Simplicial monoid actions and associated monoid constructions

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Abstract For X a pointed simplicial set with a simplicial monoid action, we construct a simplicial monoid and give a sufficient condition for its classifying space to be the homotopy cofiber of the inclusion of X into its reduced Borel construction. The known formulae for the respective classifying spaces of the four classical constructions of James, Milnor, Carlsson and Wu are special cases of this result. Thus, we unify categorially the four above-mentioned constructions. We also give the classical constructions and an example as the applications of the main result.

1 Introduction

James' [5] construction on a pointed simplicial set X is the simplicial monoid obtained by applying the reduced free monoid functor in each dimension. Similarly, Milnor's [7] construction is the simplicial group obtained by applying the reduced free group functor in each dimension. The geometric realization of both these constructions is the loop-suspension space $\Omega\Sigma X$.

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The monoid or group structure on $\Omega\Sigma X$ facilitates the application of algebraic techniques to study its homotopy and homology. For example, the word length filtration on the free monoid is used to prove James Splitting Theorem (see the original article [5] or pages 143–144 of [10]), from which the homology of the loop space of spheres can be computed. Milnor's construction is fibrant, allowing combinatorial computation of its homotopy groups, and is used with the EHP spectral sequence to investigate the homotopy groups of spheres.

Given an action of a discrete group on a pointed simplicial set X , Carlsson [1] constructed a simplicial group whose classifying space is the homotopy cofiber of the inclusion of X into its reduced Borel construction (see Corollary 4.2). As a special case, Milnor's construction can be recovered as Carlsson's construction on trivial actions of the additive group of integers.

Carlsson's construction and its classifying space are of interest to homotopy theory. As Carlsson [1] noted, certain quotients of a Thom complex by its lowest cell are the geometric realization of his construction. This includes all stunted projective spaces. An application of Carlsson's construction to homotopy theory is to give a representation of the homotopy groups of simply connected suspension spaces to certain combinatorial groups as centers [12].

Wu [11] computed the classifying space of the tensor product of a pointed simplicial set by a simplicial monoid (see Corollary 4.3). The tensor product is related to the earlier mentioned constructions. For instance, James' construction is the tensor product of X by the discrete additive monoid of nonnegative integers. Taking the tensor product with a discrete group recovers Carlsson's construction on trivial group actions.

Each of the above four constructions has a common categorical description. Namely, given a pointed simplicial set X acted upon by a simplicial monoid M (see Section 6 in [2] or Section 3.2 in [4]), each construction is the universal simplicial monoid on the associated reduced simplicial action category. In detail, the action category $X//M$ associated to a right M -set X has an object for each element of X and a morphism $x \rightarrow y$ for each monoid element $m \in M$ such that $y = xm$. If X is a pointed right M -set with its basepoint fixed by the M -action, then define its *reduced action category* as the cokernel of $*//M \rightarrow X//M$. The *universal monoid functor* $U: \mathbf{Cat} \rightarrow \mathbf{Mon}$ is left adjoint to the functor $\mathbf{Mon} \hookrightarrow \mathbf{Cat}$ that views a monoid as a small category with one object (see Sect. 2). Our observation is that all four constructions above are obtained by applying U to the reduced action category in each dimension.

To this end, we define $J^M[X]$ as the monoid $U(\text{coker}(*//M \rightarrow X//M))$. Concretely, writing $J[\bullet]$ for the reduced free monoid functor (see Sect. 2), this monoid $J^M[X]$ can be given as

$$\frac{J[X \wedge M]}{\langle \forall x \in X \forall m, k \in M (x \wedge m) \cdot (xm \wedge k) \sim (x \wedge mk) \rangle}. \quad (1.1)$$

In the simplicial case where X is a pointed right M -simplicial set, apply the above definition dimensionwise to obtain a simplicial monoid $J^M[X]$.

