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A nontrivial product in the E_2-term of the Adams spectral sequence for the sphere spectrum

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Abstract Let p be a prime greater than five and A the mod p Steenrod algebra. In this paper, we prove that the product $h_0 h_n \tilde{\delta}_{s+4} \in \mathrm{Ext}_A^{s+6,t(s,n)+s}(\mathbb{Z}/p, \mathbb{Z}/p)$ is nontrivial in the Adams E_2-term for $n \geq 5$ or $n = 2$, and trivial for $n = 3, 4$, where $0 \leq s < p-4$ and $t(s,n) = 2(p-1)[(s+2) + (s+2)p + (s+3)p^2 + (s+4)p^3 + p^n]$.

Keywords Steenrod algebra · Cohomology · Adams spectral sequence · May spectral sequence

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1 Introduction and statement of results

Computing the stable homotopy groups of spheres is an important task of algebraic topology. These groups are not fully-understood subjects yet. Homotopy groups of spheres are interesting because they are pretty fundamental and surprisingly compli-

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cated. Most modern methods for computing homotopy groups of spheres are based on spectral sequences. For example, we have the classical Adams spectral sequence (Adams SS)

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(\mathbb{Z}/p, \mathbb{Z}/p) \Rightarrow \pi_{t-s}(S)$$

(cf. [1]) based on the Eilenberg-MacLane spectrum $K\mathbb{Z}/p$, where S is the sphere spectrum localized at an odd prime p, A is the mod p Steenrod algebra and the differential is

$$d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}.$$

We also have the Adams-Novikov spectral sequence (cf. [9]) based on the Brown-Peterson spectrum BP.

Let S denote the sphere spectrum localized at a prime p greater than three. Let M be the Moore spectrum modulo the prime p given by the cofibration

$$S \xrightarrow{p} S \xrightarrow{i} M \xrightarrow{j} \Sigma S. \tag{1.1}$$

Let $\alpha : \Sigma^q M \longrightarrow M$ be the Adams map and $V(1)$ be its cofibre given by the cofibration

$$\Sigma^q M \xrightarrow{\alpha} M \xrightarrow{i'} V(1) \xrightarrow{j'} \Sigma^{q+1} M. \tag{1.2}$$

This spectrum $V(1)$ is known to be the Smith-Toda spectrum. Here $q = 2(p-1)$ as usual. Smith [10] showed when $p \geq 5$ there is a periodic map

$$\beta : \Sigma^{(p+1)q} V(1) \longrightarrow V(1)$$

which induces multiplication by v_2 in $K(2)$-theory. Let $V(2)$ be the cofibre of β : $\sum^{(p+1)q} V(1) \longrightarrow V(1)$ given by the cofibration

$$\Sigma^{(p+1)q} V(1) \xrightarrow{\beta} V(1) \xrightarrow{\bar{i}} V(2) \xrightarrow{\bar{j}} \Sigma^{(p+1)q+1} V(1). \tag{1.3}$$

When $p \geq 7$, Toda [11] proved that there exists the v_3-map $\gamma : \sum^{(p^2+p+1)q} V(2) \longrightarrow V(2)$ and $V(3)$ is its cofibre given by the cofibration

$$\Sigma^{(p^2+p+1)q} V(2) \xrightarrow{\gamma} V(2) \xrightarrow{\bar{\bar{i}}} V(3) \xrightarrow{\bar{\bar{j}}} \Sigma^{(p^2+p+1)q+1} V(2). \tag{1.4}$$

Remark 1.1 When $p \geq 11$, up till now we do not know if $V(4)$ exists, since a self-map on $V(3)$ inducing multiplication by v_4 is not known to exist.

Throughout this paper, we fix $q = 2(p-1)$. For computing the stable homotopy groups of spheres with the classical Adams SS, we must compute the E_2-term of the Adams SS $\mathrm{Ext}_A^{*,*}(\mathbb{Z}/p, \mathbb{Z}/p)$. There are two best methods of computing the stable homotopy groups of spheres with the classical Adams SS, i.e., the May spectral sequence and the lambda algebra. The known results on $\mathrm{Ext}_A^{*,*}(\mathbb{Z}/p, \mathbb{Z}/p)$ are as follows. $\mathrm{Ext}_A^{0,*}(\mathbb{Z}/p, \mathbb{Z}/p) = \mathbb{Z}/p$ by its definition. From [8], we have $\mathrm{Ext}_A^{1,*}(\mathbb{Z}/p, \mathbb{Z}/p)$ has $\mathbb{Z}/p$-basis consisting of $a_0 \in \mathrm{Ext}_A^{1,1}(\mathbb{Z}/p, \mathbb{Z}/p)$ and $h_i \in \mathrm{Ext}_A^{1,p^i q}(\mathbb{Z}/p, \mathbb{Z}/p)$ for all $i \geq 0$ and $\mathrm{Ext}_A^{2,*}(\mathbb{Z}/p, \mathbb{Z}/p)$ has $\mathbb{Z}/p$-basis consisting of α_2, a_0^2, $a_0 h_i (i > 0)$, $g_i (i \geq 0)$, $k_i (i \geq 0)$, $b_i (i \geq 0)$, and $h_i h_j (j \geq i+2, i \geq 0)$ whose internal degrees are $2q+1$, 2, $p^i q + 1$, $p^{i+1}q + 2p^i q$, $2p^{i+1}q + p^i q$, $p^{i+1}q$ and $p^i q + p^j q$, respectively. In 1980, Aikawa [2] determined $\mathrm{Ext}_A^{3,*}(\mathbb{Z}/p, \mathbb{Z}/p)$ by λ-algebra.

Remark 1.2 There is an important unresolved problem in stable homotopy theory: the convergence of $h_0 h_n$ in the Adams spectral sequence. In 1984, Cohen and Goerss [3] claimed that $h_0 h_n$ is a permanent cycle in the Adams spectral sequence for all primes bigger than 3. Later, a flaw in Cohen and Goerss [3] was found by N. Minami, and it appears to be fatal to their proof. So this problem is still open.

In 1998, X. Wang and Q. Zheng [12] proved the following theorem.

Theorem 1.1 [12] *For $p \geq 7$ and $0 \leq s < p-4$, there exists the fourth Greek letter family element $\tilde{\delta}_{s+4} \neq 0 \in \mathrm{Ext}_A^{s+4, t_1(s)+s}(\mathbb{Z}/p, \mathbb{Z}/p)$, where $t_1(s) = 2(p-1)[(s+1) + (s+2)p + (s+3)p^2 + (s+4)p^3]$.*

Note that we write $\tilde{\delta}_{s+4}$ for $\tilde{\alpha}_{s+4}^{(4)}$ which is described in [12].

