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(Non-)Koszulness of operads for n-ary algebras, galgalim and other curiosities

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Abstract We investigate operads for various n-ary algebras. As a useful tool we introduce galgalim—analogs of the Lie-hedra for n-ary algebras. We then focus on algebras with one anti-associative operation. We describe the relevant part of the deformation cohomology for this type of algebra using the minimal model for the anti-associative operad. We also discuss free partially associative algebras and formulate some open problems.

Keywords Operad · Koszulness · Partial and total associativity · Minimal model

Mathematics Subject Classfication 8D50 · 55P48

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1 Introduction

We study Koszulness of operads for various n-ary algebras, i.e. algebras with an n-multilinear operation satisfying a specific version of associativity. In Sect. 2 we recall basic notions of quadratic duality and Koszulness for quadratic operads and prove a couple of related statements, emphasizing specific features of the non-binary case.

In Sect. 3 we introduce four families of operads—operads for totally resp. partially associative n-algebras, and the operadic suspensions of these operads. In Sect. 4 we define galgalim that, in some sense, generalize the classical Stasheff's associahedra to the realms of partially associative n-algebras. We will see that galgalim encode some properties of free partially associative algebras.

In Sect. 5 we formulate and prove results concerning Koszulness of operads for n-ary algebras. They are summed up in the table of Fig. 3. We will then, in Sect. 6, focus on the particular case of algebras with one anti-associative operation, i.e. an operation $a, b \mapsto ab$ satisfying $a(bc)+(ab)c = 0$ for each a, b and c. The corresponding operad $\widetilde{\mathcal{A}ss}$ is not Koszul, so the deformation cohomology differs from the "standard" one. We describe the relevant part of the deformation cohomology based on the minimal model of $\widetilde{\mathcal{A}ss}$.

In Sect. 7 we give a description of the free partially associative algebras which, in the Koszul cases, coincides with the one given in [10]. Section 8 formulates open problems.

Conventions. The basic reference for operads, quadratic duality and Koszulness is the classical [9] or more recent [14], our notation and terminology will also be based on [20] and [24]. We will work with operads in the category of chain complexes over a field $\mathbf{k}$ of characteristic zero though, in the light of [7], most if not all results remain valid over the ring of integers.

2 Duality for quadratic operads revisited

In this section we recall necessary notions and results related to Koszul duality of quadratic operads generated by operations of the same arity, emphasizing specific features of the non-binary case. Most of the ideas mentioned here are implicitly present in the classical sources [8,9], in explicit form they appeared in Chapter 7 of [14].

Fix a natural $n \geq 2$ and assume $E = \{E(a)\}_{a\geq 2}$ is a Σ-module such that $E(a) = 0$ if $a \neq n$ and that, moreover, $E(n)$ is finite-dimensional. We will study operads $\mathcal{P}$ of the form $\mathcal{P} = \Gamma(E)/(R)$, where $\Gamma(E)$ is the free operad generated by E and (R) the operadic ideal generated by a subspace $R \subset \Gamma(E)(2n-1)$. Operads of this type are particular examples of *quadratic* operads in the sense of [14, Chapter 7], they are *binary quadratic* if $n = 2$.[1] Let $E^\vee = \{E^\vee(a)\}_{a\geq 2}$ be a Σ-module with

$$E^\vee(a) := \begin{cases} \mathrm{sgn}_a \otimes \uparrow^{a-2} E(a)^\#, & \text{if } a = n \text{ and} \\ 0, & \text{otherwise} \end{cases}$$

where $\uparrow^{a-2}$ denotes the suspension iterated $a-2$ times, sgn_a the signum representation of the symmetric group Σ_a, and # the linear dual of a graded vector space with the induced representation. Recall that $V^\# := Hom(V, \mathbf{k})$, so $(V^\#)_d = (V_{-d})^\#$. There is a non-degenerate, Σ_{2n-1}-equivariant pairing

$$\langle -|-\rangle : \Gamma(E^\vee)(2n-1) \otimes \Gamma(E)(2n-1) \to \mathrm{sgn}_{2n-1} \tag{1}$$

determined by requiring that

$$\langle \uparrow^{n-2} e' \circ_i \uparrow^{n-2} f' \mid e'' \circ_j f'' \rangle := \delta_{ij}(-1)^{(i+1)(n+1)} e'(e'') f'(f'') \in \mathbf{k} \cong \mathrm{sgn}_{2n-1},$$

for arbitrary $e', f' \in E(n)^\#$, $e'', f'' \in E(n)$. The following definition is a particular case of the Koszul dual of a quadratic operad $\mathcal{P}$ as defined in [14, Chapter 7] when $\mathcal{P}$ is generated by operations of arity n.

Definition 1 The *Koszul* or *quadratic* dual of a quadratic operad $\mathcal{P} = \Gamma(E)/(R)$ as above is the quotient

$$\mathcal{P}^! := \Gamma(E^\vee)/(R^\perp),$$

where $R^\perp \subset \Gamma(E^\vee)(2n-1)$ is the annihilator of $R \subset \Gamma(E)(2n-1)$ in the pairing (1), and $(R^\perp)$ the operadic ideal generated by $R^\perp$.

Remark 2 If $\mathcal{P}$ is a quadratic operad generated by an operation of arity n and degree d, then the generating operation of $\mathcal{P}^!$ has the same arity but degree $-d+n-2$, i.e. for $n \neq 2(d+1)$ the Koszul duality *does not* preserve the degree of the generating operation. As in the binary case, the quadratic dual is a reflection, $(\mathcal{P}^!)^! \cong \mathcal{P}$.

Recall that the *operadic suspension* $\mathbf{s}E$ of a Σ-module $E = \{E(a)\}_{a\geq 1}$ is the Σ-module $\mathbf{s}E = \{\mathbf{s}E(a)\}_{a\geq 1}$, where $\mathbf{s}E(a) := \mathrm{sgn}_a \otimes \uparrow^{a-1} E(a)$. It is a standard fact that, for a dg-operad $\mathcal{P} = \{\mathcal{P}(a)\}_{a\geq 1}$, the operadic suspension $\mathbf{s}\mathcal{P} = \{\mathbf{s}\mathcal{P}(a)\}_{a\geq 1}$ of the underlying Σ-module is has a natural dg-operad structure. The operadic suspension therefore extends from Σ-modules to an endofunctor on the category of dg-operads. Likewise, the operadic suspension $\mathbf{s}\mathcal{C}$ of a dg-cooperad $\mathcal{C}$ is a dg-cooperad. We denote

[1] In the original paper [9], *quadratic* always meant *binary quadratic* in the terminology of the present note.

by $\mathbf{s}^{-1}$ the inverse operation and call it, if necessary, the operadic *desuspension*. In the following proposition, $\mathcal{P}^{\#}$ denotes the componentwise linear dual of a dg-operad with components of finite type, with the obvious cooperad structure.

Proposition 3 *The free operad functor commutes with the operadic suspension,* $\mathbf{s}\Gamma = \Gamma\mathbf{s}$. *For a dg-operad* $\mathcal{P}$ *with components of finite type, one has a natural isomorphism*

$$(\mathbf{s}\mathcal{P})^{\#} \cong \mathbf{s}^{-1}(\mathcal{P}^{\#})$$

of dg-cooperads. Finally, if $\mathcal{P}$ *is a quadratic operad as in Definition 1, its operadic suspension* $\mathbf{s}\mathcal{P}$ *is again quadratic and one has a natural isomorphism of quadratic operads*

$$(\mathbf{s}\mathcal{P})^{!} \cong \mathbf{s}^{-1}(\mathcal{P}^{!}). \tag{2}$$

Proof The first, rather nontrivial, claim of the proposition is the content of [24, Proposition II.3.20]. The second claim is obvious and the third can be verified directly. □

The *cobar construction* [24, Definition II.3.9] of a coaugmented cooperad $\mathcal{C}$ is a dg-operad $\Omega(\mathcal{C})$ of the form $\Omega(\mathcal{C}) = (\Gamma(\downarrow \overline{\mathbf{s}\mathcal{C}}), \partial_{\Omega})$. Here $\mathbf{s}$ denotes the cooperadic suspension recalled above, $\overline{\mathbf{s}\mathcal{C}}$ the coaugmentation coideal of the coaugmented cooperad $\mathbf{s}\mathcal{C}$, and $\downarrow$ the component-wise desuspension. The differential ∂_{Ω} is induced by the structure operations of the cooperad $\mathcal{C}$. If $\mathcal{P} = \{\mathcal{P}(a)\}_{a\geq 1}$ is an augmented operad with finite-dimensional components, the component-wise linear dual $\mathcal{P}^{\#} = \{\mathcal{P}(a)^{\#}\}_{a\geq 1}$ is a coaugmented cooperad. The composition $\mathsf{D}(\mathcal{P}) := \Omega(\mathcal{P}^{\#})$ of the linear dual with the cobar construction is the *dual operad* of [9, (3.2.12)]. In section II.3.3 of the monograph [24], $\mathsf{D}(-)$ was called the *dual bar construction*. We will use the latter terminology.

For $\mathcal{P}$ quadratic, there exists a natural map $\mathsf{D}(\mathcal{P}^{!}) \to \mathcal{P}$ of dg-operads. The following definition is a straightforward extension of [9, Definition 4.1.3], allowing that the quadratic operad $\mathcal{P}$ may not be binary (i.e. generated by operations of arity two), see also [14, Corollary 7.4.3] for the general case.

Definition 4 A quadratic operad $\mathcal{P}$ is *Koszul* if the natural map $\mathsf{D}(\mathcal{P}) \to \mathcal{P}^{!}$ is a homology isomorphism.

Let us close this section by formulating a couple of properties of quadratic operads.

Proposition 5 *A quadratic operad as in Definition 1 is Koszul if and only if its operadic suspension* $\mathbf{s}\mathcal{P}$ *is Koszul, i.e. the operadic suspension preserves Koszulness.*

Proof Assume that $\mathcal{P}$ is Koszul. This, by definition, means that the map $\rho : \mathsf{D}(\mathcal{P}) \to \mathcal{P}^{!}$ is a homology isomorphism. Since the operadic desuspension obviously preserves homology isomorphisms, the desuspension of ρ,

$$\mathbf{s}^{-1}\rho : \mathbf{s}^{-1}\mathsf{D}(\mathcal{P}) \to \mathbf{s}^{-1}(\mathcal{P}^{!}) \tag{3}$$

is a homology isomorphism, too. Expanding the definition of the dual bar construction, one readily sees that the properties of the operadic (de)suspension stated in Proposition 3 imply that

$$\mathbf{s}^{-1}\mathsf{D}(\mathcal{P}) \cong \mathsf{D}(\mathbf{s}\mathcal{P})$$

Combining this isomorphism with (3) and (2), we obtain a homology isomorphism $\mathsf{D}(\mathbf{s}\mathcal{P}) \to (\mathbf{s}\mathcal{P})^!$, which coincides with the canonical map for the quadratic operad $\mathbf{s}\mathcal{P}$. This shows that $\mathbf{s}\mathcal{P}$ is Koszul. To prove that the Koszulness of $\mathbf{s}P$ implies the Koszulness of $\mathcal{P}$, all one needs to do is to reverse the steps of the proof of the above implication. □

Observe that quadratic operads $\mathcal{P}$ as we introduced them at the beginning of this section have the properties that $\mathcal{P}(1) \cong \mathbf{k}$ and that $\mathcal{P}(a)$ is finite-dimensional for each $a \geq 1$. This means that they are *admissible* in the sense of [9, (3.1.5)]. Therefore, all the properties of the dual bar construction $\mathsf{D}(-)$ proved in [9] apply to our case. Namely, the contravariant functor $\mathsf{D}(-)$ preserves homology isomorphisms [9, Theorem 3.2.7b] and the canonical map $\mathsf{D}\big(\mathsf{D}(\mathcal{P})\big) \to \mathcal{P}$ is a homology isomorphism. We also have the following extension of [9, Proposition 4.1.4a] to the non-binary case, see also [14, Proposition 7.4.4].

