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Principal ∞-bundles: general theory

Thomas Nikolaus · Urs Schreiber · Danny Stevenson

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Abstract The theory of principal bundles makes sense in any ∞-topos, such as the ∞-topos of topological, of smooth, or of otherwise geometric ∞-groupoids/∞-stacks, and more generally in slices of these. It provides a natural geometric model for structured higher nonabelian cohomology and controls general fiber bundles in terms of associated bundles. For suitable choices of structure ∞-group G these *G-principal ∞-bundles* reproduce various higher structures that have been considered in the literature and further generalize these to a full geometric model for twisted higher nonabelian sheaf cohomology. We discuss here this general abstract theory of principal ∞-bundles, observing that it is intimately related to the axioms that characterize ∞-toposes. A central result is a natural equivalence between principal ∞-bundles and intrinsic nonabelian cocycles, implying the classification of principal ∞-bundles by nonabelian sheaf hyper-cohomology. We observe that the theory of geometric fiber ∞-bundles *associated* to principal ∞-bundles subsumes a theory of ∞-*gerbes* and of *twisted ∞-bundles*, with twists deriving from *local coefficient ∞-bundles*, which we define, relate to extensions of principal ∞-bundles and show to be classified by

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T. Nikolaus
Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg, Germany
e-mail: Thomas1.Nikolaus@mathematik.uni-regensburg.de

U. Schreiber (✉)
Mathematics Institute, Radboud Universiteit Nijmegen, 6500 HC Nijmegen, The Netherlands
e-mail: urs.schreiber@gmail.com

D. Stevenson
School of Mathematical Sciences, University of Adelaide,
North Terrace Campus, Adelaide, SA 5005, Australia
e-mail: daniel.stevenson@adelaide.edu.au

a corresponding notion of *twisted cohomology*, identified with the cohomology of a corresponding slice ∞-topos.

Keywords Nonabelian cohomology · Higher topos theory · Principal bundles

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1 Overview

The concept of a *G-principal bundle* for a topological or Lie group G is fundamental in classical topology and differential geometry, e.g. [10]. More generally, for G a *geometric* group in the sense of a *sheaf of groups* over some site, the notion of G-principal bundle or *G-torsor* is fundamental in *topos theory* [12,19]. Its relevance rests in the fact that G-principal bundles constitute natural geometric representatives of cocycles in degree 1 nonabelian cohomology $H^1(-, G)$ and that general fiber bundles are *associated* to principal bundles.

In recent years it has become clear that various applications, notably in "*String-geometry*" [27,28], involve a notion of principal bundles where geometric groups G are generalized to *geometric grouplike A_∞-spaces*, in other words *geometric ∞-groups*: geometric objects that are equipped with a group structure up to higher coherent homotopy. The resulting *principal ∞-bundles* should be natural geometric representatives of geometric nonabelian *hypercohomology*: Čech cohomology with coefficients in arbitrary positive degree.

In the *absence* of geometry, these principal ∞-bundles are essentially just the classical *simplicial* principal bundles of simplicial sets [16] (this we discuss in Section 4.1 of [21]). However, in the presence of non-trivial geometry the situation is both more subtle and richer, and plain simplicial principal bundles can only serve as a specific *presentation* for the general notion (section 3.7.2 of [21]).

For the case of *principal 2-bundles*, which is the first step after ordinary principal bundles, aspects of a geometric definition and theory have been proposed and developed by various authors, see section 1 of [21] for references and see [22] for a comprehensive discussion. Notably the notion of a *bundle gerbe* [20] is, when regarded as an extension of a Čech-groupoid, almost manifestly that of a principal 2-bundle, even though this perspective is not prominent in the respective literature.

The oldest definition of geometric 2-bundles is conceptually different, but closely related: Giraud's *G-gerbes* [9] are by definition not principal 2-bundles but are fiber 2-bundles *associated* to $\mathbf{Aut}(\mathbf{B}G)$-principal 2-bundles, where $\mathbf{B}G$ is the *geometric moduli stack* of G-principal bundles. This means that G-gerbes provide the universal *local coefficients*, in the sense of *twisted cohomology*, for G-principal bundles.

From the definition of principal 2-bundles/bundle gerbes it is fairly clear that these ought to be just the first step (or second step) in an infinite tower of higher analogs. Accordingly, definitions of *principal 3-bundles* have been considered in the literature, mostly in the guise of *bundle 2-gerbes* [31]. The older notion of Breen's *G -2-gerbes* [6] (also discussed by Brylinski-MacLaughlin), is, as before, not that of a principal 3-bundle, but that of a fiber 3-bundle which is *associated* to an $\mathbf{Aut}(\mathbf{B}G)$-principal 3-bundle, where now $\mathbf{B}G$ is the *geometric moduli 2-stack* of G-principal 2-bundles .

Generally, for every $n \in \mathbb{N}$ and every geometric n-group G, it is natural to consider the theory of G-principal n-bundles *twisted* by an $\mathbf{Aut}(\mathbf{B}G)$-principal $(n+1)$-bundle, hence by the associated *G-n-gerbe*. A complete theory of principal bundles therefore needs to involve the notion of principal n-bundles and also that of twisted principal n-bundles in the limit as $n \to \infty$.

As n increases, the piecemeal conceptualization of principal n-bundles quickly becomes tedious and their structure opaque, without a general theory of higher geometric structures. In recent years such a theory—long conjectured and with many precursors—has materialized in a comprehensive and elegant form, now known as *∞-topos theory* [13,25,32]. Whereas an ordinary topos is a category of *sheaves* over some site,[1] an ∞-topos is an *∞-category* of *∞-sheaves* or equivalently of *∞-stacks* over some *∞-site*, where the prefix "∞"- indicates that all these notions are generalized to structures up to *coherent higher homotopy* (as in the older terminology of A_∞-, C_∞-, E_∞- and L_∞-algebras, all of which re-appear as algebraic structures in ∞-topos theory). In as far as an ordinary topos is a context for general *geometry*, an ∞-topos is a context for what is called *higher geometry* or *derived geometry*: the pairing of the notion of *geometry* with that of *homotopy*. (Here "derived" alludes to "derived category" and "derived functor" in homological algebra, but refers in fact to a nonabelian generalization of these concepts.) Therefore we may refer to objects of an ∞-topos also as *geometric homotopy types*.

As a simple instance of this pairing, one observes that for any geometric abelian group (sheaf of abelian groups) A, the higher degree (sheaf) cohomology $H^{n+1}(-, A)$ in ordinary geometry may equivalently be understood as the degree-1 cohomology $H^1(-, \mathbf{B}^n A)$ in higher geometry, where $\mathbf{B}^n A$ is the geometric ∞-group obtained by successively *delooping* A geometrically. More generally, there are geometric ∞-groups G not of this abelian form. The general degree-1 geometric cohomology $H^1(X, G)$ is a nonabelian and simplicial generalization of *sheaf hypercohomology*, whose cocycles are morphisms $X \to \mathbf{B}G$ into the geometric delooping of G. Indeed, delooping plays a central role in

[1] Throughout *topos* here stands for *Grothendieck topos*, as opposed to the more general notion of *elementary topos*.

∞-topos theory; a fundamental fact of ∞-topos theory (recalled as Theorem 2.19 below) says that, quite generally, under internal looping and delooping, ∞-groups G in an ∞-topos $\mathbf{H}$ are equivalent to connected and *pointed* objects in $\mathbf{H}$:

$$\{\text{groups in } \mathbf{H}\} \underset{\text{delooping } \mathbf{B}}{\overset{\text{looping } \Omega}{\underset{\longrightarrow}{\overset{\longleftarrow}{\simeq}}}} \left\{ \begin{array}{c} \text{pointed connected} \\ \text{objects in } \mathbf{H} \end{array} \right\} ..$$

We will see that this equivalence of ∞-categories plays a key role in the theory of principal ∞-bundles.

Topos theory is renowned for providing a general convenient context for the development of geometric structures. In some sense, ∞-topos theory provides an even more convenient context, due to the fact that ∞ *-(co)limits* or *homotopy (co)limits* in an ∞-topos exist, and refine the corresponding naive (co)limits. This convenience manifests itself in the central definition of principal ∞-bundles (Definition 3.4 below): whereas the traditional definition of a G-principal bundle over X as a quotient map $P \to P/G \simeq X$ requires the additional clause that the quotient be *locally trivial*, ∞-topos theory comes pre-installed with the correct homotopy quotient for higher geometry, and as a result the local triviality of $P \to P//G =: X$ is automatic; we discuss this in more detail in Sect. 3.1 below. Hence conceptually, G-principal ∞-bundles are in fact simpler than their traditional analogs, and so their theory is stronger.

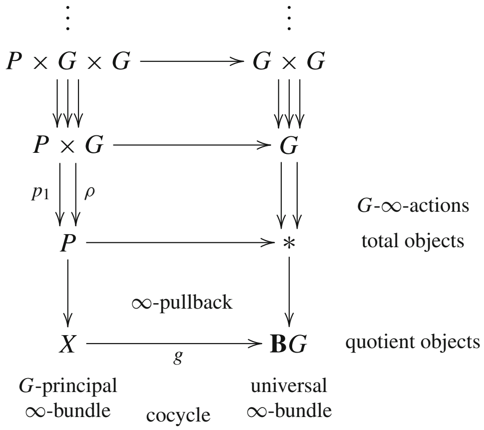
A central theorem of topos theory is *Giraud's theorem*, which intrinsically characterizes toposes as those presentable categories that satisfy three simple conditions: 1. coproducts are disjoint, 2. colimits are preserved by pullback, and 3. quotients are effective. The analog of this characterization turns out to remain true essentially verbatim in ∞-topos theory: this is the *Giraud-Toën-Vezzosi-Rezk-Lurie* characterization of ∞-toposes, recalled as Definition 2.1 below. We will show that given an ∞-topos $\mathbf{H}$, the second and the third of these axioms lead directly to the *classification theorem* for principal ∞-bundles (Theorem 3.17 below) which states that there is an equivalence of ∞-groupoids

$$G\mathrm{Bund}(X) \qquad \simeq \qquad \mathbf{H}(X, \mathbf{B}G)$$

$$\left\{ \begin{array}{c} G\text{-principal } \infty\text{-bundles} \\ \text{over } X \end{array} \quad \begin{array}{c} P \\ \downarrow \\ X \end{array} \right\} \simeq \{\text{cocycles} \quad g : X \to \mathbf{B}G\}$$

between the ∞-groupoid of G-principal ∞-bundles on X, and the mapping space $\mathbf{H}(X, \mathbf{B}G)$.

The mechanism underlying the proof of this theorem is summarized in the following diagram, which is supposed to indicate that the geometric G-principal ∞-bundle corresponding to a cocycle is nothing but the corresponding homotopy fiber:



The fact that all geometric G-principal ∞-bundles arise this way, up to equivalence, is quite useful in applications, and also sheds helpful light on various existing constructions and provides more examples.

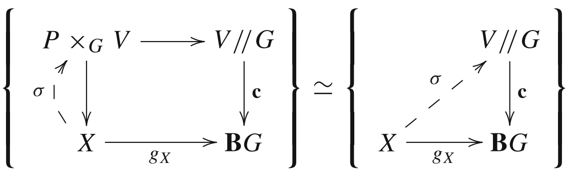
Notably, the implication that *every* geometric ∞-action $\rho : V \times G \to V$ of an ∞-group G on an object V has a classifying morphism $\mathbf{c} : V//G \to \mathbf{B}G$, tightly connects the theory of associated ∞-bundles with that of principal ∞-bundles (Sect. 4.1 below): the fiber sequence

$$\begin{array}{ccc} V & \longrightarrow & V//G \\ & & \downarrow{\scriptstyle \mathbf{c}} \\ & & \mathbf{B}G \end{array}$$

is found to be the V-fiber ∞-bundle which is ρ-*associated* to the universal G-principal ∞-bundle $* \to \mathbf{B}G$. Again, using the ∞-Giraud axioms, an ∞-pullback of $\mathbf{c}$ along a cocycle $g_X : X \to \mathbf{B}G$ is identified with the ∞-bundle $P \times_G V$ that is ρ-*associated* to the principal ∞-bundle $P \to X$ classified by g_X (Proposition 4.7) and every V-fiber ∞-bundle arises this way, associated to an $\mathbf{Aut}(V)$-principal ∞-bundle (Theorem 4.11).

Using this, we may observe that the space $\Gamma_X(P \times_G V)$ of *sections* of $P \times_G V$ is equivalently the space $\mathbf{H}_{/\mathbf{B}G}(g_X, \mathbf{c})$ of cocycles $\sigma : g_X \to \mathbf{c}$ in the slice ∞-topos $\mathbf{H}_{/\mathbf{B}G}$:

$$\Gamma_X(P \times_G V) \qquad \simeq \qquad \mathbf{H}_{/\mathbf{B}G}(g_X, \mathbf{c})$$



Moreover, by the above classification theorem of G-principal ∞-bundles, g_X trivializes over some cover $U \longrightarrow\!\!\!\!\!\rightarrow X$, and so the universal property of the ∞-pullback implies that *locally* a section σ is a V-valued function

$$
\begin{array}{ccccc}
 & & V & \longrightarrow & V//G \\
 & {\scriptstyle \sigma|_U} \nearrow & & & \big\downarrow {\scriptstyle \mathbf{c}} \\
U & \underset{\text{cover}}{\longrightarrow\!\!\!\!\!\rightarrow} & X & \overset{g_X}{\longrightarrow} & \mathbf{B}G\,.
\end{array}
$$

For V an ordinary space, hence a *0-truncated* object in the ∞-topos, this is simply the familiar statement about sections of associated bundles. But in higher geometry V, being an object of the ∞-topos and hence an ∞-stack, may more generally itself be a higher moduli ∞-stack, classifying some geometric structures, which makes the general theory of sections more interesting. Specifically, if V is a pointed connected object, then it is of the form $\mathbf{B}\Omega V$ and this means that it is locally a cocycle for an ΩV-principal ∞-bundle, and so globally is a *twisted ΩV-principal ∞-bundle*. This identifies $\mathbf{H}_{/\mathbf{B}G}(-, \mathbf{c})$ as the *twisted cohomology* induced by the *local coefficient bundle* $\mathbf{c}$ with *local coefficients* V. This yields a geometric and unstable analogue of the picture of twisted cohomology discussed in [1].

Given V, the most general twisting group is the *automorphism ∞-group* $\mathbf{Aut}(V) \hookrightarrow [V, V]_{\mathbf{H}}$, formed in the ∞-topos (Definition 4.9). If V is pointed connected and hence of the form $V = \mathbf{B}G$, this means that the most general universal local coefficient bundle is

$$
\begin{array}{ccc}
\mathbf{B}G & \longrightarrow & (\mathbf{B}G)//\mathbf{Aut}(\mathbf{B}G) \\
 & & \big\downarrow {\scriptstyle \mathbf{c}_{\mathbf{B}G}} \\
 & & \mathbf{B}\mathbf{Aut}(\mathbf{B}G).
\end{array}
$$

The corresponding associated twisting ∞-bundles are *G-∞-gerbes*: fiber ∞-bundles with typical fiber the moduli ∞-stack $\mathbf{B}G$. These are the universal local coefficients for twists of G-principal ∞-bundles.

While twisted cohomology in $\mathbf{H}$ is hence identified simply with ordinary cohomology in a slice of $\mathbf{H}$, the corresponding geometric representatives, the ∞-bundles, do not translate to the slice quite as directly. The reason is that a universal local coefficient bundle $\mathbf{c}$ as above is rarely a pointed connected object in the slice (if it is, then it is essentially trivial) and so the theory of principal ∞-bundles does not directly apply to these coefficients. In Sect. 4.3 we show that what does translate is a notion of *twisted ∞-bundles*, a generalization of the twisted bundles known from twisted K-theory: given a section $\sigma : g_X \to \mathbf{c}$ as above, the following pasting diagram of ∞-pullbacks

$$
\begin{array}{ccccccl}
Q & \longrightarrow & * & & & & P\text{-twisted } \Omega V\text{-principal } \infty\text{-bundle} \\
\downarrow & & \downarrow & & & & \\
P & \longrightarrow & V & \longrightarrow & * & & G\text{-principal } \infty\text{-bundle} \\
\downarrow & & \downarrow & & \downarrow & & \\
X & \xrightarrow{\sigma} & V//G & \xrightarrow{\mathbf{c}} & \mathbf{B}G & & \text{section of } \rho\text{-associated } V\text{-}\infty\text{-bundle}
\end{array}
\qquad (X \xrightarrow{g_X} \mathbf{B}G)
$$

naturally identifies an ΩV-principal ∞-bundle Q on the total space P of the twisting G-principal ∞-bundle, and since this is classified by a G-equivariant morphism $P \to V$ it enjoys itself a certain twisted G-equivariance with respect to the defining G-action on P. We call such $Q \to P$ the *$[g_X]$-twisted ΩV-principal bundle* classified by σ. Again, a special case of special importance is that where $V = \mathbf{B}A$ is pointed connected, which identifies the universal V-coefficient bundle with an *extension of ∞-groups*

$$
\begin{array}{ccc}
\mathbf{B}A & \longrightarrow & \mathbf{B}\hat{G} \\
 & & \downarrow \\
 & & \mathbf{B}G.
\end{array}
$$

Accordingly, P-twisted A-principal ∞-bundles are equivalently *extensions* of P to $\hat{G}$-principal ∞-bundles.

