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Third homology of SL_2 and the indecomposable K_3

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Abstract It is known that, for an infinite field F, the indecomposable part of $K_3(F)$ and the third homology of $SL_2(F)$ are closely related. In fact, there is a canonical map $\alpha : H_3(SL_2(F), \mathbb{Z})_{F^*} \to K_3(F)^{ind}$. Suslin has raised the question: Is α an isomorphism? Recently Hutchinson and Tao have shown that this map is surjective. In this article, we show that α is bijective if and only if the natural maps $H_3(GL_2(F), \mathbb{Z}) \to H_3(GL_3(F), \mathbb{Z})$ and $H_3(SL_2(F), \mathbb{Z})_{F^*} \to H_3(GL_2(F), \mathbb{Z})$ are injective.

1 Introduction

For an infinite field F, Suslin has proved that the Hurewicz homomorphism

$$h_3 : K_3(F) = \pi_3(BSL(F)^+) \longrightarrow H_3(BSL(F)^+, \mathbb{Z}) \simeq H_3(SL(F), \mathbb{Z})$$

is surjective with 2-torsion kernel. In fact, he has shown that h_3 sits in the exact sequence

$$K_2(F) \stackrel{l(-1)}{\longrightarrow} K_3(F) \longrightarrow H_3(SL(F), \mathbb{Z}) \longrightarrow 0,$$

where the homomorphism $l(-1) : K_2(F) \to K_3(F)$ coincides with multiplication by $l(-1) \in K_1(\mathbb{Z})$ [10, Lemma 5.2, Corollary 5.2]. Let

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$$\alpha : H_0(F^*, H_3(\mathrm{SL}_2(F), \mathbb{Z})) \to K_3(F)^{\mathrm{ind}}$$

be the composition of the following sequence of homomorphisms

$$H_0(F^*, H_3(\mathrm{SL}_2(F), \mathbb{Z})) \xrightarrow{\mathrm{inc}_*} H_3(\mathrm{SL}(F), \mathbb{Z}) \xrightarrow[\simeq]{\bar{h}_3^{-1}} K_3(F)/l(-1)K_2(F) \xrightarrow{p} K_3(F)^{\mathrm{ind}} := K_3(F)/K_3^M(F),$$

where inc_* is induced by the inclusion $\mathrm{inc} : \mathrm{SL}_2(F) \to \mathrm{SL}(F)$, and p is induced by the inclusion $l(-1)K_2(F) \subseteq \mathrm{im}(K_3^M(F) \to K_3(F))$. For algebraically closed fields, it was known that α is an isomorphism [1,9]. Following this, Suslin raised the following question:

Question (Suslin). Is it true that $H_0(F^*, H_3(\mathrm{SL}_2(F), \mathbb{Z}))$ coincides with $K_3(F)^{\mathrm{ind}}$? (See [9, Question 4.4]).

In other words, is α bijective for an arbitrary infinite field F? This question is true after killing 2-power torsion elements (i.e. after tensoring the both sides of this map with $\mathbb{Z}[1/2]$) or when $F^* = F^{*2} = \{a^2 | a \in F^*\}$ [6, Proposition 6.4].

Recently Hutchinson and Tao have proved that α is surjective [4, Lemma 5.1]. The following theorem is our main result, which improves an argument of Hutchinson and Tao in [4].

Theorem *Let F be an infinite field. The following conditions are equivalent.*

(i) *The homomorphism $\alpha : H_0(F^*, H_3(\mathrm{SL}_2(F), \mathbb{Z})) \to K_3(F)^{\mathrm{ind}}$ is bijective.*
(ii) *The natural homomorphisms $H_3(\mathrm{GL}_2(F), \mathbb{Z}) \to H_3(\mathrm{GL}_3(F), \mathbb{Z})$ and $H_0(F^*, H_3(\mathrm{SL}_2(F), \mathbb{Z})) \to H_3(\mathrm{GL}_2(F), \mathbb{Z})$ are injective.*

In the mean time we also establish that the kernel of the homomorphism

$$H_3(\mathrm{inc}) : H_3(\mathrm{GL}_2(F), \mathbb{Z}) \to H_3(\mathrm{GL}_3(F), \mathbb{Z})$$

is equal to

$$\mathrm{im}(H_3(\mathrm{SL}_2(F), \mathbb{Z}) \to H_3(\mathrm{GL}_2(F), \mathbb{Z})) \cap F^* \cup H_2(\mathrm{GL}_1(F), \mathbb{Z})),$$

where the cup product is induced by the natural diagonal inclusion $\mathrm{inc} : F^* \times \mathrm{GL}_1(F) \to \mathrm{GL}_2(F)$. It seems that, for an arbitrary field, not much is known about the kernel of

$$H_0\left(F^*, H_3(\mathrm{SL}_2(F), \mathbb{Z})\right) \to H_3(\mathrm{GL}_2(F), \mathbb{Z}),$$

except that it is a 2-power torsion group (see proof of Theorem 6.1 in [6]).

