1.1]. The 2-cell $f'' f' \Rightarrow f$ induces a functor

$$F: G'^{-1}_i(f') \times G''^{-1}_i(f'') \rightarrow G^{-1}_i(f),$$

where $G'^{-1}_i(f')$ and $G''^{-1}_i(f'')$ are groupoids defined similarly to $G^{-1}_i(f)$ and are contractible. Thus it suffices to show that every object $\rho = (f_m * \cdots * f_1, \pi)$ of $G^{-1}_i(f)$ is in the essential image of F . Here each f_s , $1 \leq s \leq m$ is either a morphism of $\mathcal{A}_{1i}$ or a morphism of $\mathcal{A}_{2i}$. Note that, for every morphism g of $\mathcal{A}_{ki}$, there exist g' in $\mathcal{A}'_{ki}$, g'' in $\mathcal{A}''_{ki}$, and a 2-cell $g'' g' \Rightarrow g$ in $\mathcal{A}_{ki}$. Thus, up to replacing m by $2m$, we may assume that each f_s is in $\mathcal{A}'_{1i}$, or $\mathcal{A}''_{1i}$, or $\mathcal{A}'_{2i}$, or $\mathcal{A}''_{2i}$. Next we show that for $l = 1, 2$, morphisms of $\mathcal{A}'_{li}$ can be moved to the left. Let $g''': x \rightarrow y$ be a morphism of $\mathcal{A}''_{li}$, and let $g': y \rightarrow z$ be a morphism of $\mathcal{A}'_{ki}$. Applying the assumption that $(\mathcal{B}, \mathcal{A}_i)$ is squaring, we obtain a 2-cell $g' g'' \Rightarrow h' h''$, where h' is in $\mathcal{B}'_i$, and h'' is in $\mathcal{A}''_{li}$ by the assumption (6) of Theorem 6.7 (for $l = 2$ we also need the fact that $\mathcal{B} \cap \mathcal{A}_i$ contains all equivalences in $\mathcal{C}$). If $l = 1$, then h' is in $\mathcal{A}'_{ki}$ by assumption (5) of Theorem 6.7, since $\tau_0 h'$ is a base change of $\tau_0 g'$. For $l = 2$, we may decompose h' into $h'_2 h'_1$ with h'_1 in $\mathcal{A}'_{1i}$ and h'_2 in $\mathcal{A}'_{2i}$, and we get a 2-cell $g' g'' \Rightarrow (h'' h'_2) h'_1$ in $\mathcal{G}_i$. Thus we may

assume that $\rho = (f''_{m''} * \cdots * f'_1 * f'_m * \cdots * f'_1, \pi)$, where each f'_s is in $\mathcal{A}'_{1i}$ or $\mathcal{A}'_{2i}$ and each f''_s is in $\mathcal{A}''_{1i}$ or $\mathcal{A}''_{2i}$. We may further assume that the source of f'_1 is the source of f'' . In this case ρ is in the image of F .

Next we check condition (3) of Theorem 6.10. Let f be a morphism of $Tr(\mathcal{A}_i; \mathcal{B})$. There is a 2-cell $f \Rightarrow g''g'g$ with $g''g'g$ of the form

$$\begin{array}{ccccccc} X_0 & \xrightarrow{a_0} & Y_0 & \longrightarrow & Z_0 & \longrightarrow & W_0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y_1 & \xlongequal{\quad} & Y_1 & \xrightarrow{a'_1} & Z_1 & \longrightarrow & W_1 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Z_2 & \xlongequal{\quad} & Z_2 & \xlongequal{\quad} & Z_2 & \xrightarrow{a''_2} & W_2, \end{array}$$

$\begin{matrix} D' & D'' \\ E'' \end{matrix}$

where E'' , D'' , and D' are Cartesian $(\mathcal{B}, \mathcal{A}_i)$ -squares in $\mathcal{C}$, and the arrows a_0 and a'_1 are in $\mathcal{B} \cap \mathcal{A}_i$. The three Cartesian squares are constructed successively using the assumption that $(\mathcal{B}, \mathcal{A}_i)$ is squaring. We decompose a''_2 by assumption (1) of Theorem 6.7 and decompose a_0 and a'_1 by assumption (2) of Theorem 6.7. We get a 2-cell $g \Rightarrow g_2g_1$, where g_k is in $Tr(\mathcal{A}_i; \mathcal{A}_k \cap \mathcal{A}_i)$ for $k = 1, 2$. Applying the first two parts of assumption (6) of Theorem 6.7, we get 2-cells $g' \Rightarrow g'_2g'_1$ and $g'' \Rightarrow g''_2g''_1$, where g'_k and g''_k are in $Tr(\mathcal{A}_i; \mathcal{A}_k)$ for $k = 1, 2$.

Finally we check condition (4) of Theorem 6.10. Let f be a morphism of $Sq(\mathcal{A}_i, \mathcal{A}_j; \mathcal{B})$. We let $\tau_{\rightarrow,0}, \tau_{\rightarrow,1}: Sq(\mathcal{A}_i, \mathcal{A}_j; \mathcal{B}) \rightarrow \mathcal{B}_i$ and $\tau_{0,\rightarrow}, \tau_{1,\rightarrow}: Sq(\mathcal{A}_i, \mathcal{A}_j; \mathcal{B}) \rightarrow \mathcal{B}_j$ denote the restriction 2-functors. Applying the assumption that $(\mathcal{B}, \mathcal{A}_j)$ is $\mathcal{A}_i$ -squaring in $\mathcal{C}$ (assumption (8) of Theorem 6.7), we obtain a 2-cell $f \Rightarrow f'g$, where f' is a morphism of $Sq(\mathcal{A}_i, \mathcal{A}_j; \mathcal{B})$ such that $\tau_{0,\rightarrow}f'$ and $\tau_{1,\rightarrow}f'$ are given by Cartesian $(\mathcal{B}, \mathcal{A}_j)$ -squares in $\mathcal{C}$, and g is a morphism in $Sq(\mathcal{A}_i, \mathcal{A}_j; \mathcal{B} \cap \mathcal{A}_j)$ such that $\tau_{\rightarrow,1}g$ is an identity. Applying the assumption that $(\mathcal{B}, \mathcal{A}_i)$ is $\mathcal{A}_j$ -squaring in $\mathcal{C}$, we obtain a 2-cell $f' \Rightarrow f''g'$, where f'' is a morphism of $Sq(\mathcal{A}_i, \mathcal{A}_j; \mathcal{B})$ whose image under all four restriction 2-functors are given by Cartesian squares, and g' is a morphism in $Sq(\mathcal{A}_i, \mathcal{A}_j; \mathcal{B} \cap \mathcal{A}_i)$ such that $\tau_{1,\rightarrow}g'$ is an identity and $\tau_{0,\rightarrow}g'$ is given by a Cartesian $(\mathcal{A}_j, \mathcal{B} \cap \mathcal{A}_i)$ -square in $\mathcal{C}$. Applying assumption (1) and the first two parts of assumption (6) of Theorem 6.7, we get a 2-cell $f'' \Rightarrow f''_2f''_1$, where f''_k is in $Sq(\mathcal{A}_i, \mathcal{A}_j; \mathcal{A}_k)$ for $k = 1, 2$. Applying assumption (2) and the first two parts of assumption (7) of Theorem 6.7, we get a 2-cell $g' \Rightarrow g'_2g'_1$, where g'_k is in $Sq(\mathcal{A}_i, \mathcal{A}_j; \mathcal{A}_k \cap \mathcal{A}_i)$ for $k = 1, 2$. Applying assumptions (2) and (7) of Theorem 6.7, we get a 2-cell $g \Rightarrow g_2g_1h$, where g_k is in $Sq(\mathcal{A}_i, \mathcal{A}_j; \mathcal{A}_k \cap \mathcal{A}_j)$ for $k = 1, 2$ and h is a morphism in $Sq(\mathcal{A}_i, \mathcal{A}_j; \mathcal{B} \cap \mathcal{A}_i \cap \mathcal{A}_j)$ such that $\tau_{1,\rightarrow}h$ and $\tau_{\rightarrow,1}h$ are identities. Finally applying assumption (3) of Theorem 6.7, we decompose h into morphisms of $Sq(\mathcal{A}_i, \mathcal{A}_j; \mathcal{A}_k \cap \mathcal{A}_i \cap \mathcal{A}_j)$ for $k = 1, 2$. $\square$

This finishes the first part of the article, which relates pseudo functors to gluing data. In the next part of the article (Sects. 7 through 9), we develop several tools for constructing gluing data. In Sects. 7 and 8, we introduce Cartesian gluing data, which are usually easier to construct in applications.

7 Cartesian gluing data for two pseudo functors

Let $\mathcal{C}$ be a $(2, 1)$ -category and let $\mathcal{A}$ and $\mathcal{B}$ be two locally full sub-2-categories of $\mathcal{C}$, each containing all objects of $\mathcal{C}$. Let $\mathcal{D}$ be a 2-category. We studied the 2-category $\mathcal{GD}_{\mathcal{A}, \mathcal{B}}(\mathcal{C}, \mathcal{D})$ of gluing data in Sect. 5. One way to construct such data is by taking adjoints in base change isomorphisms (see Sect. 9). In many applications, these isomorphisms only exist for Cartesian squares. In this section, we introduce a variant $\mathcal{GD}_{\mathcal{A}, \mathcal{B}}^{\text{Cart}}(\mathcal{C}, \mathcal{D})$ of $\mathcal{GD}_{\mathcal{A}, \mathcal{B}}(\mathcal{C}, \mathcal{D})$, whose objects only make use of G_D for Cartesian squares D . The main result of this section is a criterion for $\mathcal{GD}^{\text{Cart}}$ and $\mathcal{GD}$ to be isomorphic (Theorem 7.5). This is used in the construction of $Rf_!$ for Deligne-Mumford stacks in [24] to produce the desired gluing data.

The idea of using Cartesian squares as an intermediary step to construct gluing data was already used by Deligne [6, 5.1.5] and Ayoub [1, Section 1.6.5]. In [6, Section 5.1] it is possible to avoid this intermediary step by taking $\mathcal{A}$ to be spanned by *dominant* open immersions and $\mathcal{B}$ by proper morphisms, so that every $(\mathcal{A}, \mathcal{B})$ -square is Cartesian. However, the intermediary step is necessary in other applications.