In [7], X. Liu and H. Zhao made use of the above theorem to prove the following theorem as stated.

Theorem 1.2 [7] *For $p \geqslant 7$ and $4 \leqslant s < p$, the product $h_0 b_0 \tilde{\delta}_s \neq 0$ in the classical Adams spectral sequence.*

In [6], X. Liu and H. Wang considered the non-triviality of the product $h_0 h_n \tilde{\delta}_{s+4}$ and obtained the following theorem.

Theorem 1.3 [6] *Let $p \geq 7$ and $0 \leq s < p-4$. Then we have:*

(1) *the product $k_0 h_n \tilde{\delta}_{s+4}$ is nontrivial for $n \geq 5$.*
(2) *the product $k_0 h_n \tilde{\delta}_{s+4}$ is trivial for $n = 3, 4$.*
Here $t(s,n) = q[(s+1) + (s+3)p + (s+3)p^2 + (s+4)p^3 + p^n]$.

In this paper, we also consider some product involving $\tilde{\delta}_{s+4}$ and our main result can be stated as follows.

Theorem 1.4 *Let $p \geq 7$ and $0 \leq s < p-4$. Then in the cohomology of the mod p Steenrod algebra A, $\mathrm{Ext}_A^{s+6, t(s,n)+s}(\mathbb{Z}/p, \mathbb{Z}/p)$,*

(1) *the product* $h_0 h_n \tilde{\delta}_{s+4}$ *is nontrivial for* $n \geq 5$ *or* $n = 2$.
(2) *the product* $h_0 h_n \tilde{\delta}_{s+4}$ *is trivial for* $n = 3, 4$.
Here $t(s,n) = q[(s+2) + (s+2)p + (s+3)p^2 + (s+4)p^3 + p^n]$.

Remark 1.3 Here $\tilde{\delta}_s$ is the element of lowest filtration which could detect the element δ_s arising from the existence of a self-map on $V(3)$ inducing multiplication by v_4^s. Of course, such a self-map is not known to exist, so our result should be regarded as a result on the algebraic structure of $\mathrm{Ext}_A^{*,*}(\mathbb{Z}/p, \mathbb{Z}/p)$,.

The paper is arranged as follows: after recalling some knowledge on the May spectral sequence, we introduce a method of detecting generators of the E_1-term $E_1^{*,*,*}$ of the May spectral sequence in Sect. 2. Section 3 is devoted to showing Theorem 1.4.

2 The May spectral sequence

For completeness, we first recall some knowledge on the May spectral sequence in this section. From [9], there is a May spectral sequenc $\{E_r^{s,t,*}, d_r\}$ which converges to $\mathrm{Ext}_A^{s,t}(\mathbb{Z}/p, \mathbb{Z}/p)$ with E_1-term

$$E_1^{*,*,*} = E(h_{m,i} | m > 0, i \geq 0) \otimes P(b_{m,i} | m > 0, i \geq 0) \otimes P(a_n | n \geq 0), \quad (2.1)$$

where $E()$ is the exterior algebra, $P()$ is the polynomial algebra, and

$$h_{m,i} \in E_1^{1,2(p^m-1)p^i,2m-1}, b_{m,i} \in E_1^{2,2(p^m-1)p^{i+1},p(2m-1)}, a_n \in E_1^{1,2p^n-1,2n+1}.$$

One has

$$d_r : E_r^{s,t,u} \to E_r^{s+1,t,u-r} \quad (2.2)$$

and if $x \in E_r^{s,t,*}$ and $y \in E_r^{s',t',*}$, then

$$d_r(x \cdot y) = d_r(x) \cdot y + (-1)^s x \cdot d_r(y). \quad (2.3)$$

In particular, the first May differential d_1 is given by

$$d_1(h_{i,j}) = \sum_{0<k<i} h_{i-k,k+j} h_{k,j}, \quad d_1(a_i) = \sum_{0 \leq k < i} h_{i-k,k} a_k, d_1(b_{i,j}) = 0. \quad (2.4)$$

There also exists a graded commutativity in the May spectral sequence: $x \cdot y = (-1)^{ss'+tt'} y \cdot x$ for $x, y = h_{m,i}, b_{m,i}$ or a_n.

For each element $x \in E_1^{s,t,u}$, we define $\dim x = s$, $\deg x = t$, $M(x) = u$. Then we have that

$$\begin{cases} \dim h_{i,j} = \dim a_i = 1, \\ \dim b_{i,j} = 2, \\ \deg h_{i,j} = q(p^{i+j-1} + \cdots + p^j), \\ \deg b_{i,j} = q(p^{i+j} + \cdots + p^{j+1}), \\ \deg a_i = q(p^{i-1} + \cdots + 1) + 1, \\ \deg a_0 = 1, \\ M(h_{i,j}) = M(a_{i-1}) = 2i - 1, \\ M(b_{i,j}) = (2i-1)p, \end{cases} \tag{2.5}$$

where $i \geq 1$, $j \geq 0$.

Note that by the knowledge on the p-adic expression in number theory, for each integer $m \geq 0$, it can be expressed uniquely as

$$m = q(c_n p^n + c_{n-1} p^{n-1} + \cdots + c_1 p + c_0) + e,$$

where $0 \leq c_i < p$ $(0 \leq i < n)$, $p > c_n > 0$, $0 \leq e < q$.

Now we give a method used to compute generators of the May E_1-term. As regards the method, it originates from [4] and was developed in [5]. For convenience, we rewrite it here. Assume that g is a generator of the May E_1-term $E_1^{s,*,*}$ when $s < q$. We denote a_i, $h_{i,j}$ and $b_{i,j}$ by x, y and z respectively. By the graded commutativity of $E_1^{*,*,*}$, g can be written as the form

$$(x_1 \cdots x_b)(y_1 \cdots y_v)(z_1 \cdots z_l) \in E_1^{s,t+b,*},$$

where $t = (\bar{c}_0 + \bar{c}_1 + \cdots + \bar{c}_n p^n)q$ with $0 \leq \bar{c}_i < p$ $(0 \leq i < n)$, $0 < \bar{c}_n < p$, $s < q$ and $0 \leq b < q$. By (2.5), the degrees of x_i, y_i and z_i can be expressed uniquely as:

$$\begin{cases} \deg x_i = (x_{i,0} + x_{i,1} p + \cdots + x_{i,n} p^n)q + 1, \\ \deg y_i = (y_{i,0} + y_{i,1} p + \cdots + y_{i,n} p^n)q, \\ \deg z_i = (0 + z_{i,1} p + \cdots + z_{i,n} p^n)q, \end{cases}$$

and

(a) $(x_{i,0}, x_{i,1}, \ldots, x_{i,n})$ is of the form $(1, \ldots, 1, 0, \ldots, 0)$;
(b) $(y_{i,0}, y_{i,1}, \ldots, y_{i,n})$ is of the form $(0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0)$;
(c) $(0, z_{i,1}, \ldots, z_{i,n})$ is of the form $(0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0)$.