Notation

In this article by $H_i(G)$ we mean the homology of group G with integral coefficients, namely $H_i(G, \mathbb{Z})$. By GL_n (resp. SL_n) we mean the general (resp. special) linear group

$\mathrm{GL}_n(F)$ (resp. $\mathrm{SL}_n(F)$), where F is an infinite field. If $A \to A'$ is a homomorphism of abelian groups, by A'/A we mean $\mathrm{coker}(A \to A')$ and we take other liberties of this kind. Here by Σ_n we mean the symmetric group of rank n.

2 The group $H_1(F^*, H_2(\mathrm{SL}_2))$

We start this section by looking at the corresponding Lyndon/Hochschild-Serre spectral sequence of the commutative diagram of extensions

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{SL}_2 & \longrightarrow & \mathrm{GL}_2 & \xrightarrow{\det} & F^* & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{SL}_3 & \longrightarrow & \mathrm{GL}_3 & \xrightarrow{\det} & F^* & \longrightarrow & 1. \end{array}$$

So we get a morphism of spectral sequences

$$\begin{array}{ccc} \mathcal{E}^2_{p,q} = H_p(F^*, H_q(\mathrm{SL}_2)) & \Longrightarrow & H_{p+q}(\mathrm{GL}_2), \\ \downarrow & & \downarrow \\ E^2_{p,q} = H_p(F^*, H_q(\mathrm{SL}_3)) & \Longrightarrow & H_{p+q}(\mathrm{GL}_3). \end{array}$$

By an easy analysis of this spectral sequence we obtain the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} & & H_3(\mathrm{SL}_2)_{F^*} & \longrightarrow & H_3(\mathrm{GL}_2)/H_3(\mathrm{GL}_1) & \xrightarrow{\varphi} & H_1(F^*, H_2(\mathrm{SL}_2)) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H_3(\mathrm{SL}_3)_{F^*} & \longrightarrow & H_3(\mathrm{GL}_3)/H_3(\mathrm{GL}_1) & \xrightarrow{\psi} & H_1(F^*, H_2(\mathrm{SL}_3)) & \longrightarrow & 0. \end{array}$$

The following theorem is due to Hutchinson and Tao [4, Theorem 3.2], which is very fundamental in their proof of the surjectivity of α.

Theorem 2.1 *The inclusion* $\mathrm{SL}_2 \longrightarrow \mathrm{SL}_3$ *induces a short exact sequence*

$$0 \longrightarrow H_1\left(F^*, H_2(\mathrm{SL}_2)\right) \longrightarrow H_1\left(F^*, H_2(\mathrm{SL}_3)\right) \longrightarrow k_3^M(F) \longrightarrow 0,$$

where $k_3^M(F) := K_3^M(F)/2$.

Since the action of F^* on $H_2(\mathrm{SL}_3)$ is trivial,

$$H_1\left(F^*, H_2(\mathrm{SL}_3)\right) \simeq F^* \otimes K_2^M(F).$$

So we consider $H_1(F^*, H_2(\mathrm{SL}_2))$ as a subgroup of $F^* \otimes K_2^M(F)$. It is easy to see that the map

$$H_1\left(F^*, H_2(\mathrm{SL}_3)\right) \longrightarrow k_3^M(F)$$

is induced by the natural product map $F^* \otimes K_2^M(F) \longrightarrow K_3^M(F)$. Since the n-th Milnor K-group, $K_n^M(F)$, is naturally isomorphic to the n-th tensor of F^* modulo the two families of relations

$$a_1 \otimes \cdots \otimes a_{n-1} \otimes (1 - a_{n-1}), \qquad a_i \in F^*, a_{n-1} \neq 1,$$
$$a_1 \otimes \cdots a_i \otimes a_{i+1} \cdots \otimes a_n + a_1 \otimes \cdots a_{i+1} \otimes a_i \cdots \otimes a_n, \;\; a_i \in F^*,$$

it easily follows that the kernel of the product map $F^* \otimes K_2^M(F) \longrightarrow K_3^M(F)$ is generated by elements $a \otimes \{b, c\} + b \otimes \{a, c\}$. This proves the following lemma.

Lemma 2.2 *As a subgroup of $H_1(F^*, H_2(\mathrm{SL}_3)) = F^* \otimes K_2^M(F)$, the group $H_1(F^*, H_2(\mathrm{SL}_2))$ is generated by elements $a \otimes \{b, c\} + b \otimes \{a, c\}$ and $2d \otimes \{e, f\}$.*

To go further, we need to introduce some notations. Let G be a group and set

$$\mathbf{c}(g_1, g_2, \ldots, g_n) := \sum_{\sigma \in S_n} \mathrm{sign}(\sigma)[g_{\sigma(1)}|g_{\sigma(2)}|\ldots|g_{\sigma(n)}] \in H_n(G),$$

where $g_i \in G$ pairwise commute and S_n is the symmetric group of degree n. Here we use the bar resolution of G [2, Chapter I, Section 5] to define the homology of G.