Construction 7.1 (Pseudo natural isomorphism ρ from a gluing datum) Let $(F_{\mathcal{A}}, F_{\mathcal{B}}, G)$ be an object of $\mathcal{GD}_{\mathcal{A}, \mathcal{B}}(\mathcal{C}, \mathcal{D})$, and let $F_0 = |F_{\mathcal{A}}| = |F_{\mathcal{B}}|$. Between the restrictions to $\mathcal{A} \cap \mathcal{B}$, we define an isomorphism $\rho: F_{\mathcal{B}}|_{\mathcal{A} \cap \mathcal{B}} \rightarrow F_{\mathcal{A}}|_{\mathcal{A} \cap \mathcal{B}}$ in $\mathcal{Psfun}(\mathcal{A} \cap \mathcal{B}, \mathcal{D})$ with $|\rho| = \text{id}_{F_0}$ as follows. For any morphism $f: X \rightarrow Y$ of $\mathcal{A} \cap \mathcal{B}$, let $\rho(f)$ be the composition

$$F_{\mathcal{B}}(f) \Longrightarrow F_{\mathcal{A}}(\text{id}_Y)F_{\mathcal{B}}(f) \xRightarrow{G_D} F_{\mathcal{B}}(\text{id}_Y)F_{\mathcal{A}}(f) \Longrightarrow F_{\mathcal{A}}(f),$$

where D is the left square in the diagram

$$\begin{array}{ccccc} X & \xlongequal{\quad} & X & \xrightarrow{f} & Y \\ \parallel & \swarrow \parallel & \downarrow f & \searrow \parallel & \parallel \\ X & \xrightarrow{f} & Y & \xlongequal{\quad} & Y. \end{array}$$

Denote the right square by D' . Applying axiom (a) of Remark 5.1 to the outer square and axiom (b') of Remark 5.1 to the above diagram, one sees that $\rho(f)$ is the inverse of the composition

$$F_{\mathcal{A}}(f) \Longrightarrow F_{\mathcal{A}}(f)F_{\mathcal{B}}(\text{id}_X) \xRightarrow{G_{D'}} F_{\mathcal{B}}(f)F_{\mathcal{A}}(\text{id}_X) \Longrightarrow F_{\mathcal{B}}(f).$$

For any object X of $\mathcal{C}$, applying axiom (a) of Remark 5.1 to the constant square $[1] \times [1] \rightarrow \mathcal{C}$ of value X , one finds that the following diagram commutes

$$\begin{array}{ccc} \text{id}_{F_0 X} & & \\ \parallel & \searrow & \\ F_{\mathcal{B}}(\text{id}_X) & \xRightarrow{\rho(\text{id}_X)} & F_{\mathcal{A}}(\text{id}_X). \end{array}$$

For any sequence of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, applying axioms (a), (b), (b') of Remark 5.1 to

$$\begin{array}{ccccc}
 X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\
 f \downarrow & // & \parallel & // & \parallel \\
 Y & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\
 g \downarrow & & // & & \parallel \\
 Z & \xrightarrow{f} & Z & \xrightarrow{g} & Z
 \end{array}$$

one finds that the following diagram commutes

$$\begin{array}{ccc}
 F_{\mathcal{B}}(g)F_{\mathcal{B}}(f) & \xrightarrow{\rho(g)\rho(f)} & F_{\mathcal{A}}(g)F_{\mathcal{A}}(f) \\
 \parallel \downarrow & & \parallel \downarrow \\
 F_{\mathcal{B}}(gf) & \xrightarrow{\rho(gf)} & F_{\mathcal{A}}(gf).
 \end{array}$$

Therefore, ρ is a pseudo natural isomorphism.

Remark 7.2 The pseudo natural isomorphism ρ of Construction 7.1 has the following properties:

- (c) If D is an $(\mathcal{A}, \mathcal{B})$ -square (3.8) such that p, q are morphisms of $\mathcal{A} \cap \mathcal{B}$, then the following hexagon commutes

$$\begin{array}{ccccc}
 F_{\mathcal{A}}(i)F_{\mathcal{B}}(q) & \xrightarrow{\rho(q)} & F_{\mathcal{A}}(i)F_{\mathcal{A}}(q) & \Longrightarrow & F_{\mathcal{A}}(iq) \\
 G_D \parallel \downarrow & & & & \parallel \downarrow F_{\mathcal{A}}(\alpha) \\
 F_{\mathcal{B}}(p)F_{\mathcal{A}}(j) & \xrightarrow{\rho(p)} & F_{\mathcal{A}}(p)F_{\mathcal{A}}(j) & \Longrightarrow & F_{\mathcal{A}}(pj).
 \end{array}$$

- (c') If D is an $(\mathcal{A}, \mathcal{B})$ -square (3.8) such that i, j are morphisms of $\mathcal{A} \cap \mathcal{B}$, then the following hexagon commutes

$$\begin{array}{ccccc}
 F_{\mathcal{A}}(i)F_{\mathcal{B}}(q) & \xrightarrow{\rho(i)^{-1}} & F_{\mathcal{B}}(i)F_{\mathcal{B}}(q) & \Longrightarrow & F_{\mathcal{B}}(iq) \\
 G_D \parallel \downarrow & & & & \parallel \downarrow F_{\mathcal{B}}(\alpha) \\
 F_{\mathcal{B}}(p)F_{\mathcal{A}}(j) & \xrightarrow{\rho(j)^{-1}} & F_{\mathcal{B}}(p)F_{\mathcal{B}}(j) & \Longrightarrow & F_{\mathcal{B}}(pj).
 \end{array}$$

In fact, any square D in (c) can be decomposed as

$$\begin{array}{ccccc}
 X & \xrightarrow{j} & Y & \xlongequal{\quad} & Y \\
 \parallel & \nearrow j & \parallel & \nearrow p & \downarrow p \\
 X & \xrightarrow{j} & Y & \xrightarrow{p} & W \\
 \parallel & \nearrow \alpha & \parallel & \nearrow i & \parallel \\
 X & \xrightarrow{q} & Z & \xrightarrow{i} & W \\
 \downarrow q & \nearrow & \parallel & \nearrow i & \parallel \\
 Z & \xlongequal{\quad} & Z & \xrightarrow{i} & W.
 \end{array}$$

Denote the upper left, upper right, middle, lower left, lower right squares by D_1 , D_2 , D_3 , D_4 and D_5 , respectively. Then G_{D_2} can be identified with $\rho(p)^{-1}$ and G_{D_4} can be identified with $\rho(q)$. By axiom (a) of Remark 5.1, G_{D_1} and G_{D_5} can be identified with identities and G_{D_3} can be identified with $F_A(\alpha)$. Hence axioms (b) and (b') of Remark 5.1 imply that the hexagon in (c) commutes. Similarly, axioms (a'), (b) and (b') of Remark 5.1 imply (c').

Definition 7.3 (*Cartesian gluing data*) Define the 2-category $\mathcal{GD}_{\mathcal{A}, \mathcal{B}}^{\text{Cart}}(\mathcal{C}, \mathcal{D})$ of Cartesian gluing data as follows. An *object* of this category is a quadruple $(F_A, F_B, (G_D), \rho)$ consisting of an object F_A of $\mathcal{PsFun}(\mathcal{A}, \mathcal{D})$, an object F_B of $\mathcal{PsFun}(\mathcal{B}, \mathcal{D})$, a family of invertible 2-cells of $\mathcal{D}$

$$G_D: F_A(i)F_B(q) \Rightarrow F_B(p)F_A(j),$$

D running over Cartesian $(\mathcal{A}, \mathcal{B})$ -squares in $\mathcal{C}$ of the form (3.8), and an isomorphism $\rho: F_B|_{\mathcal{A} \cap \mathcal{B}} \rightarrow F_A|_{\mathcal{A} \cap \mathcal{B}}$ in $\mathcal{PsFun}(\mathcal{A} \cap \mathcal{B}, \mathcal{D})$, such that $|\rho| = \text{id}_{F_0}$, where $F_0 = |F_A| = |F_B|$, and satisfying conditions (b), (b') of Remark 5.1 and conditions (c), (c') of Remark 7.2 for Cartesian $(\mathcal{A}, \mathcal{B})$ -squares.

A *morphism* $(F_A, F_B, G, \rho) \rightarrow (F'_A, F'_B, G', \rho')$ of $\mathcal{GD}_{\mathcal{A}, \mathcal{B}}^{\text{Cart}}(\mathcal{C}, \mathcal{D})$ is a pair (α_A, α_B) consisting of a morphism $\alpha_A: F_A \rightarrow F'_A$ of $\mathcal{PsFun}(\mathcal{A}, \mathcal{D})$ and a morphism $\alpha_B: F_B \rightarrow F'_B$ of $\mathcal{PsFun}(\mathcal{B}, \mathcal{D})$, such that $|\alpha_A| = |\alpha_B|$, satisfying condition (m) of Remark 5.1 for Cartesian $(\mathcal{A}, \mathcal{B})$ -squares, and the following condition

(n) The following square commutes

$$\begin{array}{ccc}
 F_B|_{\mathcal{A} \cap \mathcal{B}} & \xrightarrow{\rho} & F_A|_{\mathcal{A} \cap \mathcal{B}} \\
 \alpha_B|_{\mathcal{A} \cap \mathcal{B}} \downarrow & & \downarrow \alpha_A|_{\mathcal{A} \cap \mathcal{B}} \\
 F'_B|_{\mathcal{A} \cap \mathcal{B}} & \xrightarrow{\rho'} & F'_A|_{\mathcal{A} \cap \mathcal{B}}.
 \end{array}$$

A *2-cell* of $\mathcal{GD}_{\mathcal{A}, \mathcal{B}}^{\text{Cart}}(\mathcal{C}, \mathcal{D})$ is a pair $(\Xi_A, \Xi_B): (\alpha_A, \alpha_B) \Rightarrow (\alpha'_A, \alpha'_B)$ consisting of a 2-cell $\Xi_A: \alpha_A \Rightarrow \alpha'_A$ of $\mathcal{PsFun}(\mathcal{A}, \mathcal{D})$ and a 2-cell $\Xi_B: \alpha_B \Rightarrow \alpha'_B$ of $\mathcal{PsFun}(\mathcal{B}, \mathcal{D})$ such that $|\Xi_A| = |\Xi_B|$.

We view $\mathcal{GD}_{\mathcal{A},\mathcal{B}}^{\text{Cart}}(\mathcal{C}, \mathcal{D})$ as a $\mathcal{D}^{\text{Ob}(\mathcal{C})}$ -2-category via the 2-functor given by

$$\begin{aligned} (F_{\mathcal{A}}, F_{\mathcal{B}}, G, \rho) &\mapsto |F_{\mathcal{A}}| = |F_{\mathcal{B}}|, \quad (\alpha_{\mathcal{A}}, \alpha_{\mathcal{B}}) \mapsto |\alpha_{\mathcal{A}}| = |\alpha_{\mathcal{B}}|, \\ (\Xi_{\mathcal{A}}, \Xi_{\mathcal{B}}) &\mapsto |\Xi_{\mathcal{A}}| = |\Xi_{\mathcal{B}}|. \end{aligned}$$

Construction 7.1 defines a $\mathcal{D}^{\text{Ob}(\mathcal{C})}$ -2-functor

$$\mathcal{GD}_{\mathcal{A},\mathcal{B}}(\mathcal{C}, \mathcal{D}) \rightarrow \mathcal{GD}_{\mathcal{A},\mathcal{B}}^{\text{Cart}}(\mathcal{C}, \mathcal{D}), \quad (7.1)$$

which is clearly 2-faithful.