A direct generalization of the previous theorem yields the classification Theorem 4.39, which identifies $[g_X]$-twisted A-principal ∞-bundles with cocycles in twisted cohomology

$$
A\mathrm{Bund}^{[g_X]}(X) \qquad \simeq \qquad \mathbf{H}_{/\mathbf{B}G}(g_X, \mathbf{c}_{\mathbf{B}G})
$$

$$
\left\{
\begin{array}{cc}
\begin{array}{c} [g_X]\text{-twisted} \\ A\text{-principal } \infty\text{-bundles} \\ \text{over } X \end{array} &
\begin{array}{c} Q \\ \downarrow \\ P = (g_X)^* * \\ \downarrow \\ X \end{array}
\end{array}
\right\}
\simeq \{\text{twisted cocycles} \quad \sigma : g_X \to \mathbf{c}_{\mathbf{B}G}\}
$$

For instance if $\mathbf{c}$ is the connecting homomorphism

$$
\begin{array}{ccc}
\mathbf{B}\hat{G} & \longrightarrow & \mathbf{B}G \\
 & & \downarrow{\scriptstyle \mathbf{c}} \\
 & & \mathbf{B}^2 A
\end{array}
$$

of a central extension of ordinary groups $A \to \hat{G} \to G$, then the corresponding twisted $\hat{G}$-bundles are those known from geometric models of twisted K-theory.

When the internal *Postnikov tower* of a coefficient object is regarded as a sequence of local coefficient bundles as above, the induced twisted ∞-bundles are decompositions of nonabelian principal ∞-bundles into ordinary principal bundles together with equivariant abelian hypercohomology cocycles on their total spaces. This construction identifies much of equivariant cohomology theory as a special case of higher nonabelian cohomology. Specifically, when applied to a Postnikov stage of the delooping of an ∞-group of internal automorphisms, the corresponding twisted cohomology reproduces the notion of Breen *G-gerbes with band* (Giraud's *lien*s); and the corresponding twisted ∞-bundles are their incarnation as equivariant *bundle gerbes* over principal bundles.

The classification statements for principal and fiber ∞-bundles in this article, Theorems 3.17 and 4.11 are not surprising, they say exactly what one would hope for. It is however useful to see how they flow naturally from the abstract axioms of ∞-topos theory, and to observe that they immediately imply a series of classical as well as recent theorems as special cases, see Remark 4.12. Also the corresponding long exact sequences in (nonabelian) cohomology, Theorem 2.27, reproduce classical theorems, see Remark 2.28. Similarly the definition and classification of lifting of principal ∞-bundles, Theorem 4.35, and of twisted principal ∞-bundles in Theorem 4.39 flows naturally from the ∞-topos theory and yet it immediately implies various constructions and results in the literature as special cases, see Remark 4.36 and Remark 4.40, respectively. In particular the notion of nonabelian twisted cohomology itself is elementary in ∞-topos theory, Sect. 4.2, and yet it sheds light on a wealth of applications, see Remark 4.22.

This should serve to indicate that the theory of (twisted) principal ∞-bundles is rich and interesting. The present article is intentionally written in general abstraction only, aiming to present the general theory of (twisted) principal ∞-bundles as elegantly as possible, true to its roots in abstract higher topos theory. We believe that this serves to usefully make transparent the overall picture. In the companion article [21] we give a complementary discussion and construct explicit presentations of the structures appearing here that lend themselves to explicit computations.

2 Preliminaries

The discussion of principal ∞-bundles in Sect. 3 below builds on the concept of an ∞-*topos* and on a handful of basic structures and notions that are present in any ∞-topos, in particular the notion of *group objects* and of *cohomology* with coefficients in these group objects. The relevant theory has been developed in [13,15,25,32]. The purpose of this section is to recall the main aspects of this theory that we need, to establish our notation, and to highlight some aspects of the general theory that are relevant to our discussion and which have perhaps not been highlighted in this way in the existing literature.

One may reason about ∞-categories in a model-independent way, using the universal constructions that hold equivalently in all models—but the reader wishing to do so is invited to think specifically of the model given by quasi-categories, due to Joyal and laid out in detail in [13]. Ordinary categories naturally embed into ∞-categories; and in terms of quasi-categories this embedding is given by sending a category to its simplicial nerve. In view of this we will freely regard 1-categories as ∞-categories—such as for instance the simplex category Δ. This allows us to define a simplicial object in an ∞-category $\mathcal{C}$ in a direct generalization of the usual notion as an ∞-functor $\Delta^{\mathrm{op}} \to \mathcal{C}$ (note that this is now automatically a simplicial object "up to coherent higher homotopy").

We are particularly concerned with those ∞-categories that are ∞-toposes. For many purposes the notion of *∞-topos* is best thought of as a generalization of the notion of a sheaf topos—the category of sheaves over some site is replaced by an *∞-category of ∞-stacks/∞ -sheaves* over some ∞-site (there is also supposed to be a more general notion of an *elementary ∞-topos*, which however we do not consider here). In this context the ∞-topos Gpd_∞ of ∞-groupoids is the natural generalization of the punctual topos Set to ∞-topos theory. A major achievement of [13,25,32] and was to provide a more intrinsic characterization of ∞-toposes, which generalizes the classical characterization of sheaf toposes (Grothendieck toposes) originally given by Giraud. We will show that the theory of principal ∞-bundles is naturally expressed in terms of these intrinsic properties, and therefore we here take these *Giraud-Toën-Vezzosi-Rezk-Lurie axioms* to be the very definition of an ∞-topos ([13], Theorem 6.1.0.6, the main ingredients will be recalled below):

Definition 2.1 (*∞-Giraud axioms*). An *∞-topos* is a presentable ∞-category $\mathbf{H}$ that satisfies the following properties.

1. *Coproducts are disjoint.* For every two objects $A, B \in \mathbf{H}$, the intersection of A and B in their coproduct is the initial object: in other words the diagram

$$\begin{array}{ccc} \emptyset & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & A \coprod B \end{array}$$

   is a pullback.
2. *Colimits are preserved by pullback.* For all morphisms $f\colon X \to B$ in $\mathbf{H}$ and all small diagrams $A\colon I \to \mathbf{H}_{/B}$, there is an equivalence

$$\varinjlim_i f^* A_i \simeq f^*(\varinjlim_i A_i)$$

   between the pullback of the colimit and the colimit over the pullbacks of its components.
3. *Quotient maps are effective epimorphisms.* Every simplicial object $A_\bullet : \Delta^{\mathrm{op}} \to \mathbf{H}$ that satisfies the groupoidal Segal property (Definition 2.13) is the Čech nerve of its quotient projection:

$$A_n \simeq A_0 \times_{\varinjlim_n A_n} A_0 \times_{\varinjlim_n A_n} \cdots \times_{\varinjlim_n A_n} A_0 \quad (n \text{ factors}).$$

Repeated application of the second and third axiom provides the proof of the classification of principal ∞-bundles, Theorem 3.17 and the universality of the universal associated ∞-bundle, Proposition 4.6.

An ordinary topos is famously characterized by the existence of a classifier object for monomorphisms, the *subobject classifier*. With hindsight, this statement already carries in it the seed of the close relation between topos theory and bundle theory, for we may think of a monomorphism $E \hookrightarrow X$ as being a *bundle of (-1)-truncated fibers* over X. The following axiomatizes the existence of arbitrary universal bundles, providing a different but equivalent definition of ∞-toposes.

Proposition 2.2 *An ∞-topos $\mathbf{H}$ is a presentable ∞-category with the following properties.*

1. *Colimits are preserved by pullback.*
2. *There are universal κ-small bundles. For every sufficiently large regular cardinal κ, there exists a morphism $\widehat{\mathrm{Obj}}_\kappa \to \mathrm{Obj}_\kappa$ in $\mathbf{H}$, such that for every other object X, pullback along morphisms $X \to \mathrm{Obj}$ constitutes an equivalence*[2]

$$\mathrm{Core}(\mathbf{H}_{/_\kappa X}) \simeq \mathbf{H}(X, \mathrm{Obj}_\kappa)$$

between the ∞-groupoid of bundles (morphisms) $E \to X$ which are κ-small over X and the ∞-groupoid of morphisms from X into Obj_κ.

This is due to Rezk and Lurie, appearing as Theorem 6.1.6.8 in [13]. We find that this second version of the axioms naturally gives the equivalence between V-fiber bundles and $\mathbf{Aut}(V)$-principal ∞-bundles in Proposition 4.10.

In addition to these axioms, a basic property of ∞-toposes (and generally of ∞-categories with pullbacks) which we will repeatedly invoke, is the following.

Proposition 2.3 (pasting law for pullbacks) *Let $\mathbf{H}$ be an ∞-category with pullbacks. If*

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ D & \longrightarrow & E & \longrightarrow & F \end{array}$$

is a diagram in $\mathbf{H}$ such that the right square is an ∞-pullback, then the left square is an ∞-pullback precisely if the outer rectangle is.

Notice that here and in all of the following

- all square diagrams are filled by a 2-cell, even if we do not indicate this notationally;
- all limits are ∞-limits/homotopy limits (hence all pullbacks are ∞-pullbacks/homotopy pullbacks), and so on;

[2] Here Core denotes the maximal ∞-groupoid inside an ∞-category.

this is the only consistent way of speaking about **H** in generality. Only in the followup article [21] do we consider presentations of **H** by 1-categorical data; there we will draw a careful distinction between 1-categorical limits and ∞-categorical/homotopy limits.

Definition 2.4 For $f : Y \to Z$ any morphism in **H** and $z : * \to Z$ a point, the *fiber* (homotopy fiber or ∞-fiber) of f over this point is the pullback $X := * \times_Z Y$

$$\begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & Z. \end{array}$$

Observation 2.5 *Let $f : Y \to Z$ in* **H** *be as above. Suppose that Y is pointed and f is a morphism of pointed objects. Then the ∞-fiber of an ∞-fiber is the loop object of the base.*

This means that we have a diagram

$$\begin{array}{ccccc} \Omega Z & \longrightarrow & X & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & Y & \xrightarrow{f} & Z \end{array}$$

where the outer rectangle is an ∞-pullback if the left square is an ∞-pullback. This follows from the pasting law, Proposition 2.3.

2.1 Epimorphisms and monomorphisms

In an ∞-topos there is an infinite tower of notions of epimorphisms and monomorphisms: the n-connected and n-truncated morphisms for all $-2 \leq n \leq \infty$ [13,25].

Definition 2.6 For $n \in \mathbb{N}$ an ∞-groupoid is called *n-truncated* (or: an *n-type*) if all its homotopy groups in degree greater than n are trivial. It is called *(-1)-truncated* if it is either empty or contractible and *(-2)-truncated* if it is non-empty and contractible. An object in an arbitrary ∞-category is n-truncated for $-2 \leq n < \infty$ if all hom-∞-groupoids into it are n-truncated. A morphism of an ∞-category is called n-truncated if it is so as an object in the slice over its codomain (which means internally that its homotopy fibers are n-truncated). A (-1)-truncated morphism is also called a *monomorphism*. The full embedding of the n-truncated objects of an ∞-topos is reflective, and the reflector $\tau_{\leq n}$ is called the *n-truncation* operation.

This is the topic of section 5.5.6 in [13].

Remark 2.7 In a general ∞-topos every object has (groups of) *homotopy sheaves* generalizing the homotopy groups for bare ∞-groupoids. If one knows that an object X in an ∞-topos is truncated at all (for some possibly large truncation degree) then it is n-truncated if all its homotopy sheaves $\pi_k(X)$ vanish in degree $k > n$.

This is the content of Proposition 6.5.1.7 in [13].

Definition 2.8 Let $\mathbf{H}$ be an ∞-topos. For $X \to Y$ any morphism in $\mathbf{H}$, there is a simplicial object $\check{C}(X \to Y)$ in $\mathbf{H}$ (the *Čech nerve* of $f : X \to Y$) which in degree n is the $(n+1)$-fold ∞-fiber product of X over Y with itself

$$\check{C}(X \to Y) : [n] \mapsto X^{\times_Y^{n+1}}$$

A morphism $f : X \to Y$ in $\mathbf{H}$ is an *effective epimorphism* if it is the colimit of its own Čech nerve (under the natural map from the Cech nerve to Y):

$$f : X \to \varinjlim \check{C}(X \to Y).$$

Write $\mathrm{Epi}(\mathbf{H}) \subset \mathbf{H}^I$ for the collection of effective epimorphisms.

Definition 2.9 For $n \in \mathbb{N}$ a morphism in an ∞-topos is called *n-connected* if it is an effective epimorphism and all its homotopy sheaves are trivial in degree greater than n when it is regarded as an object in the slice ∞-topos over its codomain. Any effective epimorphism is called (-1)-connected. An object X is called n-connected if the canonical morphism $X \to *$ is n-connected.

This is the topic of section 6.5.1 in [13].

Proposition 2.10 *A morphism $f : X \to Y$ in the ∞-topos $\mathbf{H}$ is an effective epimorphism precisely if its 0-truncation $\tau_0 f : \tau_0 X \to \tau_0 Y$ is an epimorphism (necessarily effective) in the 1-topos $\tau_{\leq 0}\mathbf{H}$.*

This is Proposition 7.2.1.14 in [13].

Proposition 2.11 *The classes* $(\mathrm{Epi}(\mathbf{H}), \mathrm{Mono}(\mathbf{H}))$ *constitute an orthogonal factorization system.*

This is Proposition 8.5 in [25] and Example 5.2.8.16 in [13].

Definition 2.12 For $f : X \to Y$ a morphism in $\mathbf{H}$, we write its epi/mono factorization given by Proposition 2.11 as

$$f : X \twoheadrightarrow \mathrm{im}(f) \hookrightarrow Y$$

and we call $\mathrm{im}(f) \hookrightarrow Y$ the ∞-*image* of f.

2.2 Groupoids and groups

In any ∞-topos $\mathbf{H}$ we may consider groupoids *internal* to $\mathbf{H}$, in the sense of internal category theory (as exposed for instance in the introduction of [14]).

Such a *groupoid object* $\mathcal{G}$ in $\mathbf{H}$ is an $\mathbf{H}$-object $\mathcal{G}_0$ "of $\mathcal{G}$-objects" together with an $\mathbf{H}$-object $\mathcal{G}_1$ "of $\mathcal{G}$-morphisms" equipped with source and target assigning morphisms $s, t : \mathcal{G}_1 \to \mathcal{G}_0$, an identity-assigning morphism $i : \mathcal{G}_0 \to \mathcal{G}_1$ and a composition morphism $\mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \to \mathcal{G}_1$ which together satisfy all the axioms of a groupoid (unitality, associativity, existence of inverses) up to coherent homotopy in $\mathbf{H}$. One way to formalize what it means for these axioms to hold up to coherent homotopy is as follows.

One notes that ordinary groupoids, i.e. groupoid objects internal to Set, are characterized by the fact that their nerves are simplicial sets $\mathcal{G}_\bullet : \Delta^{\mathrm{op}} \to \mathrm{Set}$ with the property that the groupoidal Segal maps

$$\mathcal{G}_n \to \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \times_{\mathcal{G}_0} \cdots \times_{\mathcal{G}_0} \mathcal{G}_1$$

are isomorphisms for all $n \geq 2$. This last condition is stated precisely in Definition 2.13 below, and clearly gives a characterization of groupoids that makes sense more generally, in particular it makes sense internally to higher categories: a groupoid object in $\mathbf{H}$ is an ∞-functor $\mathcal{G} : \Delta^{\mathrm{op}} \to \mathbf{H}$ such that all groupoidal Segal morphisms are equivalences in $\mathbf{H}$. These ∞-functors $\mathcal{G}$ form the objects of an ∞-category $\mathrm{Grpd}(\mathbf{H})$ of groupoid objects in $\mathbf{H}$.

Here a subtlety arises that is the source of a lot of interesting structure in higher topos theory: the objects of $\mathbf{H}$ are themselves "structured ∞-groupoids". Indeed, there is a full embedding const : $\mathbf{H} \hookrightarrow \mathrm{Grpd}(\mathbf{H})$ that forms constant simplicial objects and thus regards every object $X \in \mathbf{H}$ as a groupoid object which, even though it has a trivial object of morphisms, already has a structured ∞-groupoid of objects. This embedding is in fact reflective, with the reflector given by forming the ∞-colimit over a simplicial diagram, the "geometric realization"

$$\mathbf{H} \underset{\text{const}}{\overset{\varinjlim}{\underset{\hookrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathrm{Grpd}(\mathbf{H}) \,.$$

For $\mathcal{G}$ a groupoid object in $\mathbf{H}$, the object $\varinjlim \mathcal{G}_\bullet$ in $\mathbf{H}$ may be thought of as the ∞-groupoid obtained by "gluing together the object of objects of $\mathcal{G}$ along the object of morphisms of $\mathcal{G}$". This idea that groupoid objects in an ∞-topos are like structured ∞-groupoids together with gluing information is formalized by the statement recalled as Theorem 2.15 below, which says that groupoid objects in $\mathbf{H}$ are equivalent to the *effective epimorphisms* $Y \twoheadrightarrow X$ in $\mathbf{H}$, the intrinsic notion of *cover* (of X by Y) in $\mathbf{H}$. The effective epimorphism/cover corresponding to a groupoid object $\mathcal{G}$ is the colimiting cocone $\mathcal{G}_0 \twoheadrightarrow \varinjlim \mathcal{G}_\bullet$.

After this preliminary discussion we state the following definition of groupoid object in an ∞-topos (this definition appears in [13] as Definition 6.1.2.7, using Proposition 6.1.2.6).