Proposition 6 *A quadratic operad $\mathcal{P}$ is Koszul if and only if so is $\mathcal{P}^!$, i.e. the quadratic duality preserves Koszulness.*

Proof A verbatim transcription of the corresponding statement of [9]. Suppose that $\mathcal{P}$ is Koszul and let $\rho : \mathsf{D}(\mathcal{P}) \to \mathcal{P}^!$ be the canonical map. One then has the composition

$$\mathsf{D}(\mathcal{P}^!) \xrightarrow{\mathsf{D}(\rho)} \mathsf{D}\big(\mathsf{D}(\mathcal{P})\big) \to \mathcal{P}$$

which is, due to the properties of the dual bar construction recalled above, a homology isomorphism. It is immediate that this composition coincides with the canonical map $\mathsf{D}(\mathcal{P}) \to \mathcal{P}^!$ for $\mathcal{P}^!$. So the Koszulness of $\mathcal{P}$ implies the Koszulness of $\mathcal{P}^!$. The opposite implication is obtained by applying the above arguments to $\mathcal{P}^!$ instead of $\mathcal{P}$. □

The *Poincaré* or *generating series* of a graded operad $\mathcal{P}_* = \{\mathcal{P}_*(a)\}_{a\geq 1}$ with finite-dimensional components is defined by

$$g_{\mathcal{P}}(t) := \sum_{a\geq 1} \frac{1}{a!}\chi(\mathcal{P}(a))t^a,$$

where $\chi(\mathcal{P}(a))$ denotes the Euler characteristic of the graded vector space $\mathcal{P}_*(n)$,

$$\chi(\mathcal{P}(a)) := \sum_i (-1)^i \dim(\mathcal{P}_i(a)).$$

Recall that each operad $\mathcal{P}$ with $\mathcal{P}(1) = \mathbf{k}$ admits a *minimal model*, unique up to isomorphism [24, II.3.10]. This is, by definition, a homology isomorphism $(\mathcal{P}, 0) \xleftarrow{\rho} (\Gamma(M), \partial)$ from the free operad $\Gamma(M)$ on a collection $M = \{M(a)\}_{a\geq 2}$, equipped with a differential ∂, to the operad $\mathcal{P}$ with the trivial differential. The *minimality* requires that $\partial(M)$ consists of decomposable elements of the free operad $\Gamma(M)$. The following proposition relates the generating series of $\mathcal{P}$ to the generating series of the collection of generators of its minimal model.

Proposition 7 *Let $\mathcal{P}$ be an arbitrary operad with $\mathcal{P}(1) = \mathbf{k}$ and finite-dimensional pieces. Let $(\mathcal{P}, 0) \stackrel{\rho}{\leftarrow} (\Gamma(M), \partial)$ be its minimal model. The Poincaré series $g_{\mathcal{P}}(t)$ of $\mathcal{P}$ is related to the generating function*

$$g_M(t) := -t + \sum_{a \geq 2} \frac{1}{a!} \chi(M(a)) t^a$$

of the Σ-module $M = \{M(a)\}_{a \geq 2}$ by the functional equation

$$g_{\mathcal{P}}(-g_M(t)) = t. \tag{4}$$

In other words, $g_M(t)$ is the formal inverse of $g_{\mathcal{P}}(t)$ taken with the opposite sign.

Proof This proposition has been known to experts for many years; a standard homological proof is referred to e.g. in [3, Remark 2.3]. An interested reader may find a detailed self-contained proof which is a modification of the proof of [9, Theorem 3.3.2] in the preprint version [21] of the present paper. □

The following important criterion of Koszulness, which is a verbatim generalization of [9, Theorem 3.3.2], follows easily from Proposition 7.

Theorem 8 *If a quadratic operad $\mathcal{P}$ is Koszul, then its Poincaré series and the Poincaré series of its dual $\mathcal{P}^!$ are tied by the functional equation*

$$g_{\mathcal{P}}\big(- g_{\mathcal{P}^!}(-t)\big) = t. \tag{5}$$

Proof If $\mathcal{P}$ is quadratic Koszul, then its minimal model is isomorphic to the dual bar construction $\mathsf{D}(\mathcal{P}^!)$ of its Koszul dual $\mathcal{P}^!$. The dual bar construction is, as a graded operad, generated by the Σ-collection $\downarrow \overline{\mathbf{s}\mathcal{P}^!}^{\#} = \{\uparrow^{a-2} \mathcal{P}^!(a)^{\#}\}_{a \geq 2}$. So, in the Koszul case

$$g_M(t) = g_{\downarrow \overline{\mathbf{s}\mathcal{P}^!}^{\#}}(t) = g_{\mathcal{P}^!}(-t),$$

which, substituted in (4), gives (5). □

Formula (5) can be obtained by taking $y := -1$ in [14, Theorem 7.5.1]. On the other hand, it is easy to show that for quadratic operads generated by operations of the same arity, (5) implies the formula of [14, Theorem 7.5.1] for arbitrary y, so the Koszulity test provided by [14, Theorem 7.5.1] is not stronger than ours.

Let us close this section with another criterion for Koszulness. Denote by $\Gamma^2(M)$ the subcollection of $\Gamma(M)$ spanned by expressions with precisely two instances of elements of the generating collection M or, equivalently, by M-decorated trees with two vertices. We say that the minimal model $(\Gamma(M), \partial)$ of $\mathcal{P}$ is *quadratic* if $\partial(M) \subset \Gamma^2(M)$.

Fact 9 *A quadratic Koszul operad has a quadratic minimal model.*

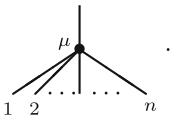
Indeed, if $\mathcal{P}$ is Koszul, by Proposition 6 so is $\mathcal{P}^!$. This, by definition, means that the natural map $\mathsf{D}(\mathcal{P}^!) \to \mathcal{P}$ is a homology isomorphism, therefore it is a quadratic minimal model of $\mathcal{P}$.

We are aware that Fact 9 is a very simple-minded Koszulness test. Yet, we will see in Sect. 6 that the non-Koszulness of the operad $\widetilde{\mathcal{A}ss}$ for anti-associative algebras can be proved by showing that it does not admit a quadratic minimal model. It is also possible that the non-Koszulness of the operads $t\mathcal{A}ss^n_d$ introduced in the following section can be, for $n \geq 8$ and d odd, established using Fact 9, while the Ginzburg–Kapranov test (Theorem 8) may not be determinative. See also a discussion in [22].

3 Four families of n-ary algebras

We introduce four families of quadratic operads and describe their Koszul duals. These families cover most of examples of 'n-ary algebras' with one operation without symmetry which we were able to find in the literature.

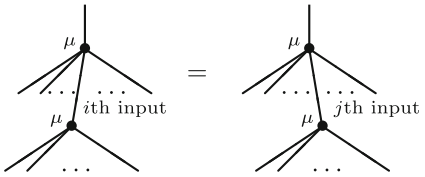
Let V be a graded vector space, $n \geq 2$, and $\mu : V^{\otimes n} \to V$ a degree d multilinear operation symbolized by



We say that $A = (V, \mu)$ is a *degree d totally associative n-ary algebra* if, for each $1 \leq i, j \leq n$,

$$\mu\left(\mathrm{ll}^{\otimes i-1} \otimes \mu \otimes \mathrm{ll}^{\otimes n-i}\right) = \mu\left(\mathrm{ll}^{\otimes j-1} \otimes \mu \otimes \mathrm{ll}^{\otimes n-j}\right),$$

where $\mathrm{ll} : V \to V$ is the identity map. Graphically, we demand that



for each i, j for which the above compositions make sense. Observe that degree 0 totally associative 2-algebras are ordinary associative algebras.

In the following definitions, $\Gamma(\mu)$ will denote the free operad on the Σ-module E_μ with

$$E_\mu(a) = \begin{cases} \text{the regular representation } \mathbf{k}[\Sigma_n] \text{ generated by } \mu & \text{if } a = n \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Definition 10 We denote $t\mathcal{A}ss^n_d$ the operad for totally associative n-ary algebras with operation in degree d, that is,

$$t\mathcal{A}ss^n_d := \Gamma(\mu)/(R_{t\mathcal{A}ss^n_d})$$

with μ an arity n generator of degree d and

$$R_{t\mathcal{A}ss^n_d} := Span\left\{\mu \circ_i \mu - \mu \circ_j \mu, \text{ for } i, j = 1, \ldots, n\right\}.$$

We call $A = (V, \mu)$ a *degree d partially associative n-ary algebra* if the following single axiom is satisfied:

$$\sum_{i=1}^{n} (-1)^{(i+1)(n-1)} \mu\left(1\!\!1^{\otimes i-1} \otimes \mu \otimes 1\!\!1^{\otimes n-i}\right) = 0. \tag{6}$$

Degree 0 partially associative 2-ary algebras are classical associative algebras. A more interesting observation is that degree $(n-2)$ partially associative n-ary algebras are the same as A_∞-algebras $A = (V, \mu_1, \mu_2, \ldots)$ [15, Sect. 1.4] which are "meager" in that they satisfy $\mu_k = 0$ for $k \neq n$. Symmetrizations of these *meager A_∞-algebras* are *Lie n-algebras* in the sense of [12].

Definition 11 We denote $p\mathcal{A}ss^n_d$ the operad for partially associative n-ary algebras with operation in degree d. Explicitly,

$$p\mathcal{A}ss^n_d := \Gamma(\mu)/\left(\sum_{i=1}^{n} (-1)^{(i+1)(n-1)} \mu \circ_i \mu\right)$$

with μ a generator of degree d and arity n.

It follows from the above remarks that $t\mathcal{A}ss^2_0 = p\mathcal{A}ss^2_0 = \mathcal{A}ss$, where $\mathcal{A}ss$ is the operad for associative algebras. We are going to introduce the remaining two families of operads. Recall that $\mathbf{s}$ denotes the operadic suspension and $\mathbf{s}^{-1}$ the obvious inverse operation.

Definition 12 We define $\widetilde{t\mathcal{A}ss^n_d} := \mathbf{s}t\mathcal{A}ss^n_{d-n+1}$ and $\widetilde{p\mathcal{A}ss^n_d} := \mathbf{s}^{-1}p\mathcal{A}ss^n_{d+n-1}$.

We leave as an exercise to verify that $\widetilde{t\mathcal{A}ss^n_d}$-algebras are structures $A = (V, \mu)$, where $\mu : V^{\otimes n} \to V$ is a degree d linear map satisfying, for each $1 \leq i, j \leq n$,

$$(-1)^{i(n+1)} \mu\left(1\!\!1^{\otimes i-1} \otimes \mu \otimes 1\!\!1^{\otimes n-i}\right) = (-1)^{j(n+1)} \mu\left(1\!\!1^{\otimes j-1} \otimes \mu \otimes 1\!\!1^{\otimes n-j}\right).$$

Likewise, $\widetilde{p\mathcal{A}ss^n_d}$-algebras are similar structures, but this time satisfying

$$\sum_{i=1}^{n} \mu\left(1\!\!1^{\otimes i-1} \otimes \mu \otimes 1\!\!1^{\otimes n-i}\right) = 0.$$

Definition 13 Let $\widetilde{\mathcal{A}ss} := \widetilde{t\mathcal{A}ss^2_0} = \widetilde{p\mathcal{A}ss^2_0}$. Explicitly, $\widetilde{\mathcal{A}ss}$-algebras are structures $A = (V, \mu)$ with a degree 0 bilinear operation $\mu : V \otimes V \to V$ satisfying

$$\mu(\mu \otimes 1\!\!1) + \mu(1\!\!1 \otimes \mu) = 0$$

or, in elements

$$a(bc) + (ab)c = 0, \tag{7}$$

for $a, b, c \in V$. We call these objects *anti-associative* algebras.

Anti-associative algebras can be viewed as associative algebras with the associativity taken with the opposite sign which explains their name. Similarly, $t\mathcal{A}ss^2_1 = p\mathcal{A}ss^2_1$-algebras are associative algebras with operation of degree 1. The corresponding, essentially equivalent, operads are the simplest examples of non-Koszul operads, as we will see in Sect. 5. The proof of the following proposition is an exercise.

Proposition 14 *For each $n \geq 2$ and d, $(t\mathcal{A}ss^n_d)^! = p\mathcal{A}ss^n_{-d+n-2}$, $(p\mathcal{A}ss^n_d)^! = t\mathcal{A}ss^n_{-d+n-2}$, $(\widetilde{t\mathcal{A}ss^n_d})^! = \widetilde{p\mathcal{A}ss}^n_{-d+n-2}$ and $(\widetilde{p\mathcal{A}ss^n_d})^! = \widetilde{t\mathcal{A}ss}^n_{-d+n-2}$.*

4 Sundry facts about *n*-ary algebras

In this section we discuss two constructions (galgalim and higher associahedra) that, in some sense, generalize classical Stasheff's associahedra to the realms of partially resp. totally associative n-algebras. We also show how galgalim encode some properties of free partially associative algebras. Necessary facts about the associahedra can be gained from [24, II.1.6] or from the original source [25].

Galgalim. This part is devoted to degree 0 partially associative n-algebras, i.e. to algebras over the operad $p\mathcal{A}ss^n_0$. The fact that, for $n \geq 3$, their defining axiom (6) has more than two terms rules out the existence of an analog of the Stasheff associahedra – the edges of such a hypothetical polyhedra ought to have more than two end-points. One can, however, still draw some graphs that visualize the relations among the axioms, similar to the Lie-hedron constructed in [23]. Their nature is somehow dual to the nature of the associahedra in that their vertices are indexed by the defining *relations*, while their edges are labelled by the iterated structure *operations*.