Under a certain sufficient condition, we are able to compute the classifying space of $J^M[X]$. For a simplicial monoid K , its *classifying space* BK is the diagonal of the bisimplicial set obtained by taking the nerve of M in each dimension. The following

theorem generalizes the results of Carlsson [1] and Wu [5]. The constructions of Milnor and James are further special cases so the fact that both their classifying spaces are ΣX is also contained in this theorem. Given $x \in X_n$, let $\text{Stab}(x)$ be the submonoid of M_n that fixes x . Let $M^\times$ denote the group of invertible elements of M . Let $X \rtimes_M EM$ denote the reduced Borel construction (see Sect. 3). A *section* of the composite $M_n^\times \hookrightarrow M_n \rightarrow M_n/\text{Stab}(x)$ is a set map $j : M_n/\text{Stab}(x) \rightarrow M_n^\times$ such that the composite $M_n/\text{Stab}(x) \xrightarrow{j} M_n^\times \hookrightarrow M_n \rightarrow M_n/\text{Stab}(x)$ is the identity map of $M_n/\text{Stab}(x)$.

Theorem 1.1 *Let X be a pointed right M -simplicial set where M is a simplicial monoid. Suppose that the basepoint of X is fixed by the M -action. If a section of the composite $M_n^\times \hookrightarrow M_n \rightarrow M_n/\text{Stab}(x)$ exists for each $x \in X_n$, then there is a cofiber sequence $X \rightarrow X \rtimes_M EM \rightarrow BJ^M[X]$.*

This sufficient condition is equivalent to saying that any two n -simplices in the same orbit differ by the action of an invertible monoid element.

We remark that this theorem is a strict generalization of the results of Carlsson and Wu. The application and examples of the main theorem are given in Sect. 4. Certain nontrivial simplicial monoid actions that satisfy the sufficient condition are given in Example 4.1.

We end this introduction with a Venn diagram to illustrate the relationship among constructions, classical and those we have introduced.

2 An abstract cofiber sequence

The following notation shall be used throughout this article. Let $J(S)$ denote the free monoid on a set S . In the pointed setting, the *reduced free monoid* on a pointed set S , denoted by $J[S]$, is defined as the cokernel of $J(*) \rightarrow J(S)$.

Similarly, let $F(S)$ denote the free group on a set S . The *reduced free group* on a pointed set S , denoted by $F[S]$, is defined as the cokernel of $F(*) \rightarrow F(S)$.

If X is a pointed simplicial set, then dimensionwise application of the above constructions gives a simplicial monoid $J[X]$ and a simplicial group $F[X]$. These are known as James' construction and Milnor's construction respectively.

If C is a small category, then we let $\text{Ob}(C)$ denote its set of objects and $\text{Mor}(C)$ denote its set of morphisms. If f is a morphism, then we write $t(f)$ and $s(f)$ for its source and target respectively. Large categories are denoted in **sans serif**. A category in ordinary font is understood as being small.

Now let us describe the abstract setting of our problem. The functor $\text{Mon} \hookrightarrow \text{Cat}$ that views a monoid as a small category with one object has a left adjoint. Recall from Sect. 1 that we denoted this left adjoint by $U: \text{Cat} \rightarrow \text{Mon}$. This functor carries a small category C to its *universal monoid* $U(C)$. This monoid $U(C)$ can be constructed as

$$\frac{J(\text{Mor}(C))}{\langle \forall f, g \in \text{Mor}(C) \ (t(f) = s(g) \implies f \cdot g \sim g \circ f) \rangle}$$

The unit map η_C of this adjunction fits into a cokernel sequence $\text{Ob}(C) \rightarrow C \xrightarrow{\eta_C} U(C)$, where $\text{Ob}(C)$ is viewed as a discrete subcategory of C . (See Section 4.1 in [4] for a reference on the universal monoid).

The nerve NC of a small category C is a simplicial set with an n -simplex for each sequence $x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} x_n$ of n composable morphisms in C . Taking the nerve of the above cokernel sequence gives a constant composite map $N\text{Ob}(C) \rightarrow NC \xrightarrow{N\eta_C} NU(C)$.

