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Combination of universal spaces for proper actions

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Abstract Given a group action on a simplicial complex such that each simplex stabiliser admits a cocompact model for its universal space for proper actions, we give conditions implying the existence of a cocompact model for the universal space for proper actions of the whole group. This is used to generalise previous combination results for boundaries of groups and hyperbolic groups.

Keywords Universal space for proper actions · Classifying space for proper actions · Complexes of groups · Complexes of spaces · Hyperbolic groups

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1 Introduction

A central problem of geometric group theory is to understand, given a group action on a simplicial complex, to what extent the properties of the group come from properties of the stabilisers of simplices. A space which encodes many informations about a group is its *universal space for proper actions* (sometimes called *classifying space* for proper actions in the literature). In this article, we focus on the existence of a *cocompact* model for the universal space for proper actions of a group, that is to say, a model such that the quotient space under the group action is compact. In the case of an amalgamated product or HNN extension, Scott and Wall [9] construct an explicit model for the universal space for proper actions as a *tree of spaces* over the Bass–Serre

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tree of the splitting. We prove the following generalisation for complexes of groups of arbitrary dimension:

Theorem 1 *Let $G(\mathcal{Y})$ be a developable complex of groups over a finite simplicial complex Y, such that:*

- *for every finite subgroup of the fundamental group of $G(\mathcal{Y})$, the associated fixed-point set in the universal cover of $G(\mathcal{Y})$ is contractible,*
- *every local group admits a cocompact model for the universal space for proper actions.*

Then there exists a cocompact model for the universal space for proper actions of the fundamental group of $G(\mathcal{Y})$, obtained as a complex of spaces over the universal cover of $G(\mathcal{Y})$, and with fibres universal spaces for proper actions of the local groups of $G(\mathcal{Y})$.

Similar combination results were proved, using different techniques, by Lück–Weiermann [6]. Namely, they prove the existence of a cocompact model for the universal space for proper actions of the fundamental group of a complex of groups in the case of *simple* complexes of groups (that is, complexes of groups where all the twisting elements are trivial) [6, Theorem 4.3], and for developable complexes of groups under a condition of contractible fixed-point sets for *every* subgroup of the fundamental group which fixes a vertex of the associated universal cover [6, Proposition 5.1]. One advantage of our construction lies in the very explicit description of the resulting universal space as a complex of spaces, which allows us to study the geometry of the fundamental group out of the geometry of the local groups.

Certain compactifications of cocompact models for the universal space for proper actions, when they exist, have proved to be very useful as they imply for instance the Novikov conjecture for the group [3]. In [8], the author studied the asymptotic topology of a group acting cocompactly on a non-positively curved simplicial complex by means of the asymptotic topology of the stabilisers of simplices. As an example, the following theorem was proved:

Theorem (M.[8]) *Let $G(\mathcal{Y})$ be a non-positively curved complex of groups over a finite piecewise-Euclidean complex Y, such that there exists a complex of universal spaces compatible with $G(\mathcal{Y})$. Let G be the fundamental group of $G(\mathcal{Y})$ and X be a universal cover of $G(\mathcal{Y})$. Assume that:*

- *The universal cover X is hyperbolic (for the associated piecewise-Euclidean structure),*
- *The local groups are hyperbolic and all the local maps are quasiconvex embeddings,*
- *The action of G on X is acylindrical.*

Then G is hyperbolic and the local groups embed in G as quasiconvex subgroups. □

To prove such a theorem, the first step is to combine the universal spaces for proper actions of the various stabilisers of simplices (in this particular case, Rips complexes) into a universal space for proper actions of the fundamental group of the complex of groups; the appropriate data used to construct such a universal space is the notion of a *complex of universal spaces compatible with a complex of groups* mentioned in the previous theorem, and introduced in [8]. Such a universal space is then constructed as *complex of spaces* over the universal cover of the complex of groups. With such a model at hand, we can try to understand the asymptotic topology of the whole space by means of the asymptotic topology of its smaller pieces.

Constructing a complex of universal spaces compatible with a given complex of groups is a non-trivial problem. This can be carried out by ad hoc constructions when the combinatorics of the underlying complex of groups is quite simple (for instance, in the case of a simple complex of hyperbolic groups [8], or in the case of metric small cancellation over a graph of groups [7]), but can prove to be much harder in general. There are many examples of groups acting on simplicial complexes for which even the simplicial structure of the quotient complex is hard to describe: the action of the mapping class group of a hyperbolic surface on its curve complex, the action of a group admitting a codimension one subgroup on the associated CAT(0) cube complex, etc. Thus, if ones wants to use the previous combination theorem to study groups through their non-proper actions on simplicial complexes in general, it would be preferable to have a way to construct compatible complexes of universal spaces which does not rely on the combinatorics of the associated complex of groups.

In this article, we give a general procedure for constructing such objects.

Theorem 2 *Let $G(\mathcal{Y})$ be a complex of groups over a finite simplicial complex Y. Then there exists a compatible complex of universal spaces, and we can require the local maps to be embeddings.*

In particular, each theorem of [8] holds with the assumption of the existence of a compatible complex of universal spaces removed. For instance, the previous combination theorem can be reformulated in the following way:

Corollary *Let G be a group acting without inversion and cocompactly on a piecewise-Euclidean simplicial complex such that:*

- *the complex X is hyperbolic and CAT(0),*
- *the stabilisers of simplices are hyperbolic and they embed into one another as quasiconvex subgroups,*
- *the action of G on X is acylindrical.*

Then G is hyperbolic. Furthermore, the stabilisers of simplices embed in G as quasiconvex subgroups.

To obtain compatible complexes of universal spaces, we follow a construction due to Haefliger [5]: given a complex of groups $G(\mathcal{Y})$ over a simplicial complex Y with contractible universal cover, he constructs an Eilenberg–MacLane space for the fundamental group of $G(\mathcal{Y})$ as a complex of spaces over Y. As we want to allow groups with torsion, the point of view adopted here is slightly different. Instead of reasoning over Y, we will be working over some particular subcomplexes of the universal cover of $G(\mathcal{Y})$, called *blocks*. The fibres will not be Eilenberg–MacLane spaces but universal spaces for proper actions, and all the constructions will be made equivariant.

We briefly mention a generalisation of this result. Let $\mathcal{F}$ be a family of subgroups of G that is stable under conjugation and taking subgroups. Recall that a model for the universal space with stabilisers in $\mathcal{F}$ for a group G, usually denoted $E_{\mathcal{F}}G$, is a G-CW complex X such that for every subgroup F of G, the fixed-point set X^F is contractible if F is in $\mathcal{F}$ and empty otherwise. A group G is of type $\mathcal{F}$-F_n, $n \geq 0$, if there exists a model of $E_{\mathcal{F}}G$ whose n-skeleton is finite modulo the action of the group (in the case where $\mathcal{F}$ is the family of finite subgroups of G, this is also denoted $\underline{F}_n$ or Bredon-F_n in the literature). The construction described in this article carries over to the following situation:

Corollary *Let $G(\mathcal{Y})$ be a developable complex of groups over a finite simplicial complex Y, with fundamental group G and universal cover X, let $\mathcal{F}$ be a family of subgroups of G that is stable under conjugation and taking subgroups, and let $n \geq 0$ be an integer. Suppose that:*

- *for every subgroup $F \in \mathcal{F}$, the associated fixed-point set X^F is contractible,*
- *for every simplex σ of Y, the local group G_σ is of type $\mathcal{F}$-$F_{n\text{-}\dim\sigma}$.*

Then G is of type $\mathcal{F}$-F_n and a model of $E_{\mathcal{F}}G$ is obtained as a complex of spaces over X with models of $E_{\mathcal{F}}G_\sigma$ as fibres.

