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Symplectic cobordism in small dimensions and a series of elements of order four

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Abstract We present the structure of symplectic cobordism ring MSp_* in dimensions up to 51 and give a construction of an infinite series of elements Γ_i, $i = 1, 3, 4, \ldots,$ of order four in this ring, where dim $\Gamma_i = 8i + 95$. The key element of the series is Γ_1 in dimension 103.

Keywords Symplectic cobordism ring · Adams–Novikov spectral sequence · Massey product

Mathematics Subject Classification Primary 55N22 · 55T15 · 57R90

Contents

0 Introduction 32
1 Action of Landweber–Novikov operations on the Ray's elements 35
2 Modified algebraic spectral sequence 42
2.1 Preliminary information 42
2.2 Projections of Ray's elements into the MASS and action of Landweber–Novikov operations on the elements $\mathbf{c_i}$ 44

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2.3 Matrix Massey products in MASS . . . 68
2.4 The cell $E_2^{0,1,t}$ for $t < 108$ . . . 69
2.5 The cell $E_2^{2,0,t}$, for $t < 108$ . . . 70
2.6 The cell $E_2^{1,1,t}$ for $t < 108$ . . . 72
2.7 On the action of differentials $d_r (r > 1)$ . . . 73
3 The Adams–Novikov spectral sequence for $t - s < 56$ . . . 73
3.1 Algebraic structure . . . 73
3.2 Computations of the action of the differential d_3 of the Adams–Novikov spectral sequence 100
4 Construction of the elements of order four in MSp_* . . . 116
5 Tables . . . 118

0 Introduction

Traditionally it is considered that roots of cobordism are contained in the work of Poincaré [27] where he introduced homology based on the notion of (what we would say now) smooth manifold. Only some time later as an answer to the critics of Heegaar Poincaré turned to definition of homology using simplicial subdivision. On the contemporary level cobordism theory started in the works of Pontriagin [28,29] and Rohlin [34].

The principal possibility to consider cobordism of manifolds with a stable normal quaternionic bundle appeared in the work of Thom [39] when he first defined cobordism for manifolds with an action of a subgroup of the orthogonal group G in the stable normal bundle and proved the variant of Pontriagin-Thom theorem which establishes connections between cobordism classes of such manifolds and homotopy groups of Thom complexes MG which he also constructed. The introduction of cobordism theory associated with such group G gave a possibility to unify the known cases of non-orientable ($O(n)$), orientable ($SO(n)$) and framed (trivial group e) cobordisms. It also raised a question about the other classical groups. The theory associated to the unitary group was first studied by Milnor [21] and Novikov [24,25]. It actually became one of the main objets of Algebraic Topology. Cobordism theories associated with other groups, for example $Spin(n)$, also found their applications in Mathematics [37]. The question about the symplectic cobordism, which corresponds to the group $G = Sp(n)$ is very natural, in the same sense that we have real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$ and quaternions $\mathbb{H}$, so what constructions and facts that are valid for the first two objects have any meaning for the third one. Unfortunately the case of symplectic cobordism is very difficult. The present work is one of the illustrations to this fact. Formally it was John Milnor who asked first the question about symplectic cobordism in [21] and the initial results on this cobordism were obtained in the work of Novikov [24,25]. The main method in the works of Milnor and Novikov was the Adams spectral sequence. S. P. Novikov proved that for $p > 2$ this spectral sequence is trivial, so the symplectic cobordism ring MSp_* is such that $MSp_* \otimes Z\left[\frac{1}{2}\right]$ is the polynomial algebra over $Z\left[\frac{1}{2}\right]$ with one $4k$-dimensional generator for any natural number k.

The next step after Novikov in the study of symplectic cobordism was made by Liulevichius [19]. He also applied the Adams spectral sequence and calculated the symplectic cobordism ring in dimensions up to 6. Then Stong [36] gave a construction of symplectic manifolds, in particular, mod p generators for odd primes p. Later Nadiradze constructed free involutions on Stong manifolds, whose orbit spaces provide examples of symplectic manifolds [22].

The subsequent calculations in the Adams spectral sequence were continued by Segal [35]. This spectral sequence is very complicated, even the calculation of its initial term is a nontrivial algebraic problem, it is studied in the work of Ivanovskii [13].

In the beginning of 1970-ties Ray [33] applied the Atiah-Hirzebruch spectral sequence to the study of symplectic cobordism. He also constructed in [31] an infinite series of elements of order two in the symplectic cobordism ring MSp_*. These elements $\theta_1 \in MSp_1$, $\Phi_i \in MSp_{8i-3}$, $i = 1, 2, \ldots$, are multiplicatively indecomposable and close under the action of operations S_ω from the Landweber–Novikov algebra $\mathbb{A}^{MSp}$ [18,26]. They will play the key role in our investigations.

The fundamental works of Kochman [14,15,17] summarize the application of the Adams spectral sequence to the study of symplectic cobordism. In his works Kochman computed E_2 and E_3-terms, calculated the image of MSp_* in the unoriented cobordism ring MO_*, found an element of order four in degree 111 and also computed the first 100 stems.

The other methods of Algebraic Topology were also applied to the study of symplectic cobordisms. After the work of Quillen [30] the formal group techniques entered into the Algebraic Topology. Later Buchstaber [10,11] constructed the theory of multi-valued formal groups and applied it to the study of symplectic cobordism. This direction was developed further in the works of Nadiradze and Bakuradze [2–5,7].

Having in mind the very complicated work in the classical Adams spectral sequence the second author turned to the Adams–Novikov spectral sequence [40,41]. The arguments are evident: this spectral sequence is based on the complex cobordism, which is “closer” to the symplectic cobordism than the ordinary homology theory used in the classical Adams spectral sequence. This is not justified completely: still the Adams–Novikov spectral sequence for the spectrum MSp is very complicated. May be this depicts the nature of symplectic cobordism itself. Anyway, the Adams–Novikov spectral sequence is the principal tool of calculations of the present work. In [40] the second author studied the algebraic Novikov spectral sequence that converges to the initial term of the Adams–Novikov spectral sequence and in [41] calculated the symplectic cobordism ring up to dimension 31 where he found a counter-example to Ray’s conjecture that “*KO*-theory decides symplectic cobordism” [32].

The techniques of cobordism with singularities developed by Sullivan [38] and Baas [1] was applied by the second author to the study of the symplectic cobordism [42]. The main result is that if we take as a sequence of permitted “singularities types” the following subsequence of Ray’s elements

$$\Sigma = (\theta_1, \Phi_1, \ldots, \Phi_{2^i}, \ldots),$$

then the corresponding cobordism theory will be without torsion: that is similar to the complex cobordism. This result shows that the elements of the sequence (Σ) serve as building blocks for construction of the whole torsion of symplectic cobordism.

The classical cobordism graded rings consist of finitely generated abelian groups in each dimension. The complex cobordism ring have no elements of finite order and in the rings of the unoriented, oriented, special unitary and *Spin* cobordism all the elements of finite order have order two [37]. So the natural question about the existence of elements of order four in symplectic cobordism arises. It is known [41] that in small dimensions the ideal of the elements of finite order Tors MSp_* contains only elements of order two.

The main purpose of this work is to present the structure of MSp_* in dimensions up to 51 (see Table 18 at the end of the work) and a construction of an infinite series of elements Γ_i, $i = 1, 3, 4, \ldots$, of order four in the symplectic cobordism ring, where $\dim \Gamma_i = 8i + 95$. The key element of the series is Γ_1 in dimension 103. So, we are proving (in Sect. 4) the following fact.

Main Theorem (i) *There exists an indecomposable element* $\Omega_1 \in MSp_{49}$ *of order two in the symplectic cobordism ring, such that the product* $\theta_1\Phi_{6+i}\Omega_1 \neq 0$.

(ii) *Let* $\Gamma_i \in \langle\Phi_{6+i}, 2, \Omega_1\rangle$, *for* $i = 1, 3, 4, \ldots$. *Then the elements* Γ_i *have order four and* $2\Gamma_i = \theta_1\Phi_{6+i}\Omega_1 \neq 0$.

Relations between the Ray's elements and the free generators in low dimensions are also given in Table 18. These relations were studied from the point of view of characteristic classes by Bakuradze et al. [6].

The present work was finished in the middle of 90-ties and the main result was announced in [44]. About the same time there appeared the works of Botvinnik and Kochman [8,9] asserting the existence of higher torsion in MSp_*. There is no intersection between these papers and the present work. Several years were spent for verification of our calculations. At that time the interest to symplectic cobordisms largely diminished, even the term "symplectic cobordism" is using now in a different sense: as cobordism of manifolds with symplectic structure, i.e. closed non-degenerate differential 2-form [12]. On the other hand there exists an opinion that our claim of the series of order four is not justified by calculations. So finally we decided to submit this text of our work. Otherwise a lot of labor would be lost.

The work is organized as follows. In Sect. 1 we prove the necessary facts about the action of Landweber–Novikov operations on the Ray's elements. This action is the essential tool in our calculations. Results are placed in Table 9 at section Tables of the work.

In Sect. 2 we study the so-called modified algebraic spectral sequence (MASS) which converges to the initial term of the Adams–Novikov spectral sequence. First of all we precise the projections of Ray's elements in terms of the generators of MASS. Together with the results of the previous section this gives the possibility to fix the action of Landweber–Novikov operations on the generators of MASS. This action is presented in Table 10. Then we study matrix Massey products in MASS and describe the following cells of the MASS: $E_2^{0,1,t}$, $E_2^{2,0,t}$ and $E_2^{1,1,t}$ for $t < 108$, results are given in Tables 11, 12 and 13. In Sect. 2.7 we prove a theorem that in these dimensions there is no higher differentials and $E_2 = E_\infty$; relations for E_∞ are presented in Table 14.

In Sect. 3 we start with algebraic description of the initial term of the Adams–Novikov spectral sequence for *MSp* for the topological dimensions up to 56. The graphical description of this term takes a lot of volume in Table 15. In Table 16 multiplicative relations are given. Section 2 is devoted to the calculations of the differential d_3. The main tool is the action of Landweber–Novikov operations. Results are presented in Table 17. After that we reconstruct the term E_4 and see that it coincides with E_∞ what gives the possibility to determine symplectic cobordism ring in dimensions up to 51, Table 18 depicts the result.

In Sect. 4 we prove the Main Theorem based on the given calculations.

1 Action of Landweber–Novikov operations on the Ray's elements

As we had already mentioned in the Introduction, Ray constructed in [31] a series of elements in the symplectic cobordism ring $\theta_1 \in MSp_1$, $\Phi_i \in MSp_{8i-3}$, $i = 1, 2, \ldots,$ which are multiplicatively indecomposable and close under the action of operations S_ω from the Landweber–Novikov algebra $\mathbb{A}^{MSp}$. For our purposes we need exact values of these operations on the elements Φ_i. Kochman proved in [16] a formula giving a possibility to calculate the action of an arbitrary Landweber–Novikov operation S_ω on any Ray's element Φ_i. Let us remind Kochman's construction. Denote by $h : \pi_*(MSp) \to MSp_*(MSp)$ the generalized Hurewicz homomorphism, $\{1, b_0, \ldots, b_n, \ldots\}$ is the canonical $\pi_*(MSp)$-basis for $MSp_*(MSp)$; $\deg b_n = 4n$, then the equality $h(x) = \sum_E x_E b_E$ is equivalent to the following assertion: the action of the Landweber–Novikov operation S_E on x is equal to $S_E(x) = x_E$. Here we have $E = (e_1, \ldots, e_n)$, $e_i \in \mathbb{N}$, $b_E = b_1^{e_1} \cdots b_n^{e_n}$ and $x, x_E \in \pi_*(MSp)$. The following formula [16] is valid for $n = 2m$:

$$h(\Phi_m) = \sum_{i=1}^{m} \left[b_{2m-2i} + \sum_{h=0}^{m-i-1} b_{2h} \chi(B)^{2h+2i}_{2m-2h-2i} \right] \Phi_i + \sum_{k=0}^{m-1} b_{2k} \chi(B)^{2k+1}_{2m-2k-1} \Phi_0. \tag{1.1}$$

Here $B = 1 + b_1 + \cdots + b_t + \cdots$; B^k_{n-k} denotes the component of B^k of degree $4n - 4k$; χ is the conjugation in the Hopf algebra $MSp_*(MSp)$:

$$\chi(B)^t_{n-t} = \sum_{r \geq 0} \sum_{n > q_r > \cdots > q_1 > t} (-1)^{r+1} B^{q_r}_{n-q_r} B^{q_{r-1}}_{q_r - q_{r-1}} \cdots B^{q_1}_{q_2 - q_1} B^t_{q_1 - t}$$

Let us calculate the value of some operations S_ω on the elements Φ_n, using the formula (1.1).

Lemma 1.1 (a) *Let mk be even and $mk < 2n - 1$. Then the coefficient $\alpha^n_{m;k}$ before b_k^m in $\chi(B)^{2n-km}_{km}$ is equal to the expression:*

$$\alpha^n_{m;k} = \sum_{r \geq 0} \sum_{m > i_r > \cdots > i_1 > 0} (-1)^{r+1} C^{m-i_r}_{2n-(m-i_r)k} C^{i_r - i_{r-1}}_{2n-(m-i_{r-1})k} \cdots C^{i_2 - i_1}_{2n-(m-i_1)k} C^{i_1}_{2n-mk}.$$

(*Here C^m_n denote as usual the binomial coefficient, if $m > n$, then $C^m_n = 0$*).

(b) *If* $mk = 2n - 1$, *then the coefficient* $\gamma^n_{m;k}$ *before* b^m_k *in* $\chi(B)_{2k-1}$ *is given by the formula:*

$$\gamma^n_{m;k} = \sum_{r\geq 0} \sum_{m>i_r>\cdots>i_2>1} (-1)^{r+1} C^{m-i_r}_{ki_r+1} C^{i_r-i_{r-1}}_{ki_{r-1}+1} \cdots C^{i_3-i_2}_{ki_2+1} C^{i_2-1}_{k+1}.$$

Proof Consider the case (a). It is evident that the monomial b^m_k appears in the decomposition of $\chi(B)^{2n-km}_{km}$ only in the following expression:

$$\sum_{r\geq 0} \sum_{m>i_r>\cdots>i_1>0} (-1)^{r+1} B^{2n-k(m-i_r)}_{k(m-i_r)} B^{2n-k(m-i_{r-1})}_{k(i_r-i_{r-1})} \cdots B^{2n-k(m-i_1)}_{k(i_2-i_1)} B^{2n-km}_{ki_1}.$$

Analogous expression appears in the case (b). □

Lemma 1.2 *The result of the action of the operation* $S_{k,\ldots,k}$ (*k is taken m times*) *on the element* Φ_n *is equal to*:

(a) $\alpha^n_{m;k}\Phi_{n-mk/2}$, *if* k *is even,* m *is even and* $mk < 2n - 1$;
(b) $(\alpha^n_{m;k} + \alpha^n_{m-1;k})\Phi_{n-mk/2}$, *if* k *is even and* $mk < 2n - 1$;
(c) $\gamma^n_{m;k}\theta_1$, *if* $mk = 2n - 1$.