Since $\text{Ob}(C)$ is a discrete category, viewing $\text{Ob}(C)$ as a simplicial set constant in each dimension, there is an isomorphism $\text{Ob}(C) \rightarrow N\text{Ob}(C)$ of simplicial sets. Let ρ_C denote the composite $\text{Ob}(C) \xrightarrow{\cong} N\text{Ob}(C) \rightarrow NC$. Concretely, this map ρ_C sends an n -simplex x of $\text{Ob}(C)$ to the n -simplex $x \xrightarrow{\text{id}_x} x \xrightarrow{\text{id}_x} \dots \xrightarrow{\text{id}_x} x$ of NC . From the above, the following composite is a constant map:

$$\text{Ob}(C) \xrightarrow{\rho_C} NC \xrightarrow{N\eta_C} NU(C) \quad (2.1)$$

In this section, we will prove the following sufficient condition for (2.1) to be a cofiber sequence. A category is *totally disconnected* if each component has only one object.

Proposition 2.1 *If C is categorially equivalent to a totally disconnected category, then $\text{Ob}(C) \xrightarrow{\rho_C} NC \xrightarrow{N\eta_C} NU(C)$ is a cofiber sequence.*

Let C be a small category. Suppose that C is categorially equivalent to a category with one object. Then, for any object x of C , there is a categorial equivalence between C and the full subcategory $\text{End}_C(x)$ whose only object is x . Thus, there exists a functor $\pi: C \rightarrow \text{End}_C(x)$ that, together with the inclusion $\iota: \text{End}_C(x) \rightarrow C$, witnesses this categorial equivalence. That is to say, there exists a natural isomorphism from the composite $\iota \circ \pi$ to the identity functor id_C .

Lemma 2.2 Suppose that C is categorially equivalent to a category with one object. Let x be an object of C , let $\pi : C \rightarrow \text{End}_C(x)$ be a functor and let $\iota : \text{End}_C(x) \rightarrow C$ be the inclusion. Then each natural isomorphism from $\iota \circ \pi$ to id_C induces a monoid isomorphism $U(C) \rightarrow \text{End}_C(x) * F[\text{Ob}(C)]$.

Here x is taken as the basepoint of $\text{Ob}(C)$.

Proof Suppose that θ is a natural isomorphism from $\iota \circ \pi$ to id_C . Applied to each object y of C , we obtain an isomorphism $\theta_y : x \rightarrow y$. In particular, θ_x is the identity morphism of x .

We wish to describe a functor $\eta : C \rightarrow \text{End}_C(x) * F[\text{Ob}(C)]$. To do so, observe that if $d : y \rightarrow z$ is a morphism of C , then the composite $\theta_z^{-1} \circ d \circ \theta_y$ is an endomorphism of x . This allows us to define η as carrying a morphism $d : y \rightarrow z$ of C to a word $y^{-1} \cdot (\theta_z^{-1} \circ d \circ \theta_y) \cdot z$. A straightforward check shows that η is initial among functors from C to categories with one object, which completes the proof. $\square$

The classifying space of topological monoids commutes, up to homotopy, with the wedge sum (see [3]). The *wedge sum*, written $\bigvee$, refers to the coproduct of pointed objects. Monoids are uniquely pointed, hence the wedge sum of pointed monoids coincides with the coproduct of monoids: this is usually called the *free product* $*$. The particular result that we will use is that, if $\{M_i\}$ is a family of discrete monoids, then the inclusions $M_j \rightarrow *M_i$ induce a weak homotopy equivalence of simplicial sets

$$\bigvee N M_i \rightarrow N(*M_i) \quad (2.2)$$

Lemma 2.3 If C is categorially equivalent to a category with one object, then $\text{Ob}(C) \xrightarrow{\rho_C} NC \xrightarrow{N\eta_C} NU(C)$ is a cofiber sequence.

