On Hilali's conjecture related to Halperin's

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Abstract In this paper, we focus on Hilali's conjecture that, for any simply-connected elliptic CW-complex X , the total sum of the rational Betti numbers is at least as large as the total rank of its rational homotopy. We investigate this conjecture for *coformal spaces* and suggest some research directions to resolve it completely. Finally, we put up a bridge between the *Hilali conjecture* and that of Halperin: the *total rank conjecture* and use it to establish the latter holds for all manifolds of dimension less than 16 and whose total rank is equal to 4.

Keywords Rational homotopy theory · Minimal models · Coformal spaces · Total rank conjecture

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1 Introduction

Through this paper, X denotes a simply-connected elliptic CW-complex with $(\Lambda V, d)$ its Sullivan minimal model. For more details on rational minimal models, we refer the reader to the standard book [8]. X is said to be *elliptic* when both $H^*(X; \mathbb{Q})$ and $\pi_*(X) \otimes \mathbb{Q}$ are of finite dimension. For these spaces, Hilali conjectured in 1990 [13] that

Hilali Conjecture *If X is a simply-connected elliptic CW-complex, then*

$$\dim \pi_*(X) \otimes \mathbb{Q} \leq \dim H^*(X; \mathbb{Q}).$$

In terms of Sullivan minimal models, one can reformulate this conjecture algebraically as follows:

Hilali Conjecture (Algebraic version) *If $(\Lambda V, d)$ is a 1-connected elliptic Sullivan model, then*

$$\dim V \leq \dim H^*(\Lambda V, d).$$

The Hilali conjecture now belongs to rational homotopy theory folklore, since it gave rise to a lot of research interests and since it holds for many interesting elliptic spaces such:

- Pure spaces, when $dV^{\text{even}} = 0$ and $dV^{\text{odd}} \subset \Lambda^{\geq 2}V$ (Theorem E, [13]);
- Hyper elliptic spaces, when $dV^{\text{even}} = 0$ (Theorem 1.2, [4]);
- Two stages spaces, when $V = U \oplus W$ with $dU = 0$ and $dW \subset \Lambda^{\geq 2}U$ (Theorem 1, [1]);
- Formal spaces, when $(\Lambda V, d) \cong (H^*(X; \mathbb{Q}), 0)$ (Theorem 2, [14]).

Our main purpose in this paper is to investigate the case of *coformal* spaces, those verifying

$$dV \subset \Lambda^2 V,$$

(i.e., the differential is purely quadratic). We will also propose some research directions to resolve completely the coformal case by induction on the nilpotency degree of the associated *homotopy Lie algebra*. We will test these ideas on some informative examples.

In fact, resolving completely the coformal case would be a decisive step towards the definitive resolution of the Hilali conjecture, since the case when the differential is homogeneous of length at least 3 (i.e., $dV \subset \Lambda^k V$ with $k \geq 3$.) is already done (see Proposition 2, [14]). After resolving the case $k = 2$ (i.e., X is coformal), a possible approach would be to consider the filtration

$$\Lambda V \supset \Lambda^{\geq 1} V \supset \dots \Lambda^{\geq p} V \supset \dots$$

with

$$d\Lambda^{\geq p}V \subset \Lambda^{\geq(p+1)}V.$$

Therefore, the Hilali conjecture holds for the first term in the associated Milnor-Moore spectral sequence

$$E_1 = H(E_0) = (\Lambda V, d_2) \implies H^*(\Lambda V, d),$$

and one should check that

$$\dim V \leq \dim E_i \quad \text{for any } i.$$

In 1985, Halperin (see [11, 12]) conjectured in the topological context of maximal free torus actions on topological manifolds, that:

Halperin Conjecture (Toral rank conjecture) *If X is a nilpotent CW-complex, then*

$$\dim H^*(X; \mathbb{Q}) \geq 2^{\text{rk}_0(X)}.$$

Here $\text{rk}_0(X)$, called the *toral rank* of X , is defined to be the largest rank among that of the tori T^n that act almost freely on X . Recall that an action is said to be *almost free*, when all its isotropy groups are finite.

This long-standing open question is actually known to hold in some cases similar to those of the Hilali conjecture, namely for:

- Pure spaces ([11], Proposition 1.5);
- hyper elliptic spaces ([13], Theorem B);
- Two stages spaces, under some addition conditions [15];
- $\text{rk}_0(X) \leq 3$ [3].