Definition 2.13 ([13], *Definition* 6.1.2.7). A *groupoid object* in an ∞-topos $\mathbf{H}$ is a simplicial object

$$\mathcal{G} : \Delta^{\mathrm{op}} \to \mathbf{H}$$

all of whose groupoidal Segal maps are equivalences: in other words, for every $n \in \mathbb{N}$ and every partition $[k] \cup [k'] = [n]$ into two subsets such that $[k] \cap [k'] = \{*\}$, the canonical diagram

$$\begin{array}{ccc} \mathcal{G}_n & \longrightarrow & \mathcal{G}_k \\ \downarrow & & \downarrow \\ \mathcal{G}_{k'} & \longrightarrow & \mathcal{G}_0 \end{array}$$

is an ∞-pullback diagram. We write

$$\mathrm{Grpd}(\mathbf{H}) \subset \mathrm{Func}(\Delta^{\mathrm{op}}, \mathbf{H})$$

for the full subcategory of the ∞-category of simplicial objects in $\mathbf{H}$ on the groupoid objects.

The following example is fundamental. In fact the third ∞-Giraud axiom says that up to equivalence, all groupoid objects are of this form.

Example 2.14 For $X \to Y$ any morphism in $\mathbf{H}$, the Čech nerve $\check{C}(X \to Y)$ of $X \to Y$ (Definition 2.8) is a groupoid object. This example appears in [13] as Proposition 6.1.2.11.

The following statement refines the third ∞-Giraud axiom, Definition 2.1.

Theorem 2.15 *There is a natural equivalence of ∞-categories*

$$\mathrm{Grpd}(\mathbf{H}) \simeq (\mathbf{H}^{\Delta[1]})_{\mathrm{eff}},$$

where $(\mathbf{H}^{\Delta[1]})_{\mathrm{eff}}$ is the full sub-∞-category of the arrow category $\mathbf{H}^{\Delta[1]}$ of $\mathbf{H}$ on the effective epimorphisms, Definition 2.8.

This appears below Corollary 6.2.3.5 in [13].

In addition, every ∞-topos $\mathbf{H}$ comes with a notion of ∞-*group objects* that generalize both the ordinary notion of group objects in a topos as well as that of grouplike A_∞-spaces in Grpd_∞.

There is an evident definition of what an ∞-group object in $\mathbf{H}$ should be, and then there is a theorem saying that this is equivalent to a certain kind of simplicial object in $\mathbf{H}$. This theorem is part of what, we find, makes the theory of groups, group actions and principal bundles in an ∞-topos be so well behaved, and we will mostly work with this simplicial incarnation of group objects. But the evident definition that the reasoning starts with is of course this: a group object is an object which is equipped with an associative and unital product operation such that for each element there is an inverse. Now in the homotopy-theoretic context of ∞-topos theory an associative unital structure means an associative unital structure *up to coherent homotopy* and the technical term for this is A_∞-*structure*, famous from the theory of loop spaces, see [15] for a comprehensive modern account. Moreover, statements about elements here are supposed to be statements about connected components, and hence we ask for such A_∞ structures such that on connected components the product operation is invertible (such A_∞-structures are traditionally also called "groupal" or "grouplike").

Therefore the manifest definition of ∞-group objects in $\mathbf{H}$ is the following (this appears as Definition 5.1.3.2 together with Remark 5.1.3.3 in [15]).

Definition 2.16 An ∞-*group* in $\mathbf{H}$ is an A_∞-algebra G in $\mathbf{H}$ such that the sheaf of connected components $\pi_0(G)$ is a group object in $\tau_{\leq 0}\mathbf{H}$. Write $\mathrm{Grp}(\mathbf{H})$ for the ∞-category of ∞-groups in $\mathbf{H}$.

As in classical algebraic topology, the fundamental examples of such ∞-groups arise from forming loops, and there is a central de-looping theorem saying that, up to equivalence, in fact all ∞-groups arise this way:

Definition 2.17 Write

- $\mathbf{H}^{*/}$ for the ∞-category of pointed objects in $\mathbf{H}$;
- $\mathbf{H}_{\geq 1}$ for the full sub-∞-category of $\mathbf{H}$ on the connected objects;
- $\mathbf{H}^{*/}_{\geq 1}$ for the full sub-∞-category of the pointed objects on the connected objects.

Definition 2.18 Write

$$\Omega : \mathbf{H}^{*/} \to \mathbf{H}$$

for the ∞-functor that sends a pointed object $* \to X$ to its *loop space object*, i.e. the ∞-pullback

$$\begin{array}{ccc} \Omega X & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & X. \end{array}$$

Theorem 2.19 *(Lurie). Every loop space object canonically has the structure of an ∞-group, and this construction extends to an ∞-functor*

$$\Omega : \mathbf{H}^{*/} \to \mathrm{Grp}(\mathbf{H}).$$

This ∞-functor constitutes part of an equivalence of ∞-categories

$$(\Omega \dashv \mathbf{B}) : \ \mathrm{Grp}(\mathbf{H}) \underset{\mathbf{B}}{\overset{\Omega}{\underset{\longrightarrow}{\overset{\longleftarrow}{\simeq}}}} \mathbf{H}^{*/}_{\geq 1} .$$

This is Lemma 7.2.2.1 in [13]. (See also Theorem 5.1.3.6 of [15] where this is the equivalence denoted ϕ_0 in the proof.) For $\mathbf{H} = \mathrm{Grpd}_\infty$ this reduces to various classical theorems in homotopy theory, for instance the construction of classifying spaces (Kan and Milnor) and de-looping theorems (May and Segal).

Definition 2.20 We call the inverse $\mathbf{B} : \mathrm{Grp}(\mathbf{H}) \to \mathbf{H}^{*/}_{\geq 1}$ in Theorem 2.19 above the *delooping* functor of $\mathbf{H}$. By convenient abuse of notation we write $\mathbf{B}$ also for the composite $\mathbf{B} : \mathrm{Grpd}(\mathbf{H}) \to \mathbf{H}^{*/}_{\geq 1} \to \mathbf{H}$ with the functor that forgets the basepoint and the connectivity.

Remark 2.21 Even if the connected objects involved admit an essentially unique point, the homotopy type of the full hom-∞-groupoid $\mathbf{H}^{*/}(\mathbf{B}G, \mathbf{B}H)$ of pointed objects in general differs from the hom ∞-groupoid $\mathbf{H}(\mathbf{B}G, \mathbf{B}H)$ of the underlying unpointed objects. For instance let $\mathbf{H} := \mathrm{Grpd}_\infty$ and let G be an ordinary group, regarded as a group object in Grpd_∞. Then $\mathbf{H}^{*/}(\mathbf{B}G, \mathbf{B}G) \simeq \mathrm{Aut}(G)$ is the ordinary automorphism group of G, but $\mathbf{H}(\mathbf{B}G, \mathbf{B}G) = \mathrm{Aut}(\mathbf{B}G)$ is the automorphism 2-group of G, we discuss this further around Example 4.50 below.

Now observe that for X a pointed connected object, then the point inclusion $* \to X$ is an effective epimorphism and the loop space object ΩX in def. 2.18 is the first stage of the corresponding Čech nerve, as in the discussion of groupoid objects above in 2.2. This suggests that, moreover, group objects in $\mathbf{H}$ should be equivalent to those groupoid objects whose degree-0 piece is equivalent to the point. This is indeed the case, and this is central to the development of our discussion:

Proposition 2.22 (Lurie). *∞-groups G in $\mathbf{H}$ are equivalently those groupoid objects $\mathcal{G}$ in $\mathbf{H}$ (Definition 2.13) for which $\mathcal{G}_0 \simeq *$.*

This is the statement of the compound equivalence $\phi_3\phi_2\phi_1$ in the proof of Theorem 5.1.3.6 in [15].

Remark 2.23 This means that for G an ∞-group object, the Čech nerve extension of its delooping fiber sequence $G \to * \to \mathbf{B}G$ is the simplicial object

$$\cdots \mathrel{\substack{\to\\[-2pt]\to\\[-2pt]\to\\[-2pt]\to}} G \times G \mathrel{\substack{\to\\[-2pt]\to\\[-2pt]\to}} G \rightrightarrows * \twoheadrightarrow \mathbf{B}G$$

that exhibits G as a groupoid object over $*$. In particular it means that for G an ∞-group, the essentially unique morphism $* \to \mathbf{B}G$ is an effective epimorphism.

2.3 Cohomology

There is an intrinsic notion of *cohomology* in every ∞-topos $\mathbf{H}$: it is simply given by the connected components of mapping spaces. Of course such mapping spaces exist in every ∞-category, but we need some extra conditions on $\mathbf{H}$ in order for them to behave like cohomology sets. For instance, if $\mathbf{H}$ has pullbacks then there is a notion of long exact sequences in cohomology. Our main theorem (Theorem 3.17 below) will show that the second and third ∞-Giraud axioms imply that this intrinsic notion of cohomology has the property that it *classifies* certain geometric structures in the ∞-topos.

Definition 2.24 For $X, A \in \mathbf{H}$ two objects, we say that

$$H^0(X, A) := \pi_0 \mathbf{H}(X, A)$$

is the *cohomology set* of X with coefficients in A. In particular if G is an ∞-group we write

$$H^1(X, G) := H^0(X, \mathbf{B}G) = \pi_0 \mathbf{H}(X, \mathbf{B}G)$$

for cohomology with coefficients in the delooping $\mathbf{B}G$ of G. Generally, if $K \in \mathbf{H}$ has a specified n-fold delooping $\mathbf{B}^n K$ for some non-negative integer n, we write

$$H^n(X, K) := H^0(X, \mathbf{B}^n K) = \pi_0 \mathbf{H}(X, \mathbf{B}^n K).$$

In the context of cohomology on X with coefficients in A we say that

- the hom-space $\mathbf{H}(X, A)$ is the *cocycle ∞-groupoid*;
- an object $g : X \to A$ in $\mathbf{H}(X, A)$ is a *cocycle*;
- a morphism: $g \Rightarrow h$ in $\mathbf{H}(X, A)$ is a *coboundary* between cocycles.
- a morphism $c : A \to B$ in $\mathbf{H}$ represents the *universal characteristic class* (cohomology operation)

$$[c] : H^0(-, A) \to H^0(-, B).$$

If $X \simeq Y//G$ is a homotopy quotient, then the cohomology of X is equivariant cohomology of Y. Similarly, for general X this notion of cohomology incorporates various local notions of equivariance (for instance X might be an orbifold which is only locally equivalent to a global quotient).

Remark 2.25 Of special interest is the cohomology defined by a slice ∞-topos

$$\mathcal{X} := \mathbf{H}_{/X}$$

over some $X \in \mathbf{H}$. Such a slice is canonically equipped with the étale geometric morphism ([13], Remark 6.3.5.10)

$$((p_X)_! \dashv (p_X)^* \dashv (p_X)_*) : \mathbf{H}_{/X} \underset{(p_X)_*}{\overset{(p_X)_!}{\underset{\longleftarrow (p_X)^* \longrightarrow}{\longrightarrow}}} \mathbf{H} \,,$$

where $p_X : X \to *$ is the canonical morphism, $(p_X)_!$ simply forgets the morphism to X and where $(p_X)^* = X \times (-)$ forms the product with X. Accordingly we have $(p_X)^*(*_{\mathbf{H}}) \simeq *_{\mathcal{X}}$ and $(p_X)_!(*_{\mathcal{X}}) = X \in \mathbf{H}$, saying that the terminal object $*_{\mathcal{X}}$ in $\mathcal{X}$, which is the pullback of the terminal object $*_{\mathbf{H}}$ of $\mathbf{H}$ to $\mathcal{X}$, is identified with X itself. Therefore cohomology over X with coefficients of the form $(p_X)^*A$ is equivalently the cohomology in $\mathbf{H}$ of X with coefficients in A:

$$\mathcal{X}(X, (p_X)^*A) \simeq \mathbf{H}(X, A).$$

But for a general coefficient object $A \in \mathcal{X}$ the A-cohomology over X in $\mathcal{X}$ is a *twisted* cohomology of X in $\mathbf{H}$. This we discuss below in Sect. 4.2.

Typically one thinks of a morphism $A \to B$ in $\mathbf{H}$ as presenting a *characteristic class* of A if B is “simpler” than A, notably if B is an Eilenberg-MacLane object

$B = \mathbf{B}^n K$ for K a 0-truncated abelian group in $\mathbf{H}$. In this case the characteristic class may be regarded as being in the degree-n K-cohomology of A

$$[c] \in H^n(A, K).$$

Definition 2.26 For every morphism $c : \mathbf{B}G \to \mathbf{B}H \in \mathbf{H}$ define the *long fiber sequence to the left*

$$\cdots \to \Omega G \to \Omega H \to F \to G \to H \to \mathbf{B}F \to \mathbf{B}G \xrightarrow{c} \mathbf{B}H$$

by the consecutive pasting diagrams of ∞-pullbacks

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & & & \\
\cdots & & F & \longrightarrow & G & \longrightarrow & * & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & * & \longrightarrow & H & \longrightarrow & \mathbf{B}F & \longrightarrow & * \\
 & & & & \downarrow & & \downarrow & & \downarrow \\
 & & & & * & \longrightarrow & \mathbf{B}G & \xrightarrow{c} & \mathbf{B}H
\end{array}$$

We have the following basic fact.

Theorem 2.27 *1. In the long fiber sequence to the left of $c : \mathbf{B}G \to \mathbf{B}H$ after n iterations all terms are equivalent to the point if H and G are n-truncated.*
2. For every object $X \in \mathbf{H}$ we have a long exact sequence of pointed cohomology sets

$$\cdots \to H^0(X, G) \to H^0(X, H) \to H^1(X, F) \to H^1(X, G) \to H^1(X, H).$$

Remark 2.28 For the special case that G is a 1-truncated ∞-group (or *2-group*) classified by a 3-cocycle $\mathbf{c}$, Theorem 2.27 is a classical result due to [5]. The first and only nontrivial stage of the internal Postnikov tower

$$\begin{array}{ccccc}
\mathbf{B}^2 A & \longrightarrow & \mathbf{B}G & & \\
 & & \downarrow & & \\
 & & \mathbf{B}H & \xrightarrow{\mathbf{c}} & \mathbf{B}^3 A
\end{array}$$

of the delooped 2-group (with $H := \tau_0 G \in \tau_{\leq 0}\mathrm{Grp}(\mathbf{H})$ an ordinary group object and $A := \pi_1 G \in \tau_{\leq 0}\mathrm{Grp}(\mathbf{H})$ an ordinary abelian group object) yields the long exact sequence of pointed cohomology sets

$$0 \to H^1(-, A) \to H^0(-, G) \to H^0(-, H)$$
$$\to H^2(-, A) \to H^1(-, G) \to H^1(-, H) \to H^3(-, A)$$

(see also [23].) Notably, the last morphism gives the obstructions against lifting traditional nonabelian cohomology $H^1(-, H)$ to nonabelian cohomology $H^1(-, G)$ with values in the 2-group. This we discuss further in Sect. 4.3.

Generally, to every cocycle $g : X \to \mathbf{B}G$ is canonically associated its ∞-fiber $P \to X$ in $\mathbf{H}$, the ∞-pullback

$$\begin{array}{ccc} P & \longrightarrow & * \\ \downarrow & & \downarrow \\ X & \xrightarrow{g} & \mathbf{B}G. \end{array}$$

We now discuss how each such P canonically has the structure of a *G-principal ∞-bundle* and that $\mathbf{B}G$ is the *fine moduli object* (the *moduli ∞-stack*) for G-principal ∞-bundles.[3]

3 Principal bundles

We define here G-principal ∞-bundles in any ∞-topos $\mathbf{H}$, discuss their basic properties and show that they are classified by the intrinsic G-cohomology in $\mathbf{H}$, as discussed in Definition 2.24.

3.1 Introduction and survey

Let G be a topological group, or Lie group or some similar such object. The traditional definition of *G-principal bundle* is the following: there is a map

$$P \to X := P/G$$

which is the quotient projection induced by a *free* action

$$\rho : P \times G \to P$$

of G on a space (or manifold, depending on context) P, such that there is a cover $U \to X$ over which the quotient projection is isomorphic to the trivial one $U \times G \to U$.

In higher geometry, if G is a topological or smooth ∞-group, the quotient projection must be replaced by the ∞-quotient (homotopy quotient) projection

[3] The concept of (fine) moduli stacks is historically most commonly associated with algebraic geometry, but the problem which they solve, namely the classification of structures including their (auto-)equivalences, is universal. Specifically, if $\mathbf{H}$ is the ∞-topos over a site of schemes then it contains the moduli stacks as they appear in algebraic geometry.

$$P \to X := P//G$$

for the action of G on a topological or smooth ∞-groupoid (or ∞-stack) P. It is a remarkable fact that this single condition on the map $P \to X$ already implies that G acts freely on P and that $P \to X$ is locally trivial, when the latter notions are understood in the context of higher geometry. We will therefore define a G-principal ∞-bundle to be such a map $P \to X$.

As motivation for this, notice that if a Lie group G acts properly, but not freely, then the quotient $P \to X := P/G$ differs from the homotopy quotient, which instead is locally the quotient stack by the non-free part of the group action (an orbifold, if the stabilizers are finite).

Conversely this means that in the context of higher geometry a non-free action may also be principal: with respect not to a base space, but with respect to a base groupoid/stack. In the example just discussed, we have that the projection $P \to X//G_{\mathrm{stab}}$ exhibits P as a G-principal bundle over the action groupoid $P//G \simeq X//G_{\mathrm{stab}}$. For instance if $P = V$ is a vector space equipped with a G-representation, then $V \to V//G$ is a G-principal bundle over a groupoid/stack. In other words, the traditional requirement of freeness in a principal action is not so much a characterization of principality as such, as rather a condition that ensures that the base of a principal action is a 0-truncated object in higher geometry.