By the graded commutativity of $E_1^{*,*,*}$, $g = (x_1 \cdots x_b)(y_1 \cdots y_v)(z_1 \cdots z_l) \in E_1^{s,t+b,*}$ can be arranged in the following way:

$$\begin{cases} \text{(i)} \quad \text{If } i > j, \text{ we put } a_i \text{ on the left side of } a_j, \\ \text{(ii)} \quad \text{If } j < k, \text{ we put } h_{i,j} \text{ on the left side of } h_{w,k}, \\ \text{(iii)} \quad \text{If } i > w, \text{ we put } h_{i,j} \text{ on the left side of } h_{w,j}, \\ \text{(iv)} \quad \text{Apply the rules (ii) and (iii) to } b_{i,j}. \end{cases}$$

Then from (a)-(c) and (i)-(iv), the factors $x_{i,j}$, $y_{i,j}$ and $z_{i,j}$ in g must satisfy the following conditions:

$$\begin{cases} (1) & x_{1,j} \geq x_{2,j} \geq \cdots \geq x_{b,j}, \\ (2) & x_{i,0} \geq x_{i,1} \geq \cdots \geq x_{i,n}, \\ (3) & \text{If } y_{i,j-1} = 0 \text{ and } y_{i,j} = 1, \text{ then for all k} < j \; y_{i,k} = 0, \\ (4) & \text{If } y_{i,j} = 1 \text{ and } y_{i,j+1} = 0, \text{ then for all } k > j \; y_{i,k} = 0, \\ (5) & y_{1,0} \geq y_{2,0} \geq \cdots \geq y_{v,0}, \\ (6) & \text{If } y_{i,0} = y_{i+1,0}, \; y_{i,1} = y_{i+1,1}, \ldots, y_{i,j} = y_{i+1,j}, \text{ then } y_{i,j+1} \geq y_{i+1,j+1}, \\ (7) & \text{Apply the similar rules (3)} \sim \text{(6) to } z_{i,j}. \end{cases} \tag{2.6}$$

From $\deg g = \sum_{i=1}^{b} \deg x_i + \sum_{i=1}^{v} \deg y_i + \sum_{i=1}^{l} \deg z_i$, by the properties of the p-adic number we get the following group of equations

$$\begin{cases} x_{1,0} + \cdots + x_{b,0} + y_{1,0} + \cdots + y_{v,0} + 0 + \cdots + 0 = \bar{c}_0 + k_0 p, \\ x_{1,1} + \cdots + x_{b,1} + y_{1,1} + \cdots + y_{v,1} + z_{1,1} + \cdots + z_{l,1} = \bar{c}_1 + k_1 p - k_0, \\ \cdots \\ x_{1,n-1} + \cdots + x_{b,n-1} + y_{1,n-1} + \cdots + y_{v,n-1} + z_{1,n-1} + \cdots + z_{l,n-1} = \bar{c}_{n-1} + k_{n-1} p - k_{n-2}, \\ x_{1,n} + \cdots + x_{b,n} + y_{1,n} + \cdots + y_{v,n} + z_{1,n} + \cdots + z_{l,n} = \bar{c}_n - k_{n-1}, \end{cases} \tag{2.7}$$

where $k_i \geq 0$ for $0 \leq i \leq n-1$. From the above group of equations, we get two integer sequences $K = (k_0, k_1, \ldots, k_{n-1})$ and $S = (\bar{c}_0 + k_0 p, \bar{c}_1 + k_1 p - k_0, \ldots, \bar{c}_n - k_{n-1})$ denoted by $(c_0, c_1, \ldots, c_n)$ which are determined by $(k_0, k_1, \ldots, k_{n-1})$ and $(\bar{c}_0, \bar{c}_1, \ldots, \bar{c}_n)$. We want to get the solutions of the group of Eq. 2.7 which satisfy the conditions 2.6. The determination of $E_1^{s,t+b,*}$ is reduced to the following steps:

(1) List up all the possible (b, v, l) such that $b + v + 2l = s$.
(2) For each given (b, v, l), list all the sequences $K = (k_0, k_1, \ldots, k_{n-1})$ and $S = (c_0, c_1, \ldots, c_n)$ such that $c_i \leq b + v + l$ for all $0 \leq i \leq n$.
(3) For each given (b, v, l), $K = (k_0, k_1, \ldots, k_{n-1})$ and $S = (c_0, c_1, \ldots, c_n)$, solve the group of Eq. 2.7 by virtue of 2.6, then determine all the generators of $E_1^{s,t+b,*}$ by setting the corresponding second degrees.

3 Proof of Theorem 1.4

In this section we first give three lemmas which are needed in the proof of Theorem 1.4. Then we will give the proof of Theorem 1.4.

Lemma 3.1 [7, Lemma 3.1]. *For $p \geq 7$ and $0 \leq s < p - 4$. Then the fourth Greek letter family element $\tilde{\delta}_{s+4} \in \mathrm{Ext}_A^{s+4,t_1(s)+s}(\mathbb{Z}/p, \mathbb{Z}/p)$ is represented by*

$$a_4^s h_{4,0} h_{3,1} h_{2,2} h_{1,3} \in E_1^{s+4,t_1(s)+s,*}$$

in the E_1-term of the May spectral sequence, where $t_1(s) = [(s+1) + (s+2)p + (s+3)p^2 + (s+4)p^3]q$.