Proof is evident.

Remark 1.1 It follows from the equality $C^{2n-1}_{2m} = \frac{2m}{2(m-n)-1} C^{2n-1}_{2m-1}$ that

$$C^{2n-1}_{2m} \equiv 0 \mod 2.$$

Consider an arbitrary summand in the decomposition $\alpha^n_{m;k}$, when $m = 2s-1$, $k = 2q$:

$$(-1)^{r+1} C^{m-i_r}_{2n-k(m-i_r)} C^{i_r-i_{r-1}}_{2n-k(m-i_{r-1})} \cdots C^{i_2-i_1}_{2n-k(m-i_1)} C^{i_1}_{2n-km}.$$

For this summand to be non-equal to zero mod 2 the fulfillment of the following conditions is necessary:

$$i_1 \equiv 0 \mod 2, \quad i_2 \equiv 0 \mod 2, \ldots, m \equiv 0 \mod 2.$$

But it is not true, so $\alpha^n_{2s-1;2q} \equiv 0 \mod 2$.

Corollary 1.1 *If* $n > k$ *then we have:*

$$S_{k,k}\Phi_n \equiv (n-k)\Phi_{n-k} \mod 2.$$

Proof For any k we have $\alpha^n_{2;k} = -C^2_{2(n-k)} + C^1_{2n-k} C^1_{2(n-k)} \equiv (n-k) \mod 2$. If $k = 2s$, then $\alpha^n_{1;k} = -C^1_{2(n-s)} \equiv 0 \mod 2$. □

Corollary 1.2 *The is the formula:*

$$S_{k,k,k}\Phi_n \equiv \begin{cases} (n-k)\Phi_{n-3s} & \mod 2, \ \textit{if}\ k=2s, 3s<n, \\ (n-k)\theta_1 & \mod 2, \ \textit{if}\ 2n+1=3k, \\ & 0, \ \textit{in the other cases}. \end{cases}$$

Proof Let $k = 2s, 3s < n$ then result of the action of the operation $S_{k,k,k}$ on the element Φ_n is equal to $(\alpha^n_{3;k} + \alpha^n_{2;k})\Phi_{n-3s}$. From the Corollary 1.1 we have $\alpha^n_{2;k} \equiv (n-k) \mod 2$, but $\alpha^n_{3,2s} \equiv 0 \mod 2$ according to the Remark 1.1. Let $3k = 2n+1$, then $\gamma^3_k = C^2_{2(n-k)} - C^1_{2n-k}C^1_{2(n-k)} \equiv (n-k) \mod 2$. □

Definition 1.1 Let us define the following function of the integer arguments $\mu(n; k)$ by the formula:

$$\mu(n;k) = \begin{cases} 1, & \text{if } n \text{ is odd, } k \text{ is even,} \\ 0, & \text{in the other cases.} \end{cases}$$

Corollary 1.3 *Let* $2k < n$, *then we have the formula:*

$$S_{k,k,k,k}\Phi_n \equiv (C^4_{2(n-2k)} + \mu(n,k))\Phi_{n-2k} \mod 2.$$

Proof According to Remark 1.1 $\alpha^n_{3;k} \equiv 0 \mod 2$, if k is even. In the decomposition of the number $\alpha^n_{4;k}$ all the summands except $C^4_{2(n-2k)}$ and $C^2_{2(n-k)}C^2_{2(n-2k)} = (n-k)(n-2k)$ are even. We have $(n-k)(n-2k) \equiv 1 \mod 2$ only if n is odd and k is even. □

Corollary 1.4 (a) *If* $k = 2s$ *and* $5k < 2n-1$, *then we have the formula:*

$$S_{k,k,k,k,k}\Phi_n \equiv (C^4_{2(n-2k)} + n)\Phi_{n-5s} \mod 2.$$

(b) *If* $5k = 2n-1$, *then we have the identity:*

$$S_{k,k,k,k,k}\Phi_n \equiv C^4_{k+1}\theta_1 \mod 2.$$

Proof (a) According to Remark 1.1 $\alpha^n_{5,2s} \equiv 0 \mod 2$. We have, $S_{k,k,k,k,k}\Phi_n \equiv \alpha^n_{4;k}\Phi_{n-5s} \mod 2$. According to Corollary 1.3 $\alpha^n_{4,k} \equiv (C^4_{2(n-2k)} + (n-k)(n-2k)) \mod 2$. However for $k = 2s$ we have $(n-2s)(n-4s) \equiv n \mod 2$.

(b) In the decomposition of the number $\gamma^n_{5;k}$ the majority of summands contain factors C^1_{k+1} or C^1_{3k+1}, C^3_{k+1}, and because k is even, these factors are even. So we have $\gamma^n_{5;k} \equiv (C^4_{k+1} - C^2_{3k+1}C^2_{k+1}) \mod 2$. Let $k = 2s-1$, then $C^2_{3k+1}C^2_{k+1} = C^2_{2(3s-1)}C^2_{2s} \equiv (3s-1)s \equiv 0 \mod 2$. □

Corollary 1.5 *Let* $3k < n$ *then we have the formula:*

$$S_{k,k,k,k,k,k}\Phi_n \equiv \left(\frac{(n+1)}{2}\mu(n;k) + \frac{(n-2)}{2}\mu(k;n)\right)\Phi_{n-3k} \mod 2.$$

Proof If k is even, then according to Remark 1.1 we have $\alpha^n_{5;k} \equiv 0 \mod 2$, and there is the following comparison mod 2:

$$\begin{aligned}
\alpha^n_{6;k} &\equiv -C^6_{2(n-3k)} + C^4_{2(n-2k)}C^2_{2(n-3k)} + C^2_{2(n-k)}C^4_{2(n-3k)} \\
&\quad -C^2_{2(n-k)}C^2_{2(n-2k)}C^2_{2(n-3k)} \\
&\equiv -\frac{(n-3k)(n-3k-1)(n-3k-2)}{2} + \frac{(n-2k)(n-2k-1)(n-3k)}{2} \\
&\quad + \frac{(n-3k)(n-3k-1)(n-k)}{2} - (n-k)(n-2k)(n-3k) \\
&\equiv \frac{n-3k}{2}(-(n-3k-1)(n-3k-2) \\
&\quad + (n-2k)(n-2k-1) + (n-3k-1)(n-k)) - \mu(n;k) \\
&\equiv \frac{n-3k}{2}((n-3k-1)(k+1)2 + (n-2k-1)(n-2)) - \mu(n;k) \\
&\equiv 1/2(n-3k)(n-2k-1)(n-2k) - \mu(n;k) \mod 2.
\end{aligned}$$

There is also such a comparison mod 2:

$$(1/2)(n-3k)(n-2k)(n-2k-1) \equiv \begin{cases} (n-2k)/2, & \text{if } n \text{ is even and } k \text{ is odd,} \\ (n-2k-1)/2, & \text{if } k \text{ is even and } n \text{ is odd,} \\ 0, & \text{in the other cases.} \end{cases}$$

Hence we can write $\alpha^n_{6;k} \equiv \mu(n;k)((n-2k-1)/2) + \mu(k;n)((n-2k)/2) - \mu(n;k) \equiv \mu(n;k)((n+1)/2) - \mu(k;n)((n-2)/2) \mod 2$. In all our calculations we supposed that $3k \leq n-3$. Let now $3k = n-2$, then we have $\alpha^n_{6;k} \equiv 2(k+2) - 3(k+1)(k+2) \equiv 0 \mod 2$. It follows from the expression $3k = n-2$ that $k \equiv n \mod 2$. Let $3k = n-1$, then: $\alpha^n_{6;k} \equiv -C^4_{2(k+1)} + (k+1) \mod 2$. Consider separately various cases. Let $k = 1$, then: $n = 4$, $\alpha^4_{6;1} \equiv 1 \mod 2$. This coincides with the value given by the formula. Let now $k > 1$. We have: $\alpha^k_{6;k} \equiv -(k+1)k/2 + k + 1 \equiv (k+1)(k-2) \mod 2$. If k is even (then n is odd), this coincides with $(k-2)/2 \equiv (n-3)/2 \equiv (n+1)/2 \mod 2$. If k is odd (n is even), this is comparable with $(k+1)/2 \equiv 3(k+1)/2 \equiv (n-2)/2 \mod 2$. □

Corollary 1.6 *Let $k = 2s$, $7k < 2n-1$, then there is the formula:*

$$S_{k,k,k,k,k,k,k}\Phi_n \equiv n[(n+1)/2]\Phi_{n-7s} \mod 2,$$

($[x]$ denotes the integer part of the number x).

Proof We have $\alpha^n_{7;2s} \equiv 0 \mod 2$, and the formula for $\alpha^n_{6;2s}$ from Corollary 1.5 becomes (because of relations $\mu(n;2s) \equiv n \mod 2$, $\mu(2s;n) \equiv 0 \mod 2$), the following comparison $\alpha^n_{6;2s} \equiv n[(n+1)/2] \mod 2$. □

Corollary 1.7 *Let $4k < n$, then we have the formula:*

$$S_{k,k,k,k,k,k,k,k}\Phi_n \equiv (\mu(n;k) + (n/2)\mu(n+1;k) + C^8_{2(n-4k)})\Phi_{n-4k} \mod 2.$$

Proof Let us calculate the value of $\alpha^n_{8;k}$. Having in mind Remark 1.1 we have the comparison $\mod 2$ (if $4k \le n-4$):

$$\begin{aligned}\alpha^n_{8;k} &\equiv -C^8_{2(n-4k)} + C^6_{2(n-3k)}C^2_{2(n-4k)} + C^4_{2(n-2k)}C^4_{2(n-4k)}\\ &+C^2_{2(n-k)}C^6_{2(n-4k)} - C^4_{2(n-2k)}C^2_{2(n-3k)}C^2_{2(n-4k)}\\ &-C^2_{2(n-k)}C^4_{2(n-3k)}C^2_{2(n-4k)} - C^2_{2(n-k)}C^2_{2(n-2k)}C^4_{2(n-4k)}\\ &+C^2_{2(n-k)}C^2_{2(n-2k)}C^2_{2(n-3k)}C^2_{2(n-4k)} \equiv C^8_{2(n-4k)}\\ &+\mu(n,k) + (n-2k)(n-2k-1)(n-2k+1)(n-4k)/4.\end{aligned}$$

If $n = 2s+1$, then $\frac{(n-2k)(n-2k-1)(n-2k+1)(n-4k)}{4} \equiv s(s+1) \equiv 0 \mod 2$. If $n = 2s$, then

$$\begin{aligned}\frac{(n-2k)(n-2k-1)(n-2k+1)(n-4k)}{4} &\equiv (n/2-k)(n/2-2k)\\ &\equiv \begin{cases} n/2, & \text{if } k \text{ is even,}\\ 0, & \text{if } k \text{ is odd.}\end{cases}\end{aligned}$$

So, $(n-2k)(n-2k-1)(n-2k+1)(n-4k)/4 \equiv \mu(n+1;k)(n/2)$. Let $n-4k=3$, we have

$$\begin{aligned}\alpha^{4k+3}_{8;k} &\equiv C^6_{2(k+3)} + C^4_{2(2k+3)} + C^2_{2(3k+3)}\\ &-C^4_{2(2k+3)}C^2_{2(k+3)} - C^4_{2(k+3)}C^2_{2(3k+3)} - C^2_{2(2k+3)}C^2_{2(3k+3)}\\ &+C^2_{2(2k+3)}C^2_{2(3k+3)}C^2_{2(k+3)} \equiv k+1 \equiv \mu(4k+3;k) \mod 2.\end{aligned}$$

Let $n-4k=2$, then

$$\alpha^{4k+2}_{8;k} \equiv C^4_{2(2k+2)} - C^2_{2(2k+2)}C^2_{2(3k+2)} \equiv (k+1)(2k+1) \equiv \mu(n+1;k)(n/2) \mod 2.$$

Finally, let $n-4k=1$, then we get

$$\alpha^{4k+1}_{8;k} \equiv k+1 \equiv \mu(n;k) \mod 2.$$

□

Corollary 1.8 *If $k = 2s$ and $9k < 2n$, then*

$$S_{k,k,k,k,k,k,k,k,k}\Phi_n \equiv (n + (n+1)[n/2] + C^8_{2(n-4k)})\Phi_{n-9s} \mod 2.$$

The proof follows directly form the previous formula because of the condition $k = 2s$.

Corollary 1.9 *Let $5k < n$, then we have:*

$$S_{k,k,k,k,k,k,k,k,k,k}\Phi_n \equiv (\mu(n;k) + (n-k)C^8_{2(n-4k)})\Phi_{n-5k} \mod 2.$$

Proof If $5k < n-5$, then we have

$$\begin{aligned}
\alpha^n_{10;k} \equiv & -C^{10}_{2(n-5k)} + C^8_{2(n-4k)}C^2_{2(n-5k)} + C^6_{2(n-3k)}C^4_{2(n-5k)} \\
& +C^4_{2(n-2k)}C^6_{2(n-5k)} + C^2_{2(n-k)}C^8_{2(n-5k)} - C^6_{2(n-3k)}C^2_{2(n-4k)}C^2_{2(n-5k)} \\
& -C^4_{2(n-2k)}C^4_{2(n-4k)}C^2_{2(n-5k)} - C^4_{2(n-2k)}C^2_{2(n-3k)}C^2_{2(n-4k)}C^2_{2(n-5k)} \\
& -C^2_{2(n-k)}C^6_{2(n-4k)}C^2_{2(n-5k)} + C^2_{2(n-k)}C^4_{2(n-3k)}C^2_{2(n-4k)}C^2_{2(n-5k)} \\
& -C^4_{2(n-2k)}C^2_{2(n-3k)}C^4_{2(n-5k)} + C^2_{2(n-k)}C^2_{2(n-2k)}C^4_{2(n-4k)}C^2_{2(n-5k)} \\
& -C^2_{2(n-k)}C^4_{2(n-3k)}C^4_{2(n-5k)} + C^2_{2(n-k)}C^2_{2(n-2k)}C^2_{2(n-3k)}C^4_{2(n-5k)} \\
& -C^2_{2(n-k)}C^2_{2(n-2k)}C^2_{2(n-3k)}C^2_{2(n-4k)}C^2_{2(n-5k)} - C^2_{2(n-k)}C^2_{2(n-2k)}C^6_{2(n-5k)} \\
\equiv & \ (n-k)C^8_{2(n-4k)} - \mu(n;k) \\
& +(n-5k)(1/4)(n-2k)(n-2k+1)(n-2k-1)(n-4k) \mod 2
\end{aligned}$$