Beyond this specific class of 0-truncated examples, this means that we have the following noteworthy general statement: in higher geometry *every* ∞-action is principal with respect to *some* base, namely with respect to its ∞-quotient.

More is true: in the context of an ∞-topos every ∞-quotient projection of an ∞-group action is locally trivial, with respect to the canonical intrinsic notion of cover, hence of locality. Therefore also the condition of local triviality in the classical definition of principality becomes automatic. In fact, from the ∞-Giraud axioms, we see that the projection map $P \to P//G$ is always a cover (an *effective epimorphism*) and so, since every G-principal ∞-bundle trivializes over itself, it exhibits a local trivialization of itself; even without explicitly requiring it to be locally trivial.

As before, this means that the local triviality clause appearing in the traditional definition of principal bundles is not so much a characteristic of principality as such, as rather a condition that ensures that a given quotient taken in a category of geometric spaces coincides with the “refined” quotient obtained when regarding the situation in the ambient ∞-topos.

Another direct consequence of the ∞-Giraud axioms is the equivalence of the definition of principal bundles as quotient maps, as we have discussed so far, with the other main definition of principality: the condition that the “shear map” $(\mathrm{id}, \rho) : P \times G \to P \times_X P$ is an equivalence. It is immediate to verify in traditional 1-categorical contexts that this is equivalent to the action being properly free and exhibiting X as its quotient. Simple as this is, one may observe, in view of the above discussion, that the shear map being an equivalence is much more fundamental even: notice that $P \times G$ is the first stage of the *action groupoid object* $(P//G)_\bullet$, and that $P \times_X P$ is the first stage of the *Čech nerve groupoid object* $\check{C}(P \to X)$ of the corresponding quotient map. Accordingly, the shear map equivalence is the first stage in the equivalence of groupoid objects in the ∞-topos

$$(P//G)_\bullet \simeq \check{C}(P \to X).$$

This equivalence is just the explicit statement of the fact mentioned before: the groupoid object $(P//G)_\bullet$ is effective – as is any groupoid object in an ∞-topos – and, equivalently, its principal ∞-bundle map $P \to X$ is an effective epimorphism.

Fairly directly from this fact, finally, springs the classification theorem of principal ∞-bundles. For we have a canonical morphism of groupoid objects $(P//G)_\bullet \to (*//G)_\bullet$ induced by the terminal map $P \to *$. By the ∞-Giraud theorem the ∞-colimit over this sequence of morphisms of groupoid objects is a G-cocycle on X (Definition 2.24) canonically induced by P:

$$\varinjlim \left(\check{C}(P \to X)_\bullet \simeq (P//G)_\bullet \to (*//G)_\bullet\right) = (X \to \mathbf{B}G) \quad \in \mathbf{H}(X, \mathbf{B}G).$$

Conversely, from any such G-cocycle one obtains a G-principal ∞-bundle simply by forming its ∞-fiber: the ∞-pullback of the point inclusion $* \to \mathbf{B}G$. We show in [21] that in presentations of the ∞-topos theory by 1-categorical tools, the computation of this homotopy fiber is *presented* by the ordinary pullback of a big resolution of the point, which turns out to be nothing but the universal G-principal bundle. This appearance of the universal ∞-bundle as just a resolution of the point inclusion may be understood in light of the above discussion as follows. The classical characterization of the universal G-principal bundle $\mathbf{E}G$ is as a space that is homotopy equivalent to the point and equipped with a *free* G-action. But by the above, freeness of the action is an artefact of 0-truncation and not a characteristic of principality in higher geometry. Accordingly, in higher geometry the universal G-principal ∞-bundle for any ∞-group G may be taken to *be* the point, equipped with the trivial (maximally non-free) G-action. As such, it is a bundle not over the classifying *space* BG of G, but over the full moduli ∞-stack $\mathbf{B}G$.

The following table summarizes the relation between ∞-bundle theory and the ∞-Giraud axioms as indicated above, and as proven in the following section.

∞-Giraud axioms	Principal ∞-bundle theory
quotients are effective	every ∞-quotient $P \to X := P//G$ is principal
colimits are preserved by pullback	G-principal ∞-bundles are classified by $\mathbf{H}(X, \mathbf{B}G)$

3.2 Definition and classification

Definition 3.1 For $G \in \mathrm{Grp}(\mathbf{H})$ a group object, we say a G *-action* on an object $P \in \mathbf{H}$ is a groupoid object $(P//G)_\bullet$ (Definition 2.13) of the form

$$\cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} P \times G \times G \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} P \times G \underset{d_1}{\overset{\rho := d_0}{\rightrightarrows}} P$$

such that $d_1 : P \times G \to P$ is the projection, and such that the degreewise projections $P \times G^n \to G^n$ constitute a morphism of groupoid objects

$$\begin{array}{ccccccc}
\cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} & P \times G \times G & \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} & P \times G & \substack{\longrightarrow\\\longrightarrow} & P \\
& \downarrow & & \downarrow & & \downarrow \\
\cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} & G \times G & \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} & G & \substack{\longrightarrow\\\longrightarrow} & *
\end{array}$$

where the lower simplicial object exhibits G as a groupoid object over $*$ (see Remark 2.23).

With convenient abuse of notation we also write

$$P//G := \varinjlim (P \times G^{\times^\bullet}) \quad \in \mathbf{H}$$

for the corresponding ∞-colimit object, the ∞-*quotient* of this action.

Write

$$G\mathrm{Action}(\mathbf{H}) \hookrightarrow \mathrm{Grpd}(\mathbf{H})_{/(*//G)}$$

for the full sub-∞-category of groupoid objects over $*//G$ having as objects those that are G-actions.

Remark 3.2 The remaining face map d_0 in Definition 3.1

$$\rho := d_0 : P \times G \to P$$

is the action itself and the condition that it fits into such a simplicial diagram encodes the action property up to coherent homotopy. For instance using effectivity of groupoid objects, Definition 2.1, and the defining assumption on $(P//G)_1$ it follows that we have specified equivalences (where we abbreviate $X := P//G$)

$$(P//G)_2 \simeq (P \times_X P) \times_X P \simeq (P \times G) \times_X P \simeq P \times G \times G.$$

From this it follows that the three maps from $P \times G \times G$ to $P \times G$ here are given by, respectively, multiplication of the two group factors, action of the first group factor on P and projection on $P \times G$. The simplicial identities in $\mathbf{H}$ then give, in particular, a homotopy between, first, the composition of multiplying in the group and then acting and, second, the composition of acting with one factor and then with the other. At the next stage the simplicial identities encode that this homotopy in turn is compatible with the associativity-homotopy involved in acting with three group factors, and so on.

Remark 3.3 Using this notation in Proposition 2.22 we have

$$\mathbf{B}G \simeq *//G.$$

We list examples of ∞-actions below as Example 4.13. This is most conveniently done after establishing the theory of principal ∞-actions, to which we now turn.

Definition 3.4 Let $G \in \infty\mathrm{Grp}(\mathbf{H})$ be an ∞-group and let X be an object of $\mathbf{H}$. A *G-principal ∞-bundle* over X (or *G-torsor over X*) is

1. a morphism $P \to X$ in $\mathbf{H}$;
2. together with a G-action on P;

such that $P \to X$ is the colimiting cocone exhibiting the quotient map $X \simeq P//G$ (Definition 3.1).

A *morphism* of G-principal ∞-bundles over X is a morphism of G-actions that fixes X; the ∞-category of G-principal ∞-bundles over X is the homotopy fiber of ∞-categories

$$G\mathrm{Bund}(X) := G\mathrm{Action}(\mathbf{H}) \times_{\mathbf{H}} \{X\}$$

over X of the quotient map

$$G\mathrm{Action}(\mathbf{H}) \hookrightarrow \mathrm{Grpd}(\mathbf{H})_{/(*//G)} \longrightarrow \mathrm{Grpd}(\mathbf{H}) \xrightarrow{\lim\limits_{\rightarrow}} \mathbf{H}\,.$$

Remark 3.5 By the third ∞-Giraud axiom (Definition 2.1) this means in particular that a G-principal ∞-bundle $P \to X$ is an effective epimorphism in $\mathbf{H}$.

Remark 3.6 Even though $G\mathrm{Bund}(X)$ is by definition a priori an ∞-category, Proposition 3.16 below says that in fact it happens to be an ∞-groupoid: all its morphisms are invertible.

Proposition 3.7 *A G-principal ∞-bundle $P \to X$ satisfies the principality condition: the canonical morphism*

$$(\rho, p_1) : \;\; P \times G \xrightarrow{\simeq} P \times_X P$$

is an equivalence, where ρ is the G-action.

Proof By the third ∞-Giraud axiom (Definition 2.1) the groupoid object $P//G$ is effective, which means that it is equivalent to the Čech nerve of $P \to X$. In first degree this implies a canonical equivalence $P \times G \to P \times_X P$. Since the two face maps $d_0, d_1 : P \times_X P \to P$ in the Čech nerve are simply the projections out of the fiber product, it follows that the two components of this canonical equivalence are the two face maps $d_0, d_1 : P \times G \to P$ of $P//G$. By definition, these are the projection onto the first factor and the action itself. □

Proposition 3.8 *For $g : X \to \mathbf{B}G$ any morphism, its homotopy fiber $P \to X$ canonically carries the structure of a G-principal ∞-bundle over X.*

Proof That $P \to X$ is the fiber of $g : X \to \mathbf{B}G$ means that we have an ∞-pullback diagram

$$\begin{array}{ccc} P & \longrightarrow & * \\ \downarrow & & \downarrow \\ X & \xrightarrow{g} & \mathbf{B}G. \end{array}$$

By the pasting law for ∞-pullbacks, Proposition 2.3, this induces a compound diagram

$$\begin{array}{ccccccccc} \cdots & \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} & P \times G \times G & \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} & P \times G & \substack{\longrightarrow\\\longrightarrow} & P & \longrightarrow & X \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle g} \\ \cdots & \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} & G \times G & \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} & G & \substack{\longrightarrow\\\longrightarrow} & * & \longrightarrow & \mathbf{B}G \end{array}$$

where each square and each composite rectangle is an ∞-pullback. This exhibits the G-action on P. Since $* \to \mathbf{B}G$ is an effective epimorphism, so is its ∞-pullback $P \to X$. Since, by the ∞-Giraud theorem, ∞-colimits are preserved by ∞-pullbacks we have that $P \to X$ exhibits the ∞-colimit $X \simeq P//G$. □

Lemma 3.9 *For $P \to X$ a G-principal ∞-bundle obtained as in Proposition 3.8, and for $x : * \to X$ any point of X, we have a canonical equivalence*

$$x^*P \xrightarrow{\simeq} G$$

between the fiber x^*P and the ∞-group object G.

Proof This follows from the pasting law for ∞-pullbacks, which gives the diagram

$$\begin{array}{ccccc} G & \longrightarrow & P & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ * & \xrightarrow{x} & X & \xrightarrow{g} & \mathbf{B}G \end{array}$$

in which both squares as well as the total rectangle are ∞-pullbacks. □

Definition 3.10 The *trivial* G-principal ∞-bundle $(P \to X) \simeq (X \times G \to X)$ is, up to equivalence, the one obtained via Proposition 3.8 from the morphism $X \to * \to \mathbf{B}G$.

Proposition 3.11 *For $P \to X$ a G-principal ∞-bundle and $Y \to X$ any morphism, the ∞-pullback $Y \times_X P$ naturally inherits the structure of a G-principal ∞-bundle.*

Proof This uses the same kind of argument as in Proposition 3.8. □

In fact this is the special case of the pullback of what we will see below in Proposition 3.13 is the universal G-principal ∞-bundle $* \to \mathbf{B}G$.

Proposition 3.12 *Every G-principal ∞-bundle is locally trivial, that is there exists an effective epimorphism $U \longrightarrow\!\!\!\!\rightarrow X$ and an equivalence of G-principal ∞-bundles*

$$U \times_X P \simeq U \times G$$

from the pullback of P (Proposition 3.11) to the trivial G-principal ∞-bundle over U (Definition 3.10).

Proof For $P \to X$ a G-principal ∞-bundle, it is, by Remark 3.5, itself an effective epimorphism. The pullback of the G-bundle to its own total space along this morphism is trivial, by the principality condition (Proposition 3.7). Hence setting $U := P$ proves the claim. □

Proposition 3.13 *For every G-principal ∞-bundle $P \to X$ the square*

$$\begin{array}{ccccccc} & & P & \longrightarrow & * & & \\ & & \downarrow & & \downarrow & & \\ X & \simeq & \varinjlim_n (P \times G^{\times_n}) & \longrightarrow & \varinjlim_n G^{\times_n} & \simeq & \mathbf{B}G \end{array}$$

is an ∞-pullback diagram.

Proof Let $U \to X$ be an effective epimorphism such that $P \to X$ pulled back to U becomes the trivial G-principal ∞-bundle. By Proposition 3.12 this exists. By definition of morphism of G-actions and by functoriality of the ∞-colimit, this induces a morphism in $\mathbf{H}^{\Delta[1]}{}_{/(*\to\mathbf{B}G)}$ corresponding to the diagram

$$\begin{array}{ccccc} U \times G & \twoheadrightarrow & P & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{pt}} \\ U & \twoheadrightarrow & X & \longrightarrow & \mathbf{B}G \end{array} \quad \simeq \quad \begin{array}{ccccc} U \times G & & \twoheadrightarrow & & * \\ \downarrow & & & & \downarrow{\scriptstyle \mathrm{pt}} \\ U & \longrightarrow & * & \xrightarrow{\mathrm{pt}} & \mathbf{B}G \end{array}$$

in $\mathbf{H}$. By assumption, in this diagram the outer rectangles and the square on the very left are ∞-pullbacks. We need to show that the right square on the left is also an ∞-pullback.

Since $U \to X$ is an effective epimorphism by assumption, and since these are stable under ∞-pullback, $U \times G \to P$ is also an effective epimorphism, as indicated. This means that

$$P \simeq \varinjlim_n (U \times G)^{\times_P^{n+1}}.$$

We claim that for all $n \in \mathbb{N}$ the fiber products in the colimit on the right are naturally equivalent to $(U^{\times_X^{n+1}}) \times G$. For $n = 0$ this is clearly true. Assume then by induction that it holds for some $n \in \mathbb{N}$. Then with the pasting law (Proposition 2.3) we find an ∞-pullback diagram of the form

$$\begin{array}{ccccccc}
(U^{\times_X^{n+1}}) \times G & \simeq & (U \times G)^{\times_P^{n+1}} & \longrightarrow & (U \times G)^{\times_P^{n}} & \simeq & (U^{\times_X^{n}}) \times G \\
 & & \downarrow & & \downarrow & & \\
 & & U \times G & \longrightarrow & P & & \\
 & & \downarrow & & \downarrow & & \\
 & & U & \longrightarrow & X. & &
\end{array}$$

This completes the induction. With this the above expression for P becomes

$$\begin{aligned}
P &\simeq \varinjlim_n (U^{\times_X^{n+1}}) \times G \\
&\simeq \varinjlim_n \mathrm{pt}^* (U^{\times_X^{n+1}}) \\
&\simeq \mathrm{pt}^* \varinjlim_n (U^{\times_X^{n+1}}) \\
&\simeq \mathrm{pt}^* X,
\end{aligned}$$

where we have used that by the second ∞-Giraud axiom (Definition 2.1) we may take the ∞-pullback out of the ∞-colimit and where in the last step we used again the assumption that $U \to X$ is an effective epimorphism. □

Example 3.14 The fiber sequence

$$\begin{array}{ccc}
G & \longrightarrow & * \\
 & & \downarrow \\
 & & \mathbf{B}G
\end{array}$$

which exhibits the delooping $\mathbf{B}G$ of G according to Theorem 2.19 is a G-principal ∞-bundle over $\mathbf{B}G$, with *trivial* G-action on its total space $*$. Proposition 3.13 says that this is the *universal G-principal ∞-bundle* in that every other one arises as an ∞-pullback of this one. In particular, $\mathbf{B}G$ is a classifying object for G-principal ∞-bundles.

Below in Theorem 4.39 this relation is strengthened: every *automorphism* of a G-principal ∞-bundle, and in fact its full automorphism ∞-group arises from pullback of the above universal G-principal ∞-bundle: $\mathbf{B}G$ is the fine *moduli ∞-stack* of G-principal ∞-bundles.

The traditional definition of universal G-principal bundles in terms of contractible objects equipped with a free G-action has no intrinsic meaning in higher topos theory. Instead this appears in *presentations* of the general theory in model categories (or categories of fibrant objects) as *fibrant representatives* $\mathbf{E}G \to \mathbf{B}G$ of the above point inclusion. This we discuss in [21].