Remark 7.4 If $(\mathcal{A}, \mathcal{B})$ is squaring in $\mathcal{C}$ (Definition 6.6), then (7.1) is 2-fully faithful. In fact, for objects $(F_{\mathcal{A}}, F_{\mathcal{B}}, G)$ and $(F'_{\mathcal{A}}, F'_{\mathcal{B}}, G')$ of $\mathcal{GD}_{\mathcal{A},\mathcal{B}}(\mathcal{C}, \mathcal{D})$ and any morphism

$$(\alpha_{\mathcal{A}}, \alpha_{\mathcal{B}}): (F_{\mathcal{A}}, F_{\mathcal{B}}, G, \rho) \rightarrow (F'_{\mathcal{A}}, F'_{\mathcal{B}}, G', \rho')$$

of $\mathcal{GD}_{\mathcal{A},\mathcal{B}}^{\text{Cart}}(\mathcal{C}, \mathcal{D})$ whose source and target are respectively the images of $(F_{\mathcal{A}}, F_{\mathcal{B}}, G)$ and $(F'_{\mathcal{A}}, F'_{\mathcal{B}}, G')$ under (7.1), the pair $(\alpha_{\mathcal{A}}, \alpha_{\mathcal{B}}): (F_{\mathcal{A}}, F_{\mathcal{B}}, G) \rightarrow (F'_{\mathcal{A}}, F'_{\mathcal{B}}, G')$ is a morphism of $\mathcal{GD}_{\mathcal{A},\mathcal{B}}(\mathcal{C}, \mathcal{D})$. Indeed, for any $(\mathcal{A}, \mathcal{B})$ -square D (3.8), decomposing it as (6.4), we see that the following diagram commutes

$$\begin{array}{ccccc} & & G_D & & \\ \alpha_0(W)F_{\mathcal{A}}(i)F_{\mathcal{B}}(q) & \xrightarrow{F_{\mathcal{B}}(\gamma)} & \alpha_0(W)F_{\mathcal{A}}(i)F_{\mathcal{B}}(r)F_{\mathcal{B}}(f) & \xrightarrow{G_{D'}\rho(f)} & \alpha_0(W)F_{\mathcal{B}}(p)F_{\mathcal{A}}(k)F_{\mathcal{A}}(f) & \xrightarrow{F_{\mathcal{A}}(\delta)} & \alpha_0(W)F_{\mathcal{B}}(p)F_{\mathcal{A}}(j) \\ \downarrow \alpha_{\mathcal{A}}(i) & & \downarrow \alpha_{\mathcal{A}}(i) & & \downarrow \alpha_{\mathcal{B}}(p) & & \downarrow \alpha_{\mathcal{B}}(p) \\ F'_{\mathcal{A}}(i)\alpha_0(Z)F_{\mathcal{B}}(q) & \xrightarrow{F_{\mathcal{B}}(\gamma)} & F'_{\mathcal{A}}(i)\alpha_0(Z)F_{\mathcal{B}}(r)F_{\mathcal{B}}(f) & & F'_{\mathcal{B}}(p)\alpha_0(Y)F_{\mathcal{A}}(k)F_{\mathcal{A}}(f) & \xrightarrow{F_{\mathcal{A}}(\delta)} & F'_{\mathcal{B}}(p)\alpha_0(Y)F_{\mathcal{A}}(j) \\ \downarrow \alpha_{\mathcal{B}}(q) & & \downarrow \alpha_{\mathcal{B}}(r) & & \downarrow \alpha_{\mathcal{A}}(k) & & \downarrow \alpha_{\mathcal{A}}(j) \\ & & F'_{\mathcal{A}}(i)F'_{\mathcal{B}}(r)\alpha_0(X')F_{\mathcal{B}}(f) & \xrightarrow{G'_{D'}\rho(f)} & F'_{\mathcal{B}}(p)F'_{\mathcal{A}}(k)\alpha_0(X')F_{\mathcal{A}}(f) & & \\ & & \downarrow \alpha_{\mathcal{B}}(f) & & \downarrow \alpha_{\mathcal{A}}(f) & & \\ F'_{\mathcal{A}}(i)F'_{\mathcal{B}}(q)\alpha_0(X) & \xrightarrow{F'_{\mathcal{B}}(\gamma)} & F'_{\mathcal{A}}(i)F'_{\mathcal{B}}(r)F'_{\mathcal{B}}(f)\alpha_0(X) & \xrightarrow{G'_{D'}\rho(f)} & F'_{\mathcal{B}}(p)F'_{\mathcal{A}}(k)F'_{\mathcal{A}}(f)\alpha_0(X) & \xrightarrow{F'_{\mathcal{A}}(\delta)} & F'_{\mathcal{B}}(p)F'_{\mathcal{A}}(j)\alpha_0(X). \\ & & G'_D & & \end{array}$$

Here $\alpha_0 = |\alpha_{\mathcal{A}}| = |\alpha_{\mathcal{B}}|$, and D' is the inner square of (6.4).

Theorem 7.5 Let $\mathcal{C}$ be a $(2, 1)$ -category and let $\mathcal{A}$ and $\mathcal{B}$ be locally full sub-2-categories of $\mathcal{C}$, each containing all objects of $\mathcal{C}$. Let $\mathcal{D}$ be a 2-category. Assume that every equivalence in $\mathcal{C}$ is contained in $\mathcal{A} \cap \mathcal{B}$, and the pairs $(\mathcal{A}, \mathcal{B})$, $(\mathcal{A}, \mathcal{A} \cap \mathcal{B})$, $(\mathcal{B}, \mathcal{A} \cap \mathcal{B})$ are squaring in $\mathcal{C}$ (Definition 6.6). Then (7.1) is an isomorphism of $\mathcal{D}^{\text{Ob}(\mathcal{C})}$ -2-categories.

Proof We construct the inverse of (7.1) as follows. Let $(F_{\mathcal{A}}, F_{\mathcal{B}}, G, \rho)$ be an object of $\mathcal{GD}_{\mathcal{A},\mathcal{B}}^{\text{Cart}}(\mathcal{C}, \mathcal{D})$. For any square D (3.8), decompose it as (6.4), and denote the inner square by D' . Let $\bar{G}_D$ be the composition

$$F_{\mathcal{A}}(i)F_{\mathcal{B}}(q) \xrightarrow{F_{\mathcal{B}}(\gamma)} F_{\mathcal{A}}(i)F_{\mathcal{B}}(r)F_{\mathcal{B}}(f) \xrightarrow{G_{D'}\rho(f)} F_{\mathcal{B}}(p)F_{\mathcal{A}}(k)F_{\mathcal{A}}(f) \xrightarrow{F_{\mathcal{A}}(\delta)} F_{\mathcal{B}}(p)F_{\mathcal{A}}(j).$$

This does not depend on the choice of the decomposition. In fact, if

is another decomposition with l in $\mathcal{A}$, s in $\mathcal{B}$, g in $\mathcal{A} \cap \mathcal{B}$, and D'' Cartesian in $\mathcal{C}$, then they can be combined into

where h is an equivalence. Applying the axioms of Remarks 5.1 and 7.2 to the following decomposition of D''

we obtain the following commutative diagram

Hence the following diagram commutes

$$\begin{array}{ccccc}
 F_{\mathcal{A}}(i)F_{\mathcal{B}}(q) & & & & \\
 \downarrow F_{\mathcal{B}}(\epsilon) & \searrow F_{\mathcal{B}}(\gamma) & & & \\
 F_{\mathcal{A}}(i)F_{\mathcal{B}}(s)F_{\mathcal{B}}(g) & \xrightarrow{F_{\mathcal{B}}(\phi)} & F_{\mathcal{A}}(i)F_{\mathcal{B}}(r)F_{\mathcal{B}}(h)F_{\mathcal{B}}(g) & \xleftarrow{F_{\mathcal{B}}(\omega)} & F_{\mathcal{A}}(r)F_{\mathcal{B}}(r)F_{\mathcal{B}}(f) \\
 \downarrow \rho(g) & & \downarrow \rho(g) & \searrow \rho(hg) & \downarrow \rho(f) \\
 F_{\mathcal{A}}(i)F_{\mathcal{B}}(s)F_{\mathcal{A}}(g) & \xrightarrow{F_{\mathcal{B}}(\phi)} & F_{\mathcal{A}}(i)F_{\mathcal{B}}(r)F_{\mathcal{B}}(h)F_{\mathcal{A}}(g) & & \\
 \downarrow G_{D''} & & \downarrow \rho(h) & \searrow & \downarrow G_{D'} \\
 F_{\mathcal{B}}(p)F_{\mathcal{A}}(l)F_{\mathcal{A}}(g) & \xleftarrow{F_{\mathcal{A}}(\psi)} & F_{\mathcal{B}}(p)F_{\mathcal{A}}(k)F_{\mathcal{A}}(h)F_{\mathcal{A}}(g) & \xleftarrow{F_{\mathcal{A}}(\omega)} & F_{\mathcal{B}}(p)F_{\mathcal{A}}(k)F_{\mathcal{A}}(f) \\
 \downarrow F_{\mathcal{A}}(\zeta) & & \downarrow G_{D'} & & \downarrow G_{D'} \\
 F_{\mathcal{B}}(p)F_{\mathcal{A}}(j) & \xleftarrow{F_{\mathcal{A}}(\delta)} & & &
 \end{array}$$

Next we show that $(F_{\mathcal{A}}, F_{\mathcal{B}}, \bar{G})$ is an object of $\mathcal{GD}_{\mathcal{A}, \mathcal{B}}(\mathcal{C}, \mathcal{D})$. Axioms (a) and (a') of Remark 5.1 for $\bar{G}$ follow from axioms (c) and (c') of Remark 7.2. Let $D, D',$ and D'' be squares as in axiom (b') of Remark 5.1 for $\bar{G}$. Decompose it as

$$\begin{array}{ccccccc}
 X_1 & & & & & & \\
 \downarrow f & \nearrow j & & & & & \\
 W & \xrightarrow{l} & X_2 & & & & \\
 \downarrow g & \nearrow \eta & \downarrow h & \nearrow \gamma & & & \\
 Z_1 & \xrightarrow{k} & Z_2 & \xrightarrow{k'} & X_3 & & \\
 \downarrow q_1 & \nearrow \beta & \downarrow q_2 & \nearrow E' & \downarrow p_3 & & \\
 Y_1 & \xrightarrow{i} & Y_2 & \xrightarrow{i'} & Y_3 & &
 \end{array}$$

where horizontal arrows are morphisms of $\mathcal{A}$, vertical arrows are morphisms of $\mathcal{B}$, oblique arrows are morphisms of $\mathcal{A} \cap \mathcal{B}$, the squares E, E' and the square H containing η are Cartesian in $\mathcal{C}$. Let $E'' = E' \circ E, I = H \circ E$. Since I is the outer square of the diagram

$$\begin{array}{ccccc}
 W & \xrightarrow{l} & X_2 & = & X_2 \\
 q_1 g \downarrow & & J \downarrow p_2 & \nearrow \beta & \downarrow q_2 h \\
 Y_1 & \xrightarrow{i} & Y_2 & = & Y_2,
 \end{array}$$

axiom (b') of Remark 5.1 and axiom (c') of Remark 7.2 imply the commutativity of the following triangle

It follows that the following diagram commutes

One establishes axiom (b) of Remark 5.1 for $\bar{G}$ in a similar way.