We have:

$$\frac{(n-5k)(n-2k)(n-2k+1)(n-2k-1)(n-4k)}{4} \equiv \begin{cases} (n/2-k)n/2 \equiv 0, & \text{if } n \text{ is even and } k \text{ is odd,} \\ ((n+1)/2)((n-1)/2) \equiv 0, & \text{if } k \text{ is even and } n \text{ is odd,} \\ 0, & \text{in the rest cases.} \end{cases}$$

Cases when $n-5k < 5$ are considered by the direct substitution. □

Corollary 1.10 *Let $k = 2s$ and $11s < n$, then we have the formula:*

$$S_{k,k,k,k,k,k,k,k,k,k,k}\Phi_n \equiv (n + nC^8_{2(n-4k)})\Phi_{n-11s} \mod 2.$$

Proof If $k = 2s$, then $\alpha^n_{10;k} \equiv (nC^8_{2(n-4k)} + n) \mod 2$, and $\alpha^n_{11,k} \equiv 0 \mod 2$. □

Corollary 1.11 *Let $6k < n$, then we have the formula:*

$$\begin{aligned}
S_{k,k,k,k,k,k,k,k,k,k,k,k}\Phi_n \equiv & \ (\mu(n;k)(n^2+3)/4 + [n/2]([n/2]-k) \\
& + \frac{1}{2}(n-2k)C^5_{n-4k+1})\Phi_{n-6s} \mod 2.
\end{aligned}$$

Proof Let us calculate the value $\alpha^n_{12;k}$, supposing that $6k \leq n-6$. Cases with $6k > n-6$ are considered by the direct substitution.

$$\begin{aligned}
\alpha^n_{12;k} \equiv & -C^{12}_{2(n-6k)} + C^2_{2(n-k)}C^2_{2(n-2k)}C^2_{2(n-3k)}C^2_{2(n-4k)}C^2_{2(n-5k)}C^2_{2(n-6k)} \\
& +C^{10}_{2(n-5k)}C^2_{2(n-6k)} + C^6_{2(n-3k)}C^2_{2(n-4k)}C^2_{2(n-5k)}C^2_{2(n-6k)} \\
& +C^8_{2(n-4k)}C^4_{2(n-6k)} + C^4_{2(n-2k)}C^4_{2(n-4k)}C^2_{2(n-5k)}C^2_{2(n-6k)} \\
& +C^6_{2(n-3k)}C^6_{2(n-6k)} + C^2_{2(n-k)}C^6_{2(n-4k)}C^2_{2(n-5k)}C^2_{2(n-6k)}
\end{aligned}$$

$$
\begin{aligned}
&+C^4_{2(n-2k)}C^8_{2(n-6k)} + C^4_{2(n-2k)}C^2_{2(n-3k)}C^4_{2(n-5k)}C^2_{2(n-6k)}\\
&+C^2_{2(n-k)}C^{10}_{2(n-6k)} + C^2_{2(n-k)}C^4_{2(n-3k)}C^4_{2(n-5k)}C^2_{2(n-6k)}\\
&-C^8_{2(n-4k)}C^2_{2(n-5k)}C^2_{2(n-6k)} - C^6_{2(n-3k)}C^4_{2(n-5k)}C^2_{2(n-6k)}\\
&-C^4_{2(n-2k)}C^6_{2(n-5k)}C^2_{2(n-6k)} - C^2_{2(n-k)}C^8_{2(n-5k)}C^2_{2(n-6k)}\\
&-C^6_{2(n-3k)}C^2_{2(n-4k)}C^4_{2(n-6k)} - C^4_{2(n-2k)}C^4_{2(n-4k)}C^4_{2(n-6k)}\\
&-C^2_{2(n-k)}C^6_{2(n-4k)}C^4_{2(n-6k)} - C^4_{2(n-2k)}C^2_{2(n-3k)}C^6_{2(n-6k)}\\
&-C^2_{2(n-k)}C^6_{2(n-6k)}C^4_{2(n-3k)} - C^2_{2(n-k)}C^2_{2(n-2k)}C^8_{2(n-6k)}\\
&+C^2_{2(n-k)}C^2_{2(n-2k)}C^6_{2(n-5k)}C^2_{2(n-6k)}\\
&+C^4_{2(n-2k)}C^2_{2(n-3k)}C^2_{2(n-4k)}C^4_{2(n-6k)}\\
&+C^2_{2(n-k)}C^4_{2(n-3k)}C^2_{2(n-4k)}C^4_{2(n-6k)}\\
&+C^2_{2(n-k)}C^2_{2(n-2k)}C^4_{2(n-4k)}C^4_{2(n-6k)}\\
&+C^2_{2(n-k)}C^2_{2(n-2k)}C^2_{2(n-3k)}C^6_{2(n-6k)}\\
&-C^4_{2(n-2k)}C^2_{2(n-3k)}C^2_{2(n-4k)}C^2_{2(n-5k)}C^2_{2(n-6k)}\\
&-C^2_{2(n-k)}C^4_{2(n-3k)}C^2_{2(n-4k)}C^2_{2(n-5k)}C^2_{2(n-6k)}\\
&-C^2_{2(n-k)}C^2_{2(n-2k)}C^4_{2(n-4k)}C^2_{2(n-5k)}C^2_{2(n-6k)}\\
&-C^2_{2(n-k)}C^2_{2(n-2k)}C^2_{2(n-3k)}C^4_{2(n-5k)}C^2_{2(n-6k)}\\
&-C^2_{2(n-k)}C^2_{2(n-2k)}C^2_{2(n-3k)}C^2_{2(n-4k)}C^4_{2(n-6k)}\\
\equiv\ &\mu(n;k) + C^4_{2(n-2k)}C^4_{2(n-4k)}C^4_{2(n-6k)}\\
&+\frac{1}{4}(n-2k+1)(n-2k)(n-2k+1)(n-4k)(n-5k)(n-6k)\\
&-\frac{1}{16}(n-4k)(n-4k-1)(n-4k+1)\\
&\quad(n-4k-2)(n-4k-3)(n-6k) \mod 2.
\end{aligned}
$$

We have the following comparisons mod 2:

$$
\begin{aligned}
&\frac{1}{4}(n-2k+1)(n-2k)(n-2k+1)(n-4k)(n-5k)(n-6k)\\
&\quad\equiv \binom{\frac{n+1}{2}}{}\binom{\frac{n-1}{2}}{}\mu(n;k),\\
&\frac{1}{16}(n-4k)(n-4k-1)(n-4k+1)(n-4k-2)(n-4k-3) \equiv \left(\frac{n-2k}{2}\right)C^5_{n-4k+1},\\
&C^4_{2(n-2k)}C^4_{2(n-4k)}C^4_{2(n-6k)} \equiv \left[\frac{n}{2}\right]\left(\left[\frac{n}{2}\right]-k\right).
\end{aligned}
$$

□

Corollary 1.12 *If $k = 2s$ and $13s < n$, then:*

$$S_{k,k,k,k,k,k,k,k,k,k,k,k,k}\Phi_n \equiv (n[(n^2+3)/4] + (n/2)C^5_{n+1} + [n/2])\Phi_{n-13s} \mod 2.$$

Proof In the case $k = 2s$ we have the following comparisons:

$$\frac{n-2k}{2} \equiv \frac{n}{2} \mod 2,\ C^5_{n-4k+1} \equiv C^5_{n+1} \mod 2,\ \mu(n;k) \equiv n \mod 2.$$

□

We put the obtained results in the Table 9 where only nonzero values are depicted.

2 Modified algebraic spectral sequence

2.1 Preliminary information

For the calculation of the initial term $E_2^{*,*}$ of the Adams–Novikov spectral sequence converging to MSp_* the second author had built in the works [40,41] the so called *modified algebraic spectral sequence* (MASS) which converges to the initial term $E_2^{*,*}$ of the Adams–Novikov spectral sequence. He had also proved that its initial term $E_1^{*,*,*}$ in this case has especially simple structure: it is a three-graded algebra isomorphic to the polynomial algebra over the field $\mathbb{Z}/2$:

$$E_1^{q,s,t} \cong \mathbb{Z}/2\big[h_0, h_1, \ldots, h_i, \ldots; u_1, \ldots u_j, \ldots; c_2, c_4, \ldots, c_n, \ldots\big].$$

where $i = 0, 1, 2, \ldots; j = 1, 2, \ldots; n = 2, 4, 5, \ldots, n \neq 2^k - 1$, $\deg h_0 = (2, 0, 0)$, $\deg h_i = (1, 0, 2(2^i - 1))$, $\deg u_j = (0, 1, 2(2^j - 1))$, $\deg c_n = (0, 0, 4n)$.

For the calculation of the symplectic cobordism ring MSp_*, the second author in the papers [40–42] used the following scheme. At the beginning the modified algebraic spectral sequence, MASS, for the spectrum MSp (or algebraic spectral sequence, ASS) is calculated. The term $E_\infty^{*,*,*}$ (as well as the corresponding term of ASS) is associated to the initial term of the Adams–Novikov spectral sequence $E_2^{*,*}$. This spectral sequence in its turn converges to the ring MSp_*. Using this scheme the ring MSp_* was calculated up to dimension 31 [41]. In this section we make the calculations of the term $E_\infty^{q,s,t}$ of MASS for $t < 108$.

Let d_1 denote the first differential of the MASS and φ_n (respectively θ_1) denote the projection of the Ray's element Φ_n (respectively Φ_0) in the term $E_1^{0,1,*}$ of MASS. Let us introduce the following notations. If $n = 2m - 1$ and $m = 2^{i_1-2} + \cdots + 2^{i_q-2}$ is the digital decomposition of the number $m (2 \leq i_1 < \cdots < i_q, q \geq 3)$, then the generator c_n we denote by $c_{i_1,\ldots,i_q}$, and the element φ_m we denote by $\varphi_{i_1,\ldots,i_q}$. The element c_{2^i-1} we denote by $c_{1,i}$.

The following facts were proved in the works [40–43]. The analogous facts for the classical Adams spectral sequence were proved by Kochman [14].

(1) On the generators c_n the differential d_1 acts by the following way:
   (a) if n is even and not a power of 2, then it is possible to choose the element c_n in such a way that it becomes an infinite cycle in MASS;
   (b)
$$d_1(c_{i,j}) = u_i h_j + u_j h_i; \tag{2.1}$$
   (c)
$$d_1(c_{i_1,\dots,i_q}) = \sum_{1\le s\le t\le q} (u_{i_t} h_{i_s} + u_{i_s} h_{i_t}) c_{1,i_1} \cdots \hat{c}_{1,i_s} \cdots \hat{c}_{1,i_t} \cdots c_{1,i_q}; \tag{2.2}$$

(2) The generators h_0, u_j are infinite cycles and the following equalities hold in the algebra $E_1^{*,*,*}$:
$$u_1 = \theta_1, u_i = \varphi_{2^i-2}, i = 2, 3, \dots$$
$$\varphi_{i_1,\dots,i_q} = u_1 c_{i_1,\dots,i_q} + \sum_{i=1}^{q} u_{i_t} c_{1,i_1} \cdots \hat{c}_{1,i_t} \cdots c_{1,i_q} + \sum_J \varphi_q c_{J_q}, \tag{2.3}$$
where the elements $c_{J_q} \in E_2^{0,0,*}$ are infinite cycles, φ_q are the projections of the elements Φ_q for $q < m$.
(3) For all $j = 1, 2, \dots,$ there is the formula:
$$d_1(h_j) = h_0 u_j.$$

Let us introduce the following notation according with the formula (2.3):
$$\varphi_{i_1,\dots,i_q} = \tilde{\varphi}_{i_1,\dots,i_q} + \sum_{J_q} \varphi_q c_{J_q}.$$

Then we can choose elements $\tilde{\varphi}_n$ or elements φ_n as the generators in $E_r^{0,1,t}, r > 1$. The differentials $d_r, r > 1$, conserve the grading t, increase the grading s by 1 and increase the grading q by r.

Initial term of the Adams–Novikov spectral sequence has the following description:
$$E_2^{*,*} \cong \mathrm{Ext}_{A^{BP}}(BP^*(MSP), BP^*).$$

There exists a filtration $F^q E_2^{s,t}$, such that we have an isomorphism
$$F^q E_2^{s,t} / F^{q+1} E_2^{s,t} \cong E_\infty^{q,s,t},$$

connecting the subfactors of term $E_2^{s,t}$ of the Adams–Novikov spectral sequence and the term $E_\infty^{q,s,t}$ of MASS.

2.2 Projections of Ray's elements into the MASS and action of Landweber–Novikov operations on the elements $\mathbf{c_i}$

It is proved in the work of the second author [41] that the projection of the Ray's element Φ_3 can be chosen in the form:

$$\varphi_3 = u_1 c_5 + u_2 c_4 + u_3 c_2. \tag{2.4}$$

Lemma 2.1 *It is possible to choose the element c_{2m-2} in such a way that for the decomposition of the element φ_m from (2.3) the following condition holds:*

$$\sum_{J_q} \varphi_q c_{J_q} = u_2 c_{2m-2} + u_3 (c_{2m-4} + \text{ decomposable }) + \sum_{J_{q'}} \varphi_{q'} c_{J_{q'}}$$

(*where* $q' \geq 3$).

Proof From the condition $S_{2m-2}\varphi_m = u_2$ ($S_{2m-4}\varphi_m = u_3$ respectively) it follows that in the decomposition φ_m present the following summands $u_2 c_{2m-2}$($u_3 c_{2m-4}$ respectively). If we take if necessary instead of c_{2m-2} the element $c'_{2m-2} = c_{2m-2} +$ (decomposable elements of power $2m-2$), we get the necessary equality. □

Let us precise the form of projections of some Ray's elements Φ_i in MASS and also calculate the action of Landweber–Novikov operations on the elements c_k. In all the calculations given below in the subsections from 1 till 9 everything is made $\mod 2$ and hence all the equalities are relations $\mod 2$.

1. Applying the operation S_1 to the decomposition (2.4) we get that $S_1\varphi_3 = 0 = u_1(S_1 c_5 + c_4)$. Hence $S_1 c_5 = c_4$.

Let us apply the operation S_2 to the decomposition (2.4). We get that $S_2\varphi_3 = u_3 = u_2(S_2 c_4 + c_2) + u_3 S_2 c_2$. Hence, $S_2 c_4 = c_2$, $S_2 c_2 = 1$.

Using the operation S_3 we get that $S_3\varphi_3 = 0 = u_1(S_3 c_5 + c_2)$, or $S_3 c_5 = c_2$.

Applying the operation $S_{2,2}$ we have $S_{2,2} c_4 = 0$.

And finally using S_5 we get an equality $S_5 c_5 = 1$.

2. The projection φ_5 of the element Φ_5 has the form:

$$\varphi_5 = u_1 c_4 + u_2 c_5 + u_3 c_6 + u_4 c_2 + \beta\varphi_3 c_2^2$$

Let us apply the operation $S_{2,2}$ to this expression, we get

$$S_{2,2}\varphi_5 = \varphi_3 = u_1 S_{2,2} c_9 + u_2 S_{2,2} c_8 + u_2 S_2 c_6 + \varphi_3 + \beta\varphi_3 + u_2 \beta c_2^2.$$

Hence, $S_{2,2} c_9 = 0$, $S_{2,2} c_8 = S_2 c_6$, $\beta = 0$.