Lemma 2.3 *Let G and G' be two groups.*

(i) *If $h_1 \in G$ commutes with all the elements $g_1, \ldots, g_n \in G$, then*

$$\mathbf{c}(g_1 h_1, g_2, \ldots, g_n) = \mathbf{c}(g_1, g_2, \ldots, g_n) + \mathbf{c}(h_1, g_2, \ldots, g_n).$$

(ii) *For every $\sigma \in S_n$, $\mathbf{c}(g_{\sigma(1)}, \ldots, g_{\sigma(n)}) = \mathrm{sign}(\sigma)\mathbf{c}(g_1, \ldots, g_n)$.*

(iii) *The cup product of $\mathbf{c}(g_1, \ldots, g_p) \in H_p(G)$ and $\mathbf{c}(g'_1, \ldots, g'_q) \in H_q(G')$ is $\mathbf{c}((g_1, 1), \ldots, (g_p, 1), (1, g'_1), \ldots, (1, g'_q)) \in H_{p+q}(G \times G')$.*

Proof The proof follows from direct computations, so we leave it to the interested readers. □

3 The kernel of $H_3(\mathrm{GL}_2) \longrightarrow H_3(\mathrm{GL}_3)$

For simplicity, in the rest of this article, we use the following notation

$$k_{a,b,c} := \mathbf{c}(\mathrm{diag}(a, 1), \mathrm{diag}(1, b), \mathrm{diag}(1, c)) \in H_3(\mathrm{GL}_2).$$

The following theorem has been proved in [7, Theorem 3.1].

Theorem 3.1 *The kernel of $\mathrm{inc}_{1*} : H_3(\mathrm{GL}_2) \longrightarrow H_3(\mathrm{GL}_3)$ consists of elements of the form $\sum k_{a,b,c} + k_{b,a,c}$ such that*

$$\sum a \otimes \{b, c\} + b \otimes \{a, c\} = 0 \in F^* \otimes K_2^M(F).$$

In particular $\ker(\mathrm{inc}_{1*}) \subseteq F^* \cup H_2(\mathrm{GL}_1) \subseteq H_3(\mathrm{GL}_2)$*, where the cup product is induced by the natural diagonal inclusion* $F^* \times \mathrm{GL}_1 \longrightarrow \mathrm{GL}_2$*. Moreover* $\ker(\mathrm{inc}_{1*})$ *is a* 2*-torsion group.*

Let Ψ and Φ be the following compositions,

$$F^* \otimes K_2^M(F) \xrightarrow{\mathrm{id}_{F^*}\otimes\iota} F^* \otimes H_2(\mathrm{GL}_2) \xrightarrow{\cup} H_3\left(F^* \times \mathrm{GL}_2\right) \xrightarrow{\mathrm{inc}_*} H_3(\mathrm{GL}_3),$$
$$F^* \otimes K_2^M(F) \xrightarrow{\mathrm{id}_{F^*}\otimes\iota} F^* \otimes H_2(\mathrm{GL}_2) \xrightarrow{\cup} H_3\left(F^* \times \mathrm{GL}_2\right) \xrightarrow{\beta_*} H_3(\mathrm{GL}_2),$$

respectively, where $\iota : K_2^M(F) \simeq H_2(\mathrm{SL}_2)_{F^*} \longrightarrow H_2(\mathrm{GL}_2)$ is the natural inclusion given by the formula $\{a, b\} \mapsto \mathbf{c}(\mathrm{diag}(a, 1), \mathrm{diag}(b, b^{-1}))$ [3, Proposition A.11] and $\beta : F^* \times \mathrm{GL}_2 \longrightarrow \mathrm{GL}_2$ is given by $(a, A) \mapsto aA$. It is easy to see that

$$\Psi(a \otimes \{b, c\}) = \mathbf{c}\left(\mathrm{diag}(a, 1, 1), \mathrm{diag}(1, b, 1), \mathrm{diag}\left(1, c, c^{-1}\right)\right),$$
$$\Phi(a \otimes \{b, c\}) = \mathbf{c}\left(\mathrm{diag}(a, a), \mathrm{diag}(b, 1), \mathrm{diag}\left(c, c^{-1}\right)\right).$$

Lemma 3.2 *Let* Θ *be the composition*

$$H_3(\mathrm{GL}_2)/H_3(\mathrm{GL}_1) \xrightarrow{\varphi} H_1(F^*, H_2(\mathrm{SL}_2)) \hookrightarrow F^* \otimes K_2^M(F).$$

Then

(i) $\Theta(k_{a,b,c} + k_{b,a,c}) = a \otimes \{b, c\} + b \otimes \{a, c\}$,
(ii) $\Theta(\mathbf{c}(\mathrm{diag}(a, a), \mathrm{diag}(b, 1), \mathrm{diag}(c, c^{-1}))) = 2a \otimes \{b, c\}$,
(iii) $\Theta(k_{c,a,b}) = b \otimes \{a, c\} - a \otimes \{b, c\}$.