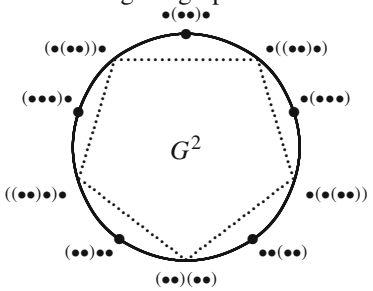
Let us start with the case $n = 2$, when $p\mathcal{A}ss^n_0$ is the operad for associative algebras, so the associahedra actually exist. There are five ways to apply a binary operation to four elements:

$$((\bullet\bullet)\bullet)\bullet,\ (\bullet(\bullet\bullet))\bullet,\ (\bullet\bullet)(\bullet\bullet),\ \bullet((\bullet\bullet)\bullet),\ \bullet(\bullet(\bullet\bullet)). \tag{8}$$

There are five relations between these expressions obtained by one instance of the axiom (6) which is, for $n = 2$, the associativity, namely

$$
\begin{aligned}
(\bullet\bullet)(\bullet\bullet) - \bullet(\bullet(\bullet\bullet)) = 0 \qquad & \text{which we denote } \bullet\bullet(\bullet\bullet),\\
\bullet(\bullet(\bullet\bullet)) - \bullet((\bullet\bullet)\bullet) = 0 \qquad & \text{which we denote } \bullet(\bullet\bullet\bullet),\\
\bullet((\bullet\bullet)\bullet) - (\bullet(\bullet\bullet))\bullet = 0 \qquad & \text{which we denote } \bullet(\bullet\bullet)\bullet,\\
(\bullet(\bullet\bullet))\,\bullet - ((\bullet\bullet)\bullet)\bullet = 0 \qquad & \text{which we denote } (\bullet\bullet\bullet)\bullet, \text{ and}\\
((\bullet\bullet)\bullet)\,\bullet - (\bullet\bullet)(\bullet\bullet) = 0 \qquad & \text{which we denote } (\bullet\bullet)\bullet\bullet.
\end{aligned}
\tag{9}
$$

We call these relations *elementary*. Observe that each symbol listed in (8) appears in precisely two elementary relations of (9). So we may draw a graph with edges labelled by the five symbols in (8) which share a common vertex if and only if their labels appear in the same relation of (9). The common vertex emerging in this way will be labelled by this relation. We get a graph with five vertices and five edges:



(10)

which is dual to the 1-skeleton of the Stasheff pentagon K_4 indicated by the dotted lines.

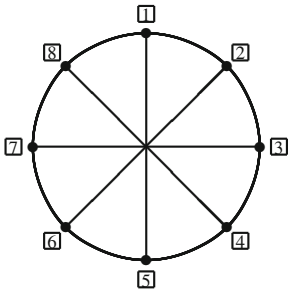
For $n = 3$, there are 12 ways to multiply 7 elements by a ternary operation:

$$
\begin{gathered}
((\bullet\bullet\bullet)\bullet\bullet)\bullet\bullet,\ (\bullet(\bullet\bullet)\bullet)\bullet\bullet,\ (\bullet\bullet(\bullet\bullet\bullet))\bullet\bullet,\ \bullet((\bullet\bullet\bullet)\bullet\bullet)\bullet,\\
\bullet(\bullet(\bullet\bullet)\bullet)\bullet,\ \bullet(\bullet\bullet(\bullet\bullet\bullet))\bullet,\ \bullet\bullet((\bullet\bullet\bullet)\bullet\bullet),\ \bullet\bullet(\bullet(\bullet\bullet)\bullet)\bullet,\\
\bullet\bullet(\bullet\bullet(\bullet\bullet\bullet)),\ (\bullet\bullet\bullet)(\bullet\bullet\bullet)\bullet,\ (\bullet\bullet\bullet)\,\bullet\,(\bullet\bullet\bullet),\ \bullet(\bullet\bullet\bullet)(\bullet\bullet\bullet)
\end{gathered}
\tag{11}
$$

and 8 elementary relations between these terms obtained by one instance of the partial associativity $(\bullet\bullet\bullet)\bullet\bullet + \bullet(\bullet\bullet\bullet)\,\bullet + \bullet\bullet(\bullet\bullet\bullet)$, namely

$$
\begin{aligned}
&\boxed{1} \text{ denoting} && (\bullet\bullet(\bullet\bullet\bullet))\bullet\bullet + \bullet(\bullet(\bullet\bullet\bullet)\bullet)\bullet + \bullet\bullet((\bullet\bullet\bullet)\bullet\bullet) = 0,\\
&\boxed{2} \text{ denoting} && \bullet\bullet((\bullet\bullet\bullet)\bullet\bullet) + \bullet\bullet(\bullet(\bullet\bullet\bullet)\bullet) + \bullet\bullet(\bullet\bullet(\bullet\bullet\bullet)) = 0,\\
&\boxed{3} \text{ denoting} && \bullet(\bullet\bullet\bullet)(\bullet\bullet\bullet) + (\bullet\bullet\bullet)\bullet(\bullet\bullet\bullet) + \bullet\bullet(\bullet\bullet(\bullet\bullet\bullet)) = 0.\\
&\boxed{4} \text{ denoting} && (\bullet(\bullet\bullet\bullet)\bullet)\bullet\bullet + \bullet((\bullet\bullet\bullet)\bullet\bullet)\bullet + \bullet(\bullet\bullet\bullet)(\bullet\bullet\bullet) = 0,\\
&\boxed{5} \text{ denoting} && \bullet((\bullet\bullet\bullet)\bullet\bullet)\bullet + \bullet(\bullet(\bullet\bullet\bullet)\bullet)\bullet + \bullet(\bullet\bullet(\bullet\bullet\bullet))\bullet = 0,\\
&\boxed{6} \text{ denoting} && (\bullet\bullet\bullet)(\bullet\bullet\bullet)\bullet + \bullet(\bullet\bullet(\bullet\bullet\bullet))\bullet + \bullet\bullet(\bullet(\bullet\bullet\bullet)\bullet) = 0,\\
&\boxed{7} \text{ denoting} && ((\bullet\bullet\bullet)\bullet\bullet)\bullet\bullet + (\bullet\bullet\bullet)\bullet(\bullet\bullet\bullet) + (\bullet\bullet\bullet)(\bullet\bullet\bullet)\bullet = 0, \text{ and}\\
&\boxed{8} \text{ denoting} && ((\bullet\bullet\bullet)\bullet\bullet)\bullet\bullet + (\bullet(\bullet\bullet\bullet)\bullet)\bullet\bullet + (\bullet\bullet(\bullet\bullet\bullet))\bullet\bullet = 0,
\end{aligned}
$$

Each element of (11) again appears in precisely 2 elementary relations. The corresponding graph with 12 edges indexed by expressions (11) and 8 vertices labelled by elementary relations is the wheel with eight spikes:



(12)

Observe that elementary relations have a left-right mirror symmetry: 1 and 5 are self-symmetric, while the mirror image of 2 is 8, the image of 3 is 7 and the image of 4 is 6. This symmetry is reflected by the left-right symmetry of (12).

For $n = 4$, there are 22 ways of applying a 4-ary operation to 10 elements, and 11 elementary relations among these elements. The resulting graph is shown in Fig. 1. The 5th galgal (case $n = 5$) has 14 vertices and 35 edges, its portrait is given in Fig. 2. We call these figures *galgalim* (plural of *galgal*), the Hebrew for wheel.

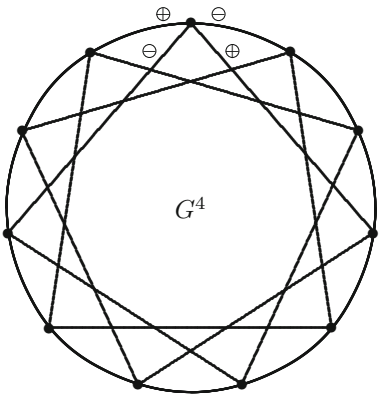
Galgalim can be used to analyze the structure of free n-ary algebras. Let us, for instance, investigate possible linear dependence of the five elementary relations (9) among binary bracketings (8) of five variables. We need to solve

$$a_1\,\bullet(\bullet\bullet)\bullet + a_2\,\bullet(\bullet\bullet\bullet) + a_3\,\bullet\bullet(\bullet\bullet) + a_4\,(\bullet\bullet)\bullet\bullet + a_5\,(\bullet\bullet\bullet)\bullet = 0, \qquad (13)$$

for some scalars $a_1, \ldots, a_5 \in \mathbf{k}$. If we view the coefficients $a_1, \ldots, a_5$ as decorations of the corresponding vertices of the 2nd galgal (10), the above relation is obviously satisfied if and only if the decorations of vertices connected by an edge agree. Therefore (13) holds if and only if

$$a_1 = a_2 = a_3 = a_4 = a_5.$$

Fig. 1 4th galgal G^4



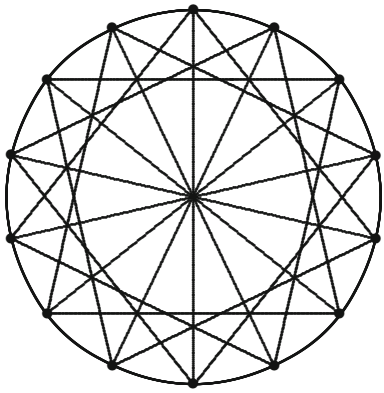


Fig. 2 5th galgal G^5 (the central point is not a vertex)

The last condition is fulfilled for instance by $(a_1, \ldots, a_5) = (1, \ldots, 1)$, so the five elementary relations (9) are *not* linearly independent. This is of course elementary and well-known.

Let us proceed to the ternary case. We have eight elementary relations which we denote, to save the space, $\boxed{1}, \ldots, \boxed{8}$. We consider the equation

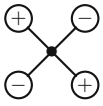
$$a_1 \boxed{1} + \cdots + a_8 \boxed{8} = 0, \tag{14}$$

with some scalars $a_1, \ldots, a_8 \in \mathbf{k}$ which we again view as decorations of the vertices of the 3th galgal G^3. Since all the terms in the elementary relations have the + signs, (14) is satisfied if and only if the decorations of two vertices connected by an edge *differ* by the sign. The presence of closed paths of odd lengths excludes this possibility. For instance, one has the circle $\boxed{1}$—$\boxed{2}$—$\boxed{3}$—$\boxed{4}$—$\boxed{5}$—$\boxed{1}$, so one requires

$$+a_1 = -a_2 = +a_3 = -a_4 = +a_5 = -a_1$$

which implies $a_1 = -a_1$, therefore $a_1 = 0$ thus $a_i = 0$ for all $1 \leq i \leq 5$. The vanishing of the remaining coefficients in (14) can be established in the same way. We conclude that elementary relations for ternary partially associative algebras are linearly independent. Observe that we did not need to know the labels of the vertices and edges of the 3rd galgal explicitly, its shape was enough to establish the linear independence of the elementary relations. We will see in Remark 29 how G^3 helps to understand free partially associative 3-algebras.

For $n = 4$, axiom (6) and thus also the elementary relations acquire nontrivial signs. Each half-edge emerging from a vertex of the 4th galgal G^4 is therefore decorated by the sign of the corresponding term in the relation labelling the vertex. Explicit calculations show that this decoration obeys the rule

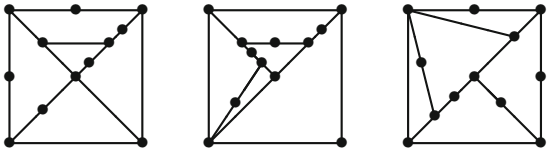


meaning that the antipodal half-edges acquire the same sign. It also turns out that the decorations possess the rotational symmetry, therefore the decorations of all half-edges are determined by the decoration of the half-edges adjacent to the upper vertex shown in Fig. 1. It is immediate to see that two half-edges of the same edge bear the opposite signs. Therefore the elementary relations are not linearly independent, but they, as in the binary case, sum up to zero.

All terms of axiom (6) and therefore also all terms of the elementary relations for 5-ary algebras have the $+$ sign. As in the ternary case, their linear independence is implied by the existence of paths of odd length in the 5th galgal G^5. We leave as an exercise to find such paths. The conclusion is that elementary relations for 5-ary degree 0 partially associative algebras are linearly independent.

Higher associahedra. Degree 0 totally associative n-algebras, i.e. algebras over the operad $t\mathcal{A}ss_0^n$, are, for $n \geq 1$, straightforward generalizations of associative algebras. Observe, for instance, that the operad $t\mathcal{A}ss_0^n$ is, for each $n \geq 2$, the linearization of an operad living in the monoidal category of sets and that this property singles degree 0 totally associative algebras out from the four families of n-ary algebras introduced in Sect. 3.