We will focus on the case when $\text{rk}_0(X) = 4$. The contents of the paper can be briefly summarized as follows: Sect. 2 is devoted to discuss the Hilali conjecture for coformal spaces. We will prove that

Proposition A *The Hilali conjecture holds for any coformal space X whose rational homotopy Lie algebra $\mathbb{L}$ is of nilpotency 1 or 2.*

In Sect. 3, we will *build* the promised bridge between the Hilali and Halperin conjectures and use it to prove:

Proposition B *The toral rank conjecture holds for any manifold of dimension less than 16, and whose rational toral rank is equal to 4.*

2 Hilali conjecture for coformal spaces

Through this section, $\mathbb{L} := \pi_{*+1}(X) \otimes \mathbb{Q}$ denotes the rational homotopy Lie algebra of X . Using the *Cartan-Eilenberg-Chevalley* construction, many connections between $\mathbb{L}$ and the related Sullivan model are established, here below some useful ones quoted from [8]:

- Page 301: $\mathcal{C}_*(\mathbb{L}) = (\Lambda s\mathbb{L}, d_0 + d_1)$ where

$$\begin{aligned} d_0(sx_1 \wedge \cdots \wedge sx_k) &= \sum_{1 \leq i \leq k} (-1)^{n_i+1} sx_1 \wedge \cdots \wedge sd_{\mathbb{L}}x_i \wedge \cdots \wedge sx_k \\ d_1(sx_1 \wedge \cdots \wedge sx_k) &= \sum_{1 \leq i < j \leq k} (-1)^{|x_i|+n_{ij}+1} s[x_i, x_j]sx_1 \wedge \cdots \wedge \hat{s}x_i \cdots \hat{s}x_j \cdots \wedge sx_k, \\ n_i &= |x_i| + |sx_1| + \cdots + |sx_{i-1}| \\ sx_1 \wedge \cdots \wedge sx_k &= (-1)^{n_{ij}} sx_i \wedge sx_j \wedge sx_1 \wedge \cdots \wedge \hat{s}x_i \cdots \hat{s}x_j \cdots \wedge sx_k \end{aligned}$$

- Page 313: $\mathcal{C}^*(\mathbb{L}) = \text{Hom}(\mathcal{C}_*(\mathbb{L}), \mathbb{Q})$;
- Page 314: $\mathcal{C}^*(\mathbb{L}) = (\Lambda V, d)$ and $V \cong \text{Hom}(s\mathbb{L}, \mathbb{Q})$;

Remark 1 When X is coformal (see Example 1, p. 315, [8]), then $d_{\mathbb{L}} = 0$ and $d = d_1$ (i.e., $d_0 = 0$). In particular

$$H^*(\Lambda V, d) = \text{Ext}_{U\mathbb{L}}(\mathbb{Q}, \mathbb{Q}).$$

By definition, we have $H_{\text{Lie}}^*(\mathbb{L}; \mathbb{Q}) := H^*(\mathcal{C}^*(\mathbb{L}))$. Since (see Lemma 1.5, [7]) the class of coformal spaces is in one to one correspondence with that of graded Lie algebras, thus the Hilali conjecture is reduced for such spaces to the following:

$$\dim \mathbb{L} \leq \dim H_{\text{Lie}}^*(\mathbb{L}; \mathbb{Q}).$$

In all the remainder of this section, X is assumed to be coformal.

Remark 2 Note that Deninger and Singhof proved in [5] the analogue of the Hilali conjecture for non graded Lie algebras.

When $V = V^{\text{odd}}$ (i.e., $\mathbb{L} = \mathbb{L}^{\text{even}}$), $\mathbb{L}$ may be considered as non graded, and one may conclude that the Hilali conjecture holds for any coformal elliptic space whose rational homotopy Lie algebra is concentrated in only even degrees.

Here above some informative examples. In the first one all degrees are even. In the second one, we put some generators of odd degree and observe how the cohomology becomes huge.