Proof (i) It is easy to see that the exact sequence

$$0 \longrightarrow H_3(\mathrm{SL}_3)_{F^*} \longrightarrow H_3(\mathrm{GL}_3)/H_3(\mathrm{GL}_1) \xrightarrow{\psi} H_1\left(F^*, H_2(\mathrm{SL}_3)\right) \longrightarrow 0$$

splits canonically by the map

$$F^* \otimes K_2^M(F) = H_1\left(F^*, H_2(\mathrm{SL}_3)\right) \longrightarrow H_3(\mathrm{GL}_3)/H_3(\mathrm{GL}_1)$$

defined by Ψ. Now consider the commutative diagram

$$\begin{array}{ccc} H_3(\mathrm{GL}_2)/H_3(\mathrm{GL}_1) & \xrightarrow{\varphi} & H_1\left(F^*, H_2(\mathrm{SL}_2)\right) \\ \downarrow & & \downarrow \\ H_3(\mathrm{GL}_3)/H_3(\mathrm{GL}_1) & \xrightarrow{\psi} & H_1\left(F^*, H_2(\mathrm{SL}_3)\right). \end{array}$$

We have

$$\begin{aligned}
\mathrm{inc}_{1*}(k_{a,b,c}) &= \mathbf{c}(\mathrm{diag}(a,1,1),\mathrm{diag}(1,b,1),\mathrm{diag}(1,c,1))\\
&= \mathbf{c}\left(\mathrm{diag}(a,1,1),\mathrm{diag}(1,b,1),\mathrm{diag}\left(1,c,c^{-1}\right)\right)\\
&\quad +\mathbf{c}(\mathrm{diag}(a,1,1),\mathrm{diag}(1,b,1),\mathrm{diag}(1,1,c))\\
&= \mathbf{c}\left(\mathrm{diag}(a,1,1),\mathrm{diag}(1,b,1),\mathrm{diag}\left(1,c,c^{-1}\right)\right)\\
&\quad -\mathbf{c}(\mathrm{diag}(b,1,1),\mathrm{diag}(1,a,1),\mathrm{diag}(1,1,c))\\
&= \mathbf{c}\left(\mathrm{diag}(a,1,1),\mathrm{diag}(1,b,1),\mathrm{diag}\left(1,c,c^{-1}\right)\right)\\
&\quad -\mathbf{c}\left(\mathrm{diag}(b,1,1),\mathrm{diag}(1,a,1),\mathrm{diag}\left(1,c^{-1},c\right)\right)\\
&\quad -\mathbf{c}(\mathrm{diag}(b,1,1),\mathrm{diag}(1,a,1),\mathrm{diag}(1,c,1))\\
&= \Psi\left(a\otimes\{b,c\}+b\otimes\{a,c\}\right)-\mathrm{inc}_{1*}(k_{b,a,c}).
\end{aligned}$$

Hence $\mathrm{inc}_{1*}(k_{a,b,c}+k_{b,a,c})=\Psi(a\otimes\{b,c\}+b\otimes\{a,c\})$. Therefore

$$\begin{aligned}
\Theta(k_{a,b,c}+k_{b,a,c}) &= \psi\circ\mathrm{inc}_{1*}(k_{a,b,c}+k_{b,a,c})\\
&= \psi\circ\Psi\left(a\otimes\{b,c\}+b\otimes\{a,c\}\right)\\
&= a\otimes\{b,c\}+b\otimes\{a,c\}.
\end{aligned}$$

(ii) Consider the composition

$$F^*\otimes K_2^M(F)\xrightarrow{\Phi} H_3(\mathrm{GL}_2)/H_3(\mathrm{GL}_1)\xrightarrow{\Theta} F^*\otimes K_2^M(F).$$

The image of $\Phi(a\otimes\{b,c\})=\mathbf{c}(\mathrm{diag}(a,a),\mathrm{diag}(b,1),\mathrm{diag}(c,c^{-1}))$ in the group $H_3(\mathrm{GL}_3)/H_3(\mathrm{GL}_1)=H_3(\mathrm{SL}_3)_{F^*}\oplus F^*\otimes K_2^M(F)$ is equal to

$$\begin{aligned}
\mathrm{inc}_{1*}\circ\Phi(a\otimes\{b,c\}) &= \mathbf{c}\left(\mathrm{diag}(a,a,1),\mathrm{diag}(b,1,1),\mathrm{diag}\left(c,c^{-1},1\right)\right)\\
&= \mathbf{c}\left(\mathrm{diag}(a,a,a^{-2}),\mathrm{diag}(b,1,1),\mathrm{diag}\left(c,c^{-1},1\right)\right)\\
&\quad +\mathbf{c}\left(\mathrm{diag}(1,1,a^{2}),\mathrm{diag}(b,1,1),\mathrm{diag}\left(c,c^{-1},1\right)\right)\\
&= \mathbf{c}\left(\mathrm{diag}(a,1,a^{-1}),\mathrm{diag}(b,1,b^{-1}),\mathrm{diag}\left(c,c^{-1},1\right)\right)\\
&\quad +\mathbf{c}\left(\mathrm{diag}(a^{2},1,1),\mathrm{diag}(1,b,1),\mathrm{diag}\left(1,c,c^{-1}\right)\right).
\end{aligned}$$

Therefore $\Theta\circ\Phi(a\otimes\{b,c\})=\psi\circ\mathrm{inc}_{1*}\circ\Phi(a\otimes\{b,c\})=2a\otimes\{b,c\}$.