The main classification Theorem 3.17 below implies in particular that every morphism in $G\mathrm{Bund}(X)$ is an equivalence. For emphasis we note how this also follows directly:

Lemma 3.15 *Let* $\mathbf{H}$ *be an* ∞*-topos and let* X *be an object of* $\mathbf{H}$*. A morphism* $f: A \to B$ *in* $\mathbf{H}_{/X}$ *is an equivalence if and only if* $p^* f$ *is an equivalence in* $\mathbf{H}_{/Y}$ *for any effective epimorphism* $p: Y \to X$ *in* $\mathbf{H}$*.*

Proof It is clear, by functoriality, that $p^* f$ is a weak equivalence if f is. Conversely, assume that $p^* f$ is a weak equivalence. Since effective epimorphisms as well as equivalences are preserved by pullback we get a simplicial diagram of the form

$$\begin{array}{ccccccc} \cdots \, \substack{\rightarrow\\\rightarrow\\\rightarrow} & p^*A \times_A p^*A & \rightrightarrows & p^*A & \twoheadrightarrow & A \\ & \downarrow \simeq & & \downarrow \simeq & & \downarrow f \\ \cdots \, \substack{\rightarrow\\\rightarrow\\\rightarrow} & p^*B \times_B p^*B & \rightrightarrows & p^*B & \twoheadrightarrow & B \end{array}$$

where the rightmost horizontal morphisms are effective epimorphisms, as indicated. By definition of effective epimorphisms this exhibits f as an ∞-colimit over equivalences, hence as an equivalence. □

Proposition 3.16 *Every morphism between* G*-actions over* X *that are* G*-principal* ∞*-bundles over* X *is an equivalence.*

Proof Since a morphism of G-principal bundles $P_1 \to P_2$ is a morphism of Čech nerves that fixes their ∞-colimit X, up to equivalence, and since $* \to \mathbf{B}G$ is an effective epimorphism, we are, by Proposition 3.13, in the situation of Lemma 3.15. □

Theorem 3.17 *For all* $X, \mathbf{B}G \in \mathbf{H}$ *there is a natural equivalence of* ∞*-groupoids*

$$G\mathrm{Bund}(X) \simeq \mathbf{H}(X, \mathbf{B}G)$$

which on vertices is the construction of Proposition 3.8: a bundle $P \to X$ *is identified with a morphism* $X \to \mathbf{B}G$ *such that* $P \to X \to \mathbf{B}G$ *is a fiber sequence.*

We therefore say

- $\mathbf{B}G$ is the *classifying object* or *moduli* ∞*-stack* for G-principal ∞-bundles;
- a morphism $c : X \to \mathbf{B}G$ is a *cocycle* for the corresponding G-principal ∞-bundle and its class $[c] \in \mathrm{H}^1(X, G)$ is its *characteristic class*.

Proof By Definitions 3.1 and 3.4 and using the refined statement of the third ∞-Giraud axiom (Theorem 2.15), the ∞-groupoid $G\mathrm{Bund}(X)$ of G-principal ∞-bundles over

X is equivalent to the fiber over X of the sub-∞-category of the slice $\mathbf{H}^{\Delta[1]}{}_{/(*\to\mathbf{B}G)}$ of the arrow ∞-topos on those squares

$$\begin{array}{ccc} P & \longrightarrow & * \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathbf{B}G \end{array}$$

that exhibit $P \to X$ as a G-principal ∞-bundle. By Proposition 3.8 and Proposition 3.13 these are ∞-pullback squares, hence objects of the full sub-∞-category

$$\mathrm{Cart}(\mathbf{H}^{\Delta[1]}{}_{/(*\to\mathbf{B}G)}) \hookrightarrow \mathbf{H}^{\Delta[1]}{}_{/(*\to\mathbf{B}G)}$$

of the slice of the arrow category over the point inclusion into $\mathbf{B}G$ on those morphisms of morphisms (hence squares) which are ∞-pullbacks ("cartesian"). This inclusion is not full, rather the morphisms of G-principal ∞-bundles over X are those morphisms of G-actions that fix the base X, up to equivalence. By the universal property of the homotopy fiber product of ∞-categories this means that

$$G\mathrm{Bund}(X) \simeq \mathrm{Cart}(\mathbf{H}^{\Delta[1]}{}_{/(*\to\mathbf{B}G)}) \times_{\mathbf{H}} \{X\}.$$

Now by the universality of the ∞-pullback in $\mathbf{H}$ the morphisms between two Cartesian squares are fixed by their value on the underlying co-span $X \to \mathbf{B}G \leftarrow *$ and since in the above $* \to \mathbf{B}G$ is held fixed, they are fully determined by their value on X, so that the above is equivalent to

$$\mathbf{H}_{/\mathbf{B}G} \times_{\mathbf{H}} \{X\}.$$

Specifically, if one is to do this argument in terms of model categories: choose a model structure for $\mathbf{H}$ in which all objects are cofibrant, choose a fibrant representative for $\mathbf{B}G$ and a fibration resolution $\mathbf{E}G \to \mathbf{B}G$ of the universal G-bundle. Then the slice model structure of the arrow model structure over this presents the slice in question and the statement follows from the analogous 1-categorical statement.

This last expression finally is equivalent to

$$\mathbf{H}(X, \mathbf{B}G).$$

To see this for instance in terms of quasi-categories: the projection $\mathbf{H}_{/\mathbf{B}G} \to \mathbf{H}$ is a fibration by Proposition 2.1.2.1 and 4.2.1.6 in [13], hence the homotopy fiber $\mathbf{H}_{/\mathbf{B}G} \times_{\mathbf{X}} \{X\}$ is the ordinary fiber of quasi-categories. This is manifestly the $\mathrm{Hom}^R_{\mathbf{H}}(X, \mathbf{B}G)$ from Proposition 1.2.2.3 of [13]. Finally, by Proposition 2.2.4.1 there, this is equivalent to $\mathbf{H}(X, \mathbf{B}G)$. □

Corollary 3.18 *Equivalence classes of G-principal ∞-bundles over X are in natural bijection with the degree-1 G-cohomology of X:*

$$G\mathrm{Bund}(X)_{/\sim} \simeq H^1(X, G).$$

Proof By Definition 2.24 this is the restriction of the equivalence $G\mathrm{Bund}(X) \simeq \mathbf{H}(X, \mathbf{B}G)$ to connected components. □

4 Twisted bundles and twisted cohomology

We show here how the general notion of cohomology in an ∞-topos, considered above in Sect. 2.3, subsumes the notion of *twisted cohomology* and we discuss the corresponding geometric structures classified by twisted cohomology: *extensions* of principal ∞-bundles and *twisted ∞-bundles*.

Whereas ordinary cohomology is given by a derived hom-∞-groupoid, twisted cohomology is given by the ∞-groupoid of *sections of a local coefficient bundle* in an ∞-topos, which in turn is an *associated ∞-bundle* induced via a representation of an ∞-group G from a G-principal ∞-bundle (this is a geometric and unstable variant of the picture of twisted cohomology developed in [1,17]).

It is fairly immediate that, given a *universal* local coefficient bundle associated to a universal principal ∞-bundle, the induced twisted cohomology is equivalently ordinary cohomology in the corresponding slice ∞-topos. This identification provides a clean formulation of the contravariance of twisted cocycles. However, a universal coefficient bundle is a pointed connected object in the slice ∞-topos only when it is a trivial bundle, so that twisted cohomology does not classify principal ∞-bundles in the slice. We show below that instead it classifies *twisted principal ∞-bundles*, which are natural structures that generalize the twisted bundles familiar from twisted K-theory. Finally, we observe that twisted cohomology in an ∞-topos equivalently classifies extensions of structure groups of principal ∞-bundles.

A wealth of structures turn out to be special cases of nonabelian twisted cohomology and of twisted principal ∞-bundles and their study benefits from the general theory of twisted cohomology.

4.1 Actions and associated ∞-bundles

Let $\mathbf{H}$ be an ∞-topos, $G \in \mathrm{Grp}(\mathbf{H})$ an ∞-group. Fix an action $\rho : V \times G \to V$ on an object $V \in \mathbf{H}$ as in Definition 3.1. We discuss the induced notion of *ρ-associated V-fiber ∞-bundles*. We show that there is a *universal* ρ-associated V-fiber bundle over $\mathbf{B}G$ and observe that under Theorem 3.17 this is effectively identified with the action itself. Accordingly, we also further discuss ∞-actions as such.

Definition 4.1 For $V, X \in \mathbf{H}$ any two objects, a *V-fiber ∞-bundle* over X is a morphism $E \to X$, such that there is an effective epimorphism $U \twoheadrightarrow X$ and an ∞-pullback of the form

$$\begin{array}{ccc} U \times V & \longrightarrow & E \\ \downarrow & & \downarrow \\ U & \twoheadrightarrow & X\,. \end{array}$$

We say that $E \to X$ locally trivializes with respect to U. As usual, we often say V-*bundle* for short.

Definition 4.2 For $P \to X$ a G-principal ∞-bundle, we write

$$P \times_G V := (P \times V)//G$$

for the ∞-quotient of the diagonal ∞-action of G on $P \times V$. Equipped with the canonical morphism $P \times_G V \to X$ we call this the ∞-bundle ρ-*associated* to P.

Remark 4.3 The diagonal G-action on $P \times V$ is the product in $G\mathrm{Action}(\mathbf{H})$ of the given actions on P and on V. Since $G\mathrm{Action}(\mathbf{H})$ is a full sub-∞-category of a slice category of a functor category, the product is given by a degreewise pullback in $\mathbf{H}$:

$$\begin{array}{ccc} P \times V \times G^{\times n} & \longrightarrow & V \times G^{\times n} \\ \downarrow & & \downarrow \\ P \times G^{\times n} & \longrightarrow & G^{\times n}. \end{array}$$

and so

$$P \times_G V \simeq \varinjlim_n (P \times V \times G^{\times n}).$$

The canonical bundle morphism of the corresponding ρ-associated ∞-bundle is the realization of the left morphism of this diagram:

$$\begin{array}{ccc} P \times_G V & := & \varinjlim_n (P \times V \times G^{\times n}) \\ \downarrow & & \downarrow \\ X & \simeq & \varinjlim_n (P \times G^{\times n}). \end{array}$$

Example 4.4 By Theorem 3.17 every ∞-group action $\rho : V \times G \to V$ has a classifying morphism $\mathbf{c}$ defined on its homotopy quotient, which fits into a fiber sequence of the form

$$\begin{array}{ccc} V & \longrightarrow & V//G \\ & & \downarrow{\scriptstyle \mathbf{c}} \\ & & \mathbf{B}G. \end{array}$$

Regarded as an ∞-bundle, this is ρ-associated to the universal G-principal ∞-bundle $* \longrightarrow \mathbf{B}G$ from Example 3.14:

$$V//G \simeq * \times_G V.$$

Lemma 4.5 *The realization functor* $\varinjlim : \mathrm{Grpd}(\mathbf{H}) \to \mathbf{H}$ *preserves the* ∞*-pullback of Remark* 4.3*:*

$$P \times_G V \simeq \varinjlim_n (P \times V \times G^{\times_n}) \simeq (\varinjlim_n P \times G^{\times_n}) \times_{(\varinjlim_n G^{\times_n})} (\varinjlim_n V \times G^{\times_n}).$$

Proof Generally, let $X \to Y \leftarrow Z \in \mathrm{Grpd}(\mathbf{H})$ be a diagram of groupoid objects, such that in the induced diagram

$$\begin{array}{ccccc} X_0 & \longrightarrow & Y_0 & \longleftarrow & Z_0 \\ \downarrow & & \downarrow & & \downarrow \\ \varinjlim_n X_n & \longrightarrow & \varinjlim_n Y_n & \longleftarrow & \varinjlim_n Z_n \end{array}$$

the left square is an ∞-pullback. By the third ∞-Giraud axiom (Definition 2.1) the vertical morphisms are effective epis, as indicated. By assumption we have a pasting of ∞-pullbacks as shown on the left of the following diagram, and by the pasting law (Proposition 2.3) this is equivalent to the pasting shown on the right:

$$\begin{array}{ccc} X_0 \times_{Y_0} Z_0 & \longrightarrow & Z_0 \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & Y_0 \\ \downarrow & & \downarrow \\ \varinjlim_n X_n & \longrightarrow & \varinjlim_n Y_n \end{array} \quad \simeq \quad \begin{array}{ccc} X_0 \times_{Y_0} Z_0 & \longrightarrow & Z_0 \\ \downarrow & & \downarrow \\ (\varinjlim_n X_n) \times_{(\varinjlim_n Y_n)} (\varinjlim_n Z_n) & \longrightarrow & \varinjlim_n Z_n \\ \downarrow & & \downarrow \\ \varinjlim_n X_n & \longrightarrow & \varinjlim_n Y_n . \end{array}$$

Since effective epimorphisms are stable under ∞-pullback, this identifies the canonical morphism

$$X_0 \times_{Y_0} Z_0 \to (\varinjlim_n X_n) \times_{(\varinjlim_n Y_n)} (\varinjlim_n Z_n)$$

as an effective epimorphism, as indicated.

Since ∞-limits commute over each other, the Čech nerve of this morphism is the groupoid object $[n] \mapsto X_n \times_{Y_n} Z_n$. Therefore the third ∞-Giraud axiom now says that $\varinjlim$ preserves the ∞-pullback of groupoid objects:

$$\varinjlim (X \times_Y Z) \simeq \varinjlim_n (X_n \times_{Y_n} Z_n) \simeq (\varinjlim_n X_n) \times_{(\varinjlim_n Y_n)} (\varinjlim_n Z_n).$$

Consider this now in the special case that $X \to Y \leftarrow Z$ is $(P \times G^{\times \bullet}) \to G^{\times \bullet} \leftarrow (V \times G^{\times \bullet})$. Theorem 3.17 implies that the initial assumption above is met, in that $P \simeq (P//G) \times_{*//G} * \simeq X \times_{\mathbf{B}G} *$, and so the claim follows. □

Proposition 4.6 *For $g_X : X \to \mathbf{B}G$ a morphism and $P \to X$ the corresponding G-principal ∞-bundle according to Theorem 3.17, there is a natural equivalence*

$$g_X^*(V//G) \simeq P \times_G V$$

over X, between the pullback of the ρ-associated ∞-bundle $V//G \xrightarrow{\mathbf{c}} \mathbf{B}G$ of Example 4.4 and the ∞-bundle ρ-associated to P by Definition 4.2.

Proof By Remark 4.3 the product action is given by the pullback

$$\begin{array}{ccc} P \times V \times G^{\times \bullet} & \longrightarrow & V \times G^{\times \bullet} \\ \downarrow & & \downarrow \\ P \times G^{\times \bullet} & \longrightarrow & G^{\times \bullet} \end{array}$$

in $\mathbf{H}^{\Delta^{\mathrm{op}}}$. By Lemma 4.5 the realization functor preserves this ∞-pullback. By Remark 4.3 it sends the left morphism to the associated bundle, and by Theorem 3.17 it sends the bottom morphism to g_X. Therefore it produces an ∞-pullback diagram of the form

$$\begin{array}{ccc} V \times_G P & \longrightarrow & V//G \\ \downarrow & & \downarrow{\scriptstyle \mathbf{c}} \\ X & \xrightarrow{g_X} & \mathbf{B}G. \end{array}$$

□

Remark 4.7 This says that $V//G \xrightarrow{\mathbf{c}} \mathbf{B}G$ is both, the V-fiber ∞-bundle ρ-associated to the universal G-principal ∞-bundle, Example 4.4, as well as the universal ∞-bundle for ρ-associated ∞-bundles.

Proposition 4.8 *Every ρ-associated ∞-bundle is a V-fiber ∞-bundle, Definition 4.1.*

Proof Let $P \times_G V \to X$ be a ρ-associated ∞-bundle. By the previous Proposition 4.6 it is the pullback $g_X^*(V//G)$ of the universal ρ-associated bundle. By Proposition 3.12 there exists an effective epimorphism $U \twoheadrightarrow X$ over which P trivializes, hence such that $g_X|_U$ factors through the point, up to equivalence. In summary and by the pasting law, Proposition 2.3, this gives a pasting of ∞-pullbacks of the form

$$\begin{array}{ccccc} U \times V & \longrightarrow & P \times_G V & \longrightarrow & V//G \\ \downarrow & & \downarrow & & \downarrow \\ U & \twoheadrightarrow & X & \xrightarrow{g_X} & \mathbf{B}G \\ & \searrow & & \nearrow & \\ & & * & & \end{array}$$

which exhibits $P \times_G V \to X$ as a V-fiber bundle by a local trivialization over U. □

So far this shows that every ρ-associated ∞-bundle is a V-fiber bundle. We want to show that, conversely, every V-fiber bundle is associated to a principal ∞-bundle.

Definition 4.9 Let $V \in \mathbf{H}$ be a κ-compact object, for some regular cardinal κ. By the characterization of Proposition 2.2, there exists an ∞-pullback square in $\mathbf{H}$ of the form

$$\begin{array}{ccc} V & \longrightarrow & \widehat{\mathrm{Obj}}_\kappa \\ \downarrow & & \downarrow \\ * & \xrightarrow{\vdash V} & \mathrm{Obj}_\kappa \end{array}$$

Write

$$\mathbf{B}\mathbf{Aut}(V) := \mathrm{im}(\vdash V)$$

for the ∞-image, Definition 2.12, of the classifying morphism $\vdash V$ of V. By definition this comes with an effective epimorphism

$$* \longrightarrow\!\!\!\!\!\rightarrow \mathbf{B}\mathbf{Aut}(V) \hookrightarrow \mathrm{Obj}_\kappa \ ,$$

and hence, by Proposition 2.22, it is the delooping of an ∞-group

$$\mathbf{Aut}(V) \in \mathrm{Grp}(\mathbf{H})$$

as indicated. We call this the *internal automorphism ∞-group* of V.