Lemma 3.2 *Let $p \geq 7$, $n \geq 2$,and $n \neq 3$ $0 < s < p - 4$. Then we have the May E_1-term*

$$E_1^{s+5,t(s,n)+s,*} = \begin{cases} 0 & n=2,5, \textit{ or } n>5 \textit{ and } 0<s<p-5, \\ M & n=4, \\ K & n>5 \textit{ and } s=p-5. \end{cases}$$

Here, $t(s,n) = [(s+2)+(s+2)p+(s+3)p^2+(s+4)p^3+p^n]q$*,* M *is the* $\mathbb{Z}/p$*-module generated by the following ten elements:*

$$\begin{cases} \mathbf{g}1 = a_4^s h_{5,0}h_{4,0}h_{2,2}b_{1,2}, \\ \mathbf{g}2 = a_4^s h_{5,0}h_{4,0}h_{1,3}b_{2,1}, \\ \mathbf{g}3 = a_5 a_4^{s-1} h_{4,0}h_{1,0}h_{3,1}h_{2,2}h_{1,3}, \\ \mathbf{g}4 = a_4^s h_{5,0}h_{1,0}h_{3,1}h_{2,2}h_{1,3}, \\ \mathbf{g}5 = a_4^s h_{4,0}h_{1,0}h_{4,1}h_{2,2}h_{1,3}, \\ \mathbf{g}6 = a_4^s h_{4,0}h_{1,0}h_{3,1}h_{3,2}h_{1,3}, \\ \mathbf{g}7 = a_4^s h_{4,0}h_{1,0}h_{3,1}h_{2,2}h_{2,3}, \\ \mathbf{g}8 = a_4^{s-1} a_1 h_{5,0}h_{4,0}h_{3,1}h_{2,2}h_{1,3}, \\ \mathbf{g}9 = a_4^s h_{4,0}h_{3,0}h_{2,2}h_{2,3}h_{1,3}, \\ \mathbf{g}10 = a_4^s h_{4,0}h_{2,0}h_{3,2}h_{2,2}h_{1,3}, \end{cases}$$

where $\mathbf{g}1 \in E_1^{s+5,t(s,4)+s,9s+p+19}$, $\mathbf{g}2 \in E_1^{s+5,t(s,4)+s,9s+3p+17}$, $\mathbf{g}i \in E_1^{s+5,t(s,4)+s,9s+19}$ $(3 \leqslant i \leqslant 10)$*, and* K *is the* $\mathbb{Z}/p$*-module generated by one element* $\mathbf{g}11 = a_n^{p-5} h_{n,0}h_{5,0}h_{n-2,2}h_{n-3,3}h_{n-4,4} \in E_1^{p,t(p-5,n)+p-5,(2n+1)(p-5)+8n+5}$.

Proof First, the case when $n=2$ can be directly calculated through the Eq. 2.7 by the virtue of 2.6,so we may now assume $n \geq 4$,and consider $g \in E_1^{s+5,t(s,n)+s,*}$, where $t(s,n) = [(s+2)+(s+2)p+(s+3)p^2+(s+4)p^3+p^n]q$ with $(\bar{c}_0, \bar{c}_1, \ldots, \bar{c}_n) = (s+2, s+2, s+3, s+4, 0, \ldots, 0, 1)$. Then $\dim g = s+5$ and $\deg g = t(s,n)+s$.

Since $s+5 < s+q$, according to the assertion in Sect. 3, the number of $x_i's$ in g is s. By the reason of dimension, all the possibilities of g can be listed as $x_1 \cdots x_s y_1 z_1 z_2$, $x_1 \cdots x_s y_1 y_2 y_3 z_1$, $x_1 \cdots x_s y_1 y_2 y_3 y_4 y_5$.

Case 1 $g = x_1 x_2 \cdots x_s y_1 z_1 z_2$. Note that $s < p-4$. Then $\sum_{i=1}^s x_{i,j} + y_{1,j} + z_{1,j} + z_{2,j} \leqslant s+3 \leqslant s+3 < p$ for all $0 \leqslant j \leqslant n$. one can easily get that the integer sequence $K = (k_0, k_1, \ldots, k_{n-1})$ in the corresponding group of Eq. 2.7 is equal to $(0, 0, \ldots, 0)$, and then $S = (c_0, c_1, c_2, c_3, c_4 \ldots, c_{n-1}, c_n) = (s+1, s+3, s+3, s+4, 0, \ldots, 0, 1)$. Since $\sum_{i=1}^s x_{i,3} + y_{1,3} + z_{1,3} + z_{2,3} \leqslant s+3 < s+4 = c_3$, the fourth equation of (3,2) has no solution. It follows that such g is impossible to exist.

Case 2 $g = x_1 x_2 \cdots x_s y_1 y_2 y_3 z_1$. Similar to Case 1, we can get that the integer sequence $K = (k_0, k_1, \ldots, k_{n-1})$ in the corresponding group of Eq. 2.7 is $(0, 0, \ldots, 0)$, and then $S = (c_0, c_1, c_2, c_3, c_4 \ldots, c_{n-1}, c_n) = (s+2, s+2, s+3, s+4, 0, \ldots, 0, 1)$.

Subcase 2.1 $n \geqslant 5$. Since $\sum_{i=1}^s x_{i,3} + y_{1,3} + y_{2,3} + y_{3,3} + z_{1,3} = s+4$, we get $x_{i,3} = y_{1,3} = y_{2,3} = y_{3,3} = z_{1,3} = 1$ for $1 \leqslant i \leqslant s$. Since $\sum_{i=1}^s x_{i,4} + y_{1,4} + y_{2,4} + y_{3,4} + z_{1,4} = 0$, we get $x_{i,4} = y_{1,4} = y_{2,4} = y_{3,4} = z_{1,4} = 0$ for $1 \leqslant i \leqslant s$. Then bythe conditions (2), (4) and (7) in 2.6, we get $x_{i,j} = y_{1,j} = y_{2,j} = y_{3,j} = z_{1,j} = 0$ for $1 \leqslant i \leqslant s$ and $5 \leqslant j \leqslant n$, which contradicts $\sum_{i=1}^{s} x_{i,n} + y_{1,n} + y_{2,n} + y_{3,n} + z_{1,n} = 1$. So the corresponding group of Eq. 2.7 has no solution. It follows that g is impossible to exist.

Subcase 2.2 $n = 4$. We solve the corresponding group of Eq. 2.7 by virtue of 2.6, and get two nontrivial generators as follows:

$$\mathbf{g}1 = a_4^s h_{5,0} h_{4,0} h_{2,2} b_{1,2}, \ \mathbf{g}2 = a_4^s h_{5,0} h_{4,0} h_{1,3} b_{2,1},$$

where $\mathbf{g}1 \in E_1^{s+5, t(s,4)+s, 9s+p+19}$, $\mathbf{g}2 \in E_1^{s+5, t(s,4)+s, 9s+3p+17}$.

Case 3 $g = x_1 x_2 \cdots x_s y_1 y_2 y_3 y_4 y_5$.