(iii) First note that

$$\begin{aligned}
\Phi(a\otimes\{b,c\}) &= \mathbf{c}\left(\mathrm{diag}(a,a),\mathrm{diag}(b,1),\mathrm{diag}\left(c,c^{-1}\right)\right)\\
&= \mathbf{c}(\mathrm{diag}(a,1),\mathrm{diag}(b,1),\mathrm{diag}(c,1))\\
&\quad -k_{c,a,b}+k_{a,b,c}+k_{b,a,c}.
\end{aligned}$$

Therefore

$$\begin{aligned}\Theta(k_{c,a,b}) &= \Theta(k_{a,b,c} + k_{b,a,c}) - \Theta(\Phi(a \otimes \{b, c\})) \\ &= b \otimes \{a, c\} - a \otimes \{b, c\}.\end{aligned}$$

□

Proposition 3.3 *Let* $\mathrm{inc}_{2*} : H_3(\mathrm{SL}_2)_{F^*} \longrightarrow H_3(\mathrm{GL}_2)$ *be induced by the natural map* $\mathrm{inc}_2 : \mathrm{SL}_2 \longrightarrow \mathrm{GL}_2$. *Then*

$$\mathrm{im}(\mathrm{inc}_{2*}) \cap \left(F^* \cup H_2(\mathrm{GL}_1)\right) = \ker(\mathrm{inc}_{1*}).$$

Proof By Theorem 3.1, the kernel of $\mathrm{inc}_{1*} : H_3(\mathrm{GL}_2) \longrightarrow H_3(\mathrm{GL}_3)$ consists of elements of the form $\sum k_{a,b,c} + k_{b,a,c}$ such that

$$\sum a \otimes \{b, c\} + b \otimes \{a, c\} = 0 \in F^* \otimes K_2^M(F).$$

By Lemma 3.2, we see that

$$\Theta\left(\sum k_{a,b,c} + k_{b,a,c}\right) = \sum a \otimes \{b, c\} + b \otimes \{a, c\} = 0.$$

Since the sequence

$$H_3(\mathrm{SL}_2)_{F^*} \xrightarrow{\mathrm{inc}_{2*}} H_3(\mathrm{GL}_2)/H_3(\mathrm{GL}_1) \longrightarrow H_1\left(F^*, H_2(\mathrm{SL}_2)\right) \longrightarrow 0,$$

is exact, $\sum k_{a,b,c} + k_{b,a,c} \in \mathrm{im}(\mathrm{inc}_{2*})$. Therefore

$$\ker(\mathrm{inc}_{1*}) \subseteq \mathrm{im}(\mathrm{inc}_{2*}) \cap \left(F^* \cup H_2(\mathrm{GL}_1)\right).$$

Now let $x \in \mathrm{im}(\mathrm{inc}_{2*}) \cap (F^* \cup H_2(\mathrm{GL}_1))$. Then x is of the following form

$$x = \sum \mathbf{c}(\mathrm{diag}(a_i, 1), \mathrm{diag}(1, b_i), \mathrm{diag}(1, c_i)).$$

Thus $\det_*(x) = \sum \mathbf{c}(a_i, b_i, c_i) = 0$, where $\det_* : H_3(\mathrm{GL}_2) \longrightarrow H_3(F^*)$ is induced by the determinant. By the inclusion $\bigwedge_{\mathbb{Z}}^3 F^* \hookrightarrow H_3(F^*)$, we have $a \wedge b \wedge c \mapsto \mathbf{c}(a, b, c)$ (see for example [10, Lemma 5.5]). Thus

$$\sum a_i \otimes b_i \otimes c_i = \sum a' \otimes a' \otimes b' + \sum a'' \otimes b'' \otimes a'' + \sum b''' \otimes a''' \otimes a'''.$$

Under the composition $F^{*\otimes 3} \longrightarrow F^* \otimes H_2(F^*) \longrightarrow H_3(\mathrm{GL}_2)$ defined by

$$a \otimes b \otimes c \mapsto a \otimes \mathbf{c}(b, c) \mapsto \mathbf{c}(\mathrm{diag}(a, 1), \mathrm{diag}(1, b), \mathrm{diag}(1, c)) = k_{a,b,c},$$

we see that x has the following form

$$x = \sum \mathbf{c}(\mathrm{diag}(a, 1), \mathrm{diag}(1, a), \mathrm{diag}(1, b)).$$