In [22] we conjectured the existence of an analog $\mathcal{K}^n = \{\mathcal{K}^n(a)\}_{a\geq 1}$ of the Stasheff associahedra for an arbitrary $n \geq 2$. We also constructed some initial pieces of the hypothetical 3-associahedra $\mathcal{K}^3$. It turned out that the inductive construction contained some choices. For example, in arity 7 we found the following three combinatorially distinct $\mathcal{K}^3(7)$'s:



They are convex 2-dimensional polyhedra with twelve vertices, sixteen edges and five 2-dimensional faces. We refer to [22] for more details.

5 Koszulness: the case study

This section is devoted to the following statement organized in the table of Fig. 3.

Theorem 15 *Let $n \leq 7$. Then the operad $t\mathcal{A}ss_d^n$ is Koszul if and only if d is even. The operad $p\mathcal{A}ss_d^n$ is Koszul if and only if n and d have the same parity. The operad $\widetilde{t\mathcal{A}ss}_d^n$ is Koszul if and only if n and d have different parities. The operad $\widetilde{p\mathcal{A}ss}_d^n$ is Koszul if and only if d is odd.*

The operads $t\mathcal{A}ss_d^n$ with d even, $p\mathcal{A}ss_d^n$ with n and d of the same parity, $\widetilde{t\mathcal{A}ss}_d^n$ with n and d of different parities, and $\widetilde{p\mathcal{A}ss}_d^n$ with d odd, are Koszul for all $n \geq 2$.

The Koszulness part ("yes" in the table of Fig. 3) will follow from [13] and relations between the operads $t\mathcal{A}ss_d^n$, $p\mathcal{A}ss_d^n$, $\widetilde{t\mathcal{A}ss}_d^n$ and $\widetilde{p\mathcal{A}ss}_d^n$, see Proposition 17. The non-Koszulness part ("no" in Fig. 3) will, for $n \leq 7$, follow in a similar fashion from

Proposition 23. We do not know how to extend our proof of Proposition 23 for $n \geq 8$, we therefore put question marks to the corresponding places in Fig. 3. See also Remark 24 and the first problem of Sect. 8.

In particular, the operads $\widetilde{\mathcal{A}ss}$ and $t\mathcal{A}ss_1^2 = p\mathcal{A}ss_1^2$ are not Koszul. Let us formulate useful

Lemma 16 *Let $\mathcal{P}_d^n$ be one of the operads above. Then $\mathcal{P}_d^n$ is Koszul if and only if $\mathcal{P}_{d+2}^n$ is Koszul, that is, only the parity of d matters.*

Proof There is a 'twisted' isomorphism

$$\varphi : \mathcal{P}_d^n \xrightarrow{\cong} \mathcal{P}_{d+2}^n, \tag{15}$$

i.e. a sequence of equivariant isomorphisms $\varphi(a) : \mathcal{P}_d^n(a) \to \mathcal{P}_{d+2}^n(a)$, $a \geq 1$, that commute with the $\circ_i$-operations such that the component $\varphi(k(n-1)+1)$ is of degree $2k$, for $k \geq 0$.

To construct such an isomorphism, consider an operation μ' of arity n and degree d, and another operation μ'' of the same arity but of degree $d+i$. We leave as an exercise to verify that the assignment $\mu' \mapsto \mu''$ extends to a twisted isomorphism $\omega : \Gamma(\mu') \to \Gamma(\mu'')$ if and only if i is even.

Let $\mathcal{P}_d^n = \Gamma(\mu')/(R')$ and $\mathcal{P}_{d+2}^n = \Gamma(\mu'')/(R'')$. It is clear that the twisted isomorphism $\omega : \Gamma(\mu') \to \Gamma(\mu'')$ preserves the ideals of relations, so it induces a twisted isomorphism (15). A moment's reflection convinces one that φ induces similar twisted isomorphisms of the Koszul duals and the bar constructions. This, by Definition 4, gives the lemma. □

Proposition 17 *The operads marked "yes" in the tables of Fig. 3 are Koszul.*

		$n \leqslant 7$		$n > 7$	
		n even	n odd	n even	n odd
$t\mathcal{A}ss_d^n$	d even	yes	yes	yes	yes
	d odd	no	no	?	?
$t\widetilde{\mathcal{A}ss}_d^n$	d even	no	yes	?	yes
	d odd	yes	no	yes	?
$p\mathcal{A}ss_d^n$	d even	yes	no	yes	?
	d odd	no	yes	?	yes
$p\widetilde{\mathcal{A}ss}_d^n$	d even	no	no	?	?
	d odd	yes	yes	yes	yes

Fig. 3 Koszulness of the operads $t\mathcal{A}ss_d^n$, $p\mathcal{A}ss_d^n$, $t\widetilde{\mathcal{A}ss}_d^n$ and $p\widetilde{\mathcal{A}ss}_d^n$. "Yes" means that the corresponding operad is Koszul, "no" that it is not Koszul

Proof As observed in [13, Sect. 7.2], the operads $t\mathcal{A}ss_0^n$ admit for all $n \geq 2$ a PBW basis[2] for the lexicographic order, so they are Koszul by [13, Theorem 6.6] (see also [10] for the case n even and $d = 0$). So, by Lemma 16, the operads $t\mathcal{A}ss_d^n$ are Koszul for all even d and $n \geq 2$, which gives the four "yes" in the first row of the table in Fig. 3.

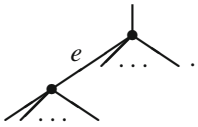
The "yes" in the 3rd row follow from the "yes" in the 1st row, the fact that an operad is Koszul if and only if its dual operad is Koszul proved in Proposition 6, and the isomorphism $(p\mathcal{A}ss_d^n)^! = t\mathcal{A}ss^n_{-d+n-2}$ established in Proposition 14. The "yes" in the remaining rows in Fig. 3 follow from the "yes" in the 1st and the 3rd rows, and Proposition 5 by which the suspension preserves Koszulness. □

Remark 18 Let us describe explicitly the lexicographic order and the PBW basis of $t\mathcal{A}ss_0^n$ mentioned in the proof of Proposition 17. We will use it later in the proof of Theorem 26.

The free operad $\Gamma(\mu)$ generated by an operation μ of arity n and degree 0 has a linear basis formed by rooted (= directed) planar trees whose vertices have n incoming edges, see e.g. [24, II.1.5] for the terminology. Let T be such a tree with u leaves. We associate to it the sequence of natural numbers $n_T = (n_1, \ldots, n_u)$, where n_i is, for $1 \leq i \leq u$, the number of vertices on the edge-path connecting the ith leaf of T with its root.

Let T', T'' be two trees as above. If T' has more leaves than T'' we put $T' > T''$. If the numbers of leaves agree, we put $T' > T''$ if and only if $n_{T'} > n_{T''}$ in the usual lexicographic order. This order is a particular case of the order defined in [13, Sect. 3.3].

A PBW basis of $t\mathcal{A}ss_0^n$ compatible with the above order consists of trees whose all internal edges are the leftmost incoming edges of their terminal vertices, i.e. they look as the edge e in



(16)

In terms of operations, the PBW basis of $t\mathcal{A}ss_0^n$ consists of the iterated compositions

$$(\cdots(\mu \circ_1 \mu) \circ_1 \cdots) \circ_1 \mu.$$

This PBW basis is generated, in the sense of [2, Theorem 3], by the Gröbner basis of $t\mathcal{A}ss_0^n$ for the reversed order[3] given by

$$\mu \circ_p \mu - \mu \circ_1 \mu, \ 1 < p \leq n.$$

The above bases are clearly PBW resp. Gröbner bases for $t\mathcal{A}ss_d^n$ with arbitrary even d.

[2] Abbreviating Poincaré–Birkhoff–Witt.

[3] The authors of [2] use different convention, so we need to reverse the order to get the expected result.

The "no" entries in Fig. 3 will be established using the Ginzburg–Kapranov criterion (5). Our first task will therefore be to describe the Poincaré series of the family $t\mathcal{A}ss^n_d$ which generates, via the duality and suspension, all the remaining operads.

Lemma 19 *The generating function for the operad $t\mathcal{A}ss^n_d$ is*

$$g_{t\mathcal{A}ss^n_d}(t) := \begin{cases} \dfrac{t}{1-t^{n-1}} & \text{if } d \text{ is even, and} \\ t - t^n + t^{2n-1} & \text{if } d \text{ is odd.} \end{cases} \tag{17}$$

Proof The components of the operad $t\mathcal{A}ss^n_d$ are trivial in arities different from $k(n-1)+1$, $k \geq 0$. The piece $t\mathcal{A}ss^n_d(k(n-1)+1)$ is generated by all possible $\circ_i$-compositions involving k instances of the generating operation μ, modulo the relations

$$\mu \circ_i \mu - \mu \circ_j \mu, \text{ for } i, j = 1, \ldots, n \tag{18}$$

which enable one to replace each $\mu \circ_i \mu$, $2 \leq i \leq n$, by $\mu \circ_1 \circ\mu$.

If the degree d is *even*, the operad $t\mathcal{A}ss^n_d$ is evenly graded, so the associativity [17, p. 1473, Eqn. (1)] of the $\circ_i$-operations does not involve signs. Therefore an arbitrary $\circ_i$-composition of k instances of μ can be brought to the form

$$\eta_k := (\cdots((\mu \circ_1 \mu) \circ_1 \mu) \circ_1 \cdots) \circ_1 \mu.$$

We see that $t\mathcal{A}ss^n_d(k(n-1)+1)$ is spanned by the set $\{\eta_k \circ \sigma;\ \sigma \in \Sigma_{k(n-1)+1}\}$, so

$$\dim(t\mathcal{A}ss^n_d(k(n-1)+1)) = (k(n-1)+1)!$$

and, by definition,

$$g_{t\mathcal{A}ss^n_d}(t) = \sum_{k\geq 0} t^{k(n-1)+1} = \frac{t}{1-t^{n-1}},$$

which verifies the even case of (17).

The *odd* case is subtler since the associativity [17, p. 1473, Eqn. (1)] may involve nontrivial signs. As in the even case we calculate that

$$\dim(t\mathcal{A}ss^n_d(k(n-1)+1)) = (k(n-1)+1)! \quad \text{for} \quad k = 0, 1, 2, \tag{19}$$

because these small arities do not require the associativity.

If $k \geq 3$, we can still to bring each $\circ_i$-composition of k instances of μ to the form of the 'canonical' generator η_k, but we may get a nontrivial sign which may moreover depend on the way we applied the associativity. Relation (18) implies that

$$(\mu \circ_1 \mu) \circ_1 \mu = (\mu \circ_n \mu) \circ_1 \mu \tag{20}$$

in $t\mathcal{A}ss^n_d(3n-2)$. Applying (18) and the associativity [17, p. 1473, Eqn. (1)] several times, we get that

$$\begin{aligned}(\mu\circ_1\mu)\circ_1\mu &= \mu\circ_1(\mu\circ_1\mu)=\mu\circ_1(\mu\circ_n\mu)=(\mu\circ_1\mu)\circ_n\mu=(\mu\circ_n\mu)\circ_n\mu\\ &=\mu\circ_n(\mu\circ_1\mu)=\mu\circ_n(\mu\circ_n\mu)=(\mu\circ_n\mu)\circ_{2n-1}\mu\\ &=(\mu\circ_1\mu)\circ_{2n-1}\mu. \end{aligned}\tag{21}$$

Since the degree of μ is odd, the first line of the associativity [17, p. 1473, Eqn. (1)] implies

$$(\mu\circ_1\mu)\circ_{2n-1}\mu=-(\mu\circ_n\mu)\circ_1\mu$$

therefore (20) and (21) combine into

$$(\mu\circ_1\mu)\circ_1\mu=-(\mu\circ_1\mu)\circ_1\mu.$$

This means that $(\mu\circ_1\mu)\circ_1\mu=0$ so $t\mathcal{A}ss^n_d(3n-2)=0$. Since $t\mathcal{A}ss^n_d(k(n-1)+1)$ is, for $k\geq 3$, generated by $t\mathcal{A}ss^n_d(3n-2)$, we conclude that $t\mathcal{A}ss^n_d(k(n-1)+1)=0$ for $k\geq 3$ which, along with (19), verifies the odd case of (17). □

Remark 20 The Poincaré series of an operad $\mathcal{P}$ and its suspension $\mathbf{s}\mathcal{P}$ are related by $g_{\mathbf{s}\mathcal{P}}(t)=-g_{\mathcal{P}}(-t)$. Lemma 19 thus implies that the generating series of the operad $\widetilde{t\mathcal{A}ss^n_d}=\mathbf{s}t\mathcal{A}ss^n_{d-n+1}$ equals

$$g_{\widetilde{t\mathcal{A}ss^n_d}}(t):=\begin{cases} t+(-1)^d t^n+t^{2n-1} & \text{if } n \text{ and } d \text{ have the same parity, and}\\ \frac{t}{1-(-1)^d t^{n-1}} & \text{if } n \text{ and } d \text{ have different parities.}\end{cases}$$

We do not know explicit formulas for the Poincaré series of $p\mathcal{A}ss^n_d$ and $\widetilde{p\mathcal{A}ss^n_d}$ except in the case $n=2$ when these operads coincide with the corresponding (anti)-associative operads.