Finally the form of φ_5 is the following:

$$\varphi_5 = u_1 c_9 + u_2 c_8 + u_3 c_6 + u_4 c_2.$$

Using the obtained decomposition let us calculate the values of the operations S_ω on c_9, c_8 and c_6.

3. The projection of the element Φ_6 has the form:

$$\varphi_6 = u_1c_{11} + u_2c_{10} + u_3c_8 + u_4c_4 + u_3(\beta_1 c_4^2 + \beta_2 c_2^4) + \beta_3\varphi_3c_6 + \beta_4 u_4 c_2^2.$$

Applying the operation $S_{4,4}$ we get: $S_{4,4}\varphi_6 = 0 = \beta_1 u_3$. Hence, $\beta_1 = 0$. Applying the operation S_6we get:

$$S_6\varphi_6 = \varphi_3 = u_1S_6c_{11} + u_2S_6c_{10} + u_3c_2 + u_2c_4 + \beta_3\varphi_3 + \beta_4 u_2 c_2^2.$$

Hence $S_6c_{11} = c_5$, $\beta_3 = 0$, $S_6c_{10} = \beta_4 c_2^2$.

Let us apply the operation $S_{2,2}$, we get:

$$S_{2,2}\varphi_6 = 0 = u_1S_{2,2}c_{11} + u_2S_{2,2}c_{10} + u_2c_2c_4 + \varphi_3c_2 + \beta_4u_4 + u_3c_2^2 + u_2c_6.$$

Hence, $S_{2,2}c_{11} = c_2c_5$, $\beta_4 = 0$, $S_{2,2}c_{10} = c_6$.

Let us apply the operation $S_{2,2,2,2}$, we have: $S_{2,2,2,2}\varphi_6 = u_3 = \beta_2 u_3$. Hence, $\beta_2 = 1$.

The final form of the projection φ_6 of the element Φ_6 is the following:

$$\varphi_6 = u_1c_{11} + u_2c_{10} + u_3c_8 + u_3c_2^4 + u_4c_4.$$

Using the obtained decomposition let us calculate the results of the action of the operation S_ω on the elements c_{11}, c_{10}.

4. The projection φ_7 of the element Φ_7 has the form:

$$\begin{aligned}\varphi_7 = {} & u_1c_{13} + u_2c_4c_8 + u_3c_2c_8 + u_4c_2c_4 + u_2c_{12} \\ & + u_3(c_{10} + \beta_1 c_5^2 + \beta_2 c_2^2 c_6) + \varphi_3(\beta_3 c_4^2 + \beta_4 c_2^4) + \beta_5 u_4 c_6 + \beta_6 \varphi_5 c_2^2.\end{aligned}$$

Applying the operation $S_{5,5}$ we have: $S_{5,5}\varphi_7 = 0 = u_3(1 + \beta_1)$, hence, $\beta_1 = 1$. Applying the operation $S_{2,2}$ we get:

$$\begin{aligned}S_{2,2}\varphi_7 = \varphi_5 = {} & u_1S_{2,2}c_{13} + u_2(c_8 + c_2^2c_4 + S_{2,2}c_{12} + c_4^2(1+\beta_3) \\ & + c_2^4(1 + \beta_2 + \beta_4)) + u_3(c_2c_4 + \beta_2c_6 + c_2^3) \\ & + \varphi_3(c_4 + c_2^2(1 + \beta_3 + \beta_5 + \beta_6)) + u_4c_2 + \beta_6\varphi_5.\end{aligned}$$

It follows from this equality that $\beta_6 = 0$, $\beta_3 = \beta_5$, $S_{2,2}c_{13} = c_4c_5 + c_2^2c_5 + c_9$, $\beta_2 = 1$, $S_{2,2}c_{12} = \beta_3c_4^2 + \beta_4c_2^4$.

Applying the operation S_6 we have:

$$S_6\varphi_7 = u_4 = u_1S_6c_{13} + u_2S_6c_{12} + u_2\beta_5c_6 + \beta_5u_4.$$

Hence, $\beta_5 = 1$, $S_6c_{13} = 0$, $S_6c_{12} = c_6$, $\beta_3 = 1$.

Let us apply the operation $S_{2,2,2,2}$, then we have:

$$S_{2,2,2,2}\varphi_7 = 0 = u_1S_{2,2,2,2}c_{13} + u_2(S_{2,2,2,2}c_{12} + c_2^2) + \varphi_3(1 + \beta_4).$$

Hence, $\beta_4 = 1$, $S_{2,2,2,2}c_{13} = 0$, $S_{2,2,2,2}c_{12} = c_2^2$.

The final form of the projection of Φ_7 will be the following:

$$\varphi_7 = u_1c_{13} + u_2c_4c_8 + u_3c_2c_8 + u_4c_2c_4 + u_2c_{12} \\ +u_3(c_{10} + c_5^2 + c_2^2c_6) + \varphi_3(c_4^2 + c_2^4) + u_4c_6.$$

Using this decomposition let us calculate the result of the action of the operation S_ω on the elements c_{13} and c_{12}.

5. The projection φ_9 of the element Φ_9 has the form:

$$\varphi_9 = u_1c_{17} + u_2c_{16} + u_3c_{14} + u_3(\beta_{12}c_2^2c_{10} + \beta_{13}c_2^2c_5^2 + \beta_{14}c_2^4c_6 \\ +\beta_{15}c_4^2c_6) + \varphi_3(\beta_1c_{12} + \beta_2c_6^2 + \beta_3c_2^6 + \beta_4c_2^2c_4^2) + u_4(\beta_5c_5^2 + \beta_6c_2^2c_6 \\ +\beta_7c_{10}) + \varphi_5(\beta_8c_4^2 + \beta_9c_2^4) + \beta_{10}\varphi_6c_6 + \beta_{11}\varphi_7c_2^2 + u_5c_2.$$

Choose the element c_{14} in such a way that $\beta_{12}, \beta_{13}, \beta_{14}, \beta_{15} = 0$.

Applying the operation S_{10}we get:

$$S_{10}\varphi_9 = u_4 = u_1S_{10}c_{17} + u_2(S_{10}c_{16} + \beta_{10}c_6) + \varphi_3c_2 + u_3(S_2c_{14} + \beta_{11}c_2^2) + \beta_7u_4.$$

It follows from here that $\beta_7 = 1$, $S_{10}c_{17} = c_2c_5$, $S_{10}c_{16} = c_2c_4 + \beta_{10}c_6$, $S_{10}c_{14} = c_2^2(1 + \beta_{11})$.

Let us apply the operation S_{12}, we have:

$$S_{12}\varphi_9 = \varphi_3 = u_1S_{12}c_{17} + u_2(S_{12}c_{16} + \beta_{11}c_2^2) + u_3c_2 + \beta_1\varphi_3S_{12}c_{12}.$$

It follows that $\beta_1 = 0$, $S_{12}c_{17} = c_5$, $S_{12}c_{16} = c_4 + \beta_{11}c_2^2$.

Let us apply the operation $S_{6,6}$, then we have: $S_{6,6}\varphi_9 = \varphi_3 = u_1S_{6,6}c_{17} + u_2(S_{6,6}c_{16} + c_2^2(\beta_6 + \beta_{11})) + \varphi_3(\beta_2 + \beta_{10})$, that gives: $\beta_2 + \beta_9 = 1$, $S_{6,6}c_{17} = 0$, $S_{6,6}c_{16} = c_2^2(\beta_6 + \beta_{11})$.

Let us apply the operation S_2, we get:

$$\begin{aligned} S_2\varphi_9 &= u_5 \\ &= u_1S_2c_{17} + u_2S_2c_{16} + u_2c_{14} + u_3S_2c_{14} \\ &\quad +\varphi_3c_{10} + u_4c_4^2 + u_4c_2^4 \\ &\quad +\varphi_7c_2 + u_5 + u_3(\beta_2c_6^2 + \beta_3c_2^6 + \beta_4c_2^2c_4^2) + \varphi_3(\beta_5c_5^2 + \beta_6c_2^2c_6) \\ &\quad +u_4\beta_6c_2^4 + u_4(\beta_8c_4^2 + \beta_9c_2^4) + \varphi_5\beta_{10}c_6 + \varphi_6\beta_{11}c_2^2 = u_1[S_2c_{17} \\ &\quad +c_5c_{10} + c_2c_{13} + c_2^5c_5 + \beta_5c_5^2 + \beta_6c_2^2c_5c_6 + c_2c_4^2c_5 + \beta_{10}c_6c_9 \\ &\quad +c_2^2c_{11}(\beta_{10} + \beta_{11})] + u_2[S_2c_{16} + c_{14} + c_2c_{12} + c_4c_{10} + c_2c_4c_8 \\ &\quad +c_2^5c_4 + c_2c_4^3 + \beta_5c_4c_5^2 + \beta_6c_2^2c_4c_6 + \beta_{10}c_6c_8 + c_2^2c_{10}(\beta_{10} + \beta_{11})] \\ &\quad +u_3[S_2c_{14} + c_2c_5^2(1 + \beta_5) + c_2^2c_8(1 + \beta_{10} + \beta_{11}) + c_2^3c_6(1 + \beta_6) \\ &\quad +c_6^2(\beta_2 + \beta_{10}) + c_2^6(1 + \beta_3 + \beta_{10} + \beta_{11}) + c_2^2c_4^2(1 + \beta_4)] \\ &\quad +u_4[c_2^2c_4(1 + \beta_{10} + \beta_{11}) + c_4^2(1 + \beta_8) + c_2^4(1 + \beta_6 + \beta_9) \\ &\quad +c_2c_6(1 + \beta_{10})] + u_5. \end{aligned}$$

We conclude from this that

$$\begin{aligned}
&\beta_8 = 1, \beta_{10} = 1, \beta_{11} = 0, \beta_5 = 1, \beta_6 = 1, \beta_9 = 0, \beta_2 = 0;\\
&S_2c_{16} = c_{14} + c_2c_{12} + c_4c_{10} + c_2c_4c_8 + c_2^5c_4 + c_4c_5^2 + c_2^2c_4c_6 + c_6c_8 + c_2^2c_{10}\\
&\qquad +c_2c_4^3;\\
&S_2c_{17} = c_2c_{13} + c_5c_{10} + c_2^5c_6 + c_5^3 + c_2^2c_5c_6 + c_6c_9 + c_2^2c_{11} + c_2c_4^2c_5;\\
&S_2c_{14} = c_6^2 + \beta_3c_2^6 + (1 + \beta_4)c_2^2c_4^2.
\end{aligned}$$

So, φ_9 has the following form:

$$\begin{aligned}
\varphi_9 = {} & u_1c_{17} + u_2c_{16} + u_3c_{14} + \varphi_3(\beta_3c_2^6 + \beta_4c_2^2c_4^2) + u_4(c_{10} + c_5^2 + c_2^2c_6)\\
& +\varphi_5c_4^2 + \varphi_6c_6 + u_5c_2.
\end{aligned}$$

To determine the coefficient β_4 let us apply the operation $S_{2,2}$:

$$\begin{aligned}
S_{2,2}\varphi_9 = \varphi_7 = {} & u_1S_{2,2}c_{17} + u_2(S_{2,2}c_{16} + c_6^2 + c_2^2c_4^2) + u_3S_{2,2}c_{14}\\
& +\varphi_3((\beta_3 + \beta_4)c_2^4 + \beta_4c_4^2) + \varphi_7.
\end{aligned}$$

We get $S_{2,2}c_{17} = (\beta_3 + \beta_4)c_2^4c_5 + \beta_4c_4^2c_5$, $S_{2,2}c_{16} = c_6^2 + c_2^2c_4^2 + \beta_4c_4^3 + (\beta_3 + \beta_4)c_2^2c_4^2$, $S_{2,2}c_{14} = (\beta_3 + \beta_4)c_2^5 + \beta_4c_2c_4^2$; and because of $d_1(c_{14}) = 0$, we have $\beta_3, \beta_4 = 0$.

The final form of φ_9 will be the following:

$$\varphi_9 = u_1c_{17} + u_2c_{16} + u_3c_{14} + u_4(c_{10} + c_5^2 + c_2^2c_6) + \varphi_5c_4^2 + \varphi_6c_6 + u_5c_2.$$

Let us calculate the result of the action of the operation S_ω on the elements c_{17}, c_{16}, c_{14}.

6. The projection φ_{10} of the element Φ_{10} has the form:

$$\begin{aligned}
\varphi_{10} = {} & u_1c_{19} + u_2c_{18} + u_3c_{16} + u_5c_4 + u_3(\beta_2c_8^2 + \beta_3c_4^4 + \beta_4c_2^8 + \beta_5c_6c_{10}\\
& +\beta_6c_2^4c_4^2 + \beta_7c_2^2c_6^2 + \beta_8c_2^2c_{12} + \beta_9c_5^2c_6) + \varphi_3(\beta_{10}c_{14} + \beta_{11}c_2^2c_{10}\\
& +\beta_{12}c_2^2c_5^2 + \beta_{13}c_4^2c_6 + \beta_{14}c_2^4c_6) + u_4(\beta_{15}c_{12} + \beta_{16}c_6^2 + \beta_{17}c_2^2c_4^2)\\
& +\varphi_5(\beta_{18}c_{10} + \beta_{19}c_5^2 + \beta_{20}c_2^2c_6) + \varphi_6(\beta_{21}c_4^2 + \beta_{22}c_2^4) + \beta_{23}\varphi_7c_6\\
& +\beta_{24}u_5c_2^2 + \beta_1u_4c_2^6.
\end{aligned}$$

Applying the operation $S_{8,8}$, we get $S_{8,8}\varphi_{10} = u_3(1 + \beta_2)$, hence, $\beta_2 = 0$.

Let us apply the operation $S_{4,4,4,4}$, we get $S_{4,4,4,4}\varphi_{10} = u_3 = u_3(1 + \beta_3 + \beta_{15})$. Hence, $\beta_3 = \beta_{15}$.

Let us apply the operation S_{14}, then:

$$S_{14}\varphi_{10} = \varphi_3 = u_1S_{14}c_{19} + u_2(S_{14}c_{18} + c_4 + \beta_{24}c_2^2) + u_3c_2 + \beta_{10}\varphi_3.$$

It follows from this that $\beta_{10} = 0$, $S_{14}c_{19} = c_5$, $S_{14}c_{18} = \beta_{24}c_2^2$.

Let us apply the operation S_{12}, then we get:

$$S_{12}\varphi_{10} = u_4 = u_1 S_{12} c_{19} + u_2(S_{12}c_{18} + \beta_{23}c_6) + u_3 c_2^2(\beta_8 + \beta_{24}) + \beta_{15}u_4.$$

Hence we have $\beta_{15} = 1$, $\beta_3 = 1$, $\beta_8 = \beta_{24}$, $S_{12}c_{19} = 0$, $S_{12}c_{18} = \beta_{23}c_6$.