Example 3 Consider $\mathbb{L}$ of dimension 3, generated by $|e_1| = 4, |e_2| = 6, |e_3| = 10$ with $[e_1, e_2] = e_3, [e_1, e_3] = [e_2, e_3] = 0$. Thus e_1, e_2 generate $\mathbb{L}/[\mathbb{L}, \mathbb{L}]$ and e_3 generates $[\mathbb{L}, \mathbb{L}]$ with

$$0 \longrightarrow \Lambda^3 s\mathbb{L} \longrightarrow \Lambda^2 s\mathbb{L} \longrightarrow s\mathbb{L} \longrightarrow 0.$$

Hence, $\partial se_1 = \partial se_2 = \partial(se_1 \wedge se_2 \wedge se_3) = \partial(se_1 \wedge se_3) = \partial(se_2 \wedge se_3) = 0$ and $\dim H(\mathcal{C}_*(\mathbb{L})) = 5$.

Example 4 Consider $\mathbb{L}$ of dimension 3, generated by $|e_1| = 3, |e_2| = 5, |e_3| = 8$ with $[e_1, e_2] = e_3, [e_1, e_1] = [e_2, e_2] = [e_3, e_3] = [e_1, e_3] = [e_2, e_3] = 0$. Thus $\mathbb{L}/[\mathbb{L}, \mathbb{L}]$ is generated by e_1 and e_2 and $[\mathbb{L}, \mathbb{L}]$ by e_3 with $\partial se_1^k = \partial se_2^k = 0$ for any k . Therefore $\dim H(\mathcal{C}_*(\mathbb{L})) = \infty$.

It is worth noting that these are examples that led us to the next result by bounding the homology dimension of the complex

$$\cdots \longrightarrow \Lambda^3 s\mathbb{L} \longrightarrow \Lambda^2 s\mathbb{L} \longrightarrow s\mathbb{L} \longrightarrow 0.$$

Proposition A *The Hilali conjecture holds for any coformal space X whose rational homotopy Lie algebra $\mathbb{L}$ is of nilpotency 1 or 2.*

Proof Put $\mathbb{L}^{k+1} = [\mathbb{L}, \mathbb{L}^k]$ for $k \geq 1$ where $\mathbb{L}^1 = \mathbb{L}$, and let n denotes the nilpotency degree of $\mathbb{L}$ (i.e., $\mathbb{L}^n \neq 0$, $\mathbb{L}^{n+1} = 0$).

- If $n = 1$, then $\mathbb{L}$ is abelian (i.e., $[\mathbb{L}, \mathbb{L}] = 0$). Therefore $d = d_1 = 0$ and hence $\dim \operatorname{Ext}_{U\mathbb{L}}(\mathbb{Q}, \mathbb{Q}) \geq \dim H_1(\mathcal{C}_*(\mathbb{L})) = \dim \mathbb{L}/[\mathbb{L}, \mathbb{L}] = \dim \mathbb{L}$.
- If $n = 2$, we consider the short exact sequence of Lie algebras

$$0 \longrightarrow \mathbb{L}^2/\mathbb{L}^3 = \mathbb{L}^2 \longrightarrow \mathbb{L}/\mathbb{L}^3 = \mathbb{L} \longrightarrow \mathbb{L}/\mathbb{L}^2 \longrightarrow 0,$$

where $\mathbb{L}^2$ and $\mathbb{L}/\mathbb{L}^2$ are both abelian. Thus $\dim \mathbb{L} = \dim \mathbb{L}^2 + \dim \mathbb{L}/\mathbb{L}^2$ and we have the exact sequence of Hopf algebras

$$\mathbb{Q} \longrightarrow U\mathbb{L}^2 \longrightarrow U\mathbb{L} \longrightarrow U\mathbb{L} \longrightarrow U\mathbb{L}/\mathbb{L}^2 \longrightarrow \mathbb{Q}.$$

which leads to the Hochschild-Serre spectral sequence (p. 464, [8]) whose second term is

$$E_2^{p,q} \cong \operatorname{Ext}_{U\mathbb{L}/\mathbb{L}^2}^p(\mathbb{Q}, \operatorname{Ext}_{U\mathbb{L}^2}^q(\mathbb{Q}, \mathbb{Q}))$$

and which converges to $\operatorname{Ext}_{U\mathbb{L}}^{p+q}(\mathbb{Q}, \mathbb{Q})$. Dually (see iii), Lemma 34.3, [8]), we have a spectral sequence whose second term