It should be noted that, although this article is written in the framework of complexes of groups over simplicial complexes for simplicity reasons, the constructions carry over without any essential change to the case of complexes of groups over *polyhedral* complexes.

The article is organised as follows. In Sect. 2, we review a few elementary facts on complexes of groups. In Sect. 3, we define the block associated to a simplex and study the induced complex of groups. In Sect. 4, we recall the definition of a complex of spaces (in the sense of Corson [2]) and of a complex of universal spaces compatible with a complex of groups [8]. Section 5 is devoted to the construction of a compatible complex of universal spaces. Finally, Sect. 6 details a construction that gives an interesting upper bound on the geometric dimension of the group.

2 Background on complexes of groups

2.1 Definitions

Complexes of groups are a high-dimensional generalisation of graphs of groups, that is, objects encoding group actions on arbitrary simplicial complexes, and were introduced by Gersten–Stallings [10], Corson [2] and Haefliger [4]. Haefliger defined a notion of complexes of groups over more combinatorial objects called *small categories without loops*. We recall in this section basic definitions and properties of complexes of groups; for a deeper treatment of these notions, we refer the reader to [1].

Definition 2.1 (*Small category without loop*) A *small category without loop* (briefly a *scwol*) is a set $\mathcal{Y}$ which is the disjoint union of a set $V(\mathcal{Y})$ called the vertex set of $\mathcal{Y}$, and a set $A(\mathcal{Y})$ called the set of edges[1] of $\mathcal{Y}$. Each edge a of $\mathcal{Y}$ comes with a choice

[1] In the literature, the set of edges is usually denotes $E(\mathcal{Y})$. Here however, as the letter E will be used to denote universal spaces (or spaces constructed out of such universal spaces), we use the French notation $A(\mathcal{Y})$ so as to avoid confusions.

of two distinct vertices of $\mathcal{Y}$, denoted $i(a)$ and $t(a)$ and called respectively the *initial* and *terminal* vertices of a.

For $k \geq 1$, let $A^{(k)}(\mathcal{Y})$ be the set of sequences $(a_k, \ldots, a_1)$ of edges of $\mathcal{Y}$ such that $i(a_{i+1}) = t(a_i)$ for $1 \leq i < k$ (the sequence of edges $a_k, \ldots, a_1$ is said to be *composable*). For $A = (a_k, \ldots, a_1) \in A^{(k)}(\mathcal{Y})$, we set $i(A) := i(a_1)$ and $t(A) := t(a_k)$. By convention, we set $A^{(0)}(\mathcal{Y}) = V(\mathcal{Y})$.

To each pair (b, a) of composable edges of $\mathcal{Y}$ is associated an edge ba, called their *concatenation* or *composition*, satisfying the following conditions:

- For every $(b, a) \in A^{(2)}(\mathcal{Y})$, we have $i(ba) = i(a)$ and $t(ba) = t(b)$;
- For every $(c, b, a) \in A^{(3)}(\mathcal{Y})$, we have $(cb)a = c(ba)$ (and the composition is simply denoted cba).

A important example of scwol is the following:

Definition 2.2 (*Simplicial scwol associated to a simplicial complex*) If Y is a simplicial complex, we associate to Y a scwol $\mathcal{Y}$ as follows:

- Vertices of $\mathcal{Y}$ correspond to simplices of Y,
- Edges of $\mathcal{Y}$ correspond to pairs of simplices (σ, σ') such that $\sigma \subset \sigma'$. For such a pair, we set $i(a) = \sigma'$ and $t(a) = \sigma$.
- For composable edges $b = (\sigma, \sigma')$ and $a = (\sigma', \sigma'')$, we set $ba = (\sigma, \sigma'')$.

The scwol $\mathcal{Y}$ is called the *simplicial scwol associated to X*.

In this article, we will often omit the distinction between a simplex σ of Y and the associated vertex of $\mathcal{Y}$.

Definition 2.3 (*Geometric/Simplicial realisation of a scwol*) Let $\mathcal{Y}$ be a scwol. For integers $k \geq 2$ and $0 \leq i \leq k$, we define maps $\partial_i : A^{(k)}(\mathcal{Y}) \to A^{(k-1)}(\mathcal{Y})$ as follows:

$$\begin{aligned}
\partial_0(a_k, \ldots, a_1) &= (a_k, \ldots, a_2) \\
\partial_i(a_k, \ldots, a_1) &= (a_k, \ldots, a_{i+1}a_i, \ldots, a_1) \quad 1 \leq i < k \\
\partial_k(a_k, \ldots, a_1) &= (a_{k-1}, \ldots, a_1).
\end{aligned}$$

For $k = 1$, we set $\partial_0 a = i(a)$ and $\partial_1(a) = t(a)$.

Let Δ^k be the standard Euclidean k-simplex, that is, the set of elements $(t_0, \ldots, t_k)$ with $t_i \geq 0$ and $\sum_i t_i = 1$. For $k \geq 1$ and $0 \leq i \leq k$, we define maps $d_i : \Delta^{k-1} \to \Delta^k$ by sending $(t_0, \ldots, t_{k-1})$ to $(t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{k-1})$.

The *geometric realisation* of the scwol $\mathcal{Y}$ is the space obtained from the disjoint union

$$\coprod_{k \geq 0, A \in A^{(k)}(\mathcal{Y})} \{A\} \times \Delta^k$$

by identifying pairs of the form $(\partial_i A, x)$ and $(A, d_i(x))$; this is a piecewise-Euclidean simplicial complex. We call the underlying simplicial complex the *simplicial realisation* of $\mathcal{Y}$.

In what follows, we will make no difference between simplicial and geometric realisations.

Remark 2.4 The simplicial realisation of the scwol associated to a simplicial complex Y is naturally isomorphic to the first barycentric subdivision Y' of Y.