Subcase 3.1 $n = 4$. Similar to Case 2, we easily get that $S = (c_0, c_1, c_2, c_3, c_4) = (s+2, s+2, s+3, s+4, 1)$. One can solve the corresponding group of Eq. 2.7 by virtue of 2.6, and get eight nontrivial generators as follows:

$$\begin{aligned} &\mathbf{g}3 = a_5 a_4^{s-1} h_{4,0} h_{1,0} h_{3,1} h_{2,2} h_{1,3}, \ \mathbf{g}4 = a_4^s h_{5,0} h_{1,0} h_{3,1} h_{2,2} h_{1,3}, \\ &\mathbf{g}5 = a_4^s h_{4,0} h_{1,0} h_{4,1} h_{2,2} h_{1,3}, \ \mathbf{g}6 = a_4^s h_{4,0} h_{1,0} h_{3,1} h_{3,2} h_{1,3}, \\ &\mathbf{g}7 = a_4^s h_{4,0} h_{1,0} h_{3,1} h_{2,2} h_{2,3}, \ \mathbf{g}8 = a_4^{s-1} a_1 h_{5,0} h_{4,0} h_{3,1} h_{2,2} h_{1,3}, \\ &\mathbf{g}9 = a_4^s h_{4,0} h_{3,0} h_{2,2} h_{2,3} h_{1,3}, \ \mathbf{g}10 = a_4^s h_{4,0} h_{2,0} h_{3,2} h_{2,2} h_{1,3}, \end{aligned}$$

where $\mathbf{g}i \in E_1^{s+5, t(s,4)+s, 9s+19}$ for $3 \leqslant i \leqslant 10$.

Subcase 3.2 $n = 5$. One can easily show there are no nontrivial elements in this specific case.

Subcase 3.3 $n > 5$ and $0 < s < p-5$. Similar to Case 2, one can get that $S = (c_0, c_1, c_2, c_3, c_4, \ldots, c_{n-1}, c_n) = (s+2, s+2, s+3, s+4, 0, \ldots, 0, 1)$. We solve the corresponding group of Eq. 2.7 by virtue of 2.6, and get a generator $a_4^s h_{4,0}^2 h_{2,2} h_{1,3} h_{1,n}$ which is trivial by $h_{4,0}^2 = 0$.

Subcase 3.4 $n > 5$ and $s = p-5$. Since $\sum_{i=1}^{s} x_{i,j} + y_{1,j} + y_{2,j} + y_{3,j} + y_{4,j} + y_{5,j} \leqslant s+5 = p$ $(0 \leqslant j \leqslant n)$, we have that all possibilities of the integer sequence $K = (k_0, k_1, \ldots, k_{n-1})$ in the corresponding group of Eq. 2.7 are

$K_1 = (0, 0, \ldots, 0)$,

$K_i = (0, 0, 0, 0, 0, \ldots, 0, 1^{(i)}, 1, \ldots, 1)$ $(5 \leqslant i \leqslant n)$,

where $1^{(i)}$ means that 1 is the i-th term of the sequence K_i. Then the corresponding sequence $S = (c_0, c_1, c_2, c_3, c_4, \ldots, c_{n-1}, c_n)$ are listed as

$S_1 = (p-3, p-3, p-2, p-1, 0, \ldots, 0, 1)$,

$S_i = (p-3, p-3, p-2, p-1, 0, \ldots, 0, p^{(i)}, p-1, \ldots, p-1, 0)$ $(5 \leqslant i \leqslant n)$.

For S_1, we only get a trivial element which's the same as the one of Subcase 3.3.

For S_5, one can solve the corresponding group of Eq. 2.7 by virtue of 2.6, and get a generator $\mathbf{g}11 = a_n^{p-5} h_{n,0} h_{5,0} h_{n-2,2} h_{n-3,3} h_{n-4,4} \in E_1^{p, t(p-5,n)+p-5, (2n+1)(p-5)+8n+5}$.

For $S_i (6 \leqslant i \leqslant n)$, it is easy to get the corresponding group of Eq. 2.7 has no solution.

Combining Cases 1-3 gives the desired result. □

The same method leads us to the case when $s = 0$:

Lemma 3.3 *Let $p \geq 7$, $n \geq 2$, and $n \neq 3$. Then the May E_1-term satisfies*

$$E_1^{5,t(0,n),*} = \begin{cases} 0 & n = 2, \text{ or } n \geq 5, \\ \bar{M} & n = 4. \end{cases}$$

Here, $t(0,n) = (2 + 2p + 3p^2 + 4p^3 + p^n)q$, $\bar{M}$ is the $\mathbb{Z}/p$-module generated by the following eight elements:

$$\begin{cases} \mathbf{g}\bar{1} = h_{5,0}h_{4,0}h_{2,2}b_{1,2}, \\ \mathbf{g}\bar{2} = h_{5,0}h_{4,0}h_{1,3}b_{2,1}, \\ \mathbf{g}\bar{4} = h_{5,0}h_{1,0}h_{3,1}h_{2,2}h_{1,3}, \\ \mathbf{g}\bar{5} = h_{4,0}h_{1,0}h_{4,1}h_{2,2}h_{1,3}, \\ \mathbf{g}\bar{6} = h_{4,0}h_{1,0}h_{3,1}h_{3,2}h_{1,3}, \\ \mathbf{g}\bar{7} = h_{4,0}h_{1,0}h_{3,1}h_{2,2}h_{2,3}, \\ \mathbf{g}\bar{9} = h_{4,0}h_{3,0}h_{2,2}h_{2,3}h_{1,3}, \\ \mathbf{g}\bar{10} = h_{4,0}h_{2,0}h_{3,2}h_{2,2}h_{1,3}, \end{cases}$$

where $\mathbf{g}\bar{1} \in E_1^{5,t(0,4),p+19}$, $\mathbf{g}\bar{2} \in E_1^{5,t(0,4)+s,3p+17}$, $\mathbf{g}\bar{i} \in E_1^{5,t(0,4)+s,19}(i \neq 1, 2)$.

Now we give the proof of Theorem 1.4.