Example 21 It easily follows from the above calculations that, for the anti-associative operad $\widetilde{\mathcal{A}ss}$, one has

$$\widetilde{\mathcal{A}ss}(1)\cong\mathbf{k},\ \widetilde{\mathcal{A}ss}(2)\cong\mathbf{k}[\Sigma_2]\quad\text{and}\quad\widetilde{\mathcal{A}ss}(3)\cong\mathbf{k}[\Sigma_3],$$

while $\widetilde{\mathcal{A}ss}(a)=0$ for $a\geq 4$.

Let us return to our task of proving the non-Koszulness of the "no" cases in the tables of Fig. 3. Our strategy will be to interpret (5) as saying that $-g_{\mathcal{P}^!}(-t)$ is a formal inverse of $g_{\mathcal{P}}(t)$ at 0. Since $g'_{\mathcal{P}}(0)=1$, this unique formal inverse exists. In the particular case of $\mathcal{P}=t\mathcal{A}ss^n_d$, with d odd, this means that $-g_{p\mathcal{A}ss^n_{-d+n-2}}(-t)$

should be compared to a formal inverse of $g_{t\mathcal{A}ss^n_d}(t) = t - t^n + t^{2n-1}$. A simple degree count shows that $g_{p\mathcal{A}ss^n_{-d+n-2}}(t)$ is of the form

$$\begin{cases} t - A_1 t^n + A_2 t^{2n-1} - A_3 t^{3n-2} + \cdots, & \text{for n even and} \\ t + A_1 t^n + A_2 t^{2n-1} + A_3 t^{3n-2} + \cdots, & \text{for n odd} \end{cases},$$

for some *non-negative* integers $A_1, A_2, A_3, \ldots$, therefore $-g_{p\mathcal{A}ss^n_{-d+n-2}}(-t)$ is in both cases the formal power series

$$t + A_1 t^n + A_2 t^{2n-1} + A_3 t^{3n-2} + \cdots \tag{22}$$

with non-negative coefficients. If we show that the formal inverse of $t - t^n + t^{2n-1}$ is not of this form, by Theorem 8 the corresponding operad $t\mathcal{A}ss^n_d$ is not Koszul.

Example 22 The Poincaré series of the operad $t\mathcal{A}ss^2_1$ is, by Lemma 19,

$$g_{t\mathcal{A}ss^2_1}(t) = t - t^2 + t^3.$$

One can compute the formal inverse of this function as

$$t + t^2 + t^3 - 4t^5 - 14t^6 - 30t^7 - 33t^8 + 55t^9 + \cdots.$$

The presence of negative coefficients implies that the operad $t\mathcal{A}ss^2_1$ is not Koszul, neither is the anti-associative operad $\widetilde{\mathcal{A}ss} = \widetilde{t\mathcal{A}ss^2_0} = \mathbf{s}^{-1}t\mathcal{A}ss^2_1$.

Likewise, the Poincaré series of the operad $t\mathcal{A}ss^3_1$ equals

$$g_{t\mathcal{A}ss^3_1}(t) = t - t^3 + t^5$$

and we computed, using `Matematica`, the initial part of the formal inverse as

$$t + t^3 + 2t^5 + 4t^7 + 5t^9 - 13t^{11} - 147t^{13} + \cdots$$

The existence of negative coefficients again implies that the operad $t\mathcal{A}ss^3_1$ is not Koszul. The formal inverse of

$$g_{t\mathcal{A}ss^4_1}(t) = t - t^4 + t^7$$

up to the first negative term is

$$t + t^4 + 3t^7 + 11t^{10} + 42t^{13} + 153t^{16} + 469t^{19} + 690t^{22} - 5967t^{25} + \cdots$$

so $t\mathcal{A}ss^4_1$ is not Koszul.

The complexity of the calculation of the relevant initial part of the inverse of $g_{t\mathcal{A}ss^n_1}(t) = t - t^n + t^{2n-1}$ grows rapidly with n. We have, however, the following:

Proposition 23 *For $n \leq 7$, the formal inverse of $t - t^n + t^{2n-1}$ has at least one negative coefficient. Therefore the operads $t\mathcal{A}ss_d^n$ for d odd and $n \leq 7$ are not Koszul.*

Proof The function $g(z) := z - z^n + z^{2n-1}$ is analytic in the complex plane $\mathbb{C}$. Its analytic inverse $g^{-1}(z)$ is a not-necessarily single-valued analytic function defined outside the points in which the derivative $g'(z)$ vanishes. Let us denote by $\mathfrak{Z}$ the set of these points, i.e.

$$\mathfrak{Z} := \{z \in \mathbb{C};\ g'(z) = 0\}.$$

The key observation is that, for $n \leq 7$, the equation $g'(z) = 0$ has no real solutions, $\mathfrak{Z} \cap \mathbb{R} = \emptyset$. Indeed, one has to solve the equation

$$g'(z) = 1 - nz^{n-1} + (2n-1)z^{2n-2} = 0 \tag{23}$$

which, after the substitution $w := z^{n-1}$ leads to the quadratic equation

$$1 - nw + (2n-1)w^2 = 0$$

whose discriminant $n^2 - 8n + 4$ is, for $n \leq 7$, negative.

Let $f(z)$ be the power series representing the branch at 0 of $g^{-1}(z)$ such that $f(0) = 0$. It is clear that $f(t)$ is precisely the formal inverse of $g(t)$ at 0. Suppose that

$$f(z) = z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots,$$

with all coefficients $a_2, a_3, a_4, \ldots$ non-negative real numbers. Since $\mathfrak{Z} \neq \emptyset$ and obviously $0 \notin \mathfrak{Z}$, the radius of convergence of $f(z)$ at 0, which equals the radius of the maximal circle centered at 0 whose interior does not contain points in $\mathfrak{Z}$, is some number r with $0 < r < \infty$. Let $\mathfrak{z} \in \mathfrak{Z}$ be such that $|\mathfrak{z}| = r$. Since all coefficients of the power series f are positive, we have

$$|f(\mathfrak{z})| \leq f(|\mathfrak{z}|) = f(r),$$

so the function $f(r)$ must have singularity at the *real* point $r \in \mathbb{R}$, i.e. $g'(z)$ must vanish at r. This contradicts the fact that $g'(z) = 0$ has no real solutions. □

Remark 24 Equation (23) has, for $n = 8$, two real solutions, $\mathfrak{z}_1 = \sqrt[7]{1/3}$ and $\mathfrak{z}_2 = \sqrt[7]{1/5}$. This means that the inverse function of $z - z^n + z^{2n-1}$ has two positive real poles and the arguments used in our proof of Proposition 23 do not apply.

We verified Proposition 23 using `Matematica`. The first negative coefficient in the inverse of $t - t^n + t^{2n-1}$ was at the power t^{57} for $n = 5$, at t^{161} for $n = 6$, and at t^{1171} for $n = 7$. For $n = 8$ we did not find any negative term of degree less than 10,000. It is indeed possible that all coefficients of the inverse of $t - t^8 + t^{15}$ are positive.

Proposition 23 together with the fact that the suspension and the !-dual preserves Koszulness (Propositions 5 and 6) imply the "no" entries of the tables in Fig. 3 for $n \leq 7$.

6 Cohomology of algebras over non-Koszul operads: an example

In this section we study anti-associative algebras introduced in Definition 13, i.e. structures $A = (V, \mu)$ with a degree-0 bilinear anti-associative multiplication $\mu : V^{\otimes 2} \to V$. We describe the 'standard' cohomology $H^*_{\widetilde{\mathcal{A}ss}}(A; A)_{\mathrm{st}}$ of an anti-associative algebra A with coefficients in itself and compare it to the relevant part of the deformation cohomology $H^*_{\widetilde{\mathcal{A}ss}}(A; A)$ based on the minimal model of the anti-associative operad $\widetilde{\mathcal{A}ss}$. Since $\widetilde{\mathcal{A}ss}$ is, by Theorem 15, not Koszul, these two cohomologies differ. While the standard cohomology has no sensible meaning, the deformation cohomology coincides with the triple cohomology [5,6] and governs deformations of anti-associative algebras.

Examples Anti-associative algebras, as algebras over a non-Koszul operad, should possess a lot of peculiar properties. Therefore, due to the 'anthropic principle,' one can hardly expect to find examples of these structures in Nature. Observe, however, that there still are 'natural' examples of the anti-associativity. For instance, the standard basis elements $\{1, e_1, \ldots, e_7\}$, see e.g. [1], of the octonions (also called the Cayley algebra) satisfy

$$(e_i e_j)e_k = -e_i(e_j e_k),$$

whenever $e_i e_j \neq \pm e_k$ and $1 \leq i, j, k \leq 7$ are distinct.

Since $\widetilde{\mathcal{A}ss}(a) = 0$ for $a \geq 4$, the product of four elements in an arbitrary anti-associative algebra is trivial. Anti-associative algebras are therefore always 3-step nilpotent. Below we classify, for $k \leq 3$, isomorphism classes of anti-associative structures on the k-dimensional vector space $V := Span(e_1, \ldots, e_k)$.

Case $k = 1$. The only 1-dimensional anti-associative algebra is the trivial one, with $e_1 \cdot e_1 = 0$.

Case $k = 2$. In dimension 2, there are two non-isomorphic anti-associative algebras: the trivial one, and the one defined by $e_1 \cdot e_1 = e_2$ and the remaining products of the basis elements trivial.

Case $k = 3$. In dimension 3, we distinguish two subclasses of anti-associative algebras. Algebras in the first subclass satisfy $v \cdot v = 0$ for all $v \in V$. There are two non-isomorphic algebras in this subclass, the trivial one, and the one with $e_1 \cdot e_2 = -e_2 \cdot e_1 = e_3$ and the remaining products of the basic elements trivial.

Algebras in the second subclass contain some v with $v \cdot v \neq 0$. Algebras with this property are either isomorphic to the one given by:

$$\begin{cases} e_1 \cdot e_1 = e_2, \\ e_1 \cdot e_2 = -e_2 \cdot e_1 = e_3, \end{cases}$$

which happens to be the free anti-associative algebra on one generator, or to an algebra belonging to one of the following two 2-dimensional families:

$$\begin{cases} e_1 \cdot e_1 = e_2, \\ e_1 \cdot e_3 = a e_2, \\ e_3 \cdot e_1 = b e_2, \\ e_3 \cdot e_3 = e_2, \end{cases} \qquad \begin{cases} e_1 \cdot e_1 = e_2, \\ e_1 \cdot e_3 = a e_2, \\ e_3 \cdot e_1 = b e_2, \end{cases}$$

where $a, b \in \mathbf{k}$.

Let us return to the main construction of this section. For each algebra over a quadratic operad $\mathcal{P}$, one has the 'standard' cohomology $H^*_{\mathcal{P}}(A; A)_{\mathrm{st}}$ defined as the cohomology of the 'standard' cochain complex

$$C^1{}_{\mathcal{P}}(A; A)_{\mathrm{st}} \xrightarrow{\delta^1_{\mathrm{st}}} C^2{}_{\mathcal{P}}(A; A)_{\mathrm{st}} \xrightarrow{\delta^2_{\mathrm{st}}} C^3{}_{\mathcal{P}}(A; A)_{\mathrm{st}} \xrightarrow{\delta^3_{\mathrm{st}}} C^4{}_{\mathcal{P}}(A; A)_{\mathrm{st}} \xrightarrow{\delta^4_{\mathrm{st}}} \cdots$$

in which $C^p{}_{\mathcal{P}}(A; A)_{\mathrm{st}} := Hom(\mathcal{P}^!(p) \otimes_{\Sigma_p} V^{\otimes p}, V)$, $p \geq 1$, and the differential δ^*_{st} is induced from the structure of $\mathcal{P}^!$ and A, see [6, Section 8] or [24, Definition II.3.99].