$$E_2^{p,q} = \operatorname{Tor}_p^{U\mathbb{L}/\mathbb{L}^2}(\mathbb{Q}, \operatorname{Tor}_q^{U\mathbb{L}^2}(\mathbb{Q}, \mathbb{Q}))$$

and which converges to $\operatorname{Tor}_{p+q}^{U\mathbb{L}}(\mathbb{Q}, \mathbb{Q})$. The canonical action of $U\mathbb{L}/\mathbb{L}^2$ on $\operatorname{Tor}_q^{U\mathbb{L}^2}(\mathbb{Q}, \mathbb{Q}) = H(\Lambda(s\mathbb{L}^2)^\#, \bar{d}) = \Lambda(s\mathbb{L}^2)^\#$ is completely described in (Example 2, p. 467, [8]). In our case, the action is trivial, thus

$$\dim \operatorname{Tor}^{U\mathbb{L}^2}(\mathbb{Q}, \mathbb{Q}) \geq \dim \mathbb{L}.$$

And we are comparing $E^2 = \operatorname{Tor}^{U(\mathbb{L}/\mathbb{L}^2)}(\mathbb{Q}, \operatorname{Tor}^{U\mathbb{L}^2}(\mathbb{Q}, \mathbb{Q}))$ to $\operatorname{Ext}_{U\mathbb{L}}(\mathbb{Q}, \mathbb{Q}) = E^\infty$. Because the action involved is trivial, we get

$$\operatorname{Tor}^{U\mathbb{L}}(\mathbb{Q}, \mathbb{Q}) = \operatorname{Tor}^{U(\mathbb{L}/\mathbb{L}^2)}(\mathbb{Q}, \mathbb{Q}) \otimes \operatorname{Tor}^{U\mathbb{L}^2}(\mathbb{Q}, \mathbb{Q}).$$

This makes $\dim E^2 \geq \dim \mathbb{L}$.

However, there may be differentials. The worst possible case is that when E^2 is only a little bit larger than $\dim \mathbb{L}$, and a differential makes it slip below $\dim \mathbb{L}$. Certainly with collapse, the result follows. In the general case, we have

$$\dim E_2^{p,q} \geq \dim E^3 \geq \cdots \geq \dim E^\infty = \dim \operatorname{Tor}_{p,q}^{U\mathbb{L}}(\mathbb{Q}, \mathbb{Q})$$

with

$$\operatorname{Tor}_*^{U\mathbb{L}^2}(\mathbb{Q}, \mathbb{Q}) = (\wedge^2 s\mathbb{L})^*.$$

Thus

$$\dim E_2^{p,q} = \dim \operatorname{Tor}_{p,q}^{U(\mathbb{L}/\mathbb{L}^2)}(\mathbb{Q}, (\wedge^2 s\mathbb{L}))^*$$

But $A := \mathbb{L}/\mathbb{L}^2$ is abelian, then $UA = \wedge A$. The resolution

$$0 \rightarrow \wedge^{\dim A} sA \otimes \wedge A \rightarrow \cdots \rightarrow \wedge^2 sA \otimes \wedge A \rightarrow \wedge sA \otimes \wedge A \rightarrow \wedge A \rightarrow \mathbb{Q}$$

enable us to compute $E_2^{p,q}$.

□

From the proof of Proposition A, a natural question arises: In the general case, are there at the E^2 -level enough permanent cycles to state that

$$\dim \operatorname{Tor}^{U\mathbb{L}^2}(\mathbb{Q}, \mathbb{Q}) \geq \dim \mathbb{L}?$$

A positive answer and an induction on the nilpotency degree will resolve completely the coformal case.

Let us recall that the *global dimension* of a graded algebra $A = (A_k)_{k \geq 0}$, with $A_0 = \mathbb{Q}$, denoted here $\operatorname{gl dim} A$ is defined to be the greatest integer k (or ∞) such that $\operatorname{Ext}_A^k(\mathbb{Q}, -) \neq 0$. As $\dim H^*(\mathcal{C}_*(\mathbb{L})) < \infty \Leftrightarrow \operatorname{gl dim} \mathbb{L} < \infty$, then the Hilali conjecture can be reformulated otherwise:

Hilali Conjecture (Lie algebra version) *If $\mathbb{L}$ is a graded Lie algebra whose dimension and global dimension are both finite, then*

$$\dim \mathbb{L} \leq \dim \operatorname{Ext}_{U\mathbb{L}}(\mathbb{Q}, \mathbb{Q}). \quad (1)$$