Definition 2.5 (*Complex of groups* [1]) Let $\mathcal{Y}$ be a scwol. A *complex of groups* $G(\mathcal{Y}) = (G_\sigma, \psi_a, g_{b,a})$ *over* $\mathcal{Y}$ consists of the following:

- for each vertex σ of $\mathcal{Y}$, a group G_σ called the *local group* at σ,
- for each edge a of $\mathcal{Y}$, an injective homomorphism $\psi_a : G_{i(a)} \to G_{t(a)}$,
- for each pair of composable edges (b, a) of $\mathcal{Y}$, a *twisting element* $g_{b,a} \in G_{t(b)}$,

which satisfy the following conditions:

- For $(b, a) \in A^{(2)}(\mathcal{Y})$, we have

$$\mathrm{Ad}(g_{b,a})\psi_{ba} = \psi_b\psi_a,$$

  where $\mathrm{Ad}(g_{b,a}) : g \mapsto g_{b,a} \cdot g \cdot g_{b,a}^{-1}$ is the conjugation by $g_{b,a}$ in $G_{t(b)}$;
- For $(c, b, a) \in A^{(3)}(\mathcal{Y})$, the following cocycle condition holds:

$$\psi_c(g_{b,a})g_{c,ba} = g_{c,b}g_{cb,a}.$$

Notation 2.6 If a is an edge of $\mathcal{Y}$ corresponding to an inclusion $\sigma \subset \sigma'$, we will sometimes write $\psi_{\sigma,\sigma'}$ in place of ψ_a. By convention, we also define $\psi_{\sigma,\sigma}$ as the identity map of G_σ.

Definition 2.7 (*Morphism of complex of groups*) Let Y, Y' be simplicial complexes, $\mathcal{Y}$, $\mathcal{Y}'$ the associated simplicial scwols, and let $G(\mathcal{Y})$ and $G(\mathcal{Y}')$ be complexes of groups over respectively Y and Y'. Let $f : Y \to Y'$ be a non-degenerate simplicial map (that is, the restriction of f to any simplex is a homeomorphism onto its image). A *morphism* $F = (F_\sigma, F(a)) : G(\mathcal{Y}) \to G(\mathcal{Y}')$ over f consists of the following:

- for each vertex σ of $\mathcal{Y}$, a *local morphism* $F_\sigma : G_\sigma \to G_{f(\sigma)}$,
- for each edge a of $\mathcal{Y}$, an element $F(a) \in G_{t(f(a))}$ such that

$$\mathrm{Ad}(F(a))\psi_{f(a)}F_{i(a)} = F_{t(a)}\psi_a,$$

  and such that for every pair (b, a) of composable edges of $\mathcal{Y}$, we have

$$F_{t(b)}(g_{b,a})F(ba) = F(b)\psi_{f(b)}(F(a))g_{f(b),f(a)}.$$

If all the local morphisms F_σ are isomorphisms and f is a simplicial isomorphism, F is called an *isomorphism*.

Definition 2.8 (*Morphism from a complex of groups to a group*) Let $G(\mathcal{Y})$ be a complex of groups over a scwol $\mathcal{Y}$ and G be a group. A *morphism* $F = (F_\sigma, F(a))$ from $G(\mathcal{Y})$ to G consists of a homomorphism $F_\sigma : G_\sigma \to G$ for each vertex σ of $\mathcal{Y}$ and an element $F(a) \in G$ for each $a \in E(\mathcal{Y})$, satisfying the following conditions:

- for every edge a of $\mathcal{Y}$, we have $F_{t(a)}\psi_a = \mathrm{Ad}(F(a))F_{i(a)}$,
- for every pair (b, a) of composable edges of $\mathcal{Y}$, we have $F_{t(b)}(g_{b,a})F(ba) = F(b)F(a)$.

We say that a morphism $F = (F_\sigma, F(a))$ from $G(\mathcal{Y})$ to G is *injective on the local groups* if all the local morphisms F_σ are injective.

2.2 Developability

Definition 2.9 (*Complex of groups associated to an action without inversion of a group on a simplicial complex, developable complex of groups* [1]) Let G be a group acting without inversion by simplicial isomorphisms on a simplicial complex X. Let Y be the quotient space and $p : X \to Y$ the quotient map. Up to a barycentric subdivision, we can assume that p restricts to a embedding on every simplex, which yields a simplicial structure for Y. Let $\mathcal{Y}$ be the associated simplicial scwol.

For each vertex σ of $\mathcal{Y}$, we choose a simplex $\widetilde{\sigma}$ of X such that $p(\widetilde{\sigma}) = \sigma$. By assumption, the restriction of p to any simplex σ of X is a homeomorphism onto its image, so that for every simplex σ' contained in σ, there is a unique simplex τ of X and contained in $\widetilde{\sigma}$, such that $p(\tau) = \sigma'$. For each edge $a = (\sigma', \sigma)$ of $\mathcal{Y}$, we choose an element $h_a \in G$ such that $h_a.\tau = \widetilde{\sigma'}$. We then define a complex of groups over Y, called a complex of groups $G(\mathcal{Y}) = (G_\sigma, \psi_a, g_{b,a})$ over Y *associated to the action of G on X*, as follows:

- for each vertex σ of $\mathcal{Y}$, the local group G_σ is the stabiliser of $\widetilde{\sigma}$,
- for every edge a of $\mathcal{Y}$, the local map $\psi_a : G_{i(a)} \to G_{t(a)}$ is defined by

$$\psi_a(g) = h_a g h_a^{-1},$$

- for every pair (b, a) of composable edges of $\mathcal{Y}$, we define the twisting element

$$g_{b,a} = h_b h_a h_{ba}^{-1}.$$

This complex of groups comes with an associated morphism $F = (F_\sigma, F(a))$ from $G(\mathcal{Y})$ to G, where $F_\sigma : G_\sigma \to G$ is the natural inclusion and $F(a) = h_a$.

A complex of groups over a simplicial complex is said to be *developable* if it is isomorphic to the complex of groups associated to an action without inversion on a simplicial complex.

We have the following characterisation of developability:

Theorem 2.10 (Theorem III.$\mathcal{C}$.2.15 of [1]) *A complex of groups $G(\mathcal{Y})$ is developable if and only if there exists a morphism from $G(\mathcal{Y})$ to some group which is injective on the local groups.* □

3 Induced complex of groups over a block

Unlike in Bass–Serre theory, not every complex of groups is developable. However, non-developability is a global phenomenon, a complex of groups being always devel-

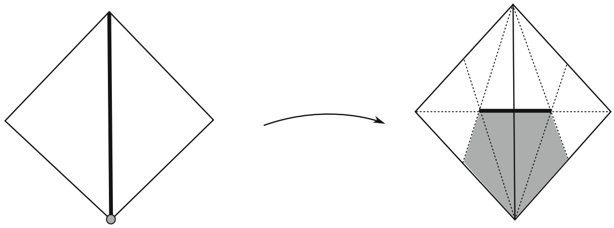

Fig. 1 On the *left*, a simplicial complex with a chosen vertex (in *grey*) and edge (in *bold*). On the *right*, the associated blocks

opable around a vertex. In this section, we describe for every simplex σ of Y, a sub-scwol associated to σ, called a *block*, such that the induced complex of groups is developable.