Proof of Theorem 1.4 We first let $s > 0$. (1) It is known that $h_{1,n} \in E_1^{1,p^n q,*}$ is a permanent cocycle and represents $h_n \in \mathrm{Ext}_A^{1,p^n q}(\mathbb{Z}/p, \mathbb{Z}/p)$ in the May spectral sequence for $n \geqslant 0$. From Lemma 3.1, $\tilde{\delta}_{s+4}$ is represented by $a_4^s h_{4,0}h_{3,1}h_{2,2}h_{1,3} \in E_1^{s+4,t_1(s)+s,*}$ in the May spectral sequence. So, we get that $h_{1,0}h_{1,n}a_4^s h_{4,0}h_{3,1}h_{2,2}$ $h_{1,3} \in E_1^{s+6,t(s,n)+s,9s+18}$ is a permanent cocycle in the May spectral sequence and represents $h_0 h_n \tilde{\delta}_{s+4} \in \mathrm{Ext}_A^{s+6,t(s,n)+s}(\mathbb{Z}/p, \mathbb{Z}/p)$.

Case 1 $0 < s < p - 5$. From Lemma 3.2, the May E_1-term $E_1^{s+5,t(s,n)+s,*} = 0$, which implies

$$E_r^{s+5,t(s,n)+s,*} = 0$$

for $r \geq 1$. Consequently, the permanent cocycle $h_{1,0}h_{1,n}a_4^s h_{4,0}h_{3,1}h_{2,2}h_{1,3}$ cannot be hit by any differential in the May spectral sequence. Thus in this case, we have

$$h_0 h_n \tilde{\delta}_{s+4} \neq 0.$$

Case 2 $s = p - 5$. By Lemma 3.2,

$$E_1^{s+5,t(s,n)+s,*} = E_1^{p,t(p-5,n),*} = \mathbb{Z}/p\{\mathbf{g}11\}.$$

Note that

$$M(\mathbf{g}11) = (2n+1)(p-5) + 8n + 5$$

and

$$M(h_{1,0}h_{1,n}a_4^{p-5}h_{4,0}h_{3,1}h_{2,2}h_{1,3}) = 9(p-5) + 18.$$

By the reason of May filtration, we have that $h_{1,0}h_{1,n}a_4^{p-5}h_{4,0}h_{3,1}h_{2,2}h_{1,3}$ is not in $d_1(E_1^{p,t(p-5,n)+p-5,(2n+1)(p-5)+8n+5})$. At the same time, from

$$\begin{aligned} d_1(\mathbf{g}11) &= d_1(a_n^{p-5}h_{n,0}h_{5,0}h_{n-2,2}h_{n-3,3}h_{n-4,4}) \\ &= a_n^{p-5}d_1(h_{n,0})h_{5,0}h_{n-2,2}h_{n-3,3}h_{n-4,4} + \cdots \\ &= a_n^{p-5}h_{n-1,1}h_{1,0}h_{5,0}h_{n-2,2}h_{n-3,3}h_{n-4,4} + \cdots \\ &\neq 0, \end{aligned}$$

we get

$$E_r^{p,t(p-5,n)+p-5,(2n+1)(p-5)+8n+5} = 0$$

for $r \geqslant 2$. It follows that $h_{1,0}h_{1,n}a_4^{p-5}h_{4,0}h_{3,1}h_{2,2}h_{1,3}$ is not in $d_r(E_r^{p,t(p-5,n)+p-5,(2n+1)(p-5)+8n+5})$ for $r \geqslant 1$. Thus $h_{1,0}h_{1,n}a_4^{p-5}h_{4,0}h_{3,1}h_{2,2}h_{1,3}$ cannot be hit by any May differential, showing that

$$h_0h_n\tilde{\delta}_{p-1} \neq 0 \in \mathrm{Ext}_A^{p+1,t(p-5,n)+p-5}(\mathbb{Z}/p, \mathbb{Z}/p).$$

This completes the proof of Theorem 1.4 (1).

(2) Since $h_0h_3\tilde{\delta}_{s+4}$ is represented in the May spectral sequence by $h_{1,0}h_{1,3}a_4^sh_{4,0}h_{3,1}h_{2,2}h_{1,3}$ which is trivial by $h_{1,3}^2 = 0$, it follows that

$$h_0h_3\tilde{\delta}_{s+4} = 0.$$

Now we prove that $h_0h_4\tilde{\delta}_{s+4} = 0$. It suffices to prove that $h_{1,0}h_{1,4}a_4^sh_{4,0}h_{3,1}h_{2,2}h_{1,3} \in E_1^{s+6,t(s,4)+s,9s+18}$ which represents $h_0h_4\tilde{\delta}_{s+4} \in \mathrm{Ext}_A^{s+6,t(s,4)+s}(\mathbb{Z}/p, \mathbb{Z}/p)$ is in $d_1(E_1^{s+5,t(s,4)+s,9s+19})$. By Lemma 3.2 we get that

$$E_1^{s+5,t(s,4)+s,9s+19} = \mathbb{Z}/p\{\mathbf{g}3, \ldots, \mathbf{g}10\}.$$

By (2.3) and (2.4), we compute the first May differential of $\mathbf{g}i\,(3 \leqslant i \leqslant 10)$ as follows:

$$\begin{aligned} d_1(\mathbf{g}3) = (-1)^s(&\underline{a_4^{s-1}a_0h_{5,0}h_{4,0}h_{1,0}h_{3,1}h_{2,2}h_{1,3}}_1 + \underline{a_4^{s-1}a_1h_{4,0}h_{1,0}h_{4,1}h_{3,1}h_{2,2}h_{1,3}}_2 \\ &-\underline{a_4^{s-1}a_2h_{4,0}h_{1,0}h_{3,1}h_{3,2}h_{2,2}h_{1,3}}_3 + \underline{a_4^{s-1}a_3h_{4,0}h_{1,0}h_{3,1}h_{2,2}h_{2,3}h_{1,3}}_4 \\ &-\underline{a_4^sh_{4,0}h_{1,0}h_{3,1}h_{2,2}h_{1,3}h_{1,4}}_5), \end{aligned}$$