For simplicity, we assume that $x = \mathbf{c}(\mathrm{diag}(a, 1), \mathrm{diag}(1, a), \mathrm{diag}(1, b))$. By Lemma 3.2, $\Theta(x) = a \otimes \{a, b\} - b \otimes \{a, a\} = 0$. Thus

$$\begin{aligned}
&\mathbf{c}\left(\mathrm{diag}(a, 1, 1), \mathrm{diag}(1, a, 1), \mathrm{diag}\left(1, b, b^{-1}\right)\right) \\
&= \Psi\left(a \otimes \{a, b\}\right) = \Psi\left(b \otimes \{a, a\}\right) \\
&= \mathbf{c}\left(\mathrm{diag}(b, 1, 1), \mathrm{diag}(1, a, 1), \mathrm{diag}\left(1, a, a^{-1}\right)\right),
\end{aligned}$$

and so

$$\begin{gathered}
+\mathbf{c}(\mathrm{diag}(a, 1, 1), \mathrm{diag}(1, a, 1), \mathrm{diag}(1, b, 1)) \\
-\mathbf{c}(\mathrm{diag}(a, 1, 1), \mathrm{diag}(1, a, 1), \mathrm{diag}(1, 1, b)) \\
= \\
-\mathbf{c}(\mathrm{diag}(b, 1, 1), \mathrm{diag}(1, a, 1), \mathrm{diag}(1, 1, a)).
\end{gathered}$$

Hence in $H_3(\mathrm{GL}_3)$ we have

$$\begin{aligned}
\mathrm{inc}_{1*}(x) &= \mathbf{c}(\mathrm{diag}(a, 1, 1), \mathrm{diag}(1, a, 1), \mathrm{diag}(1, b, 1)) \\
&= \mathbf{c}(\mathrm{diag}(a, 1, 1), \mathrm{diag}(1, a, 1), \mathrm{diag}(1, 1, b)) \\
&\quad -\mathbf{c}(\mathrm{diag}(b, 1, 1), \mathrm{diag}(1, a, 1), \mathrm{diag}(1, 1, a)) \\
&= 0
\end{aligned}$$

Therefore $x \in \ker(\mathrm{inc}_{1*})$ and this completes the proof of the proposition. □

4 The indecomposable part of the third K-group

Define the *pre-Bloch group* $\mathfrak{p}(F)$ of F as the quotient of the free abelian group $Q(F)$ generated by symbols $[a]$, $a \in F^* - \{1\}$, by the subgroup generated by elements of the form

$$[a] - [b] + \left[\frac{b}{a}\right] - \left[\frac{1 - a^{-1}}{1 - b^{-1}}\right] + \left[\frac{1 - a}{1 - b}\right],$$

where $a, b \in F^* - \{1\}$, $a \neq b$. Define

$$\lambda' : Q(F) \longrightarrow F^* \otimes F^*, \quad [a] \mapsto a \otimes (1 - a).$$

By a direct computation, we have

$$\lambda'\left([a] - [b] + \left[\frac{b}{a}\right] - \left[\frac{1 - a^{-1}}{1 - b^{-1}}\right] + \left[\frac{1 - a}{1 - b}\right]\right) = a \otimes \left(\frac{1 - a}{1 - b}\right) + \left(\frac{1 - a}{1 - b}\right) \otimes a.$$

Let $(F^* \otimes F^*)_\sigma := F^* \otimes F^* / \langle a \otimes b + b \otimes a : a, b \in F^* \rangle$. We denote the elements of $\mathfrak{p}(F)$ and $(F^* \otimes F^*)_\sigma$ represented by $[a]$ and $a \otimes b$ again by $[a]$ and $a \otimes b$, respectively. Thus we have a well-defined map

$$\lambda : \mathfrak{p}(F) \longrightarrow \left(F^* \otimes F^*\right)_\sigma, \quad [a] \mapsto a \otimes (1-a).$$

The kernel of λ is called the *Bloch group* of F and is denoted by $B(F)$. Therefore we obtain the exact sequence

$$0 \longrightarrow B(F) \longrightarrow \mathfrak{p}(F) \longrightarrow \left(F^* \otimes F^*\right)_\sigma \longrightarrow K_2^M(F) \longrightarrow 0.$$

The following remarkable theorem is due to Suslin [10, Theorem 5.2].

Theorem 4.1 *Let F be an infinite field. Then we have the exact sequence*

$$0 \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(\mu(F), \mu(F))^{\sim} \longrightarrow K_3(F)^{\mathrm{ind}} \longrightarrow B(F) \longrightarrow 0,$$

where $\mathrm{Tor}_1^{\mathbb{Z}}(\mu(F), \mu(F))^{\sim}$ *is the unique nontrivial extension of the group* $\mathrm{Tor}_1^{\mathbb{Z}}(\mu(F), \mu(F))$ *by* $\mathbb{Z}/2$ *if* $\mathrm{char}(F) \neq 2$ *and is equal to* $\mathrm{Tor}_1^{\mathbb{Z}}(\mu(F), \mu(F))$ *if* $\mathrm{char}(F) = 2$.