3.1 The block associated to a simplex.

Given a simplex σ of Y, we consider the sub-scwol $\mathcal{B}(\sigma) \subset \mathcal{Y}$ whose vertex set consists of those simplices of Y containing σ and whose set of edges consists of those edges of $\mathcal{Y}$ whose initial and terminal vertices are in $V(\mathcal{B}(\sigma))$. The simplicial realisation of $\mathcal{B}(\sigma)$ is a simplicial complex $B(\sigma)$, called the *block* associated to σ, which is isomorphic to the subcomplex $\{\sigma\} \star \mathrm{lk}(\{\sigma\}, Y')$ of the first barycentric subdivision Y' of Y (where $\{\sigma\}$ denotes the vertex of Y' corresponding to the simplex σ) (Fig. 1) .

For every inclusion of simplices $\sigma \subset \sigma'$, we have an inclusion of blocks $B(\sigma') \subset B(\sigma)$. Moreover, since the block $B(\sigma)$ is simplicially a cone over the link $\mathrm{lk}(\{\sigma\}, Y')$, it is contractible.

3.2 The local development

We denote by $G\big(\mathcal{B}(\sigma)\big)$ the induced complex of groups over $\mathcal{B}(\sigma)$, that is, the pullback of $G(\mathcal{Y})$ under the inclusion $\mathcal{B}(\sigma) \hookrightarrow \mathcal{Y}$.

We define a morphism F_σ from $G\big(\mathcal{B}(\sigma)\big)$ to G_σ as follows. For every vertex τ of $\mathcal{B}(\sigma)$ (that is, for every simplex τ of Y containing σ), the map $\big(F_\sigma\big)_\tau$ is the map $\psi_{\sigma,\tau}$. Let a be an edge of $\mathcal{B}(\sigma)$. If $t(a) = \sigma$, we define $F_\sigma(a)$ as the identity element of G_σ. Otherwise, let b be the edge from $t(a)$ to σ, and we set $F_\sigma(a) = g_{b,a}$.

This defines a morphism which is injective on the local groups, so that $G\big(\mathcal{B}(\sigma)\big)$ is developable by Theorem 2.10. We denote by $\widetilde{\mathcal{B}(\sigma)}$ the development of $\mathcal{B}(\sigma)$ associated to this morphism, and by $\widetilde{B(\sigma)}$ the simplicial realisation of $\widetilde{\mathcal{B}(\sigma)}$. We have the following description of $\widetilde{\mathcal{B}(\sigma)}$ [1, Theorem III.$\mathcal{C}$.2.13]:

$$V(\widetilde{\mathcal{B}(\sigma)}) = \coprod_{\tau \in V(\mathcal{B}(\sigma))} \Big(\psi_{\sigma,\tau}(G_\tau)\backslash G_\sigma \times \{\tau\}\Big),$$
$$A(\widetilde{\mathcal{B}(\sigma)}) = \coprod_{a \in A(\mathcal{B}(\sigma))} \Big(\psi_{\sigma,i(a)}(G_{i(a)})\backslash G_\sigma \times \{a\}\Big);$$

the initial and terminal vertices are defined as follows:

$$i\big([g], a\big) = \big([g], i(a)\big),$$
$$t\big([g], a\big) = \big([g]F_\sigma(a)^{-1}, t(a)\big);$$

the composition is

$$\big([g], b\big)\big([gF_\sigma(a)^{-1}], a\big) = \big([g], ba\big)$$

where (b, a) is a pair of composable edges of $\mathcal{B}(\sigma)$ and g an element of G_σ. We also have, for an integer $k \geq 1$:

$$A^{(k)}(\widetilde{\mathcal{B}(\sigma)}) = \coprod_{A \in A^{(k)}(\mathcal{B}(\sigma))} \Big(\psi_{\sigma,i(A)}(G_{i(A)})\backslash G_\sigma \times \{A\}\Big).$$

The simplicial realisation $\widetilde{B(\sigma)}$ has the structure of a simplicial cone, hence is contractible. If $G(\mathcal{Y})$ is developable, such a cone is simplicially isomorphic to the block associated to any lift $\widetilde{\sigma}$ of σ.

4 Complexes of spaces

Complexes of spaces are a high-dimensional generalisation of graphs of spaces. They were considered, in relation with complexes of groups, by Corson [2] and Haefliger [5]. Here, we will only be dealing with Corson's definition as it is more flexible (see Remark 4.2 for further details).

Definition 4.1 (*Complexes of spaces in the sense of Corson* [2]) Let Y be a connected simplicial complex. A complex of spaces over Y is a connected simplicial complex X together with a simplicial map $p : X \to Y$ such that for each open simplex σ of Y, $p^{-1}(\sigma)$ is a connected subcomplex of X of the form $X_\sigma \times \sigma$, where X_σ is the pre-image of the centre of the simplex σ (the *fibre* of σ), and such that for every subface τ of σ, the induced map on fundamental groups $\pi_1(X_\sigma) \to \pi_1(X_\tau)$, obtained by translating the base point along a path in X_σ, is injective. Furthermore, we require that the topology on X be coherent with subsets of the form $p^{-1}(\overline{\sigma})$, where $\overline{\sigma}$ is a closed simplex of Y.

Remark 4.2 Haefliger's definition imposes in addition that the restriction of the projection $p : X \to Y$ to the 1-skeleton of X admits a section $s : Y^{(1)} \to X^{(1)}$. This however turns out to be incompatible with the equivariance wanted in our constructions.

Let $G(\mathcal{Y})$ be a developable complex of groups over a simplicial complex Y with a contractible universal cover. Haefliger [5] and independently Corson construct an Eilenberg–MacLane space for the fundamental group of $G(\mathcal{Y})$ as a complex of spaces over Y, with fibres Eilenberg–MacLane spaces for the local groups. As we are interested in cocompact models for universal spaces for proper actions, we adopt a slightly different point of view.

Definition 4.3 (*Complex of universal spaces compatible with a complex of groups* [8]) Let $G(\mathcal{Y}) = (G_\sigma, \psi_a, g_{b,a})$ be a complex of groups over a simplicial complex Y. A *complex of universal spaces* $\underline{E}G(\mathcal{Y})$ *compatible with the complex of groups* $G(\mathcal{Y})$ consists of the following:

- For every vertex σ of $\mathcal{Y}$, a space $\underline{E}G_\sigma$ (called a *fibre*) that is a cocompact model for the universal space for proper actions of the local group G_σ,
- For every edge a of $\mathcal{Y}$, a ψ_a-equivariant map $\phi_a : \underline{E}G_{i(a)} \to \underline{E}G_{t(a)}$, that is, for every $g \in G_{i(a)}$ and every $x \in \underline{E}G_{i(a)}$, we have

$$\phi_a(g.x) = \psi_a(g).\phi_a(x),$$

and such that for every pair (b, a) of composable edges of $\mathcal{Y}$, we have:

$$g_{b,a} \cdot \phi_{ba} = \phi_b\phi_a.$$

Notation 4.4 If a is an edge of $\mathcal{Y}$ corresponding to an inclusion $\sigma \subset \sigma'$, we will sometimes write $\phi_{\sigma,\sigma'}$ in place of ϕ_a. By convention, we also define $\phi_{\sigma,\sigma}$ as the identity map of $\underline{E}G_\sigma$.