Since for non-Koszul operads the 'standard' cohomology has no sensible meaning, one needs to use the deformation (also called, in [16], the *cotangent*) cohomology based on the minimal model of $\mathcal{P}$. Recall [17, p. 1479] that the *minimal model* of an operad $\mathcal{P}$ is a homology isomorphism

$$(\mathcal{P}, 0) \xleftarrow{\rho} (\Gamma(M), \partial)$$

of dg-operads such that the image of ∂ consists of decomposable elements of the free operad $\Gamma(M)$ (the minimality). It is known [24, Section II.3.10] that each operad with $\mathcal{P}(1) \cong \mathbf{k}$ admits a minimal model unique up to isomorphism. The *deformation cohomology* $H^*_{\mathcal{P}}(A; A)$ is the cohomology of the complex

$$C^1_{\mathcal{P}}(A; A) \xrightarrow{\delta^1} C^2_{\mathcal{P}}(A; A) \xrightarrow{\delta^2} C^3_{\mathcal{P}}(A; A) \xrightarrow{\delta^3} C^4_{\mathcal{P}}(A; A) \xrightarrow{\delta^4} \cdots$$

in which $C^1_{\mathcal{P}}(A; A) := Hom(V, V)$ and

$$C^p_{\mathcal{P}}(A; A) := Hom\Big(\bigoplus_{q \geq 2} M_{p-2}(q) \otimes_{\Sigma_q} V^{\otimes q}, V\Big), \quad \text{for } p \geq 2.$$

The differential δ^* is defined by the formula which can be found in [18, Section 2] or in the introduction to [19]. If $\mathcal{P}$ is quadratic Koszul, the dual bar construction of $\mathcal{P}^!$ is, by [17, Proposition 2.6], isomorphic to the minimal model of $\mathcal{P}$, thus the standard and deformation cohomologies coincide, giving rise to the 'standard' constructions such as the Hochschild, Harrison or Chevalley–Eilenberg cohomology.

Neither $H^*_{\mathcal{P}}(A; A)_{\mathrm{st}}$ nor $H^*_{\mathcal{P}}(A; A)$ have the 0th term. A natural H^0 exists only for algebras for which the concept of unitality makes sense. This is not always the case. Assume, for example, that an anti-associative algebra $A = (V, \mu)$ has a unit, i.e. and element $1 \in V$ such that $1a = a1 = a$, for all $a \in V$. Then the anti-associativity (7) with $c = 1$ gives $ab + ab = 0$, so $ab = 0$ for each $a, b \in V$.

Let us describe the standard cohomology $H^*_{\widetilde{\mathcal{A}ss}}(A; A)_{\mathrm{st}}$ of an anti-associative algebra $A = (V, \mu)$. The operad $\widetilde{\mathcal{A}ss}$ is, by Proposition 14, self-dual and it follows

from the description of $\widetilde{\mathcal{A}ss} = \widetilde{\mathcal{A}ss}^!$ given in Example 21 that $H^*_{\widetilde{\mathcal{A}ss}}(A; A)_{\mathrm{st}}$ is the cohomology of

$$C^1_{\widetilde{\mathcal{A}ss}}(A; A)_{\mathrm{st}} \xrightarrow{\delta^1_{\mathrm{st}}} C^2_{\widetilde{\mathcal{A}ss}}(A; A)_{\mathrm{st}} \xrightarrow{\delta^2_{\mathrm{st}}} C^3_{\widetilde{\mathcal{A}ss}}(A; A)_{\mathrm{st}} \xrightarrow{\delta^3_{\mathrm{st}}} 0 \xrightarrow{0} 0 \xrightarrow{0} \cdots$$

in which $C^p_{\widetilde{\mathcal{A}ss}}(A; A) := Hom(V^{\otimes p}, V)$ for $p = 1, 2, 3$, and all higher $C^p_{\widetilde{\mathcal{A}ss}}(A; A)$'s are trivial. The two nontrivial pieces of the differential are basically the Hochschild differentials with "wrong" signs of some terms:

$$\begin{aligned} \delta^1(\varphi)(a, b) &:= a\varphi(b) - \varphi(ab) + \varphi(a)b, \text{ and} \\ \delta^2(f)(a, b, c) &:= af(b, c) + f(ab, c) + f(a, bc) + f(a, b)c, \end{aligned}$$

for $\varphi \in Hom(V, V)$, $f \in Hom(V^{\otimes 2}, V)$ and $a, b, c \in V$. We abbreviated $\mu(a, b) = ab$, $\mu(a, \varphi(b)) = a\varphi(b)$, &c. One sees, in particular, that $H^p_{\widetilde{\mathcal{A}ss}}(A; A)_{\mathrm{st}} = 0$ for $p \geq 4$.

Let us describe the relevant part of the deformation cohomology of A. It can be shown that $\widetilde{\mathcal{A}ss}$ has the minimal model

$$\left(\widetilde{\mathcal{A}ss}, 0\right) \xleftarrow{\rho} (\Gamma(M), \partial)$$

with the generating Σ-module $M = \{M(a)\}_{a \geq 2}$ such that

- $M(2)$ is generated by a degree 0 bilinear operation $\mu_2 : V \otimes V \to V$,
- $M(3)$ is generated by a degree 1 trilinear operation $\mu_3 : V^{\otimes 3} \to V$,
- $M(4) = 0$, and
- $M(5)$ is generated by four 5-linear degree 2 operations $\mu_5^1, \mu_5^2, \mu_5^3, \mu_5^4 : V^{\otimes 5} \to V$,

so the minimal model of $\widetilde{\mathcal{A}ss}$ is of the form

$$\left(\widetilde{\mathcal{A}ss}, 0\right) \xleftarrow{\alpha} \left(\Gamma\left(\mu_2, \mu_3, \mu_5^1, \mu_5^2, \mu_5^3, \mu_5^4, \ldots\right), \partial\right).$$

Notice the gap in the arity 4 generators! We do not know the exact form of the pieces $M(a)$, $a \geq 6$, of the generating Σ-module M, but we know that they do not contain elements of degrees ≤ 2. We can still, however, determine the Euler characteristic of the generating Σ-module using Proposition 4.

Inverting the generating series $g_{\widetilde{\mathcal{A}ss}}(t) = t + t^2 + t^3$, we read the Euler characteristic of the Σ-module of generators of the minimal model of $\widetilde{\mathcal{A}ss}$ as

$$\begin{aligned} &\chi(M(2)) = 1,\ \chi(M(3)) = -1,\ \chi(M(4)) = 0,\ \chi(M(5)) = 4, \\ &\chi(M(6)) = -14,\ \chi(M(7)) = 30,\ \chi(M(8)) = -33,\ \chi(M(9)) = -55, \ldots \end{aligned}$$

The differential ∂ of the relevant generators is given by:

$$
\begin{aligned}
\partial(\mu_2) &:= 0,\\
\partial(\mu_3) &:= \mu_2 \circ_1 \mu_2 + \mu_2 \circ_2 \mu_2,\\
\partial(\mu_5^1) &:= (\mu_2 \circ_2 \mu_3) \circ_4 \mu_2 - (\mu_3 \circ_3 \mu_2) \circ_4 \mu_2 + (\mu_2 \circ_1 \mu_2) \circ_3 \mu_3 - (\mu_3 \circ_1 \mu_2) \circ_3 \mu_2\\
&\quad + (\mu_2 \circ_1 \mu_3) \circ_1 \mu_2 - (\mu_3 \circ_1 \mu_2) \circ_1 \mu_2 + (\mu_2 \circ_1 \mu_3) \circ_4 \mu_2 - (\mu_3 \circ_2 \mu_2) \circ_4 \mu_2,\\
\partial(\mu_5^2) &:= (\mu_3 \circ_1 \mu_2) \circ_1 \mu_2 - (\mu_2 \circ_1 \mu_3) \circ_1 \mu_2 + (\mu_2 \circ_1 \mu_3) \circ_3 \mu_2 - (\mu_3 \circ_2 \mu_2) \circ_3 \mu_2\\
&\quad + (\mu_2 \circ_2 \mu_3) \circ_3 \mu_2 - (\mu_3 \circ_3 \mu_2) \circ_3 \mu_2 + (\mu_2 \circ_1 \mu_2) \circ_3 \mu_3 - (\mu_3 \circ_1 \mu_2) \circ_4 \mu_2,\\
\partial(\mu_5^3) &:= (\mu_3 \circ_2 \mu_2) \circ_4 \mu_2 - (\mu_2 \circ_2 \mu_3) \circ_2 \mu_2 + (\mu_3 \circ_2 \mu_2) \circ_2 \mu_2 - (\mu_2 \circ_1 \mu_2) \circ_2 \mu_3\\
&\quad + (\mu_2 \circ_1 \mu_3) \circ_3 \mu_2 - (\mu_2 \circ_1 \mu_3) \circ_1 \mu_2 + (\mu_2 \circ_1 \mu_2) \circ_1 \mu_3 - (\mu_3 \circ_1 \mu_2) \circ_2 \mu_2, \text{ and}\\
\partial(\mu_5^4) &:= (\mu_3 \circ_1 \mu_2) \circ_3 \mu_2 - (\mu_3 \circ_3 \mu_2) \circ_3 \mu_2 + (\mu_2 \circ_2 \mu_3) \circ_3 \mu_2 - (\mu_3 \circ_2 \mu_2) \circ_3 \mu_2\\
&\quad + (\mu_2 \circ_1 \mu_2) \circ_2 \mu_3 - (\mu_2 \circ_1 \mu_3) \circ_2 \mu_2 + (\mu_2 \circ_1 \mu_2) \circ_1 \mu_3 - (\mu_2 \circ_1 \mu_3) \circ_1 \mu_2.
\end{aligned}
$$

One can make the formulas clearer by using the nested bracket notation. For instance, μ_2 will be represented by $(\bullet\bullet)$, μ_3 by $(\bullet\bullet\bullet)$, μ_5^2 by $(\bullet\bullet\bullet\bullet\bullet)^2$, $\mu_3 \circ_2 \mu_2$ by $(\bullet(\bullet\bullet)\bullet)$, &c. With this shorthand, the formulas for the differential read

$$
\begin{aligned}
\partial(\bullet\bullet) &:= 0,\\
\partial(\bullet\bullet\bullet) &:= ((\bullet\bullet)\bullet) + (\bullet(\bullet\bullet)),\\
\partial(\bullet\bullet\bullet\bullet\bullet)^1 &:= (\bullet(\bullet\bullet(\bullet\bullet))) - (\bullet\bullet(\bullet(\bullet\bullet))) + ((\bullet\bullet)(\bullet\bullet\bullet)) - ((\bullet\bullet)(\bullet\bullet)\bullet)\\
&\quad + (((\bullet\bullet)\bullet\bullet)\bullet) - (((\bullet\bullet)\bullet)\bullet\bullet) + ((\bullet\bullet\bullet)(\bullet\bullet)) - (\bullet(\bullet\bullet)(\bullet\bullet)),\\
\partial(\bullet\bullet\bullet\bullet\bullet)^2 &:= (((\bullet\bullet)\bullet)\bullet\bullet) - (((\bullet\bullet)\bullet\bullet)\bullet) + ((\bullet\bullet(\bullet\bullet))\bullet) - (\bullet(\bullet(\bullet\bullet))\bullet)\\
&\quad + (\bullet(\bullet(\bullet\bullet)\bullet)) - (\bullet\bullet((\bullet\bullet)\bullet)) + ((\bullet\bullet)(\bullet\bullet\bullet)) - ((\bullet\bullet)\bullet(\bullet\bullet)),\\
\partial(\bullet\bullet\bullet\bullet\bullet)^3 &:= (\bullet(\bullet\bullet)(\bullet\bullet)) - (\bullet((\bullet\bullet)\bullet\bullet)) + (\bullet((\bullet\bullet)\bullet)\bullet) - ((\bullet(\bullet\bullet\bullet))\bullet)\\
&\quad + ((\bullet\bullet(\bullet\bullet))\bullet) - (((\bullet\bullet)\bullet\bullet)\bullet) + (((\bullet\bullet\bullet)\bullet)\bullet) - ((\bullet(\bullet\bullet))\bullet\bullet), \text{ and}\\
\partial(\bullet\bullet\bullet\bullet\bullet)^4 &:= ((\bullet\bullet)(\bullet\bullet)\bullet) - (\bullet\bullet((\bullet\bullet)\bullet)) + (\bullet(\bullet(\bullet\bullet)\bullet)) - (\bullet(\bullet(\bullet\bullet))\bullet)\\
&\quad + ((\bullet(\bullet\bullet\bullet))\bullet) - ((\bullet(\bullet\bullet)\bullet)\bullet) + (((\bullet\bullet\bullet)\bullet)\bullet) - (((\bullet\bullet)\bullet\bullet)\bullet).
\end{aligned}
$$

Let us indicate how we obtained the above formulas. We observed first that the degree-one subspace $\Gamma(\mu_2, \mu_3)(5)_1 \subset \Gamma(\mu_2, \mu_3)(5)$ is spanned by $\circ_i$-compositions of two μ_2's and one μ_3, i.e., in the bracket language, by nested bracketings of five $\bullet$'s with two binary and one ternary bracket. These elements are in one-to-one correspondence with the edges of the 5th Stasheff associahedron K_5 shown in Fig. 4, see [24, Section II.1.6].