The following theorem has been proved in [8, Theorem 4.4].

Theorem 4.2 *Let F be an infinite field. Then we have the exact sequence*

$$0 \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(\mu(F), \mu(F))^{\sim} \longrightarrow \tilde{H}_3(\mathrm{SL}_2(F)) \longrightarrow B(F) \longrightarrow 0,$$

where $\tilde{H}_3(\mathrm{SL}_2(F)) := H_3(\mathrm{GL}_2)/(H_3(\mathrm{GL}_1) + F^* \cup H_2(\mathrm{GL}_1))$.

These two theorems suggest that $K_3(F)^{\mathrm{ind}}$ and $\tilde{H}_3(\mathrm{SL}_2(F))$ should be isomorphism. But there is no natural homomorphism from one of these groups to the other one! But there is a natural map from $H_3(\mathrm{SL}_2)_{F^*}$ to both of them. Hutchinson and Tao have proved that $H_3(\mathrm{SL}_2)_{F^*} \longrightarrow K_3(F)^{\mathrm{ind}}$ is surjective [4, Lemma 5.1]. The next lemma claims that this is also true for the other map.

Lemma 4.3 *The map* $\varsigma : H_3(\mathrm{SL}_2)_{F^*} \longrightarrow \tilde{H}_3(\mathrm{SL}_2)$, *induced by the natural map* $\mathrm{SL}_2 \longrightarrow \mathrm{GL}_2$, *is surjective.*

Proof Consider the exact sequence

$$H_3(\mathrm{SL}_2)_{F^*} \longrightarrow H_3(\mathrm{GL}_2)/H_3(\mathrm{GL}_1) \xrightarrow{\varphi} H_1(F^*, H_2(\mathrm{SL}_2)) \longrightarrow 0.$$

By Lemma 3.2, we have

$$\Theta(k_{a,b,c} + k_{b,a,c}) = a \otimes \{b, c\} + b \otimes \{a, c\},$$
$$\Theta(\mathbf{c}(\mathrm{diag}(a, a), \mathrm{diag}(b, 1), \mathrm{diag}(c, c^{-1}))) = 2a \otimes \{b, c\}.$$

Since $H_1(F^*, H_2(\mathrm{SL}_2))$ as a subgroup of $H_1(F^*, H_2(\mathrm{SL}_3)) = F^* \otimes K_2^M(F)$ is generated by elements $a \otimes \{b, c\} + b \otimes \{a, c\}$ and $2d \otimes \{e, f\}$ and since the elements $a \cup \mathbf{c}(b, c) = k_{a,b,c}$ vanish in $\tilde{H}_3(\mathrm{SL}_2)$, $H_3(\mathrm{SL}_2)_{F^*} \longrightarrow \tilde{H}_3(\mathrm{SL}_2)$ must be surjective. □

Now we are ready to prove our main theorem.

Theorem 4.4 *Let F be an infinite field. The following conditions are equivalent.*

(i) *The homomorphism $\alpha : H_3(\mathrm{SL}_2)_{F^*} \longrightarrow K_3(F)^{\mathrm{ind}}$ is bijective.*
(ii) *The natural homomorphisms $\mathrm{inc}_{1*} : H_3(\mathrm{GL}_2) \longrightarrow H_3(\mathrm{GL}_3)$ and $\mathrm{inc}_{2*} : H_3(\mathrm{SL}_2)_{F^*} \longrightarrow H_3(\mathrm{GL}_2)$ are injective.*

Proof (ii) ⇒ (i) Consider the surjective map $\varsigma : H_3(\mathrm{SL}_2)_{F^*} \longrightarrow \tilde{H}_3(\mathrm{SL}_2)$ from Lemma 4.3. Let $\varsigma(x) = 0$. Then $\mathrm{inc}_{2*}(x) \in \mathrm{im}(\mathrm{inc}_{2*}) \cap F^* \cup H_3(\mathrm{GL}_1)$. But by Proposition 3.3 and the assumptions

$$\mathrm{im}(\mathrm{inc}_{2*}) \cap F^* \cup H_3(\mathrm{GL}_1) = \ker(\mathrm{inc}_{1*}) = 0.$$

From this we have $\mathrm{inc}_{2*}(x) = 0$ and hence $x = 0$. Therefore ς is an isomorphism. Now the claim follows by comparing the exact sequence of Theorem 4.2 and Suslin's Bloch-Wigner exact sequence in Theorem 4.1.