We emphasise that a complex of universal spaces compatible with the complex of groups $G(\mathcal{Y})$ is *not* a complex of spaces over Y if the twist coefficients $g_{b,a}$ are not trivial. However, the following holds:

Theorem 4.5 [8] *Let $G(\mathcal{Y})$ be a developable complex of groups over a finite simplicial complex Y, such that for every finite subgroup of the fundamental group of $G(\mathcal{Y})$, the associated fixed-point set in the universal cover of $G(\mathcal{Y})$ is contractible. Assume that there is a complex of universal spaces $\underline{E}G(\mathcal{Y})$ compatible with $G(\mathcal{Y})$. Then there exists a cocompact model for the universal space for proper actions of the fundamental group of $G(\mathcal{Y})$, obtained as a complex of spaces (in the sense of Corson) over the universal cover of $G(\mathcal{Y})$, and with fibres universal spaces for proper actions of the local groups of $G(\mathcal{Y})$.* □

5 The topological construction

From now on, we assume that we are given a complex of groups $G(\mathcal{Y})$ over a finite simplicial complex Y, such that each local group admits a cocompact model for the universal space for proper actions. For each simplex σ of Y, we choose a cocompact model for the universal space for proper actions to be a *based* G_σ-space with a chosen basepoint, that is, with a chosen G_σ-orbit and a preferred point in that orbit. For every edge $a \in A(\mathcal{Y})$, we choose a based ψ_a-equivariant continuous map φ_a from $\underline{E}G_{i(a)}$ to $\underline{E}G_{t(a)}$. Without loss of generality, we can assume that the maps ψ_a preserve basepoints.

Let $k \geq 1$ be an integer, and denote by I^k the k-dimensional cube $[0, 1]^k$.

In this section, we construct by induction CW-complexes $E_0(\sigma), E_1(\sigma), \ldots$ such that for every integer $k \geq 0$, $E_k(\sigma)$ is a complex of spaces (in the sense of Corson) over the k-skeleton of the block $\widetilde{B(\sigma)}$. In order to construct these spaces, we need the following observation:

Lemma 5.1 *Let $\sigma \subset \sigma'$ be simplices of Y, $k \geq 1$ an integer, and $f : \underline{E}G_{\sigma'} \times \partial I^k \to \underline{E}G_\sigma$ a $\psi_{\sigma,\sigma'}$-equivariant map, where $G_{\sigma'}$ acts trivially on I^k. Then f extends to a $\psi_{\sigma,\sigma'}$-equivariant map $F : \underline{E}G_{\sigma'} \times I^k \to \underline{E}G_\sigma$.*

Proof First notice that the space $\underline{E}G_{\sigma'} \times \partial I^k$ is a $G_{\sigma'}$-space whose isotropy groups are finite by definition of $\underline{E}G_{\sigma'}$. Furthermore, for each such finite subgroup H of $G_{\sigma'}$, the fixed-point set $(\underline{E}G_\sigma)^{\psi_{\sigma,\sigma'}(H)}$ is contractible by definition of $\underline{E}G_\sigma$.

We start with the case $k \geq 2$. The function f is defined on $\underline{E}G_{\sigma'} \times \partial I^k$, so in particular on the 1-skeleton of $\underline{E}G_{\sigma'} \times I^k$. Using the above remarks, it is then a standard consequence of equivariant obstruction theory that the map $f : \underline{E}G_{\sigma'} \times \partial I^k \to \underline{E}G_\sigma$ equivariantly extends to $\underline{E}G_{\sigma'} \times I^k$.

For $k = 1$, we are given two equivariant maps $f_1, f_2 : \underline{E}G_{\sigma'} \to \underline{E}G_\sigma$, which coincide on the chosen $G_{\sigma'}$-orbit by assumption. Thus we can naturally extend the map f to the 1-skeleton of $\underline{E}G_{\sigma'} \times I$, and one then concludes with the same reasoning as above. □

Let $A = (a_k, \dots, a_1)$ be an element of $A^{(k)}(\sigma)$. Following Haefliger [5], we define a polyhedral map $r_k : I^k \to \Delta^k$ as follows. The vertex $(0, \dots, 0) \in I^k$ is sent to the vertex $(0, \dots, 0, 1) \in \Delta^k$. Every other vertex of I^k can be written in a unique way as $(t_1, \dots, t_{i-1}, 1, 0, \dots, 0)$, where $t_1, \dots, t_{i-1} \in \{0, 1\}$; such a vertex is sent to the vertex $(t'_0, \dots, t'_k) \in \Delta^k$, with $t'_{k-i} = 1$ and all the other coordinates are trivial. Note that r_k realises a homeomorphism between the interior of I^k and the interior of Δ^k.

We define the space

$$E_A := \big(G_\sigma \times \underline{E}G_{i(A)}\big)/\sim \times \{A\} \times I^k,$$

where $(g, g'x) \sim (g\psi_a(g'), x)$ for every $g \in G_\sigma, x \in \underline{E}G_{i(A)}, g' \in G_{i(A)}$, and where a denotes the concatenation $a_k \dots a_1$.

Remark 5.2 We can define an equivalence relation $\sim'$ on $G_\sigma \times \underline{E}G_{i(A)} \times \{A\}$ (resp. $\sim''$ on $G_\sigma \times \underline{E}G_{i(A)} \times \{A\} \times I^k$), yielding a space E'_A (resp. E''_A), and such that the identity of $G_\sigma \times \underline{E}G_{i(A)} \times \{A\} \times I^k$ yields a homeomorphism $E_A \to E'_A$ (resp. $E_A \to E''_A$). Therefore, we will sometimes write $\big([g, x, A], (t_i)_i\big)$ or $[g, x, A, (t_i)_i]$ when speaking of an element of E_A.

Note that there is a G_σ-equivariant map p_A from E_A to the G_σ-orbit of the simplex $|A|$ in $\widetilde{B(\sigma)}$, defined by

$$p_A\big([g, x, A, (t_i)_i]\big) = [g, A, r_k\big((t_i)_i\big)].$$

Moreover, the preimage $p_A^{-1}(g|\mathring{A}|)$ of a translate of the interior of $|A|$ is homeomorphic to the product $\underline{E}G_{i(A)} \times \mathring{I}^k$.

We now construct by induction CW-complexes $E_0(\sigma), E_1(\sigma), \ldots$ such that for every integer $k \geq 0$, $E_k(\sigma)$ is a complex of spaces (in the sense of Corson) over the k-skeleton of the block $\widetilde{B(\sigma)}$.