Let us apply the operation $S_{6,6}$, this gives:

$$\begin{aligned} S_{6,6}\varphi_{10} = 0 = u_1 S_{6,6}c_{19} + u_2(S_{6,6}c_{18} + c_6(\beta_{15} + \beta_{23})) \\ + u_3 c_2^2(\beta_7 + \beta_{20}) + u_4(\beta_{16} + \beta_{23}). \end{aligned}$$

Hence, $\beta_{16} = \beta_{23}$, $\beta_7 = \beta_{20}$, $S_{6,6}c_{18} = (\beta_{15} + \beta_{23})c_6$, $S_{6,6}c_{19} = 0$.

Applying the operation $S_{4,4,4}$, we have:

$$\begin{aligned} S_{4,4,4}\varphi_{10} = 0 = u_1 S_{4,4,4}c_{19} + u_2(S_{4,4,4}c_{18} + (\beta_{13} + \beta_{23})c_6) \\ + u_3 c_2^2(\beta_8 + \beta_{17}) + u_4(\beta_{15} + \beta_{21}). \end{aligned}$$

Hence $\beta_8 = \beta_{17}$, $\beta_{15} = \beta_{21}$, $S_{4,4,4}c_{19} = 0$, $S_{4,4,4}c_{18} = (\beta_{13} + \beta_{23})c_6$.

Let us apply the operation S_{10}, we have:

$$\begin{aligned} S_{10}\varphi_{10} = \varphi_5 = u_1 S_{10}c_{19} + u_2(S_{10}c_{18} + \beta_{21}c_4^2 + \beta_{22}c_2^4) \\ + u_3(c_2c_4 + c_6(1 + \beta_5 + \beta_{23})) + \varphi_3(c_4 + c_2^2(\beta_{11} + \beta_{24})) + \beta_{18}\varphi_5. \end{aligned}$$

Hence, $\beta_{11} = \beta_{24}$, $\beta_{18} = 1$, $\beta_5 + \beta_{23} = 1$, $S_{10}c_{19} = c_4c_5$, $S_{10}c_{18} = (1 + \beta_{21})c_4^2 + \beta_{22}c_2^4$.

Applying the operation $S_{5,5}$ we have:

$$\begin{aligned} S_{5,5}\varphi_{10} = \varphi_5 = u_1 S_{5,5}c_{19} + u_2(S_{5,5}c_{18} + \beta_{21}c_4^2 + \beta_{22}c_2^4) \\ + u_3(c_2c_4 + c_6(1 + \beta_5 + \beta_9)) + \varphi_3(c_4 + c_2^2(\beta_{11} + \beta_{12} + \beta_{24})) \\ + \varphi_5(1 + \beta_{19}). \end{aligned}$$

We get from this: $\beta_{19} = 0$, $\beta_{12} = 0$, $\beta_5 + \beta_9 = 1$, $S_{5,5}c_{19} = c_4c_5$, $S_{5,5}c_{18} = (1 + \beta_{21})c_4^2 + \beta_{22}c_2^4$.

Let us apply the operation S_8, then:

$$\begin{aligned} S_8\varphi_{10} = \varphi_6 = u_1 S_8 c_{19} + u_2(S_8c_{18} + \beta_{18}c_{10} + \beta_{20}c_2^2c_6) \\ + u_3(c_8 + c_4(1 + \beta_{21}) + \beta_{22}c_2^4) \\ + \beta_{23}\varphi_3c_6 + u_4(c_4 + \beta_{24}c_2^2). \end{aligned}$$

We get: $S_8c_{19} = c_{11} + \beta_{23}c_5c_6$, $S_8c_{18} = (1 + \beta_{18})c_{10} + \beta_{20}c_2^2c_6$, $\beta_{23} = 0$, $\beta_{24} = 0$, $\beta_{21} = 1$, $\beta_{22} = 1$, $\beta_5 = 1$, $\beta_8 = 0$, $\beta_{17} = 0$, $\beta_9 = 0$, $\beta_{16} = 0$.

Applying the operation $S_{4,4}$, we see:

$$\begin{aligned} S_{4,4}\varphi_{10} = 0 = u_1 S_{4,4}c_{19} + u_2(S_{4,4}c_{18} + c_{10} + \beta_{20}c_2^2c_6) \\ + u_3\beta_6c_2^4 + \varphi_3\beta_{13}c_6 + u_3c_2^4. \end{aligned}$$

We conclude that $\beta_{13} = 0$, $\beta_6 = 1$, $S_{4,4}c_{19} = 0$, $S_{4,4}c_{18} = c_{10} + \beta_{20}c_2^2c_6$.

Using the operation $S_{2,2,2,2}$ gives the following:

$$S_{2,2,2,2}\varphi_{10} = \varphi_6 = u_1S_{2,2,2,2}c_{19} + u_2(S_{2,2,2,2}c_{18} + \beta_{20}c_2^2c_6) \\ + u_3c_2^4(\beta_7 + \beta_{14} + \beta_{20}) + \varphi_3(\beta_{14} + \beta_{20}) + \beta_1u_4c_2^2 + \varphi_6.$$

We conclude from this that $\beta_{14} = \beta_{20} = 0$, $\beta_7 = 0$, $\beta_1 = 0$, $S_{2,2,2,2}c_{18} = 0$.

For the definition of β_4 let us use the operation $S_{2,2,2,2,2,2,2,2}$:

$$S_{2,2,2,2,2,2,2,2}\varphi_{10} = u_3 = u_3(1 + \beta_4).$$

We get that $\beta_4 = 0$.

The final form of φ_{10} is the following:

$$\varphi_{10} = u_1c_{19} + u_2c_{18} + u_3c_{16} + u_3(c_8^2 + c_4^4 + c_6c_{10} + c_2^4c_4^2) \\ +u_4c_{12} + \varphi_5c_{10} + \varphi_6(c_2^4 + c_4^2) + u_5c_4.$$

Using the obtained decomposition let us calculate the results of the action of the operations S_ω on c_{19} and c_{18}.

7. Let us precise the form of the projection φ_{11} of the element Φ_{11}:

$$\varphi_{11} = u_1c_{21} + u_2c_4c_{16} + u_3c_2c_{16} + u_5c_2c_4 + u_2c_{20} + u_3(c_{18} + \beta_1c_9^2 \\ +\beta_2c_2^2c_{14} + \beta_3c_6c_{12} + \beta_4c_2^4c_{10} + \beta_5c_2^4c_5^2 + \beta_6c_2^2c_4^2c_6 + \beta_7c_2^6c_6 \\ +\beta_8c_6^3 + \beta_9c_4^2c_{10} + \beta_{10}c_4^2c_5^2) + \varphi_3(\beta_{11}c_8^2 + \beta_{12}c_4^4 + \beta_{13}c_2^2c_{12} + \beta_{14}c_2^8 \\ +\beta_{15}c_6c_{10} + \beta_{16}c_2^2c_6^2 + \beta_{17}c_6c_5^2 + \beta_{18}c_2^4c_4^2) + u_4(\beta_{19}c_{14} + \beta_{20}c_2^2c_{10} \\ +\beta_{21}c_2^2c_5^2 + \beta_{22}c_4^2c_6 + \beta_{23}c_2^4c_6) + \varphi_5(\beta_{24}c_{12} + \beta_{25}c_6^2 + \beta_{26}c_2^2c_4^2 \\ +\beta_{27}c_2^6) + \varphi_6(\beta_{28}c_{10} + \beta_{29}c_5^2 + \beta_{30}c_2^2c_6) + \varphi_7(\beta_{31}c_4^2 + \beta_{32}c_2^4) \\ +u_5\beta_{33}c_6 + \varphi_9\beta_{34}c_2^2.$$

Applying the operation $S_{9,9}$ we get $S_{9,9}\varphi_{11} = 0 = u_3(\beta_1 + 1)$, hence: $\beta_1 = 1$.

Applying the operation S_{16}, we arrive to the equality:

$$S_{16}\varphi_{11} = \varphi_3 = u_1S_{16}c_{21} + u_2c_4 + u_3c_2 + u_2(S_{16}c_{20} + \beta_{34}c_2^2),$$

which gives $S_{16}c_{21} = c_5$, $S_{16}c_{20} = \beta_{34}c_2^2$.

Using the operation $S_{8,8}$, we obtain:

$$S_{8,8}\varphi_{11} = \varphi_3 = u_1S_{8,8}c_{21} + u_2c_4 + u_3c_2 + u_2(S_{8,8}c_{20} + \beta_{34}c_2^2) + \beta_{11}\varphi_3,$$

so $\beta_{11} = 0$, $S_{8,8}c_{21} = c_5$, $S_{8,8}c_{20} = \beta_{34}c_2^2$.

Using the operation $S_{4,4,4,4}$, we have:

$$S_{4,4,4,4}\varphi_{11} = 0 = u_1S_{4,4,4,4}c_{21} + u_2S_{4,4,4,4}c_{20} + \varphi_3(\beta_{12} + \beta_{24} + \beta_{31}) \\ +u_2(\beta_{13} + \beta_{26} + \beta_{34})c_2^2,$$

Hence, $\beta_{12}+\beta_{24}+\beta_{31}=0$, $S_{4,4,4,4}c_{20}=(\beta_{13}+\beta_{26}+\beta_{34})c_2^2$, $S_{4,4,4,4}c_{21}=0$.
Using the operation $S_{2,2,2,2,2,2,2,2}$, we get

$$\begin{aligned}S_{2,2,2,2,2,2,2,2}\varphi_{11}=\varphi_3=u_1S_{2,2,2,2,2,2,2,2}c_{21}&+u_2c_2^2(\beta_3+\beta_4+\beta_6+\beta_8\\&+\beta_{13}+\beta_{15}+\beta_{16}+\beta_{18}+\beta_{24}+\beta_{26}+\beta_{28}+\beta_{34})\\&+\varphi_3(\beta_{14}+\beta_{16}+\beta_{23}+\beta_{25}+\beta_{27}+\beta_{34})\\&+u_2S_{2,2,2,2,2,2,2,2}c_{20}.\end{aligned}$$

This means that

$$\begin{aligned}S_{2,2,2,2,2,2,2,2}c_{20}=c_2^2(&\beta_3+\beta_4+\beta_6+\beta_8+\beta_{13}+\beta_{15}\\&+\beta_{16}+\beta_{18}+\beta_{24}+\beta_{26}+\beta_{28}+\beta_{34}),\end{aligned}$$
$$\beta_{14}+\beta_{16}+\beta_{23}+\beta_{25}+\beta_{27}+\beta_{34}=1,\quad S_{2,2,2,2,2,2,2,2}c_{21}=0.$$

Applying the operation S_{14}, we get:

$$S_{14}\varphi_{11}=u_4=u_1S_{14}c_{21}+u_2(S_{14}c_{20}+\beta_{33}c_6)+u_3c_2^2(1+\beta_2+\beta_{34})+u_4\beta_{19},$$

Hence, $\beta_{19}=1$, $S_{14}c_{21}=0$, $S_{14}c_{20}=\beta_{33}c_6$, $\beta_2+\beta_{34}=1$.
Let us apply the operation S_{12}, we get:

$$\begin{aligned}S_{12}\varphi_{11}=\varphi_5=u_1S_{12}c_{21}+u_2(S_{12}c_{20}+c_4^2(1+\beta_{31})+\beta_{32}c_2^4)+\varphi_5\beta_{24}\\+u_3(\beta_3+\beta_{33})c_6+\varphi_3(\beta_{13}+\beta_{34})c_2^2.\end{aligned}$$

So, $\beta_{24}=1$, $\beta_3=\beta_{33}$, $\beta_{13}=\beta_{34}$, $S_{12}c_{20}=c_4^2(1+\beta_{31})+\beta_{32}c_2^4$.
Let us act by the operation $S_{6,6}$, we get a relation:

$$\begin{aligned}S_{6,6}\varphi_{11}=\varphi_5=u_1S_{6,6}c_{21}+u_2c_2^2c_4+u_3c_2^3+u_2(S_{6,6}c_{20}+c_4^2(\beta_{19}+\beta_{22}\\+\beta_{31})+c_2^4(\beta_{23}+\beta_{32}))+u_3c_6(1+\beta_3+\beta_8+\beta_{24})\\+\varphi_5(\beta_{25}+\beta_{33})+\varphi_3c_2^2(\beta_{16}+\beta_{30}+\beta_{34}).\end{aligned}$$

Hence: $S_{6,6}c_{21}=c_5c_2^2$, $\beta_{16}+\beta_{30}+\beta_{34}=1$, $\beta_3+\beta_8+\beta_{24}=1$, $S_{6,6}c_{20}=c_4^2(\beta_{19}+\beta_{22}+\beta_{31})+c_2^4(\beta_{23}+\beta_{32})$, $\beta_{25}+\beta_{33}=1$.
Let us apply the operation $S_{4,4,4}$, we get:

$$\begin{aligned}S_{4,4,4}\varphi_{11}=\varphi_5=u_1S_{4,4,4}c_{21}+u_2\left(S_{4,4,4}c_{20}+c_4^2(1+\beta_{31})+c_2^4(\beta_{24}+\beta_{32})\right)\\+u_3c_6(\beta_3+\beta_{22})+\varphi_3c_2^2(\beta_{13}+\beta_{26}+\beta_{34})+\varphi_5(\beta_{24}+\beta_{31}).\end{aligned}$$

Hence, $S_{4,4,4}c_{21}=0$, $\beta_{24}+\beta_{31}=1$, $\beta_3+\beta_{22}=0$, $\beta_{31}=0$, $\beta_{12}=1$, $\beta_{13}+\beta_{26}+\beta_{34}=0$, $S_{4,4,4}c_{20}=c_4^2(1+\beta_{31})+c_2^4(\beta_{24}+\beta_{32})$.

Applying the operation S_{10}, we arrive to the following:

$$S_{10}\varphi_{11} = \varphi_6 = u_1 S_{10}c_{21} + u_2c_2c_4^2 + u_2c_4c_6 + u_3c_2^2c_4 + u_3c_2c_6 + \varphi_3c_2c_4 + u_2(S_{10}c_{20} + \beta_{28}c_{10} + \beta_{29}c_5^2 + \beta_{30}c_2^2c_6) + u_3(c_2^4(1+\beta_2+\beta_4+\beta_{32}) + c_4^2(\beta_9+\beta_{31})) + \varphi_3c_6(\beta_{15}+\beta_{33}) + u_4c_2^2(1+\beta_{20}+\beta_{34}) + \varphi_6\beta_{28}.$$

It follows from this: $\beta_{28} = 1$, $\beta_9 = 0$, $\beta_2+\beta_4+\beta_{32} = 1$, $\beta_{15}+\beta_{33} = 1$, $\beta_{20}+\beta_{34} = 1$, $S_{10}c_{21} = c_5c_6 + c_2c_4c_6$, $S_{10}c_{20} = c_{10} + \beta_{29}c_5^2 + \beta_{30}c_2^2c_6$.