Let $(\alpha_A, \alpha_B): (F_A, F_B, G, \rho) \rightarrow (F'_A, F'_B, G', \rho')$ be a morphism of $\mathcal{GD}_{A,B}^{\text{Cart}}(\mathcal{C}, \mathcal{D})$. Then $(\alpha_A, \alpha_B): (F_A, F_B, \bar{G}) \rightarrow (F'_A, F'_B, \bar{G}')$ is a morphism of $\mathcal{GD}_{A,B}(\mathcal{C}, \mathcal{D})$ by Remark 7.4. Let

$$(\Xi_A, \Xi_B): (\alpha_A, \alpha_B) \Rightarrow (\alpha'_A, \alpha'_B)$$

be a 2-cell of $\mathcal{GD}_{A,B}^{\text{Cart}}(\mathcal{C}, \mathcal{D})$. Then (Ξ_A, Ξ_B) is a 2-cell of $\mathcal{GD}_{A,B}(\mathcal{C}, \mathcal{D})$.

The 2-functor defined in this way is clearly the inverse of (7.1). $\square$

Theorem 7.5 relates gluing data and Cartesian gluing data for two pseudo functors. In the next section we will establish an analogue for finitely many pseudo functors.

8 Cartesian gluing data for finitely many pseudo functors

In this section, we generalize the definitions and results of the previous section to the case of finitely many pseudo functors. The main result is a general criterion for $\mathcal{GD}^{\text{Cart}}$ and $\mathcal{GD}$ to be isomorphic (Theorem 8.3).

Let $\mathcal{C}$ be a $(2, 1)$ -category, let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be locally full sub-2-categories of $\mathcal{C}$, each containing all objects of $\mathcal{C}$. Let $\mathcal{D}$ be a 2-category.

Remark 8.1 (Properties of the pseudo natural isomorphisms ρ_{ij} associated to a gluing datum) Let $((F_i), G)$ be an object of $\mathcal{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$. For $1 \leq i, j \leq n$, the triple (F_i, F_j, G_{ij}) is an object of $\mathcal{GD}_{\mathcal{A}_i, \mathcal{A}_j}(\mathcal{C}, \mathcal{D})$. Let

$$\rho_{ij}: F_j | \mathcal{A}_i \cap \mathcal{A}_j \rightarrow F_i | \mathcal{A}_i \cap \mathcal{A}_j$$

be the isomorphism in $\mathcal{PSFun}(\mathcal{A}_i \cap \mathcal{A}_j, \mathcal{D})$ associated to the triple by (7.1). Then $\rho_{ii} = \text{id}_{F_i}$ and $\rho_{ji} = \rho_{ij}^{-1}$. We claim that ρ_{ij} has the following properties:

(E) For $1 \leq i, j, k \leq n$ and any $(\mathcal{A}_i, \mathcal{A}_j \cap \mathcal{A}_k)$ -square D (6.2), the following square commutes

$$\begin{array}{ccc} F_i(a)F_k(q) & \xrightarrow{\rho_{jk}(q)} & F_i(a)F_j(q) \\ G_{ikD} \parallel \downarrow & & \parallel \downarrow G_{ijD} \\ F_k(p)F_i(b) & \xrightarrow{\rho_{jk}(p)} & F_j(p)F_i(b). \end{array}$$

(F) (cocycle condition) For $1 \leq i, j, k \leq n$, the following triangle commutes

$$\begin{array}{ccc} F_k | \mathcal{A}_{ijk} & \xrightarrow{\rho_{jk} | \mathcal{A}_{ijk}} & F_j | \mathcal{A}_{ijk} \\ & \searrow \rho_{ik} | \mathcal{A}_{ijk} & \downarrow \rho_{ij} | \mathcal{A}_{ijk} \\ & & F_i | \mathcal{A}_{ijk}. \end{array}$$

Here $\mathcal{A}_{ijk} = \mathcal{A}_i \cap \mathcal{A}_j \cap \mathcal{A}_k$.

In fact, (E) follows from axiom (D) of Remark 6.1 applied to the cube

whose top and back faces are D and whose other faces have identity 2-cells. Condition (F) follows from (E) applied to the square

for every morphism f of $\mathcal{A}_{ijk}$.

We extend Definition 7.3 as follows.

Definition 8.2 (*Cartesian gluing data for finitely many pseudo functors*) We define a 2-category $\mathcal{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}^{\text{Cart}}(\mathcal{C}, \mathcal{D})$ as follows. An object of this 2-category is a triple $((F_i)_{1 \leq i \leq n}, (G_{ij})_{1 \leq i < j \leq n}, (\rho_{ij})_{1 \leq i < j \leq n})$, where $F_i: \mathcal{A}_i \rightarrow \mathcal{D}$ is an object of $\mathcal{PsFun}(\mathcal{A}_i, \mathcal{D})$ for $1 \leq i \leq n$, and $(F_i, F_j, G_{ij}, \rho_{ij})$ is an object of $\mathcal{GD}_{\mathcal{A}_i, \mathcal{A}_j}^{\text{Cart}}(\mathcal{C}, \mathcal{D})$ for $1 \leq i < j \leq n$, satisfying condition (D) of Remark 6.1 for $1 \leq i < j < k \leq n$ and cubes with Cartesian faces, condition (E) of Remark 8.1 for $1 \leq i \leq n, 1 \leq j < k \leq n, i \neq j, k$ and Cartesian squares (here we put $G_{ij} = G_{ji}^*$ for $i > j$), and condition (F) of Remark 8.1 for $1 \leq i < j < k \leq n$.

A morphism $((F_i), G, \rho) \rightarrow ((F'_i), G', \rho')$ of $\mathcal{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}^{\text{Cart}}(\mathcal{C}, \mathcal{D})$ is a collection $(\alpha_i)_{1 \leq i \leq n}$ of morphisms $\alpha_i: F_i \rightarrow F'_i$ of $\mathcal{PsFun}(\mathcal{A}_i, \mathcal{D})$, such that for $1 \leq i < j \leq n$, the pair $(\alpha_i, \alpha_j): (F_i, F_j, G_{ij}, \rho_{ij}) \rightarrow (F'_i, F'_j, G'_{ij}, \rho'_{ij})$ is a morphism of $\mathcal{GD}_{\mathcal{A}_i, \mathcal{A}_j}^{\text{Cart}}(\mathcal{C}, \mathcal{D})$.

A 2-cell of $\mathcal{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}^{\text{Cart}}(\mathcal{C}, \mathcal{D})$ is a collection $(\Xi_i)_{1 \leq i \leq n}: (\alpha_i)_{1 \leq i \leq n} \Rightarrow (\alpha'_i)_{1 \leq i \leq n}$ of 2-cells $\Xi_i: \alpha_i \Rightarrow \alpha'_i$ of $\mathcal{PsFun}(\mathcal{A}_i, \mathcal{D})$ such that $|\Xi_1| = \dots = |\Xi_n|$.

We view $\mathcal{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}^{\text{Cart}}(\mathcal{C}, \mathcal{D})$ as a $\mathcal{D}^{\text{Ob}(\mathcal{C})}$ -2-category via the 2-functor given by

$$\begin{aligned} ((F_i), G, \rho) &\mapsto |F_1| = \dots = |F_n|, & (\alpha_i) &\mapsto |\alpha_1| = \dots = |\alpha_n|, \\ (\Xi_i) &\mapsto |\Xi_1| = \dots = |\Xi_n|. \end{aligned}$$

Remark 8.1 defines a $\mathcal{D}^{\text{Ob}(\mathcal{C})}$ -2-functor

$$\mathcal{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D}) \rightarrow \mathcal{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}^{\text{Cart}}(\mathcal{C}, \mathcal{D}), \quad (8.1)$$

which is clearly 2-faithful. If, for all $1 \leq i < j \leq n$, the pair $(\mathcal{A}_i, \mathcal{A}_j)$ is squaring in $\mathcal{C}$ (Definition 6.6), then (8.1) is 2-fully faithful by Remark 7.4.

The following generalizes Theorem 7.5.

Theorem 8.3 *Let $\mathcal{C}$ be a $(2, 1)$ -category and let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be locally full sub-2-categories of $\mathcal{C}$, each containing all objects of $\mathcal{C}$. Let $\mathcal{D}$ be a 2-category. Assume that every morphism of $\mathcal{C}$ that is an equivalence is contained in $\mathcal{A}_1 \cap \dots \cap \mathcal{A}_n$, and for $1 \leq i, j \leq n$, $i \neq j$, the pairs $(\mathcal{A}_i, \mathcal{A}_j)$ and $(\mathcal{A}_i, \mathcal{A}_i \cap \mathcal{A}_j)$ are squaring in $\mathcal{C}$ (Definition 6.6). Assume moreover that for pairwise distinct numbers $1 \leq i, j, k \leq n$, the pair $(\mathcal{A}_i, \mathcal{A}_j)$ is $\mathcal{A}_k$ -squaring and the pair $(\mathcal{A}_i \cap \mathcal{A}_j, \mathcal{A}_k)$ is squaring. Then (8.1) is an isomorphism of $\mathcal{D}^{\text{Ob}(\mathcal{C})}$ -2-categories.*

One sufficient condition for the assumptions of Theorem 8.3 to hold true is that $\mathcal{C}$ admits pseudo fiber products, and $\mathcal{A}_i$ is stable under base change in $\mathcal{C}$ and taking diagonals in $\mathcal{C}$ for all $1 \leq i \leq n$.

Proof We construct the inverse of (8.1) as follows. Let $((F_i), G, \rho)$ be an object of $\mathcal{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}^{\text{Cart}}(\mathcal{C}, \mathcal{D})$. For $1 \leq i, j \leq n$, $i \neq j$, let $(F_i, F_j, \bar{G}_{ij})$ be the image of $(F_i, F_j, G_{ij}, \rho_{ij})$ under the inverse of (7.1). To show that $((F_i), \bar{G})$ is an object of $\mathcal{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$, it suffices to check axiom (D) of Remark 6.1 for $1 \leq i < j < k \leq n$.