<u>k = 0</u>: We set

$$E_0(\sigma) = \coprod_{v \in V(\mathcal{B}(\sigma))} E_v,$$

which comes with the obvious projection to

$$\widetilde{B(\sigma)}^{(0)} = \coprod_{v \in V(\mathcal{B}(\sigma))} \{v\}.$$

<u>k = 1</u>: For an element $A \in A^{(1)}(\sigma)$ (that is, an edge a of $\mathcal{B}(\sigma)$), let $\Phi_A : \partial E_A \to E_0(\sigma)$ be the map that sends the element $[g, x, a, 0]$ to $[g, x, i(a)]$ and $[g, x, a, 1]$ to $[gF_\sigma(a)^{-1}, \varphi_a(x), t(a)]$. We can thus define the space $E_1(\sigma)$ as the quotient space

$$E_1(\sigma) := \left(E_0(\sigma) \sqcup \coprod_{A \in A^{(1)}(\sigma)} E_A \right) / (\Phi_A)_{A \in A^{(1)}(\sigma)}.$$

We check that the various maps p_A can be assembled into a map $p_1 : E_1(\sigma) \to \widetilde{B(\sigma)}^{(1)}$ that makes $E_1(\sigma)$ a complex of spaces (in the sense of Corson) over $\widetilde{B(\sigma)}^{(1)}$.

<u>k = 2</u>: We now turn to the construction of $E_2(\sigma)$. Using the same idea, we first want to define a map $\Phi_A : \partial E_A \to E_1(\sigma)$ for every $A \in A^{(2)}(\sigma)$. Let $A = (a_2, a_1)$ be such a pair of composable edges. Here, there are a priori two different ways to map $\underline{E}G_{i(a_1)}$ to $\underline{E}G_{t(a_2)}$ in a $\psi_{a_2 a_1}$-equivariant way, namely $\varphi_{a_2 a_1}$ and $g_{a_2,a_1}^{-1}\varphi_{a_2}\varphi_{a_1}$. In view of Lemma 5.1, these maps are equivariantly homotopic.

Definition 5.3 We denote by H_{a_2,a_1} an equivariant map $\underline{E}G_{i(a_1)} \times [0, 1] \to \underline{E}G_{t(a_2)}$ such that $H_{a_2,a_1}(\bullet, 0) = \varphi_{a_2 a_1}$ and $H_{a_2,a_1}(\bullet, 1) = g_{a_2,a_1}^{-1}\varphi_{a_2}\varphi_{a_1}$.

For an element $A = (a_2, a_1) \in A^{(2)}(\sigma)$, we define a map $\Phi_A : \partial E_A \to E_1(\sigma)$ as follows:

$$\begin{aligned}
\Phi_A\big([g, x, A, (t_1, t_2)]\big) &= [g, x, a_1, t_1] \text{ if } t_2 = 0,\\
\Phi_A\big([g, x, A, (t_1, t_2)]\big) &= [g, x, a_2 a_1, t_2] \text{ if } t_1 = 0,\\
\Phi_A\big([g, x, A, (t_1, t_2)]\big) &= [gF_\sigma(a_1)^{-1}, \varphi_{a_1}(x), a_2, t_2] \text{ if } t_1 = 1,\\
\Phi_A\big([g, x, A, (t_1, t_2)]\big) &= [gF_\sigma(a_2 a_1)^{-1}, H_{a_2,a_1}(x, t_1), t(a_2)] \text{ if } t_2 = 1.
\end{aligned}$$

We need to check that these definitions are compatible. The only non-trivial case to consider is when $t_1 = t_2 = 1$, for which we get

$$[gF_\sigma(a_1)^{-1}, \varphi_{a_1}(x), a_2, 1] = [gF_\sigma(a_1)^{-1}F_\sigma(a_2)^{-1}, \varphi_{a_2}\varphi_{a_1}(x), t(a_2)]$$

and

$$[gF_\sigma(a_2a_1)^{-1}, H_{a_2,a_1}(x,1), t(a_2)] = [gF_\sigma(a_2a_1)^{-1}, g_{a_2,a_1}^{-1}\varphi_{a_2}\varphi_{a_1}(x), t(a_2)]$$
$$= [gF_\sigma(a_2a_1)^{-1}\psi_{a_3a_2a_1}(g_{a_2,a_1})^{-1}, \varphi_{a_2}\varphi_{a_1}(x), t(a_2)],$$

where a_3 stands for the edge (possibly empty) corresponding to the inclusion $\sigma \subset t(a_2)$. But the cocycle condition yields

$$F_\sigma(a_2)F_\sigma(a_1) = \psi_{a_3a_2a_1}(g_{a_2,a_1})F_\sigma(a_2a_1),$$

thus

$$F_\sigma(a_2a_1)^{-1}\psi_{a_3a_2a_1}(g_{a_2,a_1})^{-1} = F_\sigma(a_1)^{-1}F_\sigma(a_2)^{-1},$$

hence the equality.

Note, as it will be important for the following steps, that the following holds:

$$\text{for every } A = (a_2, a_1) \in A^{(2)}(\sigma), \text{ we have } \mathrm{Im}(\Phi_A)_{|_{t_2=1}} \subset E_{t(A)}.$$

We now define $E_2(\sigma)$ as the quotient space

$$E_2(\sigma) := \left(E_1(\sigma) \sqcup \coprod_{A\in A^{(2)}(\sigma)} E_A \right) / (\Phi_A)_{A\in A^{(2)}(\sigma)}.$$

Here again, it is straightforward to check that the various maps p_A can be assembled into a map $p_2 : E_2(\sigma) \to \widetilde{B(\sigma)}^{(2)}$ that makes $E_2(\sigma)$ a complex of spaces (in the sense of Corson) over $\widetilde{B(\sigma)}^{(2)}$.

$\underline{k \geq 3}$: Suppose by induction that we have defined the spaces $E_0(\sigma), \dots, E_{k-1}(\sigma)$ and the maps Φ_A for every sequence of composable edges $(a_i, \dots, a_1)$, $1 \leq i < k$, satisfying the following additional condition:

$$\text{for every } A \in A^{(i)}(\sigma), 1 \leq i < k, \text{ we have } \mathrm{Im}(\Phi_A)_{|_{t_i=1}} \subset E_{t(A)} \qquad (\dagger)$$

Let $A = (a_k, \dots, a_1)$ be a sequence of composable edges of $\mathcal{B}(\sigma)$. Let $\partial' I^k$ be the closure of the boundary ∂I^k with the face $\{t_k = 1\}$ removed, and

$$\partial' E_A := \big(G_\sigma \times \underline{E}G_{i(A)}\big)/\sim \times \{A\} \times \partial' I^k$$

the associated subset of ∂E_A. We first define a map $\Phi'_A : \partial' E_A \to E_{k-1}(\sigma)$ as follows. The element $\Phi_A\big([g, x, A, (t_1, \dots, t_k)]\big)$ is defined as:

$$\Phi_{a_k,\dots,a_{i+1}a_i,\dots,a_1}\big([g, x, (a_k, \dots, a_{i+1}a_i, \dots, a_1), \quad (t_1, \dots, \hat{t_i}, \dots, t_k)]\big)$$
$$\text{for } t_i = 0 \text{ and } 1 \leq i < k,$$
$$\Phi_{a_{k-1},\dots,a_1}\big([g, x, (a_{k-1}, \quad \dots, a_1), (t_1, \dots, t_{k-1})]\big) \quad \text{for } t_k = 0,$$