Let $x_e \in \Gamma(\mu_2, \mu_3)(5)_1$ be the element indexed by an edge e of K_5. Clearly $\partial(x_e) = x_a + x_b$, where a, b are the endpoints of e and $x_a, x_b \in \Gamma(\mu_2)(5)_0$ the elements given by the nested bracketings of five $\bullet$'s with three binary brackets corresponding to these endpoints. We concluded that the ∂-cycles in $\Gamma(\mu_2, \mu_3)(5)_1$ are generated by closed edge-paths of even length in K_5; the cycle corresponding to such a path

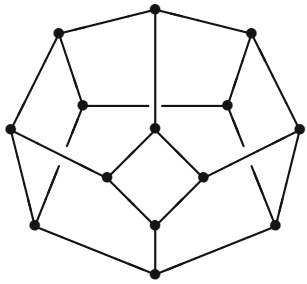


Fig. 4 Stasheff's associahedron K_5

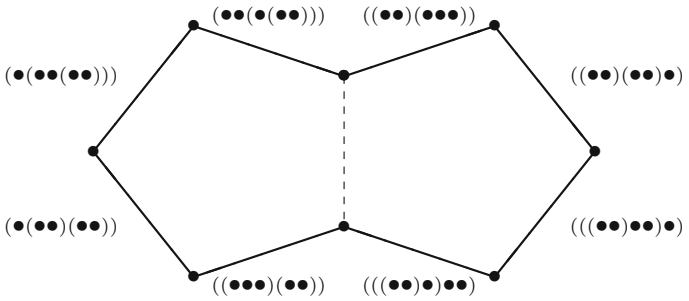


Fig. 5 An closed edge path of length 8 in K_5 defining $\partial(\mu_5^1)$

$P = (e_1, e_2, \ldots, e_{2r})$ being

$$\sum_{1 \leq i \leq 2r} (-1)^{i+1} x_{e_i}.$$

Examples of these paths are provided by two adjacent pentagons in K_5 such as the ones shown in Fig. 5.

There are also three edge paths of length 4 given by the three square faces of K_5, but the corresponding cohomology classes have already been killed by the ∂-images of the compositions $\mu_3 \circ_i \mu_3$, $i = 1, 2, 3$. We showed that there are four linearly independent edge paths of length 8 that, together with the three squares, generate all edge paths of even length in K_5. The generators $\mu_5^1, \mu_5^2, \mu_5^3, \mu_5^4$ correspond to these paths.

Also for $a \geq 6$ the 1-dimensional ∂-cycles in $\Gamma(\mu_2, \mu_3, \mu_5^1, \mu_5^2, \mu_5^3, \mu_5^4)(a)_1$ are given by closed edge paths of even length in the associahedron K_a but one can show that they are all generated by the squares and the images of the paths as in Fig. 5 under the face inclusions $K_5 \hookrightarrow K_a$. Therefore $(\Gamma(\mu_2, \mu_3, \mu_5^1, \mu_5^2, \mu_5^3, \mu_5^4), \partial)$ is acyclic in degree 1, so $\mu_5^1, \mu_5^2, \mu_5^3, \mu_5^4$ are the only degree two generators of the minimal model of $\widetilde{\mathcal{A}ss}$.

The construction extends to a minimal model $(\Gamma(M), \partial)$ of the operad $\widetilde{\mathcal{A}ss}$ whose differential is *not* quadratic. It is simple to show that there does not exist a minimal

algebra $(\Gamma(M'), \partial')$, isomorphic to $(\Gamma(M), \partial)$, with a quadratic differential. Therefore $\widetilde{\mathcal{A}ss}$ does not admit a quadratic minimal model and its non-Koszulness follows not only from the Ginzburg–Kapranov criterion, but also from Fact 9.

From the above description of the minimal model of $\widetilde{\mathcal{A}ss}$ one easily gets the relevant part

$$C^1_{\widetilde{\mathcal{A}ss}}(A;A) \xrightarrow{\delta^1} C^2_{\widetilde{\mathcal{A}ss}}(A;A) \xrightarrow{\delta^2} C^3_{\widetilde{\mathcal{A}ss}}(A;A) \xrightarrow{\delta^3} C^4_{\widetilde{\mathcal{A}ss}}(A;A) \xrightarrow{\delta^4} \cdots$$

of the complex defining the deformation cohomology of an anti-associative algebra $A = (V, \mu)$. One has

- $C^1_{\widetilde{\mathcal{A}ss}}(A;A) = Hom(V, V)$
- $C^2_{\widetilde{\mathcal{A}ss}}(A;A) = Hom(V^{\otimes 2}, V)$
- $C^3_{\widetilde{\mathcal{A}ss}}(A;A) = Hom(V^{\otimes 3}, V)$, and
- $C^4_{\widetilde{\mathcal{A}ss}}(A;A) = Hom(V^{\otimes 5}, V) \oplus Hom(V^{\otimes 5}, V) \oplus Hom(V^{\otimes 5}, V) \oplus Hom(V^{\otimes 5}, V)$.

Observe that $C^p_{\widetilde{\mathcal{A}ss}}(A;A)_{\mathrm{st}} = C^p_{\widetilde{\mathcal{A}ss}}(A;A)$ for $p = 1, 2, 3$, while $C^4_{\widetilde{\mathcal{A}ss}}(A;A)$ consists of 5-linear maps. The differential δ^p agrees with δ^p_{st} for $p = 1, 2$ while, for $g \in C^3_{\widetilde{\mathcal{A}ss}}(A;A)$, one has

$$\delta^3(g) = (\delta^3_1(g), \delta^3_2(g), \delta^3_3(g), \delta^3_4(g)),$$

where

$$\begin{aligned}
\delta^3_1(g)(a,b,c,d,e) &:= ag(b,c,de) - g(a,b,c(de)) + (ab)g(c,d,e) - g(ab,cd,e) \\
&\quad + g(ab,c,d)e - g((ab)c,d,e) + g(a,b,c)(de) - g(a,bc,de), \\
\delta^3_2(g)(a,b,c,d,e) &:= g((ab)c,d,e) - g(ab,c,d)e + g(a,b,cd)e - g(a,b(cd),e) \\
&\quad + ag(b,cd,e) - g(a,b,(cd)e) + (ab)g(c,d,e) - g(ab,c,de), \\
\delta^3_3(g)(a,b,c,d,e) &:= g(a,bc,de) - ag(bc,d,e) + g(a,(bc)d,e) - a(g(b,c,d)e) \\
&\quad + g(a,b,cd)e - g(ab,c,d)e + (g(a,b,c)d)e - g(a(bc),d,e), \text{ and} \\
\delta^3_4(g)(a,b,c,d,e) &:= g(ab,cd,e) - g(a,b,(cd)e) + ag(b,cd,e) - g(a,b(cd),e) \\
&\quad + (ag(b,c,d))e - g(a,bc,d)e + (g(a,b,c)d)e - g(ab,c,d)e,
\end{aligned}$$

for $a, b, c, d, e \in V$. The following proposition follows from [16, Section 4].

Proposition 25 *The cohomology $H^*_{\widetilde{\mathcal{A}ss}}(A;A)$ governs deformations of anti-associative algebras. This means that $H^2_{\widetilde{\mathcal{A}ss}}(A;A)$ parametrizes isomorphism classes of infinitesimal deformations and $H^3_{\widetilde{\mathcal{A}ss}}(A;A)$ contains obstructions to extensions of partial deformations.*

7 Free partially associative n-algebras

In [10], A.V. Gnedbaye described free degree d partially associative n-algebras in the situations when $d = 0$ and n was even. In this section we extend Gnedbaye's description of free $p\mathcal{A}ss^n_d$-algebras to all cases when d and n have the same parity.

Let $p\mathcal{A}ss^n_d(V)$ be the free $p\mathcal{A}ss^n_d$-algebra generated by a graded vector space V. It obviously decomposes as

$$p\mathcal{A}ss^n_d(V) = \bigoplus_{l\geq 0} p\mathcal{A}ss^n_d(V)_l,$$

where $p\mathcal{A}ss^n_d(V)_l \subset p\mathcal{A}ss^n_d(V)$ is the subspace generated by elements obtained by applying the structure n-ary multiplication μ to elements of V l-times. For instance, $p\mathcal{A}ss^n_d(V)_0 \cong V$ and $p\mathcal{A}ss^n_d(V)_1 \cong V^{\otimes n}$.

Denote by $\mathcal{T}^n_l$, $l \geq 1$, the set of planar directed (=rooted) trees with $l(n-1)+1$ leaves whose vertices have precisely n incoming edges (see [20, Section 4] or [24, II.1.5] for terminology). We extend the definition to $l = 0$ by putting $\mathcal{T}^n_0 := \{|\}$, the one-point set consisting of the exceptional tree with one leg and no internal vertex. Clearly, each tree in $\mathcal{T}^n_l$ has exactly l vertices. For each l there is a natural epimorphism

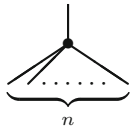
$$\omega : \mathcal{T}^n_l \times V^{\otimes l(n-1)+1} \twoheadrightarrow p\mathcal{A}ss^n_d(V)_l \tag{24}$$

given by interpreting the trees in $\mathcal{T}^n_l$ as the 'pasting schemes' for the iterated multiplication μ. More precisely, if $T \in \mathcal{T}^n_l$ and $v_1, \ldots, v_{l(n-1)+1} \in V$, then

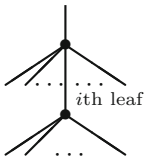
$$\omega\big(T \times (v_1, \ldots, v_{l(n-1)+1})\big) \in p\mathcal{A}ss^n_d(V)_l$$

is obtained by decorating the vertices of T by μ, the leaves of T by elements $v_1, \ldots, v_{l(n-1)+1}$, and performing the indicated composition, observing the Koszul sign rule in the nontrivially graded cases.

Let $\mathcal{S}^n_l \subset \mathcal{T}^n_l$ be the subset consisting of trees having the property that the leftmost incoming edge of each vertex is a leaf. In other words, trees in $\mathcal{S}^n_l$ have no internal edge as in (16). Since these trees correspond to the generators of partially associative algebras considered by Gnedbaye in [10], we call them *Gnedbaye's trees*. Therefore $\mathcal{S}^n_0 = \mathcal{T}^n_0 = \{|\}$, $\mathcal{S}^n_1$ is the one-point set consisting of the n-corolla



and $\mathcal{S}^n_2$ has $n-1$ elements



, $2 \leqslant i \leqslant n$.

As we already mentioned at the beginning of this section, Gnedbaye described, in [10, Proposition 12], free degree d partially associative n-algebras for $d = 0$ and n even. We extend his result to the cases where n and d are of the same parity:

Theorem 26 *Assume that n and d are of the same parity. Then the restriction (denoted by the same symbol)*

$$\omega : \mathcal{S}_l^n \times V^{\otimes l(n-1)+1} \longrightarrow p\mathcal{A}ss_d^n(V)_l \tag{25}$$

of the epimorphism (24) *is an isomorphism, for each* $l \geq 0$.

Observe that, if the parities of d and n are as in the statement, the operad $p\mathcal{A}ss_d^n$ is Koszul by Theorem 15.

Proof [Proof of Theorem 26] Hoffbeck described in [13, Theorem 5.5] an explicit rule that, from a PBW basis of a quadratic operad $\mathcal{P}$, produces a PBW basis of $\mathcal{P}^!$. We apply Hoffbeck's theorem to $\mathcal{P} = t\mathcal{A}ss_{n-d-2}^n$, in which case $\mathcal{P}^! = t\mathcal{A}ss_d^n$ by Proposition 14. Under our assumptions, $n-d-2$ is even so the PBW basis of $t\mathcal{A}ss_{n-d-2}^n$ is described in Remark 18. It is simple to apply Hoffbeck's rule to show that $\bigsqcup_{l\geq 0} \mathcal{S}_l^n$ is the corresponding PBW basis for $p\mathcal{A}ss_d^n$. This clearly implies our result. A detailed self-contained proof of Theorem 26 independent of [13] can be found in a preprint version [21] of the present paper. □

Theorem 26 gives a realization of free $p\mathcal{A}ss_d^n$-algebras in the Koszul case ($n \equiv d \bmod 2$) by putting

$$p\mathcal{A}ss_d^n(V) := \bigoplus_{l\geq 0} \mathcal{S}_l^n \times V^{\otimes l(n-1)+1}.$$

We leave as an exercise to describe the structure n-ary multiplication of $p\mathcal{A}ss_d^n(V)$ in this language, see [10].