First note that for pairwise distinct numbers $1 \leq i, j, k \leq n$, G satisfies axiom (D) of Remark 6.1 for cubes with Cartesian faces, G and ρ satisfy axiom (E) of Remark 8.1 for Cartesian squares, and ρ satisfies axiom (F) of Remark 8.1. Here for $1 \leq i < j \leq n$ we put $G_{ji} = G_{ij}^*$ and $\rho_{ji} = \rho_{ij}^{-1}$. Next we show that $\bar{G}$ satisfies axiom (E) of Remark 8.1 for pairwise distinct numbers $1 \leq i, j, k \leq n$ and all $(\mathcal{A}_i, \mathcal{A}_j \cap \mathcal{A}_k)$ -squares D (6.2). Decompose D as

$$\begin{array}{ccccc} X & & & & \\ & \searrow f & \nearrow \delta & & \\ & & X' & \xrightarrow{c} & Y \\ & \nearrow \gamma & \downarrow r & \nearrow \beta & \downarrow p \\ & & Z & \xrightarrow{a} & W \end{array}$$

(Note: The diagram also includes a curved arrow q from X to Z and a curved arrow b from X to Y .)

where c is a morphism of $\mathcal{A}_i$, r is a morphism of $\mathcal{A}_j \cap \mathcal{A}_k$, f is a morphism of $\mathcal{A}_i \cap \mathcal{A}_j \cap \mathcal{A}_k$, and the inner square D' is Cartesian. Then the following diagram commutes

$$\begin{array}{ccccccc}
 & & \tilde{G}_{ikD} & & & & \\
 F_i(a)F_k(q) & \xrightarrow[\rho_{jk}(\gamma)]{F_k(\gamma)} & F_i(a)F_k(r)F_k(f) & \xrightarrow[\rho_{jk}(f)]{\rho_{jk}(\gamma)} & F_i(a)F_k(r)F_i(f) & \xrightarrow[\rho_{jk}(p)]{G_{ikD'}} & F_k(p)F_i(c)F_i(f) \xrightarrow[\rho_{jk}(\delta)]{F_i(\delta)} F_k(p)F_i(b) \\
 \rho_{jk}(q) \downarrow & & \rho_{jk}(r) \downarrow & & \rho_{jk}(r) \downarrow & & \rho_{jk}(p) \downarrow \\
 F_i(a)F_j(q) & \xrightarrow[\rho_{jk}(\gamma)]{F_j(\gamma)} & F_i(a)F_j(r)F_j(f) & \xrightarrow[\rho_{jk}(f)]{\rho_{jk}(\gamma)} & F_i(a)F_j(r)F_i(f) & \xrightarrow[\rho_{jk}(p)]{G_{ijD'}} & F_j(p)F_i(c)F_i(f) \xrightarrow[\rho_{jk}(\delta)]{F_i(\delta)} F_j(p)F_i(b) \\
 & & \tilde{G}_{ijD} & & & &
 \end{array}$$

(F) (E)

For pairwise distinct numbers $1 \leq i, j, k \leq n$, we show axiom (D) of Remark 6.1 for $\tilde{G}$ by descending induction on the number m of pairs of Cartesian opposite faces in the cube (6.1). If $m = 3$, all the faces of the cube are Cartesian, so the assertion is identical to axiom (D) for G . If $m < 3$, by symmetry, we may assume that either the bottom face K or the top face K' is not Cartesian. Decompose the cube as (6.5). The inner cube has more than m pairs of Cartesian opposite faces, hence axiom (D) holds for the inner cube by induction hypothesis. Therefore, the following diagram commutes

$$\begin{array}{ccccccc}
 F_i(a)F_j(q)F_k(x) & \xrightarrow[\rho_{jk}(\gamma)]{F_j(\gamma)} & F_i(a)F_k(z)F_j(q') & \xrightarrow[\rho_{jk}(\gamma')]{F_j(\gamma')} & F_k(w)F_i(a')F_j(q') & & \\
 \rho_{jk}(q) \downarrow & \tilde{G}_{jkM} & \rho_{jk}(f) \downarrow & \tilde{G}_{jkI''} & \rho_{jk}(f') \downarrow & \tilde{G}_{ikJ} & \\
 F_i(a)F_j(r)F_j(f)F_k(x) & \xrightarrow[\rho_{jk}(f)]{\rho_{jk}(\gamma)} & F_i(a)F_j(r)F_k(v)F_j(f') & \xrightarrow[\rho_{jk}(f')]{\rho_{jk}(\gamma')} & F_i(a)F_k(z)F_j(r')F_j(f') & \xrightarrow[\rho_{jk}(p)]{\rho_{jk}(\gamma')} & F_k(w)F_i(a')F_j(r')F_j(f') \\
 \rho_{jk}(f) \downarrow & \tilde{G}_{ijK} & \rho_{jk}(f') \downarrow & \tilde{G}_{jkI''} & \rho_{jk}(f') \downarrow & \tilde{G}_{ikJ} & \\
 F_i(a)F_j(r)F_i(f)F_k(x) & \xrightarrow[\rho_{jk}(f)]{\rho_{jk}(\gamma)} & F_i(a)F_j(r)F_k(v)F_i(f') & \xrightarrow[\rho_{jk}(f')]{\rho_{jk}(\gamma')} & F_i(a)F_k(z)F_j(r')F_i(f') & \xrightarrow[\rho_{jk}(p)]{\rho_{jk}(\gamma')} & F_k(w)F_i(a')F_j(r')F_i(f') \\
 \rho_{jk}(f) \downarrow & \tilde{G}_{ijK} & \rho_{jk}(f') \downarrow & \tilde{G}_{jkI''} & \rho_{jk}(f') \downarrow & \tilde{G}_{ikJ} & \\
 F_j(p)F_i(c)F_i(f')F_k(x) & \xrightarrow[\rho_{jk}(f)]{\rho_{jk}(\gamma)} & F_j(p)F_i(c)F_k(v)F_i(f') & \xrightarrow[\rho_{jk}(f')]{\rho_{jk}(\gamma')} & F_j(p)F_k(y)F_i(c')F_i(f') & \xrightarrow[\rho_{jk}(p)]{\rho_{jk}(\gamma')} & F_k(w)F_j(p')F_i(c')F_i(f') \\
 \rho_{jk}(f) \downarrow & \tilde{G}_{ijK} & \rho_{jk}(f') \downarrow & \tilde{G}_{jkI''} & \rho_{jk}(f') \downarrow & \tilde{G}_{ikJ} & \\
 F_j(p)F_i(b)F_k(x) & \xrightarrow[\rho_{jk}(f)]{\rho_{jk}(\gamma)} & F_j(p)F_k(y)F_i(b') & \xrightarrow[\rho_{jk}(f')]{\rho_{jk}(\gamma')} & F_k(w)F_j(p')F_i(b') & &
 \end{array}$$

(D)

Here M is the square $X'V'XV$.

Any morphism $(\alpha_i): ((F_i), G, \rho) \rightarrow ((F'_i), G', \rho')$ of $\mathcal{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}^{\text{Cart}}(\mathcal{C}, \mathcal{D})$ is a morphism $((F_i), \tilde{G}) \rightarrow ((F'_i), \tilde{G}')$ of $\mathcal{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$. Any 2-cell $(\Xi_i): (\alpha_i) \Rightarrow (\alpha'_i)$ of $\mathcal{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}^{\text{Cart}}(\mathcal{C}, \mathcal{D})$ is a 2-cell of $\mathcal{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}(\mathcal{C}, \mathcal{D})$.

The 2-functor defined in this way is clearly the inverse of (8.1). $\square$

Remark 8.4 We can consider a simpler 2-category of gluing data

$$\mathcal{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}^{\text{Cart}}(\mathcal{C}, \mathcal{D})' = 2\text{Fun}^{\text{ps}}(\mathcal{LT}_n \mathbb{Q}_{\mathcal{A}_1, \dots, \mathcal{A}_n}^{\text{Cart}} \mathcal{C}, \mathcal{D})$$

by dropping (ρ_{ij}) from the definition of $\mathcal{GD}_{\mathcal{A}_1, \dots, \mathcal{A}_n}^{\text{Cart}}$, where $\mathbb{Q}_{\mathcal{A}_1, \dots, \mathcal{A}_n}^{\text{Cart}} \mathcal{C} \subseteq \mathbb{Q}_{\mathcal{A}_1, \dots, \mathcal{A}_n} \mathcal{C}$ is the n -fold subcategory spanned by Cartesian squares. An ∞ -categorical variant of $\mathbb{Q}_{\mathcal{A}_1, \dots, \mathcal{A}_n}^{\text{Cart}} \mathcal{C}$ is studied in [20, Section 5].

In this section and the previous one, we studied the relationship between gluing data and Cartesian gluing data. In the next section we discuss ways to construct Cartesian gluing data.

9 Gluing data from base change maps

In this section we show how to produce (Cartesian) gluing data for two sub-2-categories from base change maps (Constructions 9.3 and 9.5) by taking adjoints. We deal with the axioms individually and refer the reader to Remark 9.9 for a synthesis. A somewhat more systematic treatment is possible in the ∞ -categorical setting (for the gluing data mentioned in Remark 8.4) [19, Section 1.4]. Throughout this section, we fix a 2-category $\mathcal{D}$.

Definition 9.1 (*2-Category of adjoint pairs $\mathcal{D}^{\text{adj}}$*) We define the 2-category $\mathcal{D}^{\text{adj}}$ of adjoint pairs with the same objects as $\mathcal{D}$ by taking $\mathcal{D}^{\text{adj}}(X, Y)$ to be the category of adjoint pairs from X to Y for every pair (X, Y) of objects of $\mathcal{D}$. More explicitly, a morphism $X \rightarrow Y$ of $\mathcal{D}^{\text{adj}}$ is a quadruple (f, g, η, ϵ) consisting of morphisms $f: X \rightarrow Y, g: Y \rightarrow X$ and 2-cells $\eta: \text{id}_Y \Rightarrow fg, \epsilon: gf \Rightarrow \text{id}_X$ of $\mathcal{D}$ such that the following triangles commute

$$\begin{array}{ccc} f & \xRightarrow{\eta * f} & fgf \\ \text{id}_f \searrow & & \downarrow f * \epsilon \\ & & f \end{array} \quad \begin{array}{ccc} g & \xRightarrow{g * \eta} & gfg \\ \text{id}_g \searrow & & \downarrow \epsilon * g \\ & & g \end{array}$$

The composition of $(f_1, g_1, \eta_1, \epsilon_1): X \rightarrow Y$ and $(f_2, g_2, \eta_2, \epsilon_2): Y \rightarrow Z$ is

$$(f_2 f_1, g_1 g_2, \eta_1 \eta_2, \epsilon_1 \epsilon_2): X \rightarrow Z,$$

where $\eta_1 \eta_2$ is the composition

$$\text{id}_Z \xRightarrow{\eta_2} f_2 g_2 \xRightarrow{f_2 * \eta_1 * g_2} f_2 f_1 g_1 g_2$$

and $\epsilon_1 \epsilon_2$ is the composition

$$g_1 g_2 f_2 f_1 \xRightarrow{g_1 * \epsilon_2 * f_1} g_1 f_1 \xRightarrow{\epsilon_1} \text{id}_X.$$

The *identity morphism* of an object X is $(\text{id}_X, \text{id}_X, \text{id}_{\text{id}_X}, \text{id}_{\text{id}_X})$. A 2-cell $(f, g, \eta, \epsilon) \Rightarrow (f', g', \eta', \epsilon')$ of $\mathcal{D}^{\text{adj}}$ is a pair (α, β) of 2-cells $\alpha: f \Rightarrow f'$ and $\beta: g' \Rightarrow g$ of $\mathcal{D}$ such that the following squares commute

$$\begin{array}{ccc} \text{id}_Y & \xRightarrow{\eta} & fg \\ \eta' \downarrow & & \downarrow \alpha \\ f'g' & \xRightarrow{\beta} & f'g \end{array} \quad \begin{array}{ccc} g'f & \xRightarrow{\alpha} & g'f' \\ \beta \downarrow & & \downarrow \epsilon' \\ gf & \xRightarrow{\epsilon} & \text{id}_X \end{array}$$

The *projection 2-functors* $P_1: \mathcal{D}^{\text{adj}} \rightarrow \mathcal{D}$ and $P_2: \mathcal{D}^{\text{adj}} \rightarrow \mathcal{D}^{\text{coop}}$, sending (f, g, η, ϵ) to f and g respectively, are locally fully faithful (Definition 2.6).