By the pasting law, Proposition 2.3, the image factorization gives a pasting of ∞-pullback diagrams of the form

$$\begin{array}{ccccc} V & \longrightarrow & V/\!/\mathbf{Aut}(V) & \longrightarrow & \widehat{\mathrm{Obj}}_\kappa \\ \downarrow & & \downarrow^{\mathbf{c}_V} & & \downarrow \\ * & \xrightarrow{\vdash V} & \mathbf{B}\mathbf{Aut}(V) & \hookrightarrow & \mathrm{Obj}_\kappa \end{array}$$

By Theorem 3.17 this defines a canonical ∞-action

$$\rho_{\mathbf{Aut}(V)} : V \times \mathbf{Aut}(V) \to V$$

of $\mathbf{Aut}(V)$ on V with homotopy quotient $V/\!/\mathbf{Aut}(V)$ as indicated.

Proposition 4.10 *Every V-fiber ∞-bundle is $\rho_{\mathbf{Aut}(V)}$-associated to an* $\mathbf{Aut}(V)$*-principal ∞-bundle.*

Proof Let $E \to V$ be a V-fiber ∞-bundle. By Definition 4.1 there exists an effective epimorphism $U \longrightarrow\!\!\!\!\!\rightarrow X$ along which the bundle trivializes locally. It follows by the second Axiom in Proposition 2.2 that on U the morphism $X \xrightarrow{\vdash E} \mathrm{Obj}_\kappa$ which classifies $E \to X$ factors through the point

$$\begin{array}{ccccc} U \times V & \longrightarrow & E & \longrightarrow & \widehat{\mathrm{Obj}}_\kappa \\ \downarrow & & \downarrow & & \downarrow \\ U & \twoheadrightarrow & X & \xrightarrow{\vdash E} & \mathrm{Obj}_\kappa . \\ & \searrow & & \nearrow_{\vdash V} & \\ & & * & & \end{array}$$

Since the point inclusion, in turn, factors through its ∞-image $\mathbf{BAut}(V)$, Definition 4.9, this yields the outer commuting diagram of the following form

$$\begin{array}{ccccc} U & \longrightarrow & * & \longrightarrow & \mathbf{BAut}(V) \\ \downarrow\!\!\downarrow & & \overset{g}{\nearrow} & & \downarrow \\ X & & \xrightarrow{\quad \vdash E \quad} & & \mathrm{Obj}_\kappa \end{array}$$

By the epi/mono factorization system of Proposition 2.11 there is a diagonal lift g as indicated. Using again the pasting law and by Definition 4.9 (and the discussion following that) this factorization induces a pasting of ∞-pullbacks of the form

$$\begin{array}{ccccc} E & \longrightarrow & V//\mathbf{Aut}(V) & \longrightarrow & \widehat{\mathrm{Obj}}_\kappa \\ \downarrow & & \downarrow \mathbf{c}_V & & \downarrow \\ X & \xrightarrow{g} & \mathbf{BAut}(V) & \hookrightarrow & \mathrm{Obj}_\kappa \end{array}$$

Finally, by Proposition 4.6, this exhibits $E \to X$ as being $\rho_{\mathbf{Aut}(V)}$-associated to the $\mathbf{Aut}(V)$-principal ∞-bundle with class $[g] \in H^1(X, G)$. □

Theorem 4.11 *V-fiber ∞-bundles over $X \in \mathbf{H}$ are classified by $H^1(X, \mathbf{Aut}(V))$.*

Under this classification, the V-fiber ∞-bundle corresponding to $[g] \in H^1(X, \mathbf{Aut}(V))$ is identified, up to equivalence, with the $\rho_{\mathbf{Aut}(V)}$-associated ∞-bundle (as in Definition 4.2) to the $\mathbf{Aut}(V)$-principal ∞-bundle corresponding to $[g]$ by Theorem 3.17.

Proof By Proposition 4.10 every morphism $X \xrightarrow{\vdash E} \mathrm{Obj}_\kappa$ that classifies a small ∞-bundle $E \to X$ which happens to be a V-fiber ∞-bundle factors via some g through the moduli ∞-stack $\mathbf{BAut}(V)$ for $\mathbf{Aut}(V)$-principal ∞-bundles

$$X \xrightarrow{g} \mathbf{BAut}(V) \hookrightarrow \mathrm{Obj}_\kappa , \qquad X \xrightarrow{\vdash E} \mathrm{Obj}_\kappa$$

Therefore it only remains to show that also every homotopy $(\vdash E_1) \Rightarrow (\vdash E_2)$ factors through a homotopy $g_1 \Rightarrow g_2$. This follows by applying the epi/mono lifting property of Proposition 2.11 to the diagram

$$\begin{array}{ccc} X \coprod X & \xrightarrow{(g_1, g_2)} & \mathbf{B}\mathbf{Aut}(V) \\ \downarrow & \nearrow & \downarrow \\ X & \longrightarrow & \mathrm{Obj}_\kappa \end{array}$$

The outer diagram exhibits the original homotopy. The left morphism is an effective epi (for instance immediately by Proposition 2.10), the right morphism is a monomorphism by construction. Therefore the dashed lift exists as indicated and so the top left triangular diagram exhibits the desired factorizing homotopy. □

Remark 4.12 In the special case that $\mathbf{H} = \mathrm{Grpd}_\infty$, the classification Theorem 4.11 is classical [16,30], traditionally stated in (what in modern terminology is) the presentation of Grpd_∞ by simplicial sets or by topological spaces. Recent discussions include [3]. For $\mathbf{H}$ a general 1-localic ∞-topos (meaning: with a 1-site of definition), the statement of Theorem 4.11 appears in [34], formulated there in terms of the presentation of $\mathbf{H}$ by simplicial presheaves. (We discuss the relation of these presentations to the above general abstract result in [21].) Finally, one finds that the classification of *G-gerbes* [9] and *G-2-gerbes* in [6] is the special case of the general statement, for $V = \mathbf{B}G$ and G a 1-truncated ∞-group. This we discuss below in Sect. 4.4.

We close this section with a list of some fundamental classes of examples of ∞-actions, or equivalently, by Remark 4.7, of universal associated ∞-bundles. For doing so we use again that, by Theorem 3.17, to give an ∞-action of G on V is equivalent to giving a fiber sequence of the form $V \to V//G \to \mathbf{B}G$. Therefore the following list mainly serves to associate a traditional *name* with a given ∞-action.

Example 4.13 The following are ∞-actions.

1. For every $V \in \mathbf{H}$, the fiber sequence

$$\begin{array}{ccc} V & & \\ \downarrow{\scriptstyle (\mathrm{id}_V, \mathrm{pt}_{\mathbf{B}G})} & & \\ V \times \mathbf{B}G & \xrightarrow{p_2} & \mathbf{B}G \end{array}$$

   is the *trivial ∞-action* of G on V.
2. For every $G \in \mathrm{Grp}(\mathbf{H})$, the fiber sequence

$$\begin{array}{ccc} G & & \\ \downarrow & & \\ * & \longrightarrow & \mathbf{B}G \end{array}$$

which defines $\mathbf{B}G$ by Theorem 2.19 induces the *right action of G on itself*

$$* \simeq G//G.$$

At the same time this sequence, but now regarded as a bundle over $\mathbf{B}G$, is the universal G-principal ∞-bundle, Remark 3.14.

3. For every object $X \in \mathbf{H}$ write

$$\mathbf{L}X := X \times_{X \times X} X$$

for its *free loop space* object, the ∞-fiber product of the diagonal on X along itself

$$\begin{array}{ccc} \mathbf{L}X & \longrightarrow & X \\ {\scriptstyle \mathrm{ev}_*}\downarrow & & \downarrow \\ X & \longrightarrow & X \times X \end{array}$$

For every $G \in \mathrm{Grp}(\mathbf{H})$ there is a fiber sequence

$$\begin{array}{ccc} G & & \\ \downarrow & & \\ \mathbf{LB}G & \xrightarrow{\mathrm{ev}_*} & \mathbf{B}G \end{array}$$

This exhibits the *adjoint action of G on itself*

$$\mathbf{LB}G \simeq G//_{\mathrm{ad}}G.$$

4. For every $V \in \mathbf{H}$ there is the canonical ∞-action of the *automorphism ∞-group*

$$\begin{array}{ccc} V & & \\ \downarrow & & \\ V//\mathbf{Aut}(V) & \longrightarrow & \mathbf{BAut}(V) \end{array}$$

introduced in Definition 4.9, this exhibits the *automorphism action*.

5. For $\rho_1, \rho_2 \in \mathbf{H}_{/\mathbf{B}G}$ two G-∞-actions on objects $V_1, V_2 \in \mathbf{H}$, respectively, their internal hom $[\rho_1, \rho_2] \in \mathbf{H}_{/\mathbf{B}G}$ in the slice over $\mathbf{B}G$ is a G-∞-action on the internal hom $[V_1, V_2] \in \mathbf{H}$:

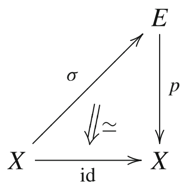
$$\begin{array}{ccc} [V_1, V_2] & & \\ \downarrow & & \\ [V_1, V_2]//G & \longrightarrow & \mathbf{B}G \end{array}$$

hence $[V_1, V_2]//G \simeq \sum_{\mathbf{B}G}[\rho_1, \rho_2]$, where $\sum_{\mathbf{B}G} : \mathbf{H}_{/\mathbf{B}G} \to \mathbf{H}$ is the left adjoint to pullback along the terminal map. (This follows by the fact that the inverse image of base change along $\mathrm{pt}_{\mathbf{B}G} : * \to \mathbf{B}G$ is a cartesian closed ∞-functor and hence preserves internal homs.[4]) This is the *conjugation ∞-action* of G on morphisms $V_1 \to V_2$ by pre- and postcomposition with the action of G on V_1 and V_2, respectively.

4.2 Sections and twisted cohomology

We discuss a general notion of *twisted cohomology* or *cohomology with local coefficients* in any ∞-topos $\mathbf{H}$, where the *local coefficient ∞-bundles* are associated ∞-bundles as discussed above, and where the cocycles are *sections* of these local coefficient bundles.

Definition 4.14 Let $p : E \to X$ be any morphism in $\mathbf{H}$, to be regarded as an ∞-bundle over X. A *section* of E is a diagram



(where for emphasis we display the presence of the homotopy filling the diagram). The *∞-groupoid of sections* of $E \xrightarrow{p} X$ is the homotopy fiber

$$\Gamma_X(E) := \mathbf{H}(X, E) \times_{\mathbf{H}(X,X)} \{\mathrm{id}_X\}$$

of the space of all morphisms $X \to E$ on those that cover the identity on X.

We record two elementary but important lemmas about spaces of sections.

Lemma 4.15 *There is a canonical identification*

$$\Gamma_X(E) \simeq \mathbf{H}_{/X}(\mathrm{id}_X, p)$$

of the space of sections of $E \to X$ with the hom-∞-groupoid in the slice ∞-topos $\mathbf{H}_{/X}$ between the identity on X and the bundle map p.

Proof For instance by Proposition 5.5.5.12 in [13]. □

[4] U.S. thanks Mike Shulman for discussion of this point.

Lemma 4.16 *Let*

$$\begin{array}{ccc} E_1 & \longrightarrow & E_2 \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} \\ B_1 & \xrightarrow{f} & B_2 \end{array}$$

be an ∞*-pullback diagram in* $\mathbf{H}$ *and let* $X \xrightarrow{g_X} B_1$ *be any morphism. Then post-composition with* f *induces a natural equivalence of hom-*∞*-groupoids*

$$\mathbf{H}_{/B_1}(g_X, p_1) \simeq \mathbf{H}_{/B_2}(f \circ g_X, p_2).$$

Proof By Proposition 5.5.5.12 in [13], the left hand side is given by the homotopy pullback

$$\begin{array}{ccc} \mathbf{H}_{/B_1}(g_X, p_1) & \longrightarrow & \mathbf{H}(X, E_1) \\ \downarrow & & \downarrow{\scriptstyle \mathbf{H}(X,p_1)} \\ \{g_X\} & \longrightarrow & \mathbf{H}(X, B_1). \end{array}$$

Since the hom-∞-functor $\mathbf{H}(X, -) : \mathbf{H} \to \mathrm{Grpd}_\infty$ preserves the ∞-pullback $E_1 \simeq f^* E_2$, this extends to a pasting of ∞-pullbacks, which by the pasting law (Proposition 2.3) is

$$\begin{array}{ccccc} \mathbf{H}_{/B_1}(g_X, p_1) & \longrightarrow & \mathbf{H}(X, E_1) & \longrightarrow & \mathbf{H}(X, E_2) \\ \downarrow & & \downarrow{\scriptstyle \mathbf{H}(X,p_1)} & & \downarrow{\scriptstyle \mathbf{H}(X,p_2)} \\ \{g_X\} & \longrightarrow & \mathbf{H}(X, B_1) & \xrightarrow[\mathbf{H}(X,f)]{} & \mathbf{H}(X, B_2) \end{array}$$

$$\simeq \quad \begin{array}{ccc} \mathbf{H}_{/B_2}(f \circ g_X, p_2) & \longrightarrow & \mathbf{H}(X, E_2) \\ \downarrow & & \downarrow{\scriptstyle \mathbf{H}(X,p_2)} \\ \{f \circ g_X\} & \longrightarrow & \mathbf{H}(X, B_2). \end{array}$$

□

Fix now an ∞-group $G \in \mathrm{Grp}(\mathbf{H})$ and an ∞-action $\rho : V \times G \to V$. Write

$$\begin{array}{ccc} V & \longrightarrow & V/\!/G \\ & & \downarrow{\scriptstyle \mathbf{c}} \\ & & \mathbf{B}G \end{array}$$

for the corresponding *universal* ρ*-associated* ∞*-bundle* as discussed in Sect. 4.1.

Proposition 4.17 *For $g_X : X \to \mathbf{B}G$ a cocycle and $P \to X$ the corresponding G-principal ∞-bundle according to Theorem 3.17, there is a natural equivalence*

$$\Gamma_X(P \times_G V) \simeq \mathbf{H}_{/\mathbf{B}G}(g_X, \mathbf{c})$$

between the space of sections of the corresponding ρ-associated V-bundle (as in Definition 4.2) and the hom-∞-groupoid of the slice ∞-topos of $\mathbf{H}$ over $\mathbf{B}G$, between g_X and $\mathbf{c}$. Schematically:

$$\left\{ \begin{array}{ccc} & & E \\ & {}^{\sigma}\nearrow \;\; \Downarrow\simeq & \downarrow p \\ X & \xrightarrow{\mathrm{id}} & X \end{array} \right\} \simeq \left\{ \begin{array}{ccc} & & V/\!/G \\ & {}^{\sigma}\nearrow \;\; \Downarrow\simeq & \downarrow \mathbf{c} \\ X & \xrightarrow{g_X} & \mathbf{B}G \end{array} \right\}$$

Proof By Lemma 4.15 and Lemma 4.16. □

Corollary 4.18 *The ∞-groupoid of sections of the associated bundle $P \times_G V$ is naturally equivalent to the ∞-groupoid of morphisms of G-actions $P \to V$:*

$$\Gamma_X(P \times_G V) \simeq G\mathrm{Action}(\mathbf{H})(P, V).$$

Proof Using Proposition 4.17 with Theorem 3.17 and with the definition of $\mathbf{c}$ shows that a section $\sigma \in \Gamma_X(P \times_G V)$ is equivalently a morphism of G-actions $\bar{\sigma} : P \to V$ from the total space of the G-principal bundle, namely the morphism on homotopy fibers induced from the commuting square underlying σ when regarded as an element of $\mathbf{H}_{/\mathbf{B}G}(g_X, \mathbf{p})$:

$$\begin{array}{ccc} P & \xrightarrow{\bar{\sigma}} & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{\sigma} & V/\!/G \\ \downarrow{\scriptstyle g_X} & & \downarrow{\scriptstyle \mathbf{c}} \\ \mathbf{B}G & \xrightarrow{=} & \mathbf{B}G \end{array} .$$

By Definition 3.1 of $G\mathrm{Action}(\mathbf{H})$ as the full sub-∞-category of the slice $\mathrm{Grpd}(\mathbf{H})_{/*/\!/G}$, this establishes the equivalence. □

Proposition 4.19 *If in the above the cocycle g_X is trivializable, in the sense that it factors through the point $* \to \mathbf{B}G$ (equivalently if its class $[g_X] \in H^1(X, G)$ is trivial) then there is an equivalence*

$$\mathbf{H}_{/\mathbf{B}G}(g_X, \mathbf{c}) \simeq \mathbf{H}(X, V).$$

Proof In this case the homotopy pullback on the right in the proof of Proposition 4.17 is

$$\begin{array}{ccccc} \mathbf{H}_{/\mathbf{B}G}(g_X, \mathbf{c}) & \simeq & \mathbf{H}(X, V) & \longrightarrow & \mathbf{H}(X, V//G) \\ & & \downarrow & & \downarrow{\scriptstyle \mathbf{H}(X,\mathbf{c})} \\ \{g_X\} & \simeq & \mathbf{H}(X, *) & \longrightarrow & \mathbf{H}(X, \mathbf{B}G) \end{array}$$

using that $V \to V//G \xrightarrow{\mathbf{c}} \mathbf{B}G$ is a fiber sequence by definition, and that $\mathbf{H}(X, -)$ preserves this fiber sequence. □

Remark 4.20 Since by Proposition 3.12 every cocycle g_X trivializes locally over some cover $U \twoheadrightarrow X$ and equivalently, by Proposition 4.8, every ∞-bundle $P \times_G V$ trivializes locally, Proposition 4.19 says that elements $\sigma \in \Gamma_X(P \times_G V) \simeq \mathbf{H}_{/\mathbf{B}G}(g_X, \mathbf{c})$ *locally* are morphisms $\sigma|_U : U \to V$ with values in V. They fail to be so *globally* to the extent that $[g_X] \in H^1(X, G)$ is non-trivial, hence to the extent that $P \times_G V \to X$ is non-trivial.