and

$$\Phi_{a_k,\ldots,a_{i+1}}\left(\Phi_{a_k,\ldots,a_{i+1}}\left([g, x, (a_k, \ldots, a_{i+1}), (t_1, \ldots, t_{k-1}, 1)]\right), (t_{i+1}, \ldots, t_k)\right)$$

for $t_i = 1$ and $1 \leq i < k$. We easily check that these definitions are compatible. Because of the condition (†), the restriction of Φ'_A to the set of elements of $\partial' E_A$ with $t_k = 1$ defines a $\psi_{a_k \ldots a_1}$-equivariant map

$$\left(G_\sigma \times \underline{E}G_{i(A)}\right)/\sim \times \{A\} \times \partial\{t_k = 1\} \to E_{t(A)}$$

that can be extended to a $\psi_{a_k \ldots a_1}$-equivariant map

$$\left(G_\sigma \times \underline{E}G_{i(A)}\right)/\sim \times \{A\} \times \{t_k = 1\} \to E_{t(A)}$$

in view of Lemma 5.1.

We thus obtain a map $\Phi_A : \partial E_A \to E_{k-1}(\sigma)$, such that $\mathrm{Im}\Phi_{|_{t_k=1}} \subset E_{t(A)}$. As usual, we use these maps to define the space $E_k(\sigma)$ as the quotient space

$$E_k(\sigma) := \left(E_{k-1}(\sigma) \sqcup \coprod_{A \in A^{(k)}(\sigma)} E_A \right) / (\Phi_A)_{A \in A^{(k)}(\sigma)}.$$

Here again, we check that the projections p_A can be combined into a map $p_k : E_k(\sigma) \to \widetilde{B(\sigma)}^{(k)}$ that turns $E_k(\sigma)$ into a complex of spaces (in the sense of Corson) over $\widetilde{B(\sigma)}^{(k)}$, which concludes the induction.

Since the complex $\widetilde{B(\sigma)}$ is of finite dimension, this procedure eventually stops, and we denote by $E(\sigma)$ the final space obtained. This space is a complex of spaces (in the sense of Corson) over the block $\widetilde{B(\sigma)}$.

Proposition 5.4 *For each simplex σ of Y, the space $E(\sigma)$ is a cocompact model for the universal space for proper actions of G_σ.*

Proof The space $E(\sigma)$ is a complex of spaces (in the sense of Corson) over the contractible block $\widetilde{B(\sigma)}$. Moreover, each fibre is contractible, being a universal space for proper actions of a local group of $G\left(\mathcal{B}(\sigma)\right)$. It thus follows from Proposition 3.1 of [2] that $E(\sigma)$ is contractible.

For every simplex τ of the block $B(\sigma)$, the action of G_τ on $\underline{E}G_\tau$ is properly discontinuous, so it is straightforward to check that the same holds for the action of G_σ on $E(\sigma)$. For every simplex τ of the block $B(\sigma)$, the action of G_τ on $\underline{E}G_\tau$ is cocompact, so we can choose a compact subspace $K_\tau \subset \underline{E}G_\tau$ that meets every G_τ-orbit. We denote by k_τ the dimension of τ and by A_τ the unique element of $A^{(k_\tau)}(\mathcal{B}(\sigma))$ corresponding to τ. It is now straightforward to check that the image of $\bigcup_{\tau \subset B(\sigma)} \{1\} \times K_\tau \times \{A_\tau\} \times I^{k_\tau}$ in $E(\sigma)$ defines a compact subspace meeting every G_σ-orbit, hence the action of G_σ on $E(\sigma)$ is cocompact.

Finally, let H be a subgroup of G_σ. Since the projection $p : E(\sigma) \to \widetilde{B(\sigma)}$ is equivariant, the fixed-point set $E(\sigma)^H$ is a complex of spaces (in the sense of Corson) over the fixed-point set $\widetilde{B(\sigma)}^H$. The latter subcomplex is contractible since it is simplicially a cone. Moreover, the fibre over a simplex τ is equivariantly homeomorphic to the fixed-point set $\underline{E}G_\tau^H$. It thus follows that $E(\sigma)^H$ is non-empty if and only if H is finite, in which case it is contractible, which concludes the proof. □

We now have a cocompact model $E(\sigma)$ for the universal space for proper actions of G_σ for every simplex σ of Y. By construction of these spaces, the embeddings of blocks $B(\sigma') \hookrightarrow B(\sigma)$ (for simplices $\sigma \subset \sigma'$ of Y) are covered by equivariant embeddings $E(\sigma') \hookrightarrow E(\sigma)$, where an element of the form $[g, x, A, (t_i)_i]$ of $E(\sigma')$ is sent to the element $[\psi_{\sigma,\sigma'}(g), x, A, (t_i)_i]$ of $E(\sigma)$. We now twist these maps in order to get a complex of universal spaces compatible with $G(\mathcal{Y})$.

Theorem 5.5 *Let $G(\mathcal{Y})$ be a complex of groups satisfying the hypotheses of Theorem 1. Then there exists a complex of universal spaces compatible with $G(\mathcal{Y})$, and we can require the local maps to be embeddings.*

Proof For every simplex σ of Y, we define the fibre of σ to be the space $E(\sigma)$. For every inclusion $\sigma \subset \sigma'$ of simplices, we define a ψ_b-equivariant embedding $\phi_b : E(\sigma') \hookrightarrow E(\sigma)$ as follows, where b stands for the edge of $\mathcal{B}(\sigma')$ corresponding to the inclusion $\sigma \subset \sigma'$. Let $A = (a_k, \ldots, a_1)$ be an element of $A^{(k)}(\mathcal{B}(\sigma'))$ and define the edge a as the concatenation $a_k \ldots a_1$. Notice that (b, a) defines a pair of composable edges. We then define the restriction of ϕ_b to the subset $E_A \subset E(\sigma')$ by setting

$$\phi_b\big([g, x, A, (t_i)_i]\big) = [\psi_b(g)g_{b,a}, x, A, (t_i)_i].$$

One checks that these maps are compatible.