Proof We may assume without loss of generality that C is categorially equivalent to $\text{End}_C(x)$ where x is some object of C . Let $\iota : \text{End}_C(x) \rightarrow C$ be the inclusion. As in the setup of Lemma 2.2, there exists a functor $\pi : C \rightarrow \text{End}_C(x)$ and a natural isomorphism θ from $\iota \circ \pi$ to id_C . Let $\tilde{\theta} : U(C) \rightarrow \text{End}_C(x) * F[\text{Ob}(C)]$ be the induced monoid isomorphism given by this Lemma.

The composite $\text{Ob}(C) \xrightarrow{\rho_C} NC \xrightarrow{N\pi} N\text{End}_C(x)$ is a constant map. Hence $\text{Ob}(C) \xrightarrow{N\pi \circ \rho_C} N\text{End}_C(x) \hookrightarrow N\text{End}_C(x) \vee \Sigma \text{Ob}(C)$ is a cofiber sequence. This cofiber sequence fits as the second row of the following commutative diagram:

$$\begin{array}{ccccc} \text{Ob}(C) & \xrightarrow{\rho_C} & NC & \xrightarrow{N\eta_C} & NU(C) \\ \parallel & & \downarrow N\pi & & \downarrow \\ \text{Ob}(C) & \xrightarrow{N\pi \circ \rho_C} & N\text{End}_C(x) & \hookrightarrow & N\text{End}_C(x) \vee \Sigma \text{Ob}(C) \end{array}$$

We wish to show that the first row of is a cofiber sequence.

In order to do so, it suffices to show that each vertical map is a weak homotopy equivalence. The first vertical map is an equality. The second vertical map $N\pi$ is the

nerve of a categorial equivalence π and hence is a homotopy equivalence. We need to explain the third map. The monoid isomorphism $\tilde{\theta}$ induces a homotopy equivalence $N\tilde{\theta} : NU(C) \rightarrow N(\text{End}_C(x) * F[\text{Ob}(C)])$. The nerve $NF[\text{Ob}(C)]$ of Milnor's construction is weakly homotopy equivalent to $\Sigma \text{Ob}(C)$. Recall from (2.2) that the nerve of monoids commute, up to weak homotopy equivalence, with the wedge sum. Hence we have a weak homotopy equivalence $N(\text{End}_C(x) * F[\text{Ob}(C)]) \rightarrow N\text{End}_C(x) \vee NF[\text{Ob}(C)]$. The third map is the composite weak homotopy equivalence

$$\begin{aligned} NU(C) &\xrightarrow{N\tilde{\theta}} N(\text{End}_C(x) * F[\text{Ob}(C)]) \\ &\rightarrow N\text{End}_C(x) \vee NF[\text{Ob}(C)] \rightarrow N\text{End}_C(x) \vee \Sigma \text{Ob}(C) \end{aligned}$$

We have thus shown that each vertical map is a weak homotopy equivalence. Therefore, the second row being a cofiber sequence implies that the first row is a cofiber sequence, as required. $\square$

Proof of Proposition 2.1 Decompose C as a coproduct $\coprod_i A_i$ of its connected components. Since C is categorially equivalent to a totally disconnected category, each A_i is categorially equivalent to a category with one object. Hence, by Lemma 2.3, each A_i satisfies a cofiber sequence $\text{Ob}(A_i) \xrightarrow{\rho_{A_i}} N(A_i) \xrightarrow{N\eta_{A_i}} NU(A_i)$. Taking the coproduct gives a cofiber sequence $\coprod \text{Ob}(A_i) \xrightarrow{\coprod \rho_{A_i}} \coprod NA_i \xrightarrow{\coprod N\eta_{A_i}} NU(A_i)$. This cofiber sequence forms the first row of a commutative diagram

$$\begin{array}{ccccc} \coprod \text{Ob}(A_i) & \xrightarrow{\coprod \rho_{A_i}} & \coprod NA_i & \xrightarrow{\coprod N\eta_{A_i}} & \coprod NU(A_i) \\ \downarrow & & \downarrow & & \downarrow \\ \text{Ob}(C) & \xrightarrow{\rho_C} & NC & \xrightarrow{N\eta_C} & NU(C) \end{array}$$

Each of the functors Ob , U and N commute with coproducts, hence all three vertical maps are isomorphisms. The first row being a cofiber sequence implies that the second row is a cofiber sequence, as required. $\square$

3 Proof of Theorem 1.1

In this section, we deduce Theorem 1.1 by applying Proposition 2.1 to action categories.