Let us act by the operation $S_{5,5}$, we get:

$$S_{5,5}\varphi_{11} = 0 = u_1S_{5,5}c_{21} + u_2c_2c_4^2 + u_2c_4c_6 + u_3c_2^2c_4 + u_3c_2c_6 + \varphi_3c_2c_4 + u_2(S_{5,5}c_{20} + \beta_{28}c_{10} + \beta_{29}c_5^2 + \beta_{30}c_2^2c_6) + u_3(c_2^4(\beta_2+\beta_4+\beta_5+1) + c_4^2(\beta_9+\beta_{10}) + \varphi_3c_6(\beta_{15}+\beta_{17}+\beta_{33}) + u_4c_2^2(\beta_{20}+\beta_{21}+1)) + \varphi_6(1+\beta_{29}).$$

We get from this $\beta_{29} = 1$, $\beta_2+\beta_4+\beta_5 = 1$, $\beta_9 = \beta_{10} = 0$, $\beta_{15}+\beta_{17}+\beta_{33} = 1$, hence, $\beta_{17} = 0$, $\beta_{20}+\beta_{21} = 1$, $S_{5,5}c_{21} = c_2c_4c_5+c_5c_6$, $S_{5,5}c_{20} = c_{10}+c_5^2+\beta_{30}c_2^2c_6$.

Let us apply the operation S_8, we get the following:

$$S_8\varphi_{11} = \varphi_7 = u_1S_8c_{21} + u_2c_4c_8 + u_3c_2c_8 + u_2c_4^3 + u_3c_2c_4^2 + u_4c_2c_4 + u_2(S_8c_{20} + c_{12} + \beta_{25}c_6^2 + \beta_{26}c_2^2c_4^2 + \beta_{27}c_2^6) + u_3(c_{10} + c_5^2 + \beta_{30}c_2^2c_6) + \varphi_3c_2^4\beta_{32} + u_4c_6\beta_{33} + \varphi_5c_2^2\beta_{34}.$$

Hence, $\beta_{34} = 0$, $\beta_{32} = 1$, $\beta_{33} = 1$, $\beta_{30} = 1$, $S_8c_{21} = c_{13} + c_5c_4^2$, $S_8c_{20} = \beta_{25}c_6^2 + \beta_{26}c_2^2c_4^2 + \beta_{27}c_2^6$, using the previous relations we get from this: β_5, β_3, β_{22}, β_2, β_4, β_{20}, β_8 are all equal to 1, and β_{13}, β_{15}, β_{16}, β_{25}, β_{21}, β_{26} all are equal to 0.

Let us apply the operation S_6, then we have:

$$S_6\varphi_{11} = u_5 = u_1S_6c_{21} + u_2c_4(c_4c_6 + c_2c_8) + u_3c_2(c_4c_6 + c_2c_8) + \varphi_5c_2c_4 + u_2(S_6c_{20} + c_{14} + c_2^2c_{10} + c_4^2c_6 + \beta_{23}c_2^4c_6) + u_3(c_2^2c_4^2(1+\beta_6) + c_2^6(\beta_7+\beta_{27})) + u_4c_2^4(1+\beta_{23}) + \varphi_6c_2^2.$$

We get that $\beta_6 = 1$, $\beta_7+\beta_{27} = 1$, $\beta_{23} = 1$, $S_6c_{21} = c_5(c_{10} + c_5^2 + c_2^2c_6) + c_2c_4c_9 + c_2^2c_{11}$, $S_6c_{20} = c_{14} + c_2^4c_6$.

Let us calculate the action of the operation S_4:

$$S_4\varphi_{11} = \varphi_9 = u_1S_4c_{21} + u_2c_{16} + u_2c_4(c_{12} + c_2c_{10} + c_4c_8 + c_4c_2^4) + u_5c_2 + u_3c_2(c_{12} + c_2c_{10} + c_4c_8 + c_4c_2^4) + \varphi_6c_2c_4 + \varphi_6c_6 + u_3(c_{14} + c_2^2c_{10} + c_4^2c_6) + u_2(S_4c_{20} + c_4^4 + \beta_{14}c_2^8 + \beta_{18}c_2^4c_4^2) + \varphi_3(c_{12} + c_2^6\beta_{27}) + u_4(c_{10} + c_5^2 + c_2^2c_6).$$

It follows from this that $\beta_{27} = 0, \beta_7 = 1, \beta_{14} = 0, S_4c_{20} = (1+\beta_{18})c_4^2c_2^4 + c_4^4, S_4c_{21} = c_{17} + c_2c_4c_{11} + c_9c_4^2 + c_5c_{12}$.

To determine the coefficient β_{18}, let us use the operation $S_{4,4}$:

$$\begin{aligned}S_{4,4}\varphi_{11} = \varphi_7 = u_1S_{4,4}c_{21} + u_2(S_{4,4}c_{20} + c_{12} + c_2c_{10} + c_4c_8 + c_4c_2^4 + c_4^3)\\ +u_3(c_2c_4^2 + c_{10} + c_5^2 + c_2^2c_6) + \varphi_3c_2^4\beta_{18} + u_4c_6 + \varphi_6c_2.\end{aligned}$$

Hence, $\beta_{18} = 0$, $S_{4,4}c_{21} = c_{13} + c_5(c_4^2 + c_2^4) + c_2c_{11}$, $S_{4,4}c_{20} = c_{12}$.

The final form of the projection φ_{11}, will be the following:

$$\begin{aligned}\varphi_{11} = u_1c_{21} + u_2c_4c_{16} + u_3c_2c_{16} + u_5c_2c_4 + u_2c_{20} + u_3(c_{18} + c_9^2 + c_2^4(c_{10}\\ +c_5^2 + c_2^2c_6) + c_2^2(c_{14} + c_4^2c_6) + c_6(c_{12} + c_6^2)) + \varphi_3c_4^4 + u_4(c_{14} + c_2^2c_{10}\\ +c_4^2c_6 + c_2^4c_6) + \varphi_5c_{12} + \varphi_6(c_{10} + c_5^2 + c_2^2c_6) + \varphi_7c_2^4 + u_5c_6.\end{aligned}$$

Using the obtained decomposition let us calculate the result of the action of the operations S_ω on the elements c_{21}, c_{20}.

8. Let us precise the form of projection in MASS φ_{12} of the element Φ_{12}:

$$\begin{aligned}\varphi_{12} = u_1c_{23} + u_2c_{22} + u_4c_{16} + u_5c_8 + u_3(c_{20} + \beta_1c_{10}^2\\ +\beta_2c_5^4 + \beta_3c_5^2c_{10} + \beta_4c_2^2c_8^2 + \beta_5c_2^2c_4^4 + \beta_6c_2^{10} + \beta_7c_2^2c_6c_{10} + \beta_8c_2^4c_6^2\\ +\beta_9c_2^4c_{12} + \beta_{10}c_2^2c_5^2c_6 + \beta_{11}c_2^6c_4^2 + \beta_{12}c_6c_{14} + \beta_{13}c_4^2c_6^2 + \beta_{14}c_4^2c_{12})\\ +\varphi_3(\beta_{15}c_{18} + \beta_{16}c_9^2 + \beta_{17}c_2^2c_{14} + \beta_{18}c_6c_{12} + \beta_{19}c_2^4c_{10} + \beta_{20}c_2^4c_5^2\\ +\beta_{21}c_4^2c_2^2c_6 + \beta_{22}c_2^6c_6 + \beta_{23}c_6^3 + \beta_{24}c_4^2c_{10} + \beta_{25}c_4^2c_5^2)\\ +u_4(\beta_{26}c_8^2 + \beta_{27}c_4^4 + \beta_{28}c_2^8 + \beta_{29}c_6c_{10} + \beta_{30}c_2^2c_6^2 + \beta_{31}c_2^2c_{12}\\ +\beta_{32}c_2^4c_4^2 + \beta_{33}c_5^2c_6) + \varphi_5(\beta_{34}c_{14} + \beta_{35}c_2^2c_{10} + \beta_{36}c_2^2c_5^2 + \beta_{37}c_4^2c_6\\ +\beta_{38}c_2^4c_6) + \varphi_6(\beta_{39}c_{12} + \beta_{40}c_6^2 + \beta_{41}c_2^6 + \beta_{42}c_2^2c_4^2)\\ +\varphi_7(\beta_{43}c_{10} + \beta_{44}c_5^2 + \beta_{45}c_2^2c_6) + u_5(\beta_{46}c_2^4 + \beta_{47}c_4^2)\\ +\varphi_9\beta_{48}c_6 + \varphi_{10}\beta_{49}c_2^2.\end{aligned}$$

It follows from the action of the operation $S_{10,10}$ that

$$S_{10,10}\varphi_{12} = 0 = u_3(\beta_1 + \beta_{43}),\ \ \beta_1 = \beta_{43}. \tag{2.5}$$

We get from the action of the operation $S_{5,5,5,5}$:

$$S_{5,5,5,5}\varphi_{12} = u_3 = u_3(\beta_1 + \beta_2 + \beta_3),\ \ \beta_1 + \beta_2 + \beta_3 = 1. \tag{2.6}$$

Applying the operation S_{18}, we get the relation:

$$S_{18}\varphi_{12} = \varphi_3 = u_1S_{18}c_{23} + u_2S_{18}c_{22} + \beta_{15}\varphi_3 + u_3c_2^2\beta_{49},$$

so, $S_{18}c_{23} = 0$, $S_{18}c_{22} = \beta_{49}c_2^2$, $\beta_{15} = 1$.

Let us act by the operation $S_{9,9}$, we get the equality:

$$S_{9,9}\varphi_{12} = \varphi_3 = u_1 S_{9,9}c_{23} + u_2 S_{9,9}c_{22} + u_2\beta_{49}c_2^2 + \varphi_3(1+\beta_{16}).$$

So, $S_{9,9}c_{23} = 0$, $S_{9,9}c_{22} = \beta_{49}c_2^2$, $\beta_{16} = 0$.

Using the operation S_{16}, we get:

$$S_{16}\varphi_{12} = u_4 = u_1 S_{16}c_{23} + u_2(S_{16}c_{22} + \beta_{48}c_6) + u_4 + u_3\beta_{49}c_2^2.$$

So, $S_{16}c_{23} = 0$, $S_{16}c_{22} = \beta_{48}c_6$, $\beta_{49} = 0$.

From the action of the operation $S_{8,8}$, we conclude that:

$$S_{8,8}\varphi_{12} = 0 = u_1 S_{8,8}c_{23} + u_2(S_{8,8}c_{22} + \beta_{48}c_6) + u_3\beta_4 c_2^2 + u_4\beta_{26}.$$

So, $S_{8,8}c_{23} = 0,\ \ S_{8,8}c_{22} = \beta_{48}c_6,\ \ \beta_4 = 0,\ \ \beta_{26} = 0$.

Using the operation $S_{4,4,4,4}$, we arrive to the equality:

$$S_{4,4,4,4}\varphi_{12} = 0 = u_1 S_{4,4,4,4}c_{23} + u_2(S_{4,4,4,4}c_{22} + (\beta_{37}+\beta_{48})c_6) \\ + u_3 c_2^2(\beta_5 + \beta_{31}) + u_4(\beta_{27}+\beta_{39}).$$

It follows from this that:

$$S_{4,4,4,4}c_{23} = 0,\ \ S_{4,4,4,4}c_{22} = (\beta_{37}+\beta_{48})c_6,\ \ \beta_5 = \beta_{31},\ \ \beta_{27} = \beta_{39}. \tag{2.7}$$

Applying the operation S_{14}, we conclude that:

$$S_{14}\varphi_{12} = \varphi_5 = u_1 S_{14}c_{23} + u_2 c_8 + u_3 c_6(1+\beta_{12}+\beta_{48}) + u_4 c_2 + \varphi_3\beta_{17}c_2^2 \\ + u_2(S_{14}c_{22} + \beta_{46}c_2^4 + \beta_{47}c_4^2) + \varphi_5\beta_{34}.$$

Hence,

$$S_{14}c_{23} = c_5,\ \ S_{14}c_{22} = \beta_{46}c_2^4 + \beta_{47}c_4^2,\ \ \beta_{17} = 0,\ \ \beta_{34} = 0,\ \ \beta_{12} = \beta_{48}. \tag{2.8}$$

From the action of the operation S_{12}, it follows that:

$$S_{12}\varphi_{12} = \varphi_6 = u_1 S_{12}c_{23} + u_3 c_8 + u_4 c_4 + u_2(S_{12}c_{22} + \beta_{43}c_{10} + \beta_{44}c_5^2 \\ + \beta_{45}c_2^2 c_6) + u_3(c_4^2(1+\beta_{14}+\beta_{47}) + c_2^4(1+\beta_9+\beta_{46})) \\ + \varphi_3 c_6(\beta_{18}+\beta_{48}) + u_4\beta_{31}c_2^2 + \varphi_6\beta_{39}.$$

We get from this

$$S_{12}c_{23} = c_{11},\ \ S_{12}c_{22} = (\beta_{43}+1)c_{10} + \beta_{44}c_5^2 + \beta_{45}c_2^2 c_6, \\ \beta_{31} = 0,\ \ \beta_{39} = 0,\ \ \beta_{14}+\beta_{47} = 1,\ \ \beta_9 = \beta_{46},\ \ \beta_{18} = \beta_{48}. \tag{2.9}$$

From the formulas (2.7), (2.9) it follows that $\beta_5 = 0$, $\beta_{27} = 0$.

Let us apply the operation $S_{6,6}$, then we have:

$$\begin{aligned}S_{6,6}\varphi_{12} = 0 = {} & u_1 S_{6,6}c_{23} + u_2(c_2c_8 + c_4c_6) + \varphi_5 c_2 + \varphi_3 c_6(1 + \beta_{18} + \beta_{23}\\ & +\beta_{48}) + u_3(c_4^2(\beta_{12} + \beta_{13} + \beta_{37}) + c_2^4(\beta_8 + \beta_{38}))\\ & +u_4c_2^2(1 + \beta_{30} + \beta_{45}) + \varphi_6(\beta_{40} + \beta_{48}) + u_2(S_{6,6}c_{22}\\ & +(1 + \beta_{29} + \beta_{43})c_{10} + (1 + \beta_{33} + \beta_{44})c_5^2\\ & +(1 + \beta_{31} + \beta_{45})c_2^2c_6).\end{aligned}$$

Hence,

$$\begin{aligned}S_{6,6}c_{23} = c_2c_9 + c_5c_6,\ & S_{6,6}c_{22} = (1 + \beta_{29} + \beta_{43})c_{10} + (1 + \beta_{33} + \beta_{44})c_5^2\\ & +(1 + \beta_{31} + \beta_{45})c_2^2c_6,\ \ \beta_{40} = \beta_{48},\ \ \beta_{18} = \beta_{48} + \beta_{23},\ \ \beta_{30} = \beta_{45},\\ & \beta_{12} + \beta_{13} = \beta_{37},\ \ \beta_8 = \beta_{38}.\end{aligned} \tag{2.10}$$

Using theses relations and the formula (2.9) we have $\beta_{23} = 0$.