Now let $\sigma \subset \sigma' \subset \sigma''$ be an inclusion of simplices of Y. Let $A = (a_k, \ldots, a_1)$ be an element of $A^{(k)}(\mathcal{B}(\sigma''))$ and define the edge a as the concatenation $a_k \ldots a_1$. Let b be the edge of $\mathcal{B}(\sigma'')$ corresponding to the inclusion $\sigma' \subset \sigma''$ and c the edge of $\mathcal{B}(\sigma')$ corresponding to $\sigma \subset \sigma'$. The map ϕ_{cb} sends an element $[g, x, A, (t_i)_i]$ to $[\psi_{cb}(g)g_{cb,a}, x, A, (t_i)_i]$, while the map $\phi_c\phi_b$ sends $[g, x, A, (t_i)_i]$ to

$$[\psi_c\psi_b(g)\psi_c(g_{b,a})g_{c,ba}, x, A, (t_i)_i] = [g_{c,b}\psi_{cb}(g)g_{c,b}^{-1}\psi_c(g_{b,a})g_{c,ba}, x, A, (t_i)_i].$$

Now the cocycle condition

$$\psi_c(g_{b,a})g_{c,ba} = g_{c,b}g_{cb,a}$$

implies that

$$\phi_c\phi_b = g_{c,b}\phi_{cb},$$

therefore turning $\big(E(\sigma), \phi_a\big)$ into a complex of universal spaces compatible with $G(\mathcal{Y})$. □

6 Geometric dimension

As recalled in the introduction, the data of a compatible complex of universal spaces allows us to construct a universal space for proper actions for the fundamental group as a complex of spaces over the universal cover [8]. Such a space is obtained from the disjoint union of spaces of the form $G \times \sigma \times E(\sigma)$ for $\sigma \subset Y$; the data of the complex of groups is used to glue these various pieces in an appropriate way. Here however, our new universal spaces $E(\sigma)$ come with an extra feature, namely they are themselves complexes of spaces over blocks of X. It is therefore possible to give a slightly different construction of the final model for the universal space for proper actions of G that involves only products of the form $\sigma \times E(\sigma)$, so as to obtain interesting bounds for the geometric dimension of G (where the geometric dimension of a group is defined as the minimal dimension of a universal space for proper actions of the group).

Before detailing the construction, we need the following description of the fundamental group of a complex of groups [4, Proposition III.C.3.7] .

Proposition 6.1 (Presentation of the fundamental group of a complex of groups) *Let $G(\mathcal{Y})$ be a complex of groups over a simplicial complex Y, and let $\mathcal{Y}$ be the associated simplicial scwol. Consider a maximal tree T in the 1-skeleton of the first barycentric subdivision of Y. We identify T with the corresponding set of edges of $A(\mathcal{Y})$. Let $A^{\pm}(\mathcal{Y})$ denote the set of oriented edges of $\mathcal{Y}$, where a positive orientation corresponds to the natural orientation of elements of $A(\mathcal{Y})$.*

The fundamental group of $G(\mathcal{Y})$ is isomorphic to the abstract group $\pi_1\big(G(\mathcal{Y}), T\big)$ generated by the set

$$\coprod_{\sigma \in V(\mathcal{Y})} G_\sigma \coprod A^{\pm}(\mathcal{Y}),$$

and subject to the following relations:

- *the relations in the groups G_σ,*
- *$(a^+)^{-1} = a^-$ and $(a^-)^{-1} = a^+$,*
- *$b^+a^+g_{b,a} = (ba)^+$ for a pair of composable edges,*
- *$\psi_a(g) = a^-ga^+$ for an element $g \in G_{i(a)}$,*
- *$a^+ = 1$ for every edge a of T.*

For an oriented edge of $A^+(\mathcal{Y})$ corresponding to an inclusion $\sigma \subset \sigma'$, we denote by $\iota_T(\sigma, \sigma')$ its image in the fundamental group of $G(\mathcal{Y})$ under the previous identification. □

We can now define a model for the universal space for proper actions of G as follows. Let

$$\underline{E}G = \left(G \times \coprod_{\sigma \subset Y} E(\sigma)\right) / \simeq$$

where

$$(g, s) \simeq \left(g\iota_T(\sigma, \sigma')^{-1}, \phi_{\sigma,\sigma'}(s)\right) \text{ if } \sigma \subset \sigma', s \in E(\sigma'), g \in G,$$
$$(gg', s) \simeq (g, g's) \text{ if } s \in E(\sigma), g' \in G_\sigma, g \in G.$$

The various projections $E(\sigma) \to \widetilde{B(\sigma)}$ can be glued together to obtain an equivariant map from $\underline{E}G$ to

$$\left(G \times \coprod_{\sigma \in V(\mathcal{Y})} \widetilde{B(\sigma)}\right) / \simeq$$

where

$$(g, x) \simeq \left(g\iota_T(\sigma, \sigma')^{-1}, \phi_{\sigma,\sigma'}(x)\right) \text{ if } \sigma \subset \sigma', x \in \sigma', g \in G,$$
$$(gg', x) \simeq (g, x) \text{ if } x \in \sigma, g' \in G_\sigma, g \in G,$$

which is equivariantly isomorphic to the first barycentric subdivision of the universal cover X of $G(\mathcal{Y})$ [1]. Furthermore, since each space $E(\sigma)$ has a structure of complex of spaces over the associated block $\widetilde{B(\sigma)}$, it follows that the space $\underline{E}G$ has a structure of complex of spaces over X. The same reasoning as in [8, Thm 2.4] immediately implies:

Proposition 6.2 *Let $G(\mathcal{Y})$ be a complex of groups satisfying the hypotheses of Theorem 1 with fundamental group G and universal cover X. Then the space $\underline{E}G$ is a cocompact model for the universal space for proper actions of G, and is a complex of spaces over X, with universal spaces for proper actions of the local groups of $G(\mathcal{Y})$ as fibres.* □

Denote by $\underline{gd}$ the geometric dimension of a group for the family of finite subgroups, that is, the minimal dimension of a model for the universal space for proper actions. The previous construction yields the following:

Corollary 6.3 *We have: $\underline{gd}(G) \leq max_{\sigma \subset Y}\left(\underline{gd}(G_\sigma) + dim(\sigma)\right)$.* □

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References

1. Bridson, M.R., Haefliger, A.: Metric spaces of non-positive curvature. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319. Springer, Berlin (1999)
2. Corson, J.M.: Complexes of groups. Proc. Lond. Math. Soc. (3), **65**(1), 199–224 (1992)
3. Farrell, F.T., Lafont, J.-F.: EZ-structures and topological applications. Comment. Math. Helv. **80**(1), 103–121 (2005)

4. Haefliger, A.: Complexes of groups and orbihedra. In: Group Theory from a Geometrical Viewpoint (Trieste, 1990), pp. 504–540. World Sci. Publl., River Edge (1991)
5. Haefliger, A.: Extension of complexes of groups. Ann. Inst. Fourier (Grenoble) **42**(1–2), 275–311 (1992)
6. Lück, W., Weiermann, M.: On the classifying space of the family of virtually cyclic subgroups. Pure Appl. Math. Q. **8**(2), 497–555 (2012)
7. Martin, A.: Complexes of groups and geometric small cancellation over graphs of groups (preprint) (2013)
8. Martin, A.: Non-positively curved complexes of groups and boundaries. Geom. Topol. **18**(1), 31–102 (2014)
9. Scott, P., Wall, T.: Topological methods in group theory. In: Homological group theory (Proc. Sympos., Durham, 1977). London Math. Soc. Lecture Note Ser., vol. 36, pp. 137–203. Cambridge Univ. Press, Cambridge (1979)
10. Stallings, J.R.: Non-positively curved triangles of groups. In: Group theory from a geometrical viewpoint (Trieste, 1990), pp. 491–503. World Sci. Publ., River Edge (1991)