Let X be a right M -set where M is a monoid. Its *action category* $X//M$ has an object for each element of X and a morphism $x \rightarrow y$ for each monoid element $m \in M$ such that $y = xm$. We denote this morphism by $m : x \rightarrow xm$.

Lemma 3.1 *Let x be an element of an M -set X . If m is invertible in M , then $m : x \rightarrow xm$ is an isomorphism.*

Proof Suppose that k is the inverse of m in M . Then $k : xm \rightarrow xmk = x$ is the inverse of the morphism $m : x \rightarrow xm$. $\square$

Lemma 3.2 *Let X be a right M -set where M is a monoid. If, for each $x \in X$, there exists a section of the composite $M^\times \hookrightarrow M \rightarrow M/\text{Stab}(x)$, then $X//M$ is equivalent to a totally disconnected category.*

Proof To show that $X//M$ is equivalent to a totally disconnected category, it suffices to show that any two elements in the same orbit are connected by an isomorphism.

First we show that, given x and y in X such that $y = xm$ for some $m \in M$, there is an isomorphism connecting x and y . By assumption, there is a section $s : M/\text{Stab}(x) \rightarrow M^\times$ of the composite $M^\times \hookrightarrow M \rightarrow M/\text{Stab}(x)$. Then $x \cdot s(\text{Stab}(x)) = xm = y$. Since the monoid element $s(\text{Stab}(x))$ is invertible, Lemma 3.1 implies that the morphism $s(\text{Stab}(x)) : x \rightarrow xm$ is an isomorphism. Therefore, we have shown that any two elements connected by a single morphism is in fact connected by an isomorphism.

Let x' and y' be two elements in the same orbit. Then there are finitely many elements $x' = x_0, x_1, \dots, x_t = y'$ such that each pair (x_i, x_{i+1}) is connected by a morphism. We have shown in the above paragraph that x_i and x_{i+1} are connected by an isomorphism. Hence x' and y' are connected by an isomorphism. This completes the proof that $X//M$ is equivalent to a totally disconnected category. $\square$

Given a pointed right M -set X whose basepoint is fixed by the M -action, recall from Sect. 1 that we define its *reduced action category* as the cokernel of $*//M \rightarrow X//M$. Let $\widehat{X//M}$ denote the reduced action category of X . Dimensionwise application of this definition gives the *reduced simplicial action category* in the simplicial case. Recall from Sect. 1 that, given a pointed right M -simplicial set X whose basepoint is fixed by the action, we defined the simplicial monoid $J^M[X]$ as the universal simplicial monoid $U(\widehat{X//M})$.

Let M be a simplicial monoid. Let EM denote a simplicial set with a free M -action. Suppose X is a simplicial M -set. The diagonal action on $X \times EM$ is free and its orbit space $(X \times EM)/M$ is thus known as the *Borel construction* of X , denoted by $X \times_M EM$. In the case where X is a pointed simplicial set with an M -action that fixes its basepoint, we define its *reduced Borel construction*, denoted by $X \rtimes_M EM$, as the homotopy cofiber of $* \times_M EM \rightarrow X \times_M EM$.

Now we prove Theorem 1.1.

Proof of Theorem 1.1 We may assume without loss of generality that, if M is a monoid and if X is a pointed M -set whose basepoint is fixed by the action such that, for each $x \in X$, there exists a section of the composite $M^\times \hookrightarrow M \rightarrow M/\text{Stab}(x)$, then there is a cofiber sequence

$$X \rightarrow X \rtimes_M EM \rightarrow NJ^M[X] \quad (3.1)$$

The general result follows from the fact that a dimensionwise weak homotopy equivalence of bisimplicial sets induces a weak homotopy equivalence of the diagonal simplicial sets.