Remark 9.2 Let $\mathcal{C}$ be a 2-category. The projection 2-functors P_1 and P_2 induce locally fully faithful 2-functors

$$\begin{aligned} P_1 &: \mathcal{P} s\mathcal{F}un(\mathcal{C}, \mathcal{D}^{\text{adj}}) \rightarrow \mathcal{P} s\mathcal{F}un(\mathcal{C}, \mathcal{D}), \\ P_2 &: \mathcal{P} s\mathcal{F}un(\mathcal{C}, \mathcal{D}^{\text{adj}}) \rightarrow \mathcal{P} s\mathcal{F}un(\mathcal{C}, \mathcal{D}^{\text{coop}}). \end{aligned}$$

An object F of $\mathcal{P} s\mathcal{F}un(\mathcal{C}, \mathcal{D})$ (resp. $\mathcal{P} s\mathcal{F}un(\mathcal{C}, \mathcal{D}^{\text{coop}})$) is in the image of P_1 (resp. P_2) if and only if for every morphism a of $\mathcal{C}$, the image $F(a)$ can be completed into an adjoint pair $(f, F(a), \eta, \epsilon)$ (resp. $(F(a), g, \eta, \epsilon)$).

In the rest of this section, we fix a 2-category $\mathcal{C}$, a pseudo functor $F: \mathcal{C} \rightarrow \mathcal{D}^{\text{coop}}$, a locally full sub-2-category $\mathcal{B}$ of $\mathcal{C}$, and a pseudo functor $B: \mathcal{B} \rightarrow \mathcal{D}^{\text{adj}}$ such that $P_2(B)$ is the restriction $F|_{\mathcal{B}}$. We do *not* assume that $\mathcal{C}$ is a $(2, 1)$ -category. We denote F by

$$f \mapsto f^*, \quad \alpha \mapsto \alpha^*,$$

and the pseudo functor $R = P_1(B): \mathcal{B} \rightarrow \mathcal{D}$ by

$$p \mapsto p_*, \quad \alpha \mapsto \alpha_*.$$

Construction 9.3 (Base change map B_D associated to a $(\mathcal{C}, \mathcal{B})$ -down-square D) In this section, it is convenient to use down-squares in $\mathcal{C}$

$$\begin{array}{ccc} X & \xrightarrow{j} & Y \\ q \downarrow & \searrow_{\alpha} & \downarrow p \\ Z & \xrightarrow{i} & W. \end{array} \quad (9.1)$$

Let D be such a square with p and q in $\mathcal{B}$. The *base change map* B_D is by definition the following 2-cell of $\mathcal{D}$

$$i^* p_* \xRightarrow{\eta_q} q_* q^* i^* p_* \Longrightarrow q_* (iq)^* p_* \xRightarrow{\alpha^*} q_* (pj)^* p_* \Longrightarrow q_* j^* p^* p_* \xRightarrow{\epsilon_p} q_* j^*.$$

Here we have as usual denoted horizontal composition of 2-cells with morphisms simply by the 2-cells.

If i and j are also morphisms of $\mathcal{B}$, then B_D is also the composition

$$i^* p_* \xRightarrow{\eta_j} i^* p_* j_* j^* \Longrightarrow i^* (pj)_* j^* \xRightarrow{\alpha_*} i^* (iq)_* j^* \Longrightarrow i^* i_* q_* j^* \xRightarrow{\epsilon_i} q_* j^*.$$

In fact, the following diagram commutes

$$\begin{array}{ccccccc}
 & & i^*p_* & \xrightarrow{\text{id}} & i^*p_* & \xrightarrow{\eta_i} & i^*i_*i^*p_* \\
 & & \uparrow \epsilon_i & & \uparrow \epsilon_i & & \uparrow \epsilon_i \\
 & & i^*p_* & \xrightarrow{\eta_i} & i^*i_*i^*p_* & \xrightarrow{\eta_q} & i^*i_*q_*q^*i^*p_* \\
 & & \downarrow \eta_p & & \downarrow \eta_p & & \downarrow \eta_p \\
 & & i^*p_*p^*p_* & \xrightarrow{\eta_j} & i^*p_*j_*j^*p^*p_* & \xrightarrow{\alpha_*} & i^*i_*q_*j^*p^*p_* \\
 & & \downarrow \epsilon_p & & \downarrow \epsilon_p & & \downarrow \epsilon_p \\
 & & i^*p_* & \xrightarrow{\eta_j} & i^*p_*j_*j^* & \xrightarrow{\alpha_*} & i^*i_*q_*j^* \\
 & & & & & & \downarrow \epsilon_i \\
 & & & & & & q_*j^*
 \end{array}$$

Proposition 9.4 (1) Let D , D' , and D'' be respectively the upper, lower, and outer squares of the diagram in C

$$\begin{array}{ccc}
 X_1 & \xrightarrow{i_1} & Y_1 \\
 q \downarrow & \swarrow \alpha & \downarrow p \\
 X_2 & \xrightarrow{i_2} & Y_2 \\
 q' \downarrow & \swarrow \alpha' & \downarrow p' \\
 X_3 & \xrightarrow{i_3} & Y_3
 \end{array}$$

where the vertical arrows are morphisms of $\mathcal{B}$. Then the following diagram commutes

$$\begin{array}{ccccc}
 i_3^*p'_*p_* & \xrightarrow{B_{D'}} & q'_*i_2^*p_* & \xrightarrow{B_D} & q'_*q_*i_1^* \\
 \parallel & & & & \parallel \\
 i_3^*(p'p)_* & \xrightarrow{B_{D''}} & (q'q)_*i_1^* & &
 \end{array}$$

(2) Let D , D' , and D'' be respectively the left, right, and outer squares of the diagram in C

$$\begin{array}{ccccc}
 X_1 & \xrightarrow{j} & X_2 & \xrightarrow{j'} & X_3 \\
 p_1 \downarrow & \swarrow \alpha & \downarrow p_2 & \swarrow \alpha' & \downarrow p_3 \\
 Y_1 & \xrightarrow{i} & Y_2 & \xrightarrow{i'} & Y_3
 \end{array}$$

where the vertical arrows are morphisms of $\mathcal{B}$. Then the following diagram commutes

$$\begin{array}{ccccc}
 i^* i'^* p_{3*} & \xRightarrow{B_{D'}} & i^* p_{2*} j'^* & \xRightarrow{B_D} & p_{1*} j^* j'^* \\
 \uparrow \eta_{q'} & & & & \uparrow \eta_q \\
 (i' i)^* p_{3*} & \xRightarrow{B_{D''}} & p_{1*} (j' j)^* & &
 \end{array}$$

Proof This is equivalent to [1, Propositions 1.1.11, 1.1.12]. We provide a proof for the sake of completeness.

(1) The following diagram commutes

$$\begin{array}{ccccccc}
 & & & B_{D'} & & & \\
 i_3^* p'_* p_* & \xRightarrow{\eta_{q'}} & q'_* q'^* i_3^* p'_* p_* & \xRightarrow{\alpha'^*} & q'_* i_2^* p'^* p'_* p_* & \xRightarrow{\epsilon_{p'}} & q'_* i_2^* p_* \\
 & \eta_q \downarrow & & \eta_q \downarrow & & \eta_q \downarrow & \\
 & q'_* q_* q'^* i_3^* p'_* p_* & \xRightarrow{\alpha'^*} & q'_* q_* q'^* i_2^* p'^* p'_* p_* & \xRightarrow{\epsilon_{p'}} & q'_* q_* q'^* i_2^* p_* & \\
 & & \alpha^* \downarrow & & \alpha^* \downarrow & & \\
 & & q'_* q_* i_1^* p'^* p'_* p_* & \xRightarrow{\epsilon_{p'}} & q'_* i_1^* p'^* p_* & \xRightarrow{\epsilon_p} & q'_* i_1^* p_* \\
 & & & & & & B_D \\
 & & & & & & \\
 & & & B_{D''} & & &
 \end{array}$$

(2) Similar to (1).

□

In the rest of this section, we further fix a locally full sub-2-category $\mathcal{A}$ of $\mathcal{C}$ and a pseudo functor $A: \mathcal{A} \rightarrow (\mathcal{D}^{\text{coop}})^{\text{adj}}$ such that $P_1(A)$ is the restriction $F|_{\mathcal{A}}$. We denote the pseudo functor $L = P_2(A): \mathcal{A} \rightarrow (\mathcal{D}^{\text{coop}})^{\text{coop}} = \mathcal{D}$ by

$$i \mapsto i_!, \quad \alpha \mapsto \alpha_!.$$

Construction 9.5 (Base change map A_D associated to an $(\mathcal{A}, \mathcal{C})$ -down-square D) Let D be a square (9.1) in $\mathcal{C}$ where i and j are morphisms of $\mathcal{A}$. The *base change map* A_D is by definition the following 2-cell in $\mathcal{D}$

$$j_! q^* \xRightarrow{\eta_i} j_! q^* i^* i_! \xRightarrow{\alpha^*} j_! (iq)^* i_! \xRightarrow{\epsilon_j} j_! j^* p^* i_! \xRightarrow{\epsilon_j} p^* i_!.$$

If p and q are also morphisms of $\mathcal{A}$, then A_D is also the composition

$$j_! q^* \xRightarrow{\eta_p} p^* p_! j_! q^* \xRightarrow{\alpha_1} p^* (pj)^* i_! q^* \xRightarrow{\epsilon_q} p^* i_! q_! q^* \xRightarrow{\epsilon_q} p^* i_!.$$

We have an analogue of Proposition 9.4 for A_D .