Remark 5 Put $n = \dim \mathbb{L}$ and $m = \operatorname{gl dim} \mathbb{L}$, we know from [9] that

$$\dim \operatorname{Ext}_{U\mathbb{L}}^k(\mathbb{Q}, \mathbb{Q}) \leq \binom{m}{k},$$

thus

$$\dim \operatorname{Ext}_{U\mathbb{L}}(\mathbb{Q}, \mathbb{Q}) \leq 2^m.$$

Therefore, a positive answer to (1) induces the inequality

$$n \leq 2^m \text{ i.e., } \ln n \leq m \ln 2.$$

The existence of a Lie algebra with a finite dimension and a finite global dimension such that

$$\ln(\dim \mathbb{L}) > (\text{gl dim } \mathbb{L}) \cdot \ln 2$$

will be a counterexample to the Hilali conjecture.

3 Hilali conjecture versus Halperin conjecture

In addition to comparing two classical rational invariants, the Hilali conjecture may be viewed as an intermediary bridge to prove the toral rank conjecture and vice-versa. This connection is possible thanks to the following results:

Let χ_c and χ_π be the cohomological and homotopic Euler-Poincaré characteristics of X , respectively defined by:

$$\begin{aligned}\chi_c &:= \sum_{k \geq 0} (-1)^k \dim H^k(\Lambda V, d); \\ \chi_\pi &:= \sum_{k \geq 0} (-1)^k \dim V^k.\end{aligned}$$

Lemma 1 (Halperin [10]) *If $(\Lambda V, d)$ is a 1-connected and elliptic model, then*

$$\chi_c \geq 0 \quad \text{and} \quad \chi_\pi \leq 0.$$

Moreover,

$$\chi_c > 0 \iff \chi_\pi = 0 \iff (\Lambda V, d) \text{ is pure.}$$

Lemma 2 (Allday-Halperin [2]) *If X is a 1-connected and finite CW-complex then*

$$\text{rk}_0(X) \leq -\chi_\pi.$$

Moreover, if the equality holds then X is pure.

The preceding lemmas combined with the definitions of χ_c and χ_π allow us to set:

$$\begin{aligned}\dim V^{\text{even}} &= n \\ \dim V^{\text{odd}} &= n + p \quad \text{with } p \geq 0 \\ \dim V &= 2n + p \\ \chi_\pi &= -p \\ \text{rk}_0(X) &= p - \varepsilon \quad \text{where } \varepsilon \geq 0\end{aligned}$$

The Hilali conjecture becomes

$$\dim H^*(\Lambda V, d) \geq 2n + p.$$

That of Halperin is now equivalent to

$$\dim H^*(\Lambda V, d) \geq 2^{p-\varepsilon}.$$

Remark 6 The promised connection is written as follows:

$$\begin{aligned} \text{Hilali conjecture} + (2n \geq 2^{p-\varepsilon} - p) &\implies \text{Toral rank conjecture} \\ \text{Toral rank conjecture} + (2n \leq 2^{p-\varepsilon} - p) &\implies \text{Hilali conjecture} \end{aligned}$$

By using Remark 6, we have

Proposition B *The toral rank conjecture holds for any manifold of dimension less than 16, and whose rational toral rank is equal to 4.*

Assume X satisfying the Hilali conjecture, then the toral rank conjecture holds when $2n + p \geq 16$. On the other hands, by Lemma 2, necessarily $p \geq 4$, with X pure if $p = 4$ (pure spaces verify the Halperin conjecture). Then one has to consider only the case when $p \geq 5$. A direct consequence of Theorem A [13], is that the toral rank conjecture holds for any manifold of dimension less than 10. In [16], the authors proved the Hilali conjecture for all manifolds of dimension less than 16. By the way, they classify their rational homotopy types, the only case when $p \geq 5$ is when $\dim X = 15$ and $\Lambda V = \Lambda\{x, y, z, t, w\}$ with $|x| = |y| = |z| = |t| = |w| = 3$, which is hyperelliptic following the proof of Theorem 1.2 [16]. By noting that hyperelliptic spaces verify the toral rank conjecture thanks to Theorem B [13], this finishes the proof.

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