By Lemma 3.2, the action category $X//M$ is equivalent to a totally disconnected category. Since $\text{Ob}(X//M) = X$, hence Corollary 2.1 gives a cofiber sequence

$$X \rightarrow N(X//M) \rightarrow NU(X//M). \quad (3.2)$$

Since the basepoint of X is fixed by the action, its basepoint $*$ is a M -subset of X . Its action category $*//M$ has only one object, there is an isomorphism $N(*//M) \rightarrow NU(*//M)$. This gives another cofiber sequence

$$* \rightarrow N(*//M) \rightarrow NU(*//M) \quad (3.3)$$

Taking the cokernel of cofiber sequences (3.3) and (3.2) implies that the second row of the following commutative diagram is a cofiber sequence:

$$\begin{array}{ccccc} X & \longrightarrow & X \rtimes_M EM & \longrightarrow & J^M[X] \\ \parallel & & \downarrow & & \downarrow \\ X & \longrightarrow & \frac{N(X//M)}{N(*//M)} & \longrightarrow & \frac{NU(X//M)}{NU(*//M)} \end{array}$$

The first row is just (3.1). To show that (3.1) is a cofiber sequence, all we are left to do is to show that the second and third vertical maps are weak homotopy equivalences.

The Borel construction $X \times_M EM$ is weakly homotopy equivalent to the nerve $N(X//M)$ of the action category. Hence, the reduced Borel construction $X \rtimes_M EM$ is weakly homotopy equivalent to the homotopy cofiber of $N(*//M) \rightarrow N(X//M)$. This proves that the second vertical map is a homotopy equivalence.

The reduced action category $\widehat{X//M}$ is defined as satisfying the cokernel sequence $*//M \rightarrow X//M \rightarrow \widehat{X//M}$. This cokernel sequence splits after taking the universal monoid functor U . Namely, there is a free product decomposition $U(X//M) \cong U(*//M) * U(\widehat{X//M})$. Since the nerve commutes with the wedge sum of monoids up to homotopy (2.2), there is a weak homotopy equivalence $NU(*//M) \vee NU(\widehat{X//M}) \rightarrow NU(X//M)$. Hence the map $NU(\widehat{X//M}) \rightarrow NU(X//M)/NU(*//M)$ is a weak homotopy equivalence. We defined $J^M[X]$ as $U(\widehat{X//M})$, which proves that the third vertical map is a weak homotopy equivalence.

Therefore (3.1) is a cofiber sequence, which completes the proof. $\square$

4 Classical constructions as examples

In this section, we demonstrate how Theorem 1.1 unifies the previously known simplicial monoid and group constructions of Carlsson, Wu, Milnor and James. Indeed, these classical constructions are special cases of the following example. This example gives a family of pointed simplicial monoid actions that satisfy Theorem 1.1. This family contains all simplicial group actions and all trivial simplicial monoid actions, yet contains certain nontrivial simplicial monoid actions, hence showing that Theorem 1.1 is a strict generalization of the previously known constructions.

Example 4.1 Begin with the action of a simplicial group G on a pointed simplicial set X . Consider the product $M \times G$ of a simplicial monoid M with G . Extend the action by letting $(m, 1)$ act trivially for all $m \in M_n$. If x and y are in the same orbit, then $y = x(m, g)$ for some m and g . Since $y = x(m, g) = x(m, 1)(1, g) = x(1, g)$, the two n -simplices x and y differ by an invertible monoid element $(1, g)$. Hence this $(M \times G)$ -action satisfies Theorem 1.1. Thus $X \rightarrow X \rtimes_{M \times G} E(M \times G) \rightarrow BJ^{M \times G}[X]$ is a cofiber sequence.