Construction 9.6 (Construction of G_D) Let D be a square (9.1) in $\mathcal{C}$ where the horizontal arrows i and j are morphisms of $\mathcal{A}$, and the vertical arrows p and q are morphisms of $\mathcal{B}$. Then

$$(B_D, A_D): (i^* p_*, p^* i_!, \eta_p \eta_i, \epsilon_p \epsilon_i) \Rightarrow (q_* j^*, j_! q^*, \eta_q \eta_j, \epsilon_q \epsilon_j)$$

is a 2-cell of $\mathcal{D}^{\text{adj}}$. In fact, the following diagrams commute

$$\begin{array}{ccccc}
 \text{id}_Z & \xrightarrow{\eta_i} & i^* i_! & \xrightarrow{\eta_p} & i^* p_* p^* i_! \\
 \eta_q \downarrow & & \downarrow \eta_q & & \downarrow \eta_q \\
 q_* q^* & \xrightarrow{\eta_i} & q_* q^* i^* i_! & & q_* q^* i^* p_* p^* i_! \\
 \eta_j \downarrow & & \searrow \alpha^* & & \searrow \alpha^* \\
 q_* j^* j_! q^* & \xrightarrow{\eta_i} & q_* j^* j_! q^* i^* i_! & \xrightarrow{\eta_p} & q_* j^* p^* p_* p^* i_! \\
 & & \searrow \alpha^* & & \downarrow \eta_j \\
 & & & & q_* j^* j_! p^* i_! \xrightarrow{\epsilon_j} q_* j^* p^* i_! \\
 & \searrow A_D & & & \downarrow \epsilon_p \\
 & & & & q_* j^* p^* i_! \xrightarrow{\epsilon_p} q_* j^* p^* i_!
 \end{array}$$

$$\begin{array}{ccccc}
 j_! q^* i^* p_* & \xrightarrow{\eta_q} & j_! q^* q_* q^* i^* p_* & \xrightarrow{\alpha^*} & j_! q^* q_* j^* p_* \\
 \eta_i \downarrow & & \downarrow \epsilon_q & & \downarrow \epsilon_q \\
 j_! q^* i^* i^* i^* p_* & \xrightarrow{\epsilon_i} & j_! q^* i^* p_* & \xrightarrow{\epsilon_p} & j_! q^* q_* j^* p_* \\
 \searrow \alpha^* & & \searrow \alpha^* & & \downarrow \epsilon_q \\
 j_! j^* p^* i_! i^* p_* & \xrightarrow{\epsilon_j} & j_! j^* p^* p_* & \xrightarrow{\epsilon_p} & j_! j^* \\
 \downarrow \epsilon_j & & \downarrow \epsilon_j & & \downarrow \epsilon_j \\
 p^* i_! i^* p_* & \xrightarrow{\epsilon_i} & p^* p_* & \xrightarrow{\epsilon_p} & \text{id}_X
 \end{array}$$

It follows that B_D is invertible if and only if A_D is. In this case, the diagram

$$\begin{array}{ccc}
 i_! q_* & \xrightarrow{\eta_j} & i_! q_* j^* j_! \xrightarrow{B_D^{-1}} i_! i^* p_* j_! \\
 \eta_p \downarrow & & \downarrow \epsilon_i \\
 p_* p^* i_! q_* & \xrightarrow{A_D^{-1}} & p_* j_! q^* q_* \xrightarrow{\epsilon_q} p_* j_!
 \end{array}$$

commutes and we define $G_D : i_! q_* \Rightarrow p_* j_!$ to be the composition. In fact, the following diagram commutes

$$\begin{array}{ccccccc}
 & & i_! q_* & \xrightarrow{\eta_j} & i_! q_* j^* j_! & & \\
 & \nearrow \text{id} & \uparrow \epsilon_q & & \downarrow B_D^{-1} & & \\
 i_! q_* & \xrightarrow{\eta_q} & i_! q_* q^* q_* & \xrightarrow{\eta_j} & i_! q_* j^* j_! q^* q_* & & \\
 \downarrow \eta_i & & \downarrow \eta_p & & \downarrow B_D^{-1} & & \\
 \text{id} \curvearrowright i_! i^* i_! q_* & \xrightarrow{\eta_p} & i_! i^* p_* p^* i_! q_* & \xrightarrow{A_D^{-1}} & i_! i^* p_* j_! q^* q_* & \xrightarrow{\epsilon_p} & i_! i^* p_* j_! \\
 \downarrow \epsilon_i & & \downarrow \eta_p & & \downarrow \epsilon_i & & \downarrow \epsilon_i \\
 i_! q_* & \xrightarrow{\eta_p} & p_* p^* a_! q_* & \xrightarrow{A_D^{-1}} & p_* j_! q^* q_* & \xrightarrow{\epsilon_q} & p_* j_!
 \end{array}$$

H

where the hexagon H commutes because the following diagram commutes

$$\begin{array}{ccccc}
 & q_* q^* & \xrightarrow{\eta_j} & q_* j^* j_! q^* & \\
 \nearrow \eta_q & & \searrow A_D & & \searrow B_D^{-1} \\
 \text{id}_Z & & q_* j^* p^* i_! & & i^* p_* j_! q^* \\
 \searrow \eta_i & & \searrow B_D^{-1} & & \searrow A_D \\
 & i^* i_! & \xrightarrow{\eta_p} & i^* p_* p^* i_! &
 \end{array}$$

Construction 9.7 (Construction of ρ_f) Let $f : X \rightarrow Y$ be a morphism of $\mathcal{A} \cap \mathcal{B}$. Then

$$(\epsilon_f^B, \eta_f^A) : (f^* f_*, f^* f_!, \eta_f^B \eta_f^A, \epsilon_f^B \epsilon_f^A) \Rightarrow (\text{id}_X, \text{id}_X, \text{id}_{\text{id}_X}, \text{id}_{\text{id}_X})$$

is a 2-cell of $\mathcal{D}^{\text{adj}}$. It follows that ϵ_f^B is invertible if and only if η_f^A is. In this case, the following diagram commutes

$$\begin{array}{ccc}
 f_! & \xrightarrow{\eta_f^B} & f_* f^* f_! \\
 (\epsilon_f^B)^{-1} \downarrow & & \downarrow (\eta_f^A)^{-1} \\
 f_! f^* f_* & \xrightarrow{\epsilon_f^A} & f_*
 \end{array}$$

and we define $\rho_f : f_! \Rightarrow f_*$ to be the composition. In fact, the following diagram commutes

Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a pair of composable morphisms of $\mathcal{A} \cap \mathcal{B}$ with ϵ_f^B and ϵ_g^B invertible. Then the following diagram commutes

The following properties of G_D and ρ_f are similar to axioms (b), (b') of Remark 5.1 and axioms (c), (c') of Remark 7.2.

Proposition 9.8

- (1) In the situation of Proposition 9.4 (1), assume that the horizontal arrows are morphisms of $\mathcal{A}$, the vertical arrows are morphisms of $\mathcal{B}$, and the 2-cells B_D and $B_{D'}$ are invertible. Then the following diagram commutes

$$\begin{array}{ccc}
 i_{3!} q'_* q_* & \xrightarrow{G_{D'}} & p_* i_{2!} q_* \xrightarrow{G_D} p'_* p_* i_{1!} \\
 \Downarrow & & \Downarrow \\
 i_{3!} (q' q)_* & \xrightarrow{G_{D''}} & (p' p)_* i_{1!}.
 \end{array}$$

- (2) In the situation of Proposition 9.4 (2), assume that the horizontal arrows are morphisms of $\mathcal{A}$, the vertical arrows are morphisms of $\mathcal{B}$, and the 2-cells B_D and $B_{D'}$ are invertible. Then the following diagram commutes

$$\begin{array}{ccccc}
 i'_! i_! p_{1*} & \xrightarrow{G_D} & i'_! p_{2*} j_! & \xrightarrow{G_{D'}} & p_{3*} j'_! j_! \\
 \Downarrow & & & & \Downarrow \\
 (i' i)_! p_{1*} & \xrightarrow{G_{D''}} & & & p_{3*} (j' j)_!
 \end{array}$$

- (3) Let D be a square (9.1) in $\mathcal{C}$ where the horizontal arrows i and j are morphisms of $\mathcal{A}$, the vertical arrows p and q are morphisms of $\mathcal{A} \cap \mathcal{B}$, and the 2-cells ϵ_p^B , ϵ_q^B and B_D are invertible. Then the following diagram commutes

$$\begin{array}{ccccc}
 i_! q_* & \xleftarrow{\rho_q} & i_! q_! & \xleftarrow{\quad} & (iq)_! \\
 G_D \Downarrow & & & & \Uparrow \alpha_! \\
 p_* j_! & \xleftarrow{\rho_p} & p_! j_! & \xleftarrow{\quad} & (pj)_!
 \end{array}$$

- (4) Let D be a square (9.1) in $\mathcal{C}$ where the horizontal arrows i and j are morphisms of $\mathcal{A} \cap \mathcal{B}$, the vertical arrows p and q are morphisms of $\mathcal{B}$, and the 2-cells ϵ_i^B , ϵ_j^B and B_D are invertible. Then the following diagram commutes

$$\begin{array}{ccccc}
 i_! q_* & \xrightarrow{\rho_i} & i_* q_* & \xrightarrow{\quad} & (iq)_* \\
 G_D \Downarrow & & & & \Uparrow \alpha_* \\
 p_* j_! & \xrightarrow{\rho_j} & p_* j_* & \xrightarrow{\quad} & (pj)_*
 \end{array}$$

Proof (1) Similar to (2).

- (2) The following diagram commutes

$$\begin{array}{ccccccc}
 & & & G_D & & & \\
 i'_! i_! p_{1*} & \xrightarrow{\eta_j} & i'_! i_! p_{1*} j'^* j_! & \xrightarrow{B_D^{-1}} & i'_! i^* p_{2*} j_! & \xrightarrow{\epsilon_i} & i'_! p_{2*} j_! \\
 & \eta_j \Downarrow & & \eta_{j'} \Downarrow & & & \eta_{j'} \Downarrow \\
 i'_! i_! p_{1*} j'^* j'^* j'_! j_! & \xrightarrow{B_D^{-1}} & i'_! i_! i^* p_{2*} j'^* j'_! j_! & \xrightarrow{\epsilon_i} & i'_! p_{2*} j'^* j'_! j_! & & \\
 & & B_{D'}^{-1} \Downarrow & & B_{D'}^{-1} \Downarrow & & \\
 & & i'_! i_! i^* p_{3*} j'_! j_* & \xrightarrow{\epsilon_i} & i'_! i'^* p_{3*} j'_! j_! & & \\
 & & & & \epsilon_{i'} \Downarrow & & \\
 & & & & p_{3*} j'_! j_! & & \\
 & & G_{D''} \nearrow & & & & G_{D'} \nearrow
 \end{array}$$

- (3) Similar to (4).