(i) ⇒ (ii) Let $\bar{F}$ be the algebraic closure of F. By a theorem of Merkurjev and Suslin, $K_3(F)^{\mathrm{ind}} \longrightarrow K_3(\bar{F})^{\mathrm{ind}}$ is injective [5, Proposition 11.3]. Thus from the commutative diagram

$$\begin{array}{ccc} H_3(\mathrm{SL}_2)_{F^*} & \longrightarrow & H_3\left(\mathrm{SL}_2(\bar{F})\right) \\ \downarrow & & \downarrow \simeq \\ K_3(F)^{\mathrm{ind}} & \longrightarrow & K_3\left(\bar{F}\right)^{\mathrm{ind}}, \end{array}$$

and the injectivity of α, we deduce the injectivity of the map $H_3(\mathrm{SL}_2)_{F^*} \longrightarrow H_3(\mathrm{SL}_2(\bar{F}))$. Now the injectivity of $H_3(\mathrm{SL}_2)_{F^*} \longrightarrow H_3(\mathrm{GL}_2)$ follows from the injectivity of $H_3(\mathrm{SL}_2(\bar{F})) \longrightarrow H_3(\mathrm{GL}_2(\bar{F}))$ [6, Theorem 6.1] and commutativity of the diagram

$$\begin{array}{ccc} H_3(\mathrm{SL}_2)_{F^*} & \longrightarrow & H_3\left(\mathrm{SL}_2\left(\bar{F}\right)\right) \\ \downarrow & & \downarrow \\ H_3(\mathrm{GL}_2) & \longrightarrow & H_3\left(\mathrm{GL}_2\left(\bar{F}\right)\right). \end{array}$$

On the other hand, by Proposition 3.3, $\ker(\mathrm{inc}_{1*}) \subseteq H_3(\mathrm{SL}_2)_{F^*} \subseteq H_3(\mathrm{GL}_2)$. Let $\mathrm{inc}_{1*}(x) = 0$. It easily follows from the commutative diagram

$$\begin{array}{ccccc} H_3(\mathrm{SL}_2)_{F^*} & \longrightarrow & H_3(\mathrm{GL}_2) & \longrightarrow & H_3(\mathrm{GL}_3) \\ \downarrow & & \downarrow & & \downarrow \\ H_3\left(\mathrm{SL}_2\left(\bar{F}\right)\right) & \longrightarrow & H_3\left(\mathrm{GL}_2\left(\bar{F}\right)\right) & \longrightarrow & H_3\left(\mathrm{GL}_3\left(\bar{F}\right)\right). \end{array}$$

that $x \in \ker(H_3(\mathrm{GL}_2(\bar{F})) \longrightarrow H_3(\mathrm{GL}_3(\bar{F}))) = 0$. Therefore $x = 0$ [6, Theorem 5.4(iii)]. □

Remark 4.5 (i) From the spectral sequence $\mathcal{E}^2_{p,q}$, one gets the exact sequence

$$\begin{aligned} H_4(\mathrm{GL}_2)/H_4(\mathrm{GL}_1) &\longrightarrow H_2\left(F^*, H_2(\mathrm{SL}_2)\right) \xrightarrow{\mathfrak{d}^2_{2,2}} H_3(\mathrm{SL}_2)_{F^*} \\ &\longrightarrow H_3(\mathrm{GL}_2)/H_3(\mathrm{GL}_1) \longrightarrow H_1\left(F^*, H_2(\mathrm{SL}_2)\right) \longrightarrow 0. \end{aligned}$$

Thus the injectivity of inc_{2*} is equivalent to triviality of the differential $\mathfrak{d}^2_{2,2}$.

(ii) Theorem 3.1 gives a clear description of elements of the kernel of inc_{1*}. But there is no such information about the kernel of inc_{2*}. It is easy to see that $s_{a,b,c} := \mathbf{c}(\mathrm{diag}(a, a^{-1}), \mathrm{diag}(b, b^{-1}), \mathrm{diag}(c, c^{-1}))$ is in the kernel of inc_{2*} and is 2-torsion:

$$\begin{aligned} s_{a,b,c} &= \mathbf{c}\left(\mathrm{diag}\left(a, a^{-1}\right), \mathrm{diag}\left(b, b^{-1}\right), \mathrm{diag}\left(c, c^{-1}\right)\right) \\ &= \mathbf{c}\left(w.\mathrm{diag}\left(a, a^{-1}\right).w^{-1}, w.\mathrm{diag}\left(b, b^{-1}\right).w^{-1}, w.\mathrm{diag}\left(c, c^{-1}\right).w^{-1}\right) \\ &= \mathbf{c}\left(\mathrm{diag}\left(a^{-1}, a\right), \mathrm{diag}\left(b^{-1}, b\right), \mathrm{diag}\left(c^{-1}, c\right)\right) \\ &= -s_{a,b,c}, \end{aligned}$$

where $w := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. But it is not clear to us why it should be zero. It is not difficult to see that $\ker(\mathrm{inc}_{2*})$ is a 2-power torsion group (see for example the proof of Theorem 6.1 in [6]).

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