Set $G = 1$ to obtain a trivial simplicial monoid action; set $M = 1$ to obtain a simplicial group action. However, in general, this $(M \times G)$ -action is neither a simplicial group action nor a trivial simplicial monoid action.

If G is a simplicial group, then each n -simplex of $J^G[X]$ is in fact invertible, hence $J^G[X]$ is a simplicial group. Given a pointed simplicial G -set X where G is a discrete simplicial group, Carlsson defined a simplicial group, which is naturally isomorphic to $J^G[X]$. The main result of his article [1] is the determination of the homotopy type of $J^G[X]$.

Corollary 4.2 (Carlsson) *If X be a pointed simplicial G -set where G is a discrete simplicial group, then the underlying simplicial set of $J^G[X]$ is weakly homotopy equivalent to $\Omega((X \rtimes_G EG)/X)$.*

Proof From Example 4.1, we have $BJ^G[X] \simeq (X \rtimes_G EG)/X$. Since $J^G[X]$ is a simplicial group, Carlsson's result follows from taking the loop space. $\square$

In [11], Wu computed the classifying space of the tensor product of a simplicial monoid by a pointed simplicial set. The category **sMon** of simplicial monoids is simplicially enriched over the category **sSet**_{*} of pointed simplicial sets. The *tensor product* of a simplicial monoid M by a pointed simplicial set X is the simplicial monoid $X \otimes M$ that satisfies a family of isomorphisms natural in simplicial monoids K :

$$\mathbf{sMon}(X \otimes M, K) \cong \mathbf{sSet}_*(X, \mathbf{sMon}(M, K)) \quad (4.1)$$

Concretely, writing $(M_n)_x$ for an isomorphic copy of M_n of each $x \in X_n$, the monoid $(X \otimes M)_n$ is the cokernel of $(M_n)_* \rightarrow *_{x \in X_n} (M_n)_x$. For an exposition of the tensor product, see [9].

Corollary 4.3 (Wu) *If M is a simplicial monoid and X is a pointed simplicial set, then $B(X \otimes M) \simeq X \wedge BM$.*

Proof The tensor product $X \otimes M$ is naturally isomorphic to $J^M[X]$ where M is taken to act trivially on X . By Example 4.1, its classifying space $B(X \otimes M) \simeq BJ^M[X]$ is the homotopy cofiber of $X \rightarrow X \rtimes_M EM$. Since the action is trivial, the reduced Borel construction $X \rtimes_M EM$ simplifies to $(X \times_M EM)/(* \times_M EM) \simeq (X \times BM)/(* \times BM)$. Hence

$$B(X \otimes M) \simeq \frac{(X \times BM)/(* \times BM)}{X} \simeq \frac{X \times BM}{X \vee BM} \simeq X \wedge BM.$$

$\square$

The constructions of James and Milnor are classical instances of the tensor product. James' construction $J[X]$ is the reduced free simplicial monoid on a pointed simplicial set X (see [5]). There is a natural isomorphism $J[X] \cong X \otimes \mathbb{N}$ where $\mathbb{N}$ is the discrete monoid of nonnegative integers under addition. Similarly, Milnor's construction $F[X]$ is the reduced free simplicial group on a pointed simplicial set X (see [7]). There is a natural isomorphism $F[X] \cong X \otimes \mathbb{Z}$ where $\mathbb{Z}$ is the discrete group of integers under addition.

Both $B\mathbb{N}$ and $B\mathbb{Z}$ are weakly homotopy equivalent to the circle S^1 . Hence Corollary 4.3 implies that $BJ[X]$ and $BF[X]$ are both weakly homotopy equivalent to $X \wedge S^1 \simeq \Sigma X$.

More recently, the simplicial group $S^1 \otimes G$ where G is a discrete simplicial group has been used to study the suspension of the Eilenberg-Mac Lane spaces [8, 9, 12]. This is because $S^1 \otimes G$ is weakly homotopy equivalent to $\Omega \Sigma K(G, 1)$. This can also be deduced from Corollary 4.3.

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