This motivates the following definition.

Definition 4.21 We say that the ∞-groupoid $\Gamma_X(P \times_G V) \simeq \mathbf{H}_{/\mathbf{B}G}(g_X, \mathbf{c})$ from Proposition 4.17 is the ∞-groupoid of $[g_X]$-*twisted cocycles* with values in V, with respect to the *local coefficient ∞-bundle* $V//G \xrightarrow{\mathbf{c}} \mathbf{B}G$.

Accordingly, its set of connected components we call the $[g_X]$-*twisted V-cohomology* with respect to the local coefficient bundle $\mathbf{c}$ and write:

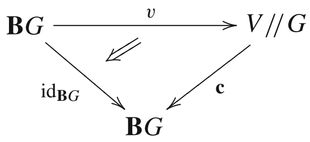
$$H^{[g_X]}(X, V) := \pi_0 \mathbf{H}_{/\mathbf{B}G}(g_X, \mathbf{c}).$$

Remark 4.22 The perspective that twisted cohomology is the theory of sections of associated bundles whose fibers are classifying spaces is maybe most famous for the case of twisted K-theory, where it was described in this form in [26]. But already the old theory of *ordinary cohomology with local coefficients* is of this form, as is made manifest in [8].

A proposal for a comprehensive theory in terms of bundles of topological spaces is in [17] and a systematic formulation in ∞-category theory and for the case of multiplicative generalized cohomology theories is in [1]. The formulation above refines this, unstably, to geometric cohomology theories/(nonabelian) sheaf hypercohomology, hence from bundles of classifying spaces to ∞-bundles of moduli ∞-stacks.

A wealth of examples and applications of such geometric nonabelian twisted cohomology of relevance in quantum field theory and in string theory is discussed in [27,28].

Example 4.23 In particular we may consider the space of sections of the universal ρ-associated V-fiber ∞-bundle itself, hence the $\mathrm{id}_{\mathbf{B}G}$-twisted cohomology with coefficients in $\mathbf{c} : V//G \to \mathbf{B}G$. A cocycle here is an (homotopy-)*invariant*



of V, under the G-action. The connected components of the hom-∞-groupoid form the ∞-*group cohomology* of G with coefficients in V:

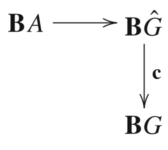
$$H_{\mathrm{Grp}}(G, V) := \pi_0 \Gamma_{\mathbf{B}G}(V//G).$$

In the case where V is in the image of a chain complex under the Dold-Kan correspondence, this statement is familiar from homological algebra: group cohomology is the derived functor of the invariants functor, which in turn is the hom-functor from the trivial G-action on the point (see the first item of example 4.13 for how $\mathrm{id}_{\mathbf{B}G}$ exhibits the trivial G-action on the point).

Remark 4.24 More generally, of special interest is the case where V is pointed connected, hence (by Theorem 2.19) of the form $V = \mathbf{B}A$ for some ∞-group A, and so (by Definition 2.24) the coefficients for degree-1 A-cohomology, and hence itself (by Theorem 3.17) the moduli ∞-stack for A-principal ∞-bundles. In this case $H^{[g_X]}(X, \mathbf{B}A)$ is *degree-1 twisted* A*-cohomology*. Generally, if $V = \mathbf{B}^n A$ it is *degree-n twisted* A*-cohomology*. In analogy with Definition 2.24 this is sometimes written

$$H^{n,[g_X]}(X, A) := H^{[g_X]}(X, \mathbf{B}^n A).$$

Moreover, in this case $V//G$ is itself pointed connected, hence of the form $\mathbf{B}\hat{G}$ for some ∞-group $\hat{G}$, and so the universal local coefficient bundle



exhibits $\hat{G}$ as an *extension of* ∞*-groups* of G by A. This case we discuss below in Sect. 4.3.

In this notation the local coefficient bundle $\mathbf{c}$ is left implicit. This convenient abuse of notation is justified to some extent by the fact that there is a *universal local coefficient bundle*:

Example 4.25 The classifying morphism of the $\mathbf{Aut}(V)$-action on some $V \in \mathbf{H}$ from Definition 4.9 according to Theorem 3.17 yields a local coefficient ∞-bundle of the form

$$\begin{array}{ccc} V & \longrightarrow & V//\mathbf{Aut}(V) \\ & & \downarrow \\ & & \mathbf{BAut}(V) \end{array}$$

which we may call the *universal local V-coefficient bundle*. In the case that V is pointed connected and hence of the form $V = \mathbf{B}G$

$$\begin{array}{ccc} \mathbf{B}G & \longrightarrow & (\mathbf{B}G)//\mathbf{Aut}(\mathbf{B}G) \\ & & \downarrow \\ & & \mathbf{BAut}(\mathbf{B}G) \end{array}$$

the universal twists of the corresponding twisted G-cohomology are the G-∞-*gerbes*. These we discuss below in Sect. 4.4.

4.3 Extensions and twisted bundles

We discuss the notion of *extensions* of ∞-groups (see Sect. 2.2), generalizing the traditional notion of group extensions. This is in fact a special case of the notion of principal ∞-bundle, Definition 3.4, for base space objects that are themselves deloopings of ∞-groups. For every extension of ∞-groups, there is the corresponding notion of *lifts of structure* ∞-*groups* of principal ∞-bundles. These are classified equivalently by trivializations of an *obstruction class* and by the twisted cohomology with coefficients in the extension itself, regarded as a local coefficient ∞-bundle.

Moreover, we show that principal ∞-bundles with an extended structure ∞-group are equivalent to principal ∞-bundles with unextended structure ∞-group but carrying a principal ∞-bundle for the *extending* ∞-group on their total space, which on fibers restricts to the given ∞-group extension. We formalize these *twisted (principal)* ∞-*bundles* and observe that they are classified by twisted cohomology, Definition 4.21.

Definition 4.26 We say a sequence of ∞-groups (Definition 2.16),

$$A \to \hat{G} \to G$$

in Grp($\mathbf{H}$) *exhibits* $\hat{G}$ *as an extension of* G *by* A if the delooping $\mathbf{B}A \to \mathbf{B}\hat{G} \to \mathbf{B}G$ is a fiber sequence in $\mathbf{H}$.

Remark 4.27 By continuing this fiber sequence to the left as

$$A \to \hat{G} \to G \to \mathbf{B}A \to \mathbf{B}\hat{G} \to \mathbf{B}G$$

this means, by Theorem 3.17, that

$$G \simeq \hat{G}//A$$

is the quotient of the extended ∞-group $\hat{G}$ by the extending ∞-group A.

Definition 4.28 A *braided ∞-group* is an ∞-group $A \in \mathrm{Grp}(\mathbf{H})$ equipped with the following equivalent structures:

1. a lift of the defining groupal $A_\infty \simeq E_1$-action to an E_2-action;
2. a group structure on the delooping $\mathbf{B}A$;
3. a double delooping $\mathbf{B}^2 A$.

Remark 4.29 The equivalence of the items in Definition 4.28 is essentially the content of theorem 5.1.3.6 in [15].

Definition 4.30 For A a braided ∞-group, Definition 4.28, a *braided-central extension* $\hat{G}$ of G by A is an extension $A \to \hat{G} \to G$, Definition 4.26, together with a prolongation of the defining fiber sequence one step further to the right:

$$\mathbf{B}A \longrightarrow \mathbf{B}\hat{G} \xrightarrow{\mathbf{p}} \mathbf{B}G \xrightarrow{\mathbf{c}} \mathbf{B}^2 A .$$

We write

$$\mathrm{Ext}(G, A) := \mathbf{H}(\mathbf{B}G, \mathbf{B}^2 A) \simeq (\mathbf{B}A)\mathrm{Bund}(\mathbf{B}G)$$

for the ∞-*groupoid of braided-central extensions* of G by A.

Example 4.31 An ordinary group (1-group) A that is braided is already abelian (by the Eckmann-Hilton argument). In this case a braided-central extension as above of a 1-group G is a central extension of G in the traditional sense.

Definition 4.32 Given an ∞-group extension $A \longrightarrow \hat{G} \xrightarrow{\Omega \mathbf{p}} G$ and given a G-principal ∞-bundle $P \to X$ in $\mathbf{H}$, we say that a *lift* $\hat{P}$ of P to a $\hat{G}$-principal ∞-bundle is a lift $\hat{g}_X$ of its classifying cocycle $g_X : X \to \mathbf{B}G$, under the equivalence of Theorem 3.17, through the extension:

$$\begin{array}{ccc} & & \mathbf{B}\hat{G} \\ & \overset{\hat{g}_X}{\nearrow} & \downarrow{\scriptstyle \mathbf{p}} \\ X & \xrightarrow[g_X]{} & \mathbf{B}G . \end{array}$$

Accordingly, the ∞-*groupoid of lifts* of P with respect to $\mathbf{p}$ is

$$\mathrm{Lift}(P, \mathbf{p}) := \mathbf{H}_{/\mathbf{B}G}(g_X, \mathbf{p}).$$

Observation 4.33 *By the universal property of the ∞-pullback, a lift exists precisely if the cohomology class*

$$[\mathbf{c}(g_X)] := [\mathbf{c} \circ g_X] \in H^2(X, A)$$

is trivial.

This is implied by Theorem 4.35, to which we turn after introducing the following terminology.

Definition 4.34 In the above situation, we call $[\mathbf{c}(g_X)]$ the *obstruction class* to the extension; and we call $[\mathbf{c}] \in H^2(\mathbf{B}G, A)$ the *universal obstruction class* of extensions through $\mathbf{p}$.

We say that a *trivialization* of the obstruction cocycle $\mathbf{c}(g_X)$ is a homotopy $\mathbf{c}(g_X) \to *_X$ in $\mathbf{H}(X, \mathbf{B}^2 A)$ (necessarily an equivalence), where $*_X : X \to * \to \mathbf{B}^2 A$ is the trivial cocycle. Accordingly, the ∞*-groupoid of trivializations of the obstruction* is

$$\mathrm{Triv}(\mathbf{c}(g_X)) := \mathbf{H}(X, \mathbf{B}^2 A)(\mathbf{c} \circ g_X, *_X).$$

We give now three different characterizations of spaces of extensions of ∞-bundles. The first two, by spaces of twisted cocycles and by spaces of trivializations of the obstruction class, are immediate consequences of the previous discussion:

Theorem 4.35 *Let $P \to X$ be a G-principal ∞-bundle corresponding by Theorem 3.17 to a cocycle $g_X : X \to \mathbf{B}G$.*

1. *There is a natural equivalence*

$$\mathrm{Lift}(P, \mathbf{p}) \simeq \mathrm{Triv}(\mathbf{c}(g_X))$$

   between the ∞-groupoid of lifts of P through $\mathbf{p}$, Definition 4.32, and the ∞-groupoid of trivializations of the obstruction class, Definition 4.34.
2. *There is a natural equivalence $\mathrm{Lift}(P, \mathbf{p}) \simeq \mathbf{H}_{/\mathbf{B}G}(g_X, \mathbf{p})$ between the ∞-groupoid of lifts and the ∞-groupoid of g_X-twisted cocycles relative to $\mathbf{p}$, Definition 4.21, hence a classification*

$$\pi_0 \mathrm{Lift}(P, \mathbf{P}) \simeq H^{1,[g_X]}(X, A)$$

   of equivalence classses of lifts by the $[g_X]$-twisted A-cohomology of X relative to the local coefficient bundle

$$\begin{array}{ccc} \mathbf{B}A & \longrightarrow & \mathbf{B}\hat{G} \\ & & \downarrow{\scriptstyle \mathbf{p}} \\ & & \mathbf{B}G. \end{array}$$

Proof The first statement is the special case of Lemma 4.16 where the ∞-pullback $E_1 \simeq f^* E_2$ in the notation there is identified with $\mathbf{B}\hat{G} \simeq \mathbf{c}^* *$. The second is evident after unwinding the definitions. □

Remark 4.36 For the special case that A is 0-truncated, we may, by the discussion in [22] identify $\mathbf{B}A$-principal ∞-bundles with A-*bundle gerbes*, [20]. Under this identification the ∞-bundle classified by the obstruction class $[\mathbf{c}(g_X)]$ above is what is called the *lifting bundle gerbe* of the lifting problem, see for instance [4] for a review. In this case the first item of Theorem 4.35 reduces to Theorem 2.1 in [33] and Theorem A (5.2.3) in [23]. The reduction of this statement to connected components, hence the special case of Observation 4.33, was shown in [5].

While, therefore, the discussion of extensions of ∞-groups and of lifts of structure ∞-groups is just a special case of the discussion in the previous sections, this special case admits geometric representatives of cocycles in the corresponding twisted cohomology by twisted principal ∞-bundles. This we turn to now.

Definition 4.37 Given an extension of ∞-groups $A \to \hat{G} \xrightarrow{\Omega \mathbf{c}} G$ and given a G-principal ∞-bundle $P \to X$, with class $[g_X] \in H^1(X, G)$, a *$[g_X]$-twisted A-principal ∞-bundle* on X is an A-principal ∞-bundle $\hat{P} \to P$ such that the cocycle $q : P \to \mathbf{B}A$ corresponding to it under Theorem 3.17 is a morphism of G-∞-actions.

The *∞-groupoid of $[g_X]$-twisted A-principal ∞-bundles on X* is

$$A\mathrm{Bund}^{[g_X]}(X) := G\mathrm{Action}(P, \mathbf{B}A) \subset \mathbf{H}(P, \mathbf{B}A).$$

Proposition 4.38 *Given an ∞-group extension $A \to \hat{G} \overset{\Omega \mathbf{c}}{\to} G$, an extension of a G-principal ∞-bundle $P \to X$ to a $\hat{G}$-principal ∞-bundle, Definition 4.32, induces an A-principal ∞-bundle $\hat{P} \to P$ fitting into a pasting diagram of ∞-pullbacks of the form*

$$\begin{array}{ccccccc}
\hat{G} & \longrightarrow & \hat{P} & \longrightarrow & * & & \\
\downarrow{\scriptstyle \Omega \mathbf{c}} & & \downarrow & & \downarrow & & \\
G & \longrightarrow & P & \xrightarrow{q} & \mathbf{B}A & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
* & \xrightarrow{x} & X & \xrightarrow{\hat{g}} & \mathbf{B}\hat{G} & \xrightarrow{\mathbf{c}} & \mathbf{B}G.
\end{array}
\qquad (g = \mathbf{c} \circ \hat{g} : X \to \mathbf{B}G)$$

In particular, it has the following properties:

1. *$\hat{P} \to P$ is a $[g_X]$-twisted A-principal bundle, Definition 4.37;*
2. *for all points $x : * \to X$ the restriction of $\hat{P} \to P$ to the fiber P_x is canonically equivalent to the ∞-group extension $\hat{G} \to G$.*

Proof This follows from repeated application of the pasting law for ∞-pullbacks, Proposition 2.3.

The bottom composite $g : X \to \mathbf{B}G$ is a cocycle for the given G-principal ∞-bundle $P \to X$ and it factors through $\hat{g} : X \to \mathbf{B}\hat{G}$ by assumption of the existence of the extension $\hat{P} \to P$.

Since also the bottom right square is an ∞-pullback by the given ∞-group extension, the pasting law asserts that the square over $\hat{g}$ is also an ∞-pullback, and then that so is the square over q. This exhibits $\hat{P}$ as an A-principal ∞-bundle over P classified by the cocycle q on P. By Corollary 4.18 this $\hat{P} \to P$ is twisted G-equivariant.

Now choose any point $x : * \to X$ of the base space as on the left of the diagram. Pulling this back upwards through the diagram and using the pasting law and the definition of loop space objects $G \simeq \Omega \mathbf{B}G \simeq * \times_{\mathbf{B}G} *$ the diagram completes by ∞-pullback squares on the left as indicated, which proves the claim. □

Theorem 4.39 *The construction of Proposition 4.38 extends to an equivalence of ∞-groupoids*

$$A\mathrm{Bund}^{[g_X]}(X) \simeq \mathbf{H}_{/\mathbf{B}G}(g_X, \mathbf{c})$$

between that of $[g_X]$-twisted A-principal bundles on X, Definition 4.37, and the cocycle ∞-groupoid of degree-1 $[g_X]$-twisted A-cohomology, Definition 4.21.