Let us apply the operation $S_{4,4,4}$, then we have:

$$\begin{aligned}S_{4,4,4}\varphi_{12} = 0 = {} & u_1 S_{4,4,4}c_{23} + u_3(c_4^2\beta_{14} + c_2^4(\beta_9 + \beta_{32})) + u_4c_2^2\beta_{42}\\ & +\varphi_3c_6(\beta_{18} + \beta_{37} + \beta_{48}) + \varphi_6(1 + \beta_{47}) + u_2(S_{4,4,4}c_{22}\\ & +(1 + \beta_{24} + \beta_{43})c_{10} + (\beta_{25} + \beta_{44})c_5^2 + (\beta_{21} + \beta_{45})c_2^2c_6).\end{aligned}$$

Hence,

$$\begin{aligned}& S_{4,4,4}c_{23}=0,\ \ S_{4,4,4}c_{22}=(1 + \beta_{24} + \beta_{43})c_{10} + (\beta_{25} + \beta_{44})c_5^2+(\beta_{21} + \beta_{45})c_2^2c_6,\\ & \beta_{47} = 1,\ \ \beta_{14} = 0,\ \ \beta_{42} = 0,\ \ \beta_9 = \beta_{32},\ \ \beta_{18} = \beta_{48} + \beta_{37}.\end{aligned} \tag{2.11}$$

From these relations and from (2.9) it follows that $\beta_{37} = 0$.

Let us apply the operation $S_{3,3,3,3}$, then it will be:

$$\begin{aligned}S_{3,3,3,3}\varphi_{12} = {} & \varphi_6 = u_1 S_{3,3,3,3}c_{23} + u_2c_{10} + u_3c_8 + u_4c_2 + u_2(c_2c_8\\ & +c_4c_6) + u_4c_2^2 + \varphi_3c_6(1 + \beta_{48}) + \varphi_5c_2 + u_2(S_{3,3,3,3}c_{22}\\ & +c_5^2 + c_2^2c_6(1 + \beta_{33})) + u_3(\beta_2 + \beta_{46})c_2^4.\end{aligned}$$

Hence,

$$\begin{aligned}& S_{3,3,3,3}c_{23} = c_{11} + c_2c_9 + c_5c_6, \quad S_{3,3,3,3}c_{22} = c_5^2 + c_2^2c_6(1 + \beta_{33}),\\ & \beta_{48} = 0,\ \ \beta_2 = \beta_{46} + 1.\end{aligned} \tag{2.12}$$

From here and from (2.9) it follows that $\beta_{18} = 0$, from (2.10) it follows that $\beta_{40} = 0$, from (2.8) it follows that $\beta_{12} = 0$, from (2.10) if follows that $\beta_{13} = 0$.

Applying the operation S_{10}, we get:

$$\begin{aligned}S_{10}\varphi_{12} = \varphi_7 \ = {} & u_1 S_{10}c_{23} + \varphi_3c_8 + u_4c_2c_4 + \varphi_3(c_2^4 + c_4^2) + u_3(c_{10} + c_5^2\\ & +c_2^2c_6) + u_4c_6 + u_2(S_{10}c_{22} + \beta_{41}c_2^6) + u_3(c_{10}\beta_{43} + c_5^2(\beta_3 + \beta_{44})\end{aligned}$$

$$+c_2^2c_6(\beta_7+\beta_{45}))+\varphi_3(c_4^2\beta_{24}+c_2^4(\beta_{19}+\beta_{46}))+u_4c_6\beta_{29}$$
$$+\varphi_5c_2^2\beta_{35}+\varphi_7\beta_{43}.$$

Hence,

$$S_{10}c_{23}=c_{13}+c_5c_8,\ S_{10}c_{22}=c_{12}+\beta_{41}c_2^6,\ \beta_{43}=0,\ \beta_{24}=0,$$
$$\beta_{29}=0,\ \beta_{35}=0,\ \beta_3=\beta_{44},\ \beta_7=\beta_{45},\ \beta_{19}=\beta_{46}. \tag{2.13}$$

From these relations and from the formula (2.5), we have: $\beta_1=0$. If we apply the operation $S_{5,5}$, we get:

$$S_{5,5}\varphi_{12}=\varphi_7\ =u_1S_{5,5}c_{23}+u_4c_2c_4+\varphi_3c_8+u_3(c_{10}+c_5^2+c_2^2c_6)+u_4c_6$$
$$+\varphi_3(c_4^2+c_2^4)+u_2(S_{5,5}c_{22}+\beta_{41}c_2^6)+u_3(c_{10}\beta_3+c_5^2\beta_3$$
$$+c_2^2c_6(\beta_7+\beta_{10}))+\varphi_3(c_4^2\beta_{25}+c_2^4(\beta_{19}+\beta_{20}+\beta_{46}))$$
$$+u_4c_6\beta_{33}+\varphi_5c_2^2\beta_{36}+\varphi_7\beta_{44}.$$

So,

$$S_{5,5}c_{23}\ =c_{13}+c_5c_8, S_{5,5}c_{22}=c_{12}+c_2^6\beta_{41},\ \beta_3=0,\ \beta_{25}=0,\ \beta_{33}=0,$$
$$\beta_{36}=0, \beta_{44}=0, \beta_7=\beta_{10}, \beta_{19}=\beta_{20}+\beta_{46}. \tag{2.14}$$

From these relations and from (2.13) it follows that $\beta_{20}=0$, from (2.6) it follows that $\beta_2=1$, from (2.12) it follows that $\beta_{46}=0$, from (2.13) it follows that $\beta_{19}=0$, from (2.9) it follows that $\beta_9=0$, from (2.11) it follows that $\beta_{32}=0$.

Let us apply the operation S_8, this leads to the relation:

$$S_8\varphi_{12}=u_5=u_1S_8c_{23}+u_5+u_2(S_8c_{22}+\beta_{38}c_2^4c_6)+u_3\beta_{41}c_2^6+\varphi_3c_2^2c_6\beta_{45}.$$

This gives us: $S_8c_{23}=0$, $S_8c_{22}=\beta_{38}c_2^4c_6$, $\beta_{41}=0$, $\beta_{45}=0$. From here and from (2.13) it follows that $\beta_7=0$, from (2.10) it follows that $\beta_{30}=0$, from (2.14) it follows that $\beta_{10}=0$.

If we use the operation $S_{4,4}$, we get:

$$S_{4,4}\varphi_{12}=0=u_1S_{4,4}c_{23}+u_3(c_2c_{10}+c_4c_8+c_4c_2^4)+u_4c_4^2+\varphi_3c_{10}$$
$$+\varphi_6c_4+u_2(S_{4,4}c_{22}+\beta_{38}c_2^4c_6)+u_3c_2^6\beta_{11}+\varphi_3c_2^2c_6\beta_{21}.$$

Hence, $S_{4,4}c_{23}=c_4c_{11}+c_5c_{10}$, $S_{4,4}c_{22}=\beta_{38}c_2^4c_6$, $\beta_{11}=0$, $\beta_{21}=0$.

Using the operation S_6, we get:

$$S_6\varphi_{12}=\varphi_9=u_1S_6c_{23}+u_2c_{16}+u_3c_{14}+u_4(c_{10}+c_5^2+c_2^2c_6)+\varphi_5c_4^2$$
$$+u_5c_2+u_4(c_4c_6+c_2c_8)+\varphi_5c_8+u_3c_2^4c_6(1+\beta_{38})+\varphi_3c_2^6\beta_{22}$$
$$+\varphi_5c_2^4\beta_{38}+u_2(S_6c_{22}+\beta_{28}c_2^8).$$

Hence, $S_6c_{23}=c_{17}+c_9c_8+c_6c_{11}$, $S_6c_{22}=c_8^2+c_6c_{10}+\beta_{28}c_2^8$, $\beta_{22}=0$, $\beta_{38}=0$. From here and from (2.10) it follows that $\beta_8=0$.

Finally we use the operation S_4.

$$S_4\varphi_{12} = \varphi_{10} = u_1 S_4 c_{23} + u_3 c_{16} + u_4(c_{12} + c_2 c_{10} + c_4 c_8 + c_4 c_2^4) + \varphi_6 c_4^2 \\ +\varphi_6 c_8 + u_5 c_4 + u_2 S_4 c_{22} + u_3(c_2^4 c_4^2 + c_4^4) + u_2 c_{18} + u_3 c_2^8 \beta_{28}.$$

Hence, $\beta_{28} = 1$, $S_4 c_{23} = c_{19} + c_8 c_{11} + c_2^4 c_{11} + c_9 c_{10}$, $S_4 c_{22} = c_2^4 c_{10}$.

Finally for the determination of the coefficient β_6, we apply the operation $S_{2,2,2,2,2,2,2,2,2,2}$:

$$S_{2,2,2,2,2,2,2,2,2,2}\varphi_{12} = 0 = u_3\beta_6.$$

So, $\beta_6 = 0$, and the projection of the element Φ_{12} has the form:

$$\varphi_{12} = u_1 c_{23} + u_4 c_{16} + u_5 c_8 + u_2 c_{22} + u_3(c_{20} + c_5^4) + \varphi_3 c_{18} + u_4 c_2^8 + u_5 c_4^2.$$

Using this decomposition let us calculate the action of the operation S_ω on the elements c_{23}, c_{22}.

9. Let us precise the form of the projection in MASS of the element Φ_{13}:

$$\varphi_{13} = u_1 c_{25} + u_2 c_8 c_{16} + u_4 c_2 c_{16} + u_5 c_2 c_{16} + u_2 c_{24} + u_3(c_{22} + \beta_1 c_{11}^2 \\ +\beta_2 c_2^2 c_{18} + \beta_3 c_2^2 c_9^2 + \beta_4 c_2^4 c_{14} + \beta_5 c_2^2 c_6 c_{12} + \beta_6 c_2^6 c_{10} + \beta_7 c_2^6 c_5^2 \\ +\beta_8 c_2^4 c_4^2 c_6 + \beta_9 c_2^8 c_6 + \beta_{10} c_2^2 c_6^3 + \beta_{11} c_2^2 c_4^2 c_{10} + \beta_{12} c_2^2 c_5^2 c_4^2 + \beta_{13} c_6 c_8^2 \\ +\beta_{14} c_6 c_4^4 + \beta_{15} c_6^2 c_{10} + \beta_{16} c_6^2 c_5^2 + \beta_{17} c_4^2 c_{14} + \beta_{18} c_{10} c_{12} + \beta_{19} c_5^2 c_{12}) \\ +\varphi_3(\beta_{20} c_{20} + \beta_{21} c_{10}^2 + \beta_{22} c_2^2 c_8^2 + \beta_{23} c_2^2 c_4^4 + \beta_{24} c_2^{10} + \beta_{25} c_2^2 c_6 c_{10} \\ +\beta_{26} c_2^4 c_6^2 + \beta_{27} c_2^4 c_{12} + \beta_{28} c_2^2 c_5^2 c_6 + \beta_{29} c_2^6 c_4^2 + \beta_{30} c_6 c_{14} + \beta_{31} c_4^2 c_6^2 \\ +\beta_{32} c_4^2 c_{12} + \beta_{33} c_5^2 c_{10} + \beta_{34} c_5^4) + u_4(\beta_{35} c_{18} + \beta_{36} c_9^2 + \beta_{37} c_2^2 c_{14} \\ +\beta_{38} c_6 c_{12} + \beta_{39} c_2^4 c_{10} + \beta_{40} c_2^4 c_5^2 + \beta_{41} c_2^2 c_4^2 c_6 + \beta_{42} c_2^6 c_6 + \beta_{43} c_6^3 \\ +\beta_{44} c_4^2 c_{10} + \beta_{45} c_4^2 c_5^2) + \varphi_5(\beta_{46} c_8^2 + \beta_{47} c_4^4 + \beta_{48} c_2^8 + \beta_{49} c_6 c_{10} \\ +\beta_{50} c_2^2 c_6^2 + \beta_{51} c_2^2 c_{12} + \beta_{52} c_5^2 c_6 + \beta_{53} c_2^4 c_4^2) + \varphi_6(\beta_{54} c_{14} + \beta_{55} c_2^2 c_{10} \\ +\beta_{56} c_2^2 c_5^2 + \beta_{57} c_4^2 c_6 + \beta_{58} c_2^4 c_6) + \varphi_7(\beta_{59} c_{12} + \beta_{60} c_6^2 + \beta_{61} c_2^2 c_4^2 \\ +\beta_{62} c_2^6) + u_5(\beta_{63} c_{10} + \beta_{64} c_5^2 + \beta_{65} c_2^2 c_6) + \varphi_9(\beta_{66} c_4^2 + \beta_{67} c_2^4) \\ +\varphi_{10}\beta_{68} c_6 + \varphi_{11}\beta_{69} c_2^2.$$

From the action of the operation $S_{11,11}$:

$$S_{11,11}\varphi_{13} = 0 = u_3(\beta_1 + 1)$$

we conclude:

$$\beta_1 = 1.$$

From the action of the operation S_{20}, we have:

$$S_{20}\varphi_{13} = \varphi_3 = u_1 S_{20}c_{25} + u_2 S_{20}c_{24} + \varphi_3\beta_{20} + u_2\beta_{69}c_2^2,$$

Hence, $\beta_{20} = 1$, $S_{20}c_{25} = 0$, $S_{20}c_{24} = \beta_{69}c_2^2$.