(4) The following diagram commutes

where the pentagon commutes because the following diagram commutes

□

Remark 9.9 If $\mathcal{C}$ is a $(2, 1)$ -category, B_D and G_D are invertible for every Cartesian $(\mathcal{A}, \mathcal{B})$ -square D , and ϵ_f^B and ρ_f are invertible for every morphism f of $\mathcal{A} \cap \mathcal{B}$, then Proposition 9.8 shows that $(L, R, (G_D), \rho)$ is an object of $\mathcal{GD}_{\mathcal{A}, \mathcal{B}}^{\text{Cart}}(\mathcal{C}, \mathcal{D})$ (Definition 7.3). Here we have used the correspondence via inverting 2-cells between down-squares (9.1) used in this section and up-squares (3.8) used in earlier sections.

We conclude this section by a couple of criteria for the axioms for morphisms of (Cartesian) gluing data to hold true.

The following property is similar to condition (n) of Definition 7.3.

Proposition 9.10 *Let D be a square (9.1) in $\mathcal{C}$ where i and j are morphisms of $\mathcal{A} \cap \mathcal{B}$ and such that the 2-cells ϵ_i^B , ϵ_j^B , and α are invertible, and let D' be the square obtained by inverting α . Then the following diagram commutes*

Proof The following diagram commutes

Proposition 9.11 *Let*

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Assume that B_K and $B_{K'}$ are invertible. Then the following diagram commutes

$$\begin{array}{ccccc}
i'_1 z'_* q_* & \xrightarrow{B_{I'}} & i'_1 q'_* x^* & \xrightarrow{G_{K'}} & p'_* j'_1 x^* \\
A_J \Downarrow & & & & \Downarrow A_{J'} \\
w^* i_1 q_* & \xrightarrow{G_K} & w^* p_* j_1 & \xrightarrow{B_I} & p'_* y^* j_1.
\end{array}$$

$$\begin{array}{ccccc}
& & & q'_* x^* j^* & \xrightarrow{\eta_{j'}} q'_* j'^* j'_! x^* j^* & \xrightarrow{B_{K'}^{-1}} i'^* p'_* j'_! x^* j^* \\
& \nearrow B_{I'} & & \downarrow \beta'^* & \downarrow \beta'^* & \downarrow \beta'^* \\
z^* q_* j^* & \xrightarrow{B_K^{-1}} z^* i^* p_* & & q'_* j'^* y^* & \xrightarrow{\eta_{j'}} q'_* j'^* j'_! y^* & \xrightarrow{B_{K'}^{-1}} i'^* p'_* j'_! y^* \\
\downarrow \eta_i & & \searrow \text{id} & \downarrow \epsilon_{j'} & & \downarrow \epsilon_{j'} \\
z^* i^* i_! q_* j^* & \xrightarrow{B_K} z^* i^* i_! i^* p_* & \xrightarrow{\epsilon_i} z^* i^* p_* & & q'_* j'^* y^* & \xrightarrow{B_{K'}^{-1}} i'^* p'_* j'_! y^* \\
\downarrow \beta^* & & \downarrow \beta^* & & \downarrow \beta^* & \\
i'^* w^* i_! q_* j^* & \xrightarrow{B_K} i'^* w^* i_! i^* p_* & \xrightarrow{\epsilon_i} i'^* w^* p_* & & & \\
& & & & & \nearrow B_I
\end{array}$$

1

This ends our discussion on the construction of gluing data. The next section is included for completeness.

10 Proof of Proposition 2.12

Proof of Proposition 2.12 (1) Let G and H be pseudo functors from $\mathcal{D}$ to $\mathcal{E}$. We need to show that the functor

$$\mathbf{PsNat}(G, H) \rightarrow \mathbf{PsNat}(GF, HF) \quad \alpha \mapsto \alpha F$$

between categories of pseudo natural transformations and modifications is fully faithful and injective on objects. Let α and β be pseudo natural transformations from G to H .

We identify the set of modifications $\alpha \Rightarrow \beta$ and the set of modifications $\alpha F \Rightarrow \beta F$ with subsets of the set of 2-cells $|\alpha| \Rightarrow |\beta|$ in $\mathcal{E}^{\text{Ob}(\mathcal{C})}$. Let $\Xi: \alpha F \Rightarrow \beta F$ be a modification. Let $g: X \rightarrow Y$ be a morphism of $\mathcal{D}$. By assumption, there exists a decomposition $g \xRightarrow{\delta} h \xRightarrow{\gamma} g$ in $\mathcal{D}$, where $h = Ff$ for some morphism f of $\mathcal{C}$. All inner cells of the diagram

$$\begin{array}{ccccccc} \alpha(Y)G(g) & \xRightarrow{G(\delta)} & \alpha(Y)G(h) & \xRightarrow{\Xi_Y} & \beta(Y)G(h) & \xRightarrow{G(\gamma)} & \beta(Y)G(g) \\ \alpha(g) \downarrow & & \downarrow \alpha(h) & & \downarrow \beta(h) & & \downarrow \beta(g) \\ H(g)\alpha(X) & \xRightarrow{H(\delta)} & H(h)\alpha(X) & \xRightarrow{\Xi_X} & H(h)\beta(X) & \xRightarrow{H(\gamma)} & H(g)\beta(X) \end{array}$$

commutes. Therefore, Ξ is a modification $\alpha \Rightarrow \beta$.

Assume $\alpha F = \beta F$. Let g and δ, γ be as above. The square

$$\begin{array}{ccccc} \alpha(Y)G(g) & \xRightarrow{G(\delta)} & \alpha(Y)G(h) & \xRightarrow{G(\gamma)} & \alpha(Y)G(g) \\ \alpha(g) \downarrow & & \downarrow \alpha(h) & & \downarrow \alpha(g) \\ H(g)\alpha(X) & \xRightarrow{H(\delta)} & H(h)\alpha(X) & \xRightarrow{H(\gamma)} & H(g)\alpha(X) \end{array}$$

commutes and the composition of each of the two lines is an identity. The same holds for β . Since $\alpha(h) = (\alpha F)(f) = (\beta F)(f) = \beta(h)$, we have

$$\alpha(g) = H(\gamma)\alpha(h)G(\delta) = H(\gamma)\beta(h)G(\delta) = \beta(g).$$

Therefore, $\alpha = \beta$.

Proof of Proposition 2.12 (2) It is clear that (c) implies (d).

Next we show that (d) implies (a). Since $\Phi_{\mathcal{C}}$ is pseudo surjective, there exists a pseudo functor $G: \mathcal{D} \rightarrow \mathcal{C}$ such that $\Phi_{\mathcal{C}}(G) = GF$ is equivalent to $\text{id}_{\mathcal{C}}$. Then $\Phi_{\mathcal{D}}(FG) = FGF$ is equivalent to $\Phi_{\mathcal{D}}(\text{id}_{\mathcal{D}}) = F$. Since $\Phi_{\mathcal{D}}$ is a local equivalence, it follows that FG is equivalent to $\text{id}_{\mathcal{D}}$. Therefore, F is a bi-equivalence.

To prove that (a) implies (b), let $G: \mathcal{D} \rightarrow \mathcal{C}$ be a pseudo functor endowed with pseudo natural equivalences $\eta: \text{id}_{\mathcal{D}} \rightarrow GF$ and $FG \rightarrow \text{id}_{\mathcal{D}}$. For every object Y of $\mathcal{D}$, let $G'Y = \text{Ob}(F)^{-1}Y$. For every morphism $g: Y \rightarrow Y'$ of $\mathcal{D}$, choose a morphism $G'g: G'Y \rightarrow G'Y'$ and an invertible 2-cell ψ_g in $\mathcal{C}$:

$$\begin{array}{ccc} G'Y & \xrightarrow{G'g} & G'Y' \\ \eta(G'Y) \downarrow & \psi_g \nearrow & \downarrow \eta(G'Y') \\ GY & \xrightarrow{Gg} & GY'. \end{array}$$

By Lemma 2.10, this determines a pseudo functor $G': \mathcal{D} \rightarrow \mathcal{C}$ such that $\text{Ob}(G'): \text{Ob}(\mathcal{D}) \rightarrow \text{Ob}(\mathcal{C})$ is the inverse of $\text{Ob}(F)$ and a pseudo natural equivalence $\psi: G' \rightarrow G$ such that $\psi(Y) = \eta(G'Y)$ for every object Y of $\mathcal{D}$. For any morphism $f: X \rightarrow X'$ of $\mathcal{C}$, η induces the following square in $\mathcal{C}$:

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \eta(X) \downarrow & \eta_f \nearrow & \downarrow \eta(X') \\ GFX & \xrightarrow{GFf} & GFX'. \end{array}$$

Since $\eta(X')$ is an equivalence in $\mathcal{C}$, there exists an invertible 2-cell $\eta'_f: f \Rightarrow G'Ff$ such that $\eta_f^{-1}\eta'_f = \psi_{Ff}^{-1}$. This defines a pseudo natural isomorphism $\eta': \text{id}_{\mathcal{C}} \rightarrow G'F$ satisfying $\eta'(X) = \text{id}_X$ for every object X of $\mathcal{C}$. Since F is locally essentially surjective, for every morphism $g: Y \rightarrow Y'$ of $\mathcal{D}$, there exist a morphism $f: G'Y \rightarrow G'Y'$ of $\mathcal{C}$ and an invertible 2-cell $\alpha: g \Rightarrow Ff$. The composition

$$\epsilon_g: FG'g \xrightarrow{FG'\alpha} FG'Ff \xrightarrow{(\eta'_{Ff})^{-1}} Ff \xrightarrow{\alpha^{-1}} g$$

does not depend on the choice of (f, α) . This defines a pseudo natural isomorphism $\epsilon: FG' \rightarrow \text{id}_{\mathcal{D}}$ such that $\epsilon(Y) = \text{id}_Y$ for every object Y of $\mathcal{D}$.

It remains to show that (b) implies (c). By Proposition 2.11, it suffices to show that the restriction of $\Phi_{\mathcal{E}}$ to $\mathcal{UPsFun}(\mathcal{D}, \mathcal{E})$ is a $\mathcal{E}^{\text{Ob}(\mathcal{C})}$ -2-equivalence. For this, we may assume that F is strictly unital. Then G is strictly unital. Consider the $\mathcal{E}^{\text{Ob}(\mathcal{C})}$ -2-functors

$$\Phi'_{\mathcal{E}}: \mathcal{UPsFun}(\mathcal{D}, \mathcal{E}) \rightarrow \mathcal{UPsFun}(\mathcal{C}, \mathcal{E}), \quad \Psi'_{\mathcal{E}}: \mathcal{UPsFun}(\mathcal{C}, \mathcal{E}) \rightarrow \mathcal{UPsFun}(\mathcal{D}, \mathcal{E})$$

induced by F and G , respectively. Then η and ϵ induce $\mathcal{E}^{\text{Ob}(\mathcal{C})}$ -2-natural isomorphisms $\text{id} \rightarrow \Phi'_{\mathcal{E}}\Psi'_{\mathcal{E}}$ and $\Psi'_{\mathcal{E}}\Phi'_{\mathcal{E}} \rightarrow \text{id}$. Thus (b) implies (c). $\square$

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