In particular the classification of $[g_X]$-twisted A-principal bundles is

$$A\mathrm{Bund}^{[g_X]}(X)_{/\sim} \simeq H^{1,[g_X]}(X, A).$$

Proof For $G = *$ the trivial group, the statement reduces to Theorem 3.17. The general proof works along the same lines as the proof of that theorem. The key step is the generalization of the proof of Proposition 3.13. This proceeds verbatim as there, only with $\mathrm{pt} : * \to \mathbf{B}G$ generalized to $i : \mathbf{B}A \to \mathbf{B}\hat{G}$. The morphism of G-actions $P \to \mathbf{B}A$ and a choice of effective epimorphism $U \to X$ over which $P \to X$ trivializes gives rise to a morphism in $\mathbf{H}^{\Delta[1]}_{/(* \to \mathbf{B}G)}$ which involves the diagram

$$\begin{array}{ccccc} U \times G & \twoheadrightarrow & P & \to & \mathbf{B}A \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle i} \\ U & \twoheadrightarrow & X & \to & \mathbf{B}\hat{G} \end{array} \quad \simeq \quad \begin{array}{ccccc} U \times G & & \twoheadrightarrow & & \mathbf{B}A \\ \downarrow & & & & \downarrow{\scriptstyle i} \\ U & \to & * & \xrightarrow{\mathrm{pt}} & \mathbf{B}\hat{G} \end{array}$$

in $\mathbf{H}$. (We are using that for the 0-connected object $\mathbf{B}\hat{G}$ every morphism $* \to \mathbf{B}G$ factors through $\mathbf{B}\hat{G} \to \mathbf{B}G$.) Here the total rectangle and the left square on the left are ∞-pullbacks, and we need to show that the right square on the left is then also an ∞-pullback. Notice that by the pasting law the rectangle on the right is indeed equivalent to the pasting of ∞-pullbacks

$$\begin{array}{ccccc} U \times G & \to & G & \to & \mathbf{B}A \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle i} \\ U & \to & * & \xrightarrow{\mathrm{pt}} & \mathbf{B}\hat{G} \end{array}$$

so that the relation

$$U^{\times_X^{n+1}} \times G \simeq i^*(U^{\times_X^{n+1}})$$

holds. With this the proof finishes as in the proof of Proposition 3.13, with pt^* generalized to i^*. □

Remark 4.40 Aspects of special cases of this theorem can be identified in the literature. For the special case of ordinary extensions of ordinary Lie groups, the equivalence of the corresponding extensions of a principal bundle with certain equivariant structures on its total space is essentially the content of [2,18]. In particular the twisted unitary bundles or *gerbe modules* of twisted K-theory [4] are equivalent to such structures.

For the case of $\mathbf{B}U(1)$-extensions of Lie groups, such as the String-2-group, the equivalence of the corresponding String-principal 2-bundles, by the above theorem, to certain bundle gerbes on the total spaces of principal bundles underlies constructions such as in [24]. Similarly the bundle gerbes on double covers considered in [29] are $\mathbf{B}U(1)$-principal 2-bundles on $\mathbb{Z}_2$-principal bundles arising by the above theorem from the extension $\mathbf{B}U(1) \to \mathbf{Aut}(\mathbf{B}U(1)) \to \mathbb{Z}_2$, a special case of the extensions that we consider in the next Sect. 4.4.

4.4 Gerbes

Recall from Remark 4.24 above that in an ∞-topos $\mathbf{H}$, those V-fiber ∞-bundles $E \to X$ whose typical fiber V is pointed connected are of special relevance. Recall that such a V is the moduli ∞-stack $V = \mathbf{B}G$ of G-principal ∞-bundles for some ∞-group G. Due to their local triviality, when regarded as objects in the slice ∞-topos $\mathbf{H}_{/X}$, these $\mathbf{B}G$-fiber ∞-bundles are themselves *connected objects*. Generally, for $\mathcal{X}$ an ∞-topos regarded as an ∞-topos of ∞-stacks over a given space X, it makes sense to consider its connected objects as ∞-bundles over X. Here we discuss these ∞-*gerbes*.

In the following discussion it is useful to consider two ∞-toposes:

1. an "ambient" ∞-topos $\mathbf{H}$ as before, to be thought of as an ∞-topos "of all geometric homotopy types" for a given notion of geometry (recall the discussion in Sect. 1), in which ∞-bundles are given by *morphisms* and the terminal object plays the role of the geometric point $*$;
2. an ∞-topos $\mathcal{X}$, to be thought of as the topos-theoretic incarnation of a single geometric homotopy type (space) X, hence as an ∞-topos of "geometric homotopy types étale over X", in which an ∞-bundle over X is given by an *object* and the terminal object plays the role of the base space X.
   In practice, $\mathcal{X}$ is the slice $\mathbf{H}_{/X}$ of the previous ambient ∞-topos over $X \in \mathbf{H}$, or the smaller ∞-topos $\mathcal{X} = \mathrm{Sh}_\infty(X)$ of (internal) ∞-stacks over X (hence étale objects over X, see section 3.10.7 of [28]).

In topos-theory literature the role of $\mathbf{H}$ above is sometimes referred to as that of a *gros* topos and then the role of $\mathcal{X}$ is referred to as that of a *petit* topos. The reader should beware that much of the classical literature on gerbes is written from the point of view of only the *petit* topos $\mathcal{X}$. For the following, recall remark 2.25 on cohomology in slice toposes.

The original definition of a *gerbe* on X as given by [9] is: a stack E (i.e. a 1-truncated ∞-stack) over X that is 1. *locally non-empty* and 2. *locally connected*. In the more intrinsic language of higher topos theory, these two conditions simply say that E is a *connected object* (Definition 6.5.1.10 in [13]): 1. the terminal morphism $E \to *$ is an effective epimorphism and 2. the 0th homotopy sheaf is trivial, $\pi_0(E) \simeq *$. This reformulation is made explicit in the literature for instance in Section 5 of [11] and in Section 7.2.2 of [13]. Therefore:

Definition 4.41 For $\mathcal{X}$ an ∞-topos, a *gerbe* in $\mathcal{X}$ is an object $E \in \mathcal{X}$ which is

1. connected;
2. 1-truncated.

For $X \in \mathbf{H}$ an object, a *gerbe E over X* is a gerbe in the slice $\mathbf{H}_{/X}$. This is an object $E \in \mathbf{H}$ together with an effective epimorphism $E \to X$ such that $\pi_i(E) = X$ for all $i \neq 1$.

Remark 4.42 Notice that conceptually this is different from the notion of *bundle gerbe* introduced in [20] (see [22] for a review). Bundle gerbes are presentations of *principal* ∞-bundles (Definition 3.4). But gerbes—at least the *G-gerbes* considered in a moment in Definition 4.48—are V-fiber ∞-bundles (Definition 4.1) hence *associated* to principal ∞-bundles (Proposition 4.10) with the special property of having pointed connected fibers. By Theorem 4.11 V-fiber ∞-bundles may be identified with their underlying $\mathbf{Aut}(V)$-principal ∞-bundles and so one may identify G-gerbes with nonabelian $\mathrm{Aut}(\mathbf{B}G)$-bundle gerbes (see also around Corollary 4.51 below), but considered generally, neither of these two notions is a special case of the other. Therefore the terminology is slightly unfortunate, but it is standard.

Definition 4.41 has various obvious generalizations. The following is considered in [13].

Definition 4.43 For $n \in \mathbb{N}$, an *EM n-gerbe* is an object $E \in \mathcal{X}$ which is

1. $(n-1)$-connected;
2. n-truncated.

Remark 4.44 This is almost the definition of an *Eilenberg-Mac Lane object* in $\mathcal{X}$, only that the condition requiring a global section $* \to E$ (hence $X \to E$) is missing. Indeed, the Eilenberg-Mac Lane objects of degree n in $\mathcal{X}$ are precisely the EM n-gerbes of *trivial class*, according to Corollary 4.51 below.

There is also an earlier established definition of *2-gerbes* in the literature [6], which is more general than EM 2-gerbes. Stated in the above fashion it reads as follows.

Definition 4.45 (*Breen* [6]) A *2-gerbe* in $\mathcal{X}$ is an object $E \in \mathcal{X}$ which is

1. connected;
2. 2-truncated.

This definition has an evident generalization to arbitrary degree, which we adopt here.

Definition 4.46 An *n-gerbe* in $\mathcal{X}$ is an object $E \in \mathcal{X}$ which is

1. connected;
2. n-truncated.

In particular an ∞-*gerbe* is a connected object.

The real interest is in those ∞-gerbes which have a prescribed typical fiber:

Remark 4.47 By the above, ∞-gerbes (and hence EM n-gerbes and 2-gerbes and hence gerbes) are much like deloopings of ∞-groups (Theorem 2.19) only that there is no requirement that there exists a global section. An ∞-gerbe for which there exists a global section $X \to E$ is called *trivializable*. By Theorem 2.19 trivializable ∞-gerbes are equivalent to ∞-group objects in $\mathcal{X}$ (and the ∞-groupoids of all of these are equivalent when transformations are required to preserve the canonical global section).

But *stalkwise* every ∞-gerbe E is of this form. For let

$$(x^* \dashv x_*) : \mathrm{Grpd}_\infty \underset{x_*}{\overset{x^*}{\rightleftarrows}} \mathcal{X}$$

be a topos point. Then the stalk $x^*E \in \mathrm{Grpd}_\infty$ of the ∞-gerbe is connected: because inverse images preserve the finite ∞-limits involved in the definition of homotopy sheaves, and preserve the terminal object. Therefore

$$\pi_0 x^* E \simeq x^* \pi_0 E \simeq x^* * \simeq *.$$

Hence for every point x we have a stalk ∞-group G_x and an equivalence

$$x^* E \simeq B G_x.$$

Therefore one is interested in the following notion.

Definition 4.48 For $G \in \mathrm{Grp}(\mathcal{X})$ an ∞-group object, a G-∞-*gerbe* is an ∞-gerbe E such that there exists

1. an effective epimorphism $U \longrightarrow\!\!\!\!\!\to X$ (onto the terminal object X of $\mathcal{X}$);
2. an equivalence $E|_U \simeq \mathbf{B}G|_U$.

Equivalently: a G-∞-gerbe is a $\mathbf{B}G$-fiber ∞-bundle over the terminal object X of $\mathcal{X}$, according to Definition 4.1.

In words this says that a G-∞-gerbe is one that locally looks like the moduli ∞-stack of G-principal ∞-bundles.

Example 4.49 For X a topological space and $\mathcal{X} = \mathrm{Sh}_\infty(X)$ the ∞-topos of ∞-sheaves over it, these notions reduce to the following.

- a 0-group object $G \in \tau_0\mathrm{Grp}(\mathcal{X}) \subset \mathrm{Grp}(\mathcal{X})$ is a sheaf of groups on X (here $\tau_0\mathrm{Grp}(\mathcal{X})$ denotes the 0-truncation of $\mathrm{Grp}(\mathcal{X})$);

- for $\{U_i \to X\}$ any open cover, the canonical morphism $\coprod_i U_i \to X$ is an effective epimorphism to the terminal object;
- $(\mathbf{B}G)|_{U_i}$ is the stack of $G|_{U_i}$-principal bundles ($G|_{U_i}$-torsors).

It is clear that one way to construct a G-∞-gerbe should be to start with an $\mathbf{Aut}(\mathbf{B}G)$-principal ∞-bundle, Remark 4.25, and then canonically *associate* a fiber ∞-bundle to it.

Example 4.50 For $G \in \tau_0\mathrm{Grp}(\mathrm{Grpd}_\infty)$ an ordinary group, $\mathbf{Aut}(\mathbf{B}G)$ is usually called the *automorphism 2-group* of G. Its underlying groupoid is equivalent to

$$\mathbf{Aut}(G) \times G \rightrightarrows \mathbf{Aut}(G),$$

the action groupoid for the action of G on $\mathbf{Aut}(G)$ via the homomorphism $\mathrm{Ad}\colon G \to \mathbf{Aut}(G)$.

Corollary 4.51 *Let $\mathcal{X}$ be an ∞-topos. Then for $G \in \mathrm{Grp}(\mathcal{X})$ any ∞-group object, G-∞-gerbes are classified by $\mathbf{Aut}(\mathbf{B}G)$-cohomology:*

$$\pi_0 G\mathrm{Gerbe} \simeq \pi_0\mathcal{X}(X, \mathbf{BAut}(\mathbf{B}G)) =: H^1(X, \mathbf{Aut}(\mathbf{B}G)).$$

Proof This is the special case of Theorem 4.11 for $V = \mathbf{B}G$. □

For the case that G is 0-truncated (an ordinary group object) this is the content of Theorem 23 in [11].

Example 4.52 For $G \in \tau_{\leq 0}\mathrm{Grp}(\mathcal{X}) \subset \mathrm{Grp}(\mathcal{X})$ an ordinary 1-group object, this reproduces the classical result of [9], which originally motivated the whole subject: by Example 4.50 in this case $\mathbf{Aut}(\mathbf{B}G)$ is the traditional automorphism 2-group and $H^1(X, \mathbf{Aut}(\mathbf{B}G))$ is Giraud's nonabelian G-cohomology that classifies G-gerbes (for arbitrary *band*, see Definition 4.59 below).

For $G \in \tau_{\leq 1}\mathrm{Grp}(\mathcal{X}) \subset \mathrm{Grp}(\mathcal{X})$ a 2-group, we recover the classification of 2-gerbes as in [6,7].

Remark 4.53 In Section 7.2.2 of [13] the special case that here we called *EM-n-gerbes* is considered. Beware that there are further differences: for instance the notion of morphisms between n-gerbes as defined in [13] is more restrictive than the notion considered here. For instance with our definition (and hence also that in [6]) each group automorphism of an abelian group object A induces an automorphism of the trivial A-2-gerbe $\mathbf{B}^2 A$. But, except for the identity, this is not admitted in [13] (manifestly so by the diagram above Lemma 7.2.2.24 there). Accordingly, the classification result in [13] is different: it involves the cohomology group $H^{n+1}(X, A)$. Notice that there is a canonical morphism

$$H^{n+1}(X, A) \to H^1(X, \mathbf{Aut}(\mathbf{B}^n A))$$

induced from the morphism $\mathbf{B}^{n+1} A \to \mathbf{Aut}(\mathbf{B}^n A)$.

We now discuss how the ∞-group extensions (Definition 4.26) given by the Postnikov stages of $\mathbf{Aut}(\mathbf{B}G)$, induce the notion of *band* of a gerbe, and how the corresponding twisted cohomology, according to Remark 4.35, reproduces the original definition of nonabelian cohomology in [9] and generalizes it to higher degree.

Definition 4.54 Fix $k \in \mathbb{N}$. For $G \in \infty\mathrm{Grp}(\mathcal{X})$ a k-truncated ∞-group object (a $(k+1)$-group), write

$$\mathbf{Out}(G) := \tau_k \mathbf{Aut}(\mathbf{B}G)$$

for the k-truncation of $\mathbf{Aut}(\mathbf{B}G)$. (Notice that this is still an ∞-group, since by Lemma 6.5.1.2 in [13] τ_n preserves all ∞-colimits and additionally all products.) We call this the *outer automorphism n-group* of G.

In other words, we write

$$\mathbf{c} : \mathbf{BAut}(\mathbf{B}G) \to \mathbf{BOut}(G)$$

for the top Postnikov stage of $\mathbf{BAut}(\mathbf{B}G)$.

Example 4.55 Let $G \in \tau_0\mathrm{Grp}(\mathrm{Grpd}_\infty)$ be a 0-truncated group object, an ordinary group. Then by Example 4.50, $\mathbf{Out}(G)$ is the coimage of $\mathrm{Ad} : G \to \mathrm{Aut}(G)$, which is the traditional group of outer automorphisms of G.

Definition 4.56 Write $\mathbf{B}^2\mathbf{Z}(G)$ for the ∞-fiber of the morphism $\mathbf{c}$ from Definition 4.54, fitting into a fiber sequence

$$\begin{array}{ccc} \mathbf{B}^2\mathbf{Z}(G) & \longrightarrow & \mathbf{BAut}(\mathbf{B}G) \ . \\ & & \downarrow{\scriptstyle \mathbf{c}} \\ & & \mathbf{BOut}(G) \end{array}$$

We call $\mathbf{Z}(G)$ the *braided center* of the ∞-group G.

Remark 4.57 To see that the fiber of $\Omega\mathbf{c}$ here is indeed the delooping of a group, notice that by theorem 2.19 one has to see that it is connected and pointed. Now the fiber of $\Omega\mathbf{c}$ is connected due to definition of $\mathbf{c}$ as a truncation map and the induced long exact sequence of (sheaves of) homotopy groups. It is moreover pointed since $\Omega\mathbf{c}$, being a morphism of groups, is a pointed morphism (the point being the neutral element) and using the universal property of the homotopy fiber.

Example 4.58 For G an ordinary group, so that $\mathbf{Aut}(\mathbf{B}G)$ is the automorphism 2-group from Example 4.50, $\mathbf{Z}(G)$ is the center of G in the traditional sense.

By Corollary 4.51 there is an induced morphism

$$\mathrm{Band} : \pi_0 G\mathrm{Gerbe} \to H^1(X, \mathbf{Out}(G)).$$

Definition 4.59 For $E \in G\mathrm{Gerbe}$ we call $\mathrm{Band}(E)$ the *band* of E.

By using Definition 4.56 in Definition 4.21, given a band $[\phi_X] \in H^1(X, \mathbf{Out}(G))$, we may regard it as a twist for twisted $\mathbf{Z}(G)$-cohomology, classifying G-gerbes with this band:

$$\pi_0 G\mathrm{Gerbe}^{[\phi_X]}(X) \simeq H^{2,[\phi_X]}(X, \mathbf{Z}(G)).$$

Remark 4.60 The original definition of *gerbe with band* in [9] is slightly more general than that of *G-gerbe* (with band) in [6]: in the former the local sheaf of groups whose delooping is locally equivalent to the gerbe need not descend to the base. These more general Giraud gerbes are 1-gerbes in the sense of Definition 4.46, but only the slightly more restrictive G-gerbes of Breen have the good property of being connected fiber ∞-bundles. From our perspective this is the decisive property of gerbes, and the notion of band is relevant only in this case.

Example 4.61 For G a 0-group this reduces to the notion of band as introduced in [9], for the case of G-gerbes as in [6].

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