$$
\begin{aligned}
d_1(\mathbf{g}4) &= (-1)^s(-s\underline{a_4^{s-1}a_0h_{5,0}h_{4,0}h_{1,0}h_{3,1}h_{2,2}h_{1,3}}_1 - \underline{a_4^s h_{2,0}h_{1,0}h_{3,1}h_{3,2}h_{2,2}h_{1,3}}_6 \\
&\quad + \underline{a_4^s h_{3,0}h_{1,0}h_{3,1}h_{2,2}h_{2,3}h_{1,3}}_7 - \underline{a_4^s h_{4,0}h_{1,0}h_{3,1}h_{2,2}h_{1,3}h_{1,4}}_5), \\
d_1(\mathbf{g}5) &= (-1)^s(-s\underline{a_4^{s-1}a_1h_{4,0}h_{1,0}h_{4,1}h_{3,1}h_{2,2}h_{1,3}}_2 - \underline{a_4^s h_{4,0}h_{1,0}h_{1,1}h_{3,2}h_{2,2}h_{1,3}}_8 \\
&\quad + \underline{a_4^s h_{4,0}h_{1,0}h_{2,1}h_{2,2}h_{2,3}h_{1,3}}_9 - \underline{a_4^s h_{4,0}h_{1,0}h_{3,1}h_{2,2}h_{1,3}h_{1,4}}_5), \\
d_1(\mathbf{g}6) &= (-1)^s(s\underline{a_4^{s-1}a_2h_{4,0}h_{1,0}h_{3,1}h_{3,2}h_{2,2}h_{1,3}}_3 + \underline{a_4^s h_{2,0}h_{1,0}h_{3,1}h_{3,2}h_{2,2}h_{1,3}}_6 \\
&\quad + \underline{a_4^s h_{4,0}h_{1,0}h_{1,1}h_{3,1}h_{2,2}h_{1,3}}_8 + \underline{a_4^s h_{4,0}h_{1,0}h_{3,1}h_{1,2}h_{2,3}h_{1,3}}_{10} \\
&\quad - \underline{a_4^s h_{4,0}h_{1,0}h_{3,1}h_{2,2}h_{1,3}h_{1,4}}_5), \\
d_1(\mathbf{g}7) &= (-1)^s(-s\underline{a_4^{s-1}a_3h_{4,0}h_{1,0}h_{3,1}h_{2,2}h_{2,3}h_{1,3}}_4 - \underline{a_4^s h_{3,0}h_{1,0}h_{3,1}h_{2,2}h_{2,3}h_{1,3}}_7 \\
&\quad - \underline{a_4^s h_{4,0}h_{1,0}h_{2,1}h_{2,2}h_{2,3}h_{1,3}}_9 - \underline{a_4^s h_{4,0}h_{1,0}h_{3,1}h_{1,2}h_{2,3}h_{1,3}}_{10} \\
&\quad - \underline{a_4^s h_{4,0}h_{1,0}h_{3,1}h_{2,2}h_{1,3}h_{1,4}}_5), \\
d_1(\mathbf{g}8) &= (-1)^s(\underline{a_4^{s-1}a_0h_{5,0}h_{4,0}h_{1,0}h_{3,1}h_{2,2}h_{1,3}}_1 - \underline{a_4^{s-1}a_1h_{4,0}h_{1,0}h_{4,1}h_{3,1}h_{2,2}h_{1,3}}_2 \\
&\quad + \underline{a_4^{s-1}a_1h_{4,0}h_{2,0}h_{3,1}h_{3,2}h_{2,2}h_{1,3}}_{11} - \underline{a_4^{s-1}a_1h_{4,0}h_{3,0}h_{3,1}h_{2,2}h_{2,3}h_{1,3}}_{12}), \\
d_1(\mathbf{g}9) &= (-1)^s(s\underline{a_4^{s-1}a_1h_{4,0}h_{3,0}h_{3,1}h_{2,2}h_{2,3}h_{1,3}}_{12} - \underline{a_4^s h_{3,0}h_{1,0}h_{3,1}h_{2,2}h_{2,3}h_{1,3}}_7 \\
&\quad + \underline{a_4^s h_{4,0}h_{1,0}h_{2,1}h_{2,2}h_{2,3}h_{1,3}}_9 - \underline{a_4^s h_{4,0}h_{2,0}h_{2,2}h_{1,2}h_{2,3}h_{1,3}}_{13}), \\
d_1(\mathbf{g}10) &= (-1)^s(s\underline{a_4^{s-1}a_1h_{4,0}h_{2,0}h_{3,1}h_{3,2}h_{2,2}h_{1,3}}_{11} - \underline{a_4^s h_{2,0}h_{1,0}h_{3,1}h_{3,2}h_{2,2}h_{1,3}}_6 \\
&\quad + \underline{a_4^s h_{4,0}h_{1,0}h_{1,1}h_{3,1}h_{2,2}h_{1,3}}_8 - \underline{a_4^s h_{4,0}h_{2,0}h_{2,2}h_{1,2}h_{2,3}h_{1,3}}_{13}).
\end{aligned}
$$

Without loss of generality, we let s be even. Then we easily get

$$
\begin{pmatrix} d_1(\mathbf{g}3) \\ d_1(\mathbf{g}4) \\ d_1(\mathbf{g}5) \\ d_1(\mathbf{g}6) \\ d_1(\mathbf{g}7) \\ d_1(\mathbf{g}8) \\ d_1(\mathbf{g}9) \\ d_1(\mathbf{g}10) \end{pmatrix}
=
\begin{pmatrix}
1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-s & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -s & 0 & 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & s & 0 & -1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -s & -1 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & s & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & s & 0 & -1
\end{pmatrix}
\begin{pmatrix} \underline{\quad}_1 \\ \underline{\quad}_2 \\ \underline{\quad}_3 \\ \underline{\quad}_4 \\ \underline{\quad}_5 \\ \underline{\quad}_6 \\ \underline{\quad}_7 \\ \underline{\quad}_8 \\ \underline{\quad}_9 \\ \underline{\quad}_{10} \\ \underline{\quad}_{11} \\ \underline{\quad}_{12} \\ \underline{\quad}_{13} \end{pmatrix}.
$$

By direct computation, we can get

$$
\underline{a_4^s h_{4,0}h_{1,0}h_{3,1}h_{2,2}h_{1,3}h_{1,4}}_5 = -(s+4)^{-1}(sd_1(\mathbf{g}3) + d_1(\mathbf{g}4) + d_1(\mathbf{g}5) + d_1(\mathbf{g}6) + d_1(\mathbf{g}7)).
$$

So $h_{1,0}h_{1,4}a_4^s h_{4,0}h_{3,1}h_{2,2}h_{1,3}$ is in $d_1(E_1^{s+5,t(s,4)+s,9s+19})$, showing that

$$h_0h_4\tilde{\delta}_{s+4} = 0.$$

This finishes the proof of Theorem 1.4 when $s > 0$.

Now we turn to the case $s = 0$. In this case, $h_{1,0}h_{1,n}h_{4,0}h_{3,1}h_{2,2}h_{1,3} \in E_1^{6,t(0,n),18}$ is a permanent cocycle in the May spectral sequence and represents $h_0h_n\tilde{\delta}_4 \in \mathrm{Ext}_A^{6,t(0,n)}(\mathbb{Z}/p, \mathbb{Z}/p)$.