Proposition 27 *The Poincaré series of the operad* $p\mathcal{A}ss_d^n$ *is, in the Koszul case (with n and d of the same parity), given by*

$$g_{p\mathcal{A}ss_d^n}(t) = \sum_{l\geq 0} (-1)^{ln} A_l^n t^{l(n-1)+1} \tag{26}$$

where the coefficients $\{A_l^n\}_{l\geq 0}$ *are defined recursively by* $A_0^n := 1$ *and*

$$A_l^n := \sum_{\substack{0\leq l_2,\cdots,l_n\leq l-1 \\ l_2+\cdots+l_{n-1}=l-1}} A_{l_2}^n \cdots A_{l_n}^n \quad \text{for } l \geq 1. \tag{27}$$

Proof One can easily check that the recursive definition (27) of the coefficients of $f(t) := g_{p\mathcal{A}ss^n_d}(t)$ is equivalent to the functional equation

$$f(t) = t\left(1 + (-1)^n f(t)^{n-1}\right)$$

which in turn immediately implies that $f(t)$ is the unique formal solution of

$$g_{t\mathcal{A}ss^n_{-d+n-2}}(-f(-t)) = t,$$

where the Poincaré series $g_{t\mathcal{A}ss^n_{-d+n-2}}(t)$ is as in the first line of (17) because $-d+n-2$ is even. Since we are in the Koszul case, the above display means, by Theorem 8, that $f(t)$ is the Poincaré series of $(t\mathcal{A}ss^n_{-d+n-2})^! = p\mathcal{A}ss^n_d$. This proves the proposition. □

The description of the Poincaré series of $p\mathcal{A}ss^n_d$ for n and d of the same parity given in Proposition 27 implies that the Poincaré series of $\widetilde{p\mathcal{A}ss^n_d}$ for d odd equals

$$g_{\widetilde{p\mathcal{A}ss^n_d}}(t) = \sum_{l\geq 0}(-1)^l A^n_l t^{l(n-1)+1},$$

with $\{A^n_l\}_{l\geq 0}$ having the meaning as in (26).

Example 28 Using `Matematica`, we calculated initial values of the series $\{A^3_l\}_{l\geq 0}$ as $A^3_0 = 1$, $A^3_1 = 1$, $A^3_2 = 2$, $A^3_3 = 5$, $A^3_4 = 14$, $A^3_5 = 42$, $A^3_6 = 132$, $A^3_7 = 429$, $A^3_8 = 1{,}430$, $A^3_9 = 4{,}862$, $A^3_{10} = 16{,}796$, &c.

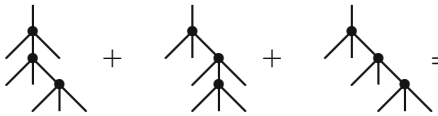
Remark 29 If n and d are of different parities, the map (25) of Theorem 26, while always being an epimorphism, need not be a monomorphism. This means that there may be "unexpected relations" in the free algebra $p\mathcal{A}ss^n_d(V)$. Consequently, the vector space

$$\bigoplus_{l\geq 1} \mathcal{S}^n_l \times V^{\otimes l(n-1)+1}$$

associated to Gnedbaye's trees *cannot be* equipped with a structure of partially associative algebra generated by $V = \mathcal{S}^n_1 \times V$. For instance, while the dimension of S^3_3 equals 5, the dimension of $p\mathcal{A}ss^3_0(7)$ equals $7! \cdot 4$, so $p\mathcal{A}ss^3_0(V)_3$ has one copy of $V^{\otimes 7}$ less than $\mathcal{S}^3_3 \times V^{\otimes 7}$. More concretely, it turns out that

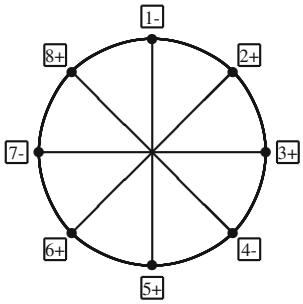
$$v_1(v_2v_3(v_4v_5v_6))v_7 + v_1v_2(v_3(v_4v_5v_6)v_7) + v_1v_2(v_3v_4(v_5v_6v_7)) = 0,$$
$$\text{for } v_1, \ldots, v_7 \in V,$$

in the free algebra $p\mathcal{A}ss^3_0(V)$ and therefore also in every degree-0 partially associative 3-algebra. In terms of Gnedbaye's trees,



(28)

This relation can be read off the corresponding galgal in (12). To do so, decorate the vertices of G^3 by $+$ or $-$ as



(29)

and take the sum of the corresponding elementary relations, with the above choice of signs. Notice that endpoints of all edges in (29) differ by sign, except for the edges

$$\boxed{2}\text{—}\boxed{3},\ \boxed{2}\text{—}\boxed{6}\ \text{ and }\ \boxed{5}\text{—}\boxed{6}.$$

The sum of the elements of the free algebra corresponding to these edges must be zero. It is is represented by Gnedbaye's trees at the left hand side of (28).

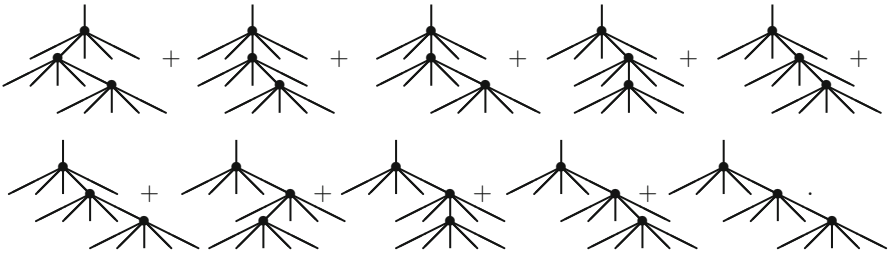
We conclude that the map (25) has, for $n = 3$, $d = 0$ and $l = 3$, a nontrivial kernel. The Poincaré series of $p\mathcal{A}ss_0^3$ was calculated in [11] as

$$g_{p\mathcal{A}ss_0^3}(t) = t + t^3 + 2t^5 + 4t^7 + 5t^9 + 6t^{11} + 7t^{13} + 8t^{15} \cdots .$$

The same phenomenon takes place also for $n = 5$. By choosing appropriate decorations of the vertices of the 5th galgal G^5 depicted in Fig. 2, one can verify that the equation

$$\begin{aligned} 0 = \ & v_1(v_2v_3v_4v_5(v_6v_7v_8v_9v_{10}))v_{11}v_{12}v_{13} + v_1v_2(v_3v_4v_5(v_6v_7v_8v_9v_{10})v_{11})v_{12}v_{13} \\ & + v_1v_2(v_3v_4v_5v_6(v_7v_8v_9v_{10}v_{11}))v_{12}v_{13} + v_1v_2v_3(v_4v_5(v_6v_7v_8v_9v_{10})v_{11}v_{12})v_{13} \\ & + v_1v_2v_3(v_4v_5v_6(v_7v_8v_9v_{10}v_{11})v_{12})v_{13} + v_1v_2v_3(v_4v_5v_6v_7(v_8v_9v_{10}v_{11}v_{12}))v_{13} \\ & + v_1v_2v_3v_4(v_5(v_6v_7v_8v_9v_{10})v_{11}v_{12}v_{13}) + v_1v_2v_3v_4(v_5v_6(v_7v_8v_9v_{10}v_{11})v_{12}v_{13}) \\ & + v_1v_2v_3v_4(v_5v_6v_7(v_8v_9v_{10}v_{11}v_{12})v_{13}) + v_1v_2v_3v_4(v_5v_6v_7v_8(v_9v_{10}v_{11}v_{12}v_{13})) \end{aligned}$$

holds for elements $v_1, \ldots, v_{13}$ of any degree-0 partially associative 5-ary algebra. In terms of Gnedbaye's trees, the right hand side is represented by the sum



We believe that the same explicit calculations can be performed for degree 0 partially associative n-algebras with an arbitrary odd n.

8 Open problems

The first question which our paper leaves open is the Koszulness of the operads $t\mathcal{A}ss^n_d$ with d odd and $n \geq 8$. The method used in the proof of Proposition 23 does not apply to these cases and indeed, our numerical tests mentioned in Remark 24 suggest that it may happen that all coefficients in the formal inverse of $t - t^n + t^{2n-1}$ are non-negative. More about numerical tests related to n-ary algebras can be found in [22].

Even if this happens, it would not necessarily mean that the operad $t\mathcal{A}ss^n_d$ is Koszul, only that subtler methods must be applied to that case. For instance, one may try to compare the coefficients of this formal inverse to the dimensions of the components of the dual operad $(t\mathcal{A}ss^n_d)^!$.

Understanding these components is, of course, equivalent to finding a basis for the free partially associative algebras in the non-Koszul cases. This problem was solved, in [11], for free $p\mathcal{A}ss^3_0$-algebras; for $n \geq 4$ it remains open.

The last problem we want to formulate here is to find more about the minimal model of the anti-associative operads $\widetilde{\mathcal{A}ss}$, or even to describe it completely. As far as we know, beyond the 'obvious' cases, no complete description of the minimal model of a non-Koszul operad is known. Since $\widetilde{\mathcal{A}ss}$ is one of the simplest non-Koszul operads, it is the first obvious candidate to attack. A related task is to find as much as information about minimal models of the remaining non-Koszul n-ary operads as possible.

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References

1. Baez, J.C.: The Octonions. Bull. Am. Math. Soc. **39**, 145–205 (2002)
2. Dotsenko, V., Khoroshkin, A.: Gröbner bases for operads. Duke Math. J. **153**(2), 363–396 (2010)
3. Dotsenko, V., Khoroshkin, A.: Quillen homology for operads via Gröbner bases. Doc. Math. **18**, 707–747 (2013)
4. Dotsenko, V., Vallette, B.: Higher Koszul Duality for Associative Algebras. 2012. arXiv:1201.6509
5. Fox, T.F.: An introduction to algebraic deformation theory. J. Pure Appl. Algebra **84**, 17–41 (1993)

6. Fox, T.F., Markl, M.: Distributive laws, bialgebras, and cohomology. In: Loday, J.-L., Stasheff, J.D., Voronov, A.A. (eds.) Operads: Proceedings of Renaissance Conferences of Contemporary Mathematics, American Mathematical Society, vol. 202, pp. 167–205 (1997)
7. Fresse, B.: Koszul duality of operads and homology of partition posets. Contemp. Math. **346**, 115–215 (2004)
8. Getzler, E., Jones, J.D.S.: Operads, homotopy algebra, and iterated integrals for double loop spaces. Preprint March 1994. arXiv:hep-th/9403055
9. Ginzburg, V., Kapranov, M.M.: Koszul duality for operads. Duke Math. J. **76**(1), 203–272 (1994)
10. Gnedbaye, A.V.: Opérades des algèbres $(k+1)$-aires. In: Loday, J.L., Stasheff, J.D., Voronov, A.A. (eds.) Operads: Proceedings of Renaissance Conferences of Contemporary Mathematics, American Mathematical Society, vol. 202, pp. 83–113 (1996)
11. Goze, N., Remm, E.: Dimension theorem for free ternary partially associative algebras and applications. J. Algebra **348**, 14–36 (2011)
12. Hanlon, P., Wachs, M.L.: On Lie k-algebras. Adv. Math. **113**, 206–236 (1995)
13. Hoffbeck, E.: A Poincaré–Birkhoff–Witt criterion for Koszul operads. Manuscripta Math. **131**(1–2), 87–110 (2010)
14. Loday, J.-L., Vallette, B.: Algebraic operads, Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), vol. 346, Springer, Heidelberg (2012)
15. Markl, M.: A cohomology theory for $A(m)$-algebras and applications. J. Pure Appl. Algebra **83**, 141–175 (1992)
16. Markl, M.: Cotangent cohomology of a category and deformations. J. Pure Appl. Algebra **113**(2), 195–218 (1996)
17. Markl, M.: Models for operads. Comm. Algebra **24**(4), 1471–1500 (1996)
18. Markl, M.: Intrinsic brackets and the L_∞-deformation theory of bialgebras. J. Homotopy Relat. Struct. **5**(1), 177–212 (2010)
19. Markl, M.: A resolution (minimal model) of the PROP for bialgebras. J. Pure Appl. Algebra **205**(2), 341–374 (2006)
20. Markl, M.: Handbook of algebra, chapter Operads and PROPs, vol. 5, pp. 87–140. Elsevier, New York (2008)
21. Markl, M., Remm, E.: Version 2 of Aug 2011. arXiv:0907.1505
22. Markl, M., Remm, E.: Operads for n-ary algebras: calculations and conjectures. Arch. Math. (Brno) **47**(5), 377–387 (2011)
23. Markl, M., Shnider, S.: Coherence constraints for operads, categories and algebras. In: Proceedings of the 20th Winter School "Geometry and Physics", Srní, Czech Republic, 15–22 Jan 2000, Supplem. ai Rend. Circ. Matem. Palermo, Ser. II, vol. 66, pp. 29–57 (2001)
24. Markl, M., Shnider, S., Stasheff, J.D.: Operads in Algebra, Topology and Physics. In: Proceedings of Mathematical Surveys and Monographs. American Mathematical Society, Providence, vol. 96 (2002)
25. Stasheff, J.D.: Homotopy associativity of H-spaces I, II. Trans. Am. Math. Soc. **108**, 275–312 (1963)