(1) From Lemma 3.3, we have that the May E_1-term

$$E_1^{5,t(0,n),*} = 0,$$

which implies that

$$E_r^{5,t(0,n),*} = 0$$

for $r \geq 1$. Consequently, the permanent cocycle $h_{1,0}h_{1,n}h_{4,0}h_{3,1}h_{2,2}h_{1,3}$ cannot be hit by any May differential. Thus in this case, we have

$$h_0h_n\tilde{\delta}_4 \neq 0.$$

(2) Since $h_0h_3\tilde{\delta}_4$ is represented in the May spectral sequence by $h_{1,0}h_{1,3}h_{4,0}h_{3,1}h_{2,2}h_{1,3}$ which is trivial by $h_{1,3}^2 = 0$, it follows that

$$h_0h_3\tilde{\delta}_4 = 0.$$

Now we prove that $h_0h_4\tilde{\delta}_4 = 0$. It suffices to prove that $h_{1,0}h_{1,4}h_{4,0}h_{3,1}h_{2,2}h_{1,3} \in E_1^{6,t(0,4),18}$ which represents $h_0h_4\tilde{\delta}_4 \in \mathrm{Ext}_A^{6,t(0,4)}(\mathbb{Z}/p, \mathbb{Z}/p)$ is in $d_1(E_1^{5,t(0,4),19})$. By Lemma 3.3 we get that

$$E_1^{5,t(0,4),19} = \mathbb{Z}/p\{\mathbf{g}\bar{i} \mid 4 \leq i \leq 10, i \neq 8\}.$$

By (2.3), we compute the first May differential of $\mathbf{g}\bar{i}$ ($4 \leq i \leq 10$ and $i \neq 8$) by using the same method as above, and similarly obtain the following equalities:

$$\begin{aligned}
d_1(\mathbf{g}\bar{4}) = & -\underline{h_{2,0}h_{1,0}h_{3,1}h_{3,2}h_{2,2}h_{1,3}}_6 + \underline{h_{3,0}h_{1,0}h_{3,1}h_{2,2}h_{2,3}h_{1,3}}_7 \\
& -\underline{h_{4,0}h_{1,0}h_{3,1}h_{2,2}h_{1,3}h_{1,4}}_5, \\
d_1(\mathbf{g}\bar{5}) = & -\underline{h_{4,0}h_{1,0}h_{1,1}h_{3,2}h_{2,2}h_{1,3}}_8 + \underline{h_{4,0}h_{1,0}h_{2,1}h_{2,2}h_{2,3}h_{1,3}}_9 \\
& -\underline{h_{4,0}h_{1,0}h_{3,1}h_{2,2}h_{1,3}h_{1,4}}_5, \\
d_1(\mathbf{g}\bar{6}) = & \underline{h_{2,0}h_{1,0}h_{3,1}h_{3,2}h_{2,2}h_{1,3}}_6 + \underline{h_{4,0}h_{1,0}h_{1,1}h_{3,1}h_{2,2}h_{1,3}}_8 \\
& +\underline{h_{4,0}h_{1,0}h_{3,1}h_{1,2}h_{2,3}h_{1,3}}_{10} - \underline{h_{4,0}h_{1,0}h_{3,1}h_{2,2}h_{1,3}h_{1,4}}_5,
\end{aligned}$$

$$
\begin{aligned}
d_1(\mathbf{g}\bar{7}) &= -\underline{h_{3,0}h_{1,0}h_{3,1}h_{2,2}h_{2,3}h_{1,3}}_7 - \underline{h_{4,0}h_{1,0}h_{2,1}h_{2,2}h_{2,3}h_{1,3}}_9 \\
&\quad -\underline{h_{4,0}h_{1,0}h_{3,1}h_{1,2}h_{2,3}h_{1,3}}_{10} - \underline{h_{4,0}h_{1,0}h_{3,1}h_{2,2}h_{1,3}h_{1,4}}_5, \\
d_1(\mathbf{g}\bar{9}) &= -\underline{h_{3,0}h_{1,0}h_{3,1}h_{2,2}h_{2,3}h_{1,3}}_7 + \underline{h_{4,0}h_{1,0}h_{2,1}h_{2,2}h_{2,3}h_{1,3}}_9 \\
&\quad -\underline{h_{4,0}h_{2,0}h_{2,2}h_{1,2}h_{2,3}h_{1,3}}_{13}, \\
d_1(\mathbf{g}\bar{1}0) &= -\underline{h_{2,0}h_{1,0}h_{3,1}h_{3,2}h_{2,2}h_{1,3}}_6 + \underline{h_{4,0}h_{1,0}h_{1,1}h_{3,1}h_{2,2}h_{1,3}}_8 \\
&\quad -\underline{h_{4,0}h_{2,0}h_{2,2}h_{1,2}h_{2,3}h_{1,3}}_{13}.
\end{aligned}
$$

Then we easily get

$$
\begin{pmatrix} d_1(\mathbf{g}\bar{4}) \\ d_1(\mathbf{g}\bar{5}) \\ d_1(\mathbf{g}\bar{6}) \\ d_1(\mathbf{g}\bar{7}) \\ d_1(\mathbf{g}\bar{9}) \\ d_1(\mathbf{g}\bar{1}0) \end{pmatrix}
=
\begin{pmatrix}
-1 & -1 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & 1 & 0 & 0 \\
-1 & 1 & 0 & 1 & 0 & 1 & 0 \\
-1 & 0 & -1 & 0 & -1 & -1 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 & -1 \\
0 & -1 & 0 & 1 & 0 & 0 & -1
\end{pmatrix}
\begin{pmatrix} \underline{\quad}_5 \\ \underline{\quad}_6 \\ \underline{\quad}_7 \\ \underline{\quad}_8 \\ \underline{\quad}_9 \\ \underline{\quad}_{10} \\ \underline{\quad}_{13} \end{pmatrix}.
$$

By direct computation, we can get

$$
\underline{h_{4,0}h_{1,0}h_{3,1}h_{2,2}h_{1,3}h_{1,4}}_5 = -4^{-1}(d_1(\mathbf{g}\bar{4}) + d_1(\mathbf{g}\bar{5}) + d_1(\mathbf{g}\bar{6}) + d_1(\mathbf{g}\bar{7})).
$$

So $h_{1,0}h_{1,4}h_{4,0}h_{3,1}h_{2,2}h_{1,3}$ is in $d_1(E_1^{5,t(0,4),19})$, showing that

$$
h_0h_4\tilde{\delta}_4 = 0.
$$

This completes the proof of the theorem. □

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