Applying the operation $S_{10,10}$, we get:

$$S_{10,10}\varphi_{13} = \varphi_3 = u_1 S_{10,10}c_{25} + u_2 S_{10,10}c_{24} + (\beta_{21} + \beta_{63})\varphi_3 + u_2c_2^2(\beta_{54} + \beta_{55} + \beta_{69}).$$

Hence,

$$S_{10,10}c_{25} = 0,\ \ S_{10,10}c_{24} = c_2^2(\beta_{54} + \beta_{55} + \beta_{69}),\ \ \beta_{21} + \beta_{63} = 1. \tag{2.15}$$

If we apply the operation S_{18}, we get:

$$S_{18}\varphi_{13} = u_4 = u_1 S_{18}c_{25} + u_2(S_{18}c_{24} + \beta_{68}c_6) + u_3c_2^2(\beta_2 + \beta_{69}) + u_4\beta_{35}.$$

Hence,

$$\beta_{35} = 1,\ \ S_{18}c_{25} = 0,\ \ S_{18}c_{24} = \beta_{68}c_6,\ \ \beta_2 = \beta_{69}. \tag{2.16}$$

From the action $S_{9,9}$, we conclude:

$$S_{9,9}\varphi_{13} = 0 = u_1 S_{9,9}c_{25} + u_2(S_{9,9}c_{24} + \beta_{68}c_6) + u_3c_2^2(\beta_2 + \beta_3) + u_4(\beta_{36} + 1),$$

hence,

$$S_{9,9}c_{25} = 0,\quad S_{9,9}c_{24} = \beta_{68}c_6,\quad \beta_{36} = 1,\quad \beta_2 = \beta_3. \tag{2.17}$$

Using the operation $S_{6,6,6}$, we get the equality:

$$S_{6,6,6}\varphi_{13} = u_4 = u_1 S_{6,6,6}c_{25} + u_2(S_{6,6,6}c_{24} + (1 + \beta_{38} + \beta_{59} + \beta_{43})c_6) + u_3c_2^2(1 + \beta_{10} + \beta_{13} + \beta_{46} + \beta_{50} + \beta_{69}) + u_4(\beta_{43} + \beta_{60}).$$

So,

$$\begin{gathered} S_{6,6,6}c_{25} = 0,\ \ S_{6,6,6}c_{24} = (1 + \beta_{38} + \beta_{53} + \beta_{43})c_6,\\ \beta_{43} + \beta_{60} = 1,\ \ \beta_{10} + \beta_{13} + \beta_{46} + \beta_{50} + \beta_{69} = 1. \end{gathered} \tag{2.18}$$

If we apply the operation S_{16}, then we get:

$$S_{16}\varphi_{13} = \varphi_5 = u_1 S_{16}c_{25} + u_2(S_{16}c_{24} + \beta_{66}c_4^2 + \beta_{67}c_2^4)) + u_2c_8 + u_4c_2 + u_3\beta_{68}c_6 + \varphi_3c_2^2\beta_{69},$$

So, it will be: $S_{16}c_{25} = c_9$, $S_{16}c_{24} = \beta_{66}c_4^2 + \beta_{67}c_2^4$, $\beta_{68} = 1$, $\beta_{69} = 0$. From these relations and from (2.16) and (2.17) we get that $\beta_2, \beta_3 = 0$.

Let us act by the operation $S_{8,8}$, we get:

$$S_{8,8}\varphi_{13} = \varphi_5 = u_1 S_{8,8}c_{25} + u_2(S_{8,8}c_{24} + (1+\beta_{66})c_4^2 + \beta_{67}c_2^4) + u_3\beta_{13}c_6 \\ +\varphi_3\beta_{22}c_2^2 + \varphi_5\beta_{46}.$$

Hence, $S_{8,8}c_{25} = 0$, $S_{8,8}c_{24} = (1+\beta_{66})c_4^2 + \beta_{67}c_2^4$, $\beta_{13} = 0$, $\beta_{22} = 0$, $\beta_{46} = 1$.

Let us apply the operation S_{14}, then it will be:

$$S_{14}\varphi_{13} = \varphi_6 = u_1 S_{14}c_{25} + u_2(S_{14}c_{24} + \beta_{63}c_{10} + \beta_{64}c_5^2 + \beta_{65}c_2^2c_6) \\ +u_3(c_4^2(1+\beta_{17}+\beta_{66}) + c_2^4(\beta_4+\beta_{67})) + \varphi_3\beta_{30}c_6 \\ +u_4c_2^2(\beta_{37}+1) + \varphi_6\beta_{54}.$$

From these relations we obtain:

$$S_{14}c_{25} = 0,\ S_{14}c_{24} = \beta_{63}c_{10} + \beta_{64}c_5^2 + \beta_{65}c_2^2c_6, \\ \beta_{30} = 0,\ \beta_{37} = 1,\ \beta_{54} = 1,\ \beta_{17}+\beta_{66} = 1,\ \beta_4 = \beta_{67}. \tag{2.19}$$

Applying the operation S_{12}, we get the equality:

$$S_{12}\varphi_{13} = \varphi_7 = u_1 S_{12}c_{25} + u_2c_4c_8 + u_3c_2c_8 + u_4c_2c_4 + \varphi_3(c_4^2 + c_2^4) + u_4c_6 \\ +u_2(S_{12}c_{24} + \beta_{59}c_{12} + \beta_{60}c_6^2 + \beta_{61}c_2^2c_4^2 + \beta_{62}c_2^6) \\ +u_3(c_{10}(1+\beta_{18}+\beta_{63}) + c_5^2(\beta_{19}+\beta_{64}) + c_2^2c_6(\beta_5+\beta_{65})) \\ +\varphi_3(c_2^4(\beta_{27}+\beta_{67}) + c_4^2(\beta_{32}+\beta_{66})) + u_4c_6\beta_{38} \\ +\varphi_5\beta_{51}c_2^2 + \varphi_7\beta_{59}.$$

This gives the result:

$$S_{12}c_{25} = c_{13},\ S_{12}c_{24} = (\beta_{59}+1)c_{12} + \beta_{60}c_6^2 + \beta_{61}c_2^2c_4^2 + \beta_{62}c_2^6, \\ \beta_{59} = 0,\ \beta_{51} = 0,\ \beta_{38} = 0,\ \beta_{19}+\beta_{64} = 1,\ \beta_5+\beta_{65} = 1, \beta_{18} = \beta_{63}, \\ \beta_{27} = \beta_{67},\ \beta_{32} = \beta_{66}. \tag{2.20}$$

Applying the operation $S_{6,6}$ we obtain:

$$S_{6,6}\varphi_{13} = \varphi_7 = u_1 S_{6,6}c_{25} + u_2c_2^2c_8 + u_3c_2^2c_6 + u_4c_2^3 + \varphi_5c_2^2(\beta_{50}+\beta_{65}) \\ +\varphi_7 + u_2(S_{6,6}c_{24} + c_2^2c_4^2(1+\beta_{41}+\beta_{61}) + c_2^6(\beta_{42}+\beta_{62}) \\ +c_6^2(\beta_{43}+\beta_{60})) + u_3(c_{10}(1+\beta_{15}+\beta_{49}) + c_5^2(\beta_{16} \\ +\beta_{52}) + c_2^2c_6(\beta_5+\beta_{10})) + \varphi_3(c_4^2(1+\beta_{31}+\beta_{57}+\beta_{66}) \\ +c_2^4(\beta_{26}+\beta_{58}+\beta_{67})) + u_4c_6(1+\beta_{43}) + \varphi_7\beta_{60}.$$

We have:

$$S_{6,6}c_{25} = c_9c_2^2,\ S_{6,6}c_{24} = c_2^2c_4^2(1+\beta_{41}+\beta_{61}) + c_2^6(\beta_{42}+\beta_{62}) \\ +c_6^2(\beta_{43}+\beta_{60}),\ \beta_{60} = 0,\ \beta_{43} = 1,\ \beta_{15}+\beta_{49} = 1,\ \beta_{16} = \beta_{52}, \\ \beta_5 = \beta_{10},\ \beta_{31}+\beta_{57}+\beta_{66} = 1,\ \beta_{26}+\beta_{58}+\beta_{67} = 0. \tag{2.21}$$

Let us calculate the result of the action of the operation S_{10} on φ_{13}, we have:

$$\begin{aligned}S_{10}\varphi_{13} = u_5 = {} & u_1 S_{10}c_{25} + u_2 c_8(c_2c_4 + c_6) + u_4 c_2(c_2c_4 + c_6) + \varphi_3 c_2 c_8 \\ & +\varphi_5 c_6 + \varphi_6 c_2^2 + u_3 c_6^2\beta_{15} + u_2(S_{10}c_{24} + c_{14} + c_2^2c_{10}\beta_{55} + c_2^2c_5^2\beta_{56} \\ & +c_4^2c_6\beta_{57} + c_2^4c_6\beta_{58}) + u_3(c_{12}(1+\beta_{18}) + c_2^6(\beta_4+\beta_6+\beta_{62}) \\ & +c_2^2c_4^2(\beta_{11}+\beta_{17}+\beta_{61})) + \varphi_3(c_{10}(1+\beta_{63}) + c_5^2(1+\beta_{33}+\beta_{64}) \\ & +c_2^2c_6(1+\beta_{25}+\beta_{65})) + u_4(c_2^4(\beta_{39}+\beta_{67}) + c_4^2(\beta_{44}+\beta_{66})) \\ & +\varphi_5c_6\beta_{49} + \varphi_6c_2^2\beta_{55} + u_5\beta_{63}.\end{aligned}$$

It follows from these relations that

$$\begin{aligned}& S_{10}c_{24} = c_{14} + c_2^2c_{10} + c_2^2c_5^2\beta_{56} + c_4^2c_6\beta_{57} + c_2^4c_6\beta_{58}, \\ & S_{10}c_{25} = c_2^2c_{11} + c_2c_5c_8 + c_6c_9,\ \beta_4+\beta_6+\beta_{62} = 1,\ \beta_{11}+\beta_{17}+\beta_{61} = 0, \\ & \beta_{44} = \beta_{66},\ \beta_{63} = 1,\ \beta_{15} = 1,\ \beta_{18} = 1,\ \beta_{55} = 0,\ \beta_{49} = 0,\ \beta_{39} = \beta_{67}, \\ & \beta_{33}+\beta_{64} = 1,\ \beta_{25}+\beta_{65} = 1. \qquad (2.22)\end{aligned}$$

From these relations and from (2.15) it follows that $\beta_{21} = 0$. Let us use now the operation $S_{5,5}$, we have:

$$\begin{aligned}S_{5,5}\varphi_{13} = 0 = {} & u_1S_{5,5}c_{25} + u_2c_8(c_2c_4+c_6) + u_4c_2(c_2c_4+c_6) + \varphi_3c_2c_8 \\ & +\varphi_5c_6 + \varphi_6c_2^2 + u_3c_6^2 + u_2(S_{5,5}c_{24} + c_{14} + c_2^2c_5^2\beta_{56} + c_4^2c_6\beta_{57} \\ & +c_2^4c_6\beta_{58}) + u_3(c_2^6(\beta_4+\beta_6+\beta_7) + c_2^2c_4^2(\beta_{11}+\beta_{12}+\beta_{17}) \\ & +c_6^2\beta_{16} + c_{12}\beta_{19}) + \varphi_3(c_5^2(1+\beta_{33}+\beta_{64}) + c_2^2c_6(1+\beta_{25}+\beta_{28} \\ & +\beta_{65})) + u_4(c_2^4(\beta_{39}+\beta_{40}) + c_4^2(\beta_{44}+\beta_{45})) + \varphi_5c_6\beta_{52} \\ & +\varphi_6c_2^2\beta_{56} + u_5(1+\beta_{64}).\end{aligned}$$

This means that:

$$\begin{aligned}& S_{5,5}c_{24} = c_{14} + c_2^2c_{10} + c_4^2c_6\beta_{57} + c_2^4c_6\beta_{58},\ S_{5,5}c_{25} = c_2c_5c_8 + c_2^2c_{11} + c_6c_9, \\ & \beta_4+\beta_6+\beta_7 = 1,\ \beta_{11}+\beta_{12}+\beta_{17} = 0,\ \beta_{25}+\beta_{28}+\beta_{65} = 1,\ \beta_{16} = 0, \\ & \beta_{19} = 0,\ \beta_{52} = 0,\ \beta_{64} = 1,\ \beta_{33} = 0,\ \beta_{56} = 0,\ \beta_{39} = \beta_{40}, \beta_{44} = \beta_{45}. \qquad (2.23)\end{aligned}$$

From these relations and from (2.22) it follows that $\beta_{28} = 0$. Let us act on φ_{13} by the operation S_8:

$$\begin{aligned}S_8\varphi_{13} = \varphi_9 = {} & u_1S_8c_{25} + u_2c_{16} + u_3c_{14} + u_4(c_{10} + c_5^2 + c_2^2c_6\beta_{65}) + \varphi_6c_6 \\ & +u_5c_2 + c_4^2(u_2c_8 + u_3c_6\beta_{57} + u_4c_2) + u_3c_2^4c_6\beta_{58} \\ & +u_2(S_8c_{24} + \beta_{47}c_4^4 + \beta_{48}c_2^8 + \beta_{50}c_2^2c_6^2 + \beta_{53}c_2^4c_4^2) \\ & +\varphi_3(c_2^6\beta_{62} + c_2^2c_4^2\beta_{61}) + \varphi_5(c_4^2\beta_{66} + c_2^4\beta_{67}).\end{aligned}$$

This means that

$$S_8c_{25} = c_{17} + c_9c_4^2,\ \ S_8c_{24} = \beta_{47}c_4^4 + \beta_{48}c_2^8 + \beta_{50}c_2^2c_6^2 + \beta_{53}c_2^4c_4^2.$$
$$\beta_{65} = 1, \beta_{57} = 1, \beta_{58} = 0, \beta_{61} = 0,\ \ \beta_{62} = 0, \beta_{66} = 0, \beta_{67} = 0,$$

From these relations and from (2.19) it follows that $\beta_{17} = 1$, $\beta_4 = 0$; from (2.20) it follows that $\beta_5 = 0$, $\beta_{27} = 0$, $\beta_{32} = 0$; from (2.21) it follows that $\beta_{10} = 0$, $\beta_{31} = 0$, $\beta_{26} = 0$; from (2.22) it follows that $\beta_6 = 1$, $\beta_{11} = 1$, $\beta_{25} = 0$, $\beta_{39} = 0$, $\beta_{44} = 0$; from (2.23) it follows $\beta_7 = 0$, $\beta_{12} = 0$, $\beta_{40} = 0$, $\beta_{45} = 0$; from (2.18) it follows that $\beta_{50} = 0$.

Applying the operation S_6, we obtain the equality:

$$\begin{aligned} S_6\varphi_{13} = \varphi_{10} = {} & u_1S_6c_{25} + u_2c_8(c_{10} + c_5^2 + c_2^2c_6 + c_4c_6 + c_2c_8) + u_4c_2(c_{10} \\ & + c_5^2 + c_2^2c_6 + c_4c_6 + c_2c_8) + \varphi_5c_2c_8 + \varphi_5(c_{10} + c_5^2 + c_2^2c_6) + \varphi_7c_6 \\ & + \varphi_3(c_4^2 + c_2^4)c_6 + u_4c_6^2 + u_2(S_6c_{24} + c_{18} + c_9^2 + c_2^2c_{14} + \beta_{41}c_2^2c_4^2c_6 \\ & + \beta_{42}c_2^6c_6 + c_6^3) + u_3(c_2^8(1 + \beta_9 + \beta_{48}) + c_2^4c_4^2(\beta_8 + \beta_{53}) \\ & + c_4^4(1 + \beta_{14} + \beta_{47})) + u_4(c_2^2c_4^2(1 + \beta_{41}) + c_2^6\beta_{42}) + \varphi_{10}. \end{aligned}$$

From these relations we obtain:

$$\begin{aligned} & S_6c_{25} = c_9(c_2c_8 + c_{10} + c_5^2 + c_2^2c_6) + c_6c_{13}, \\ & S_6c_{24} = c_{18} + c_9^2 + c_2^2c_{14} + c_6c_{12} + c_2^2c_4^2c_6, \\ & \beta_9 + \beta_{48} = 1,\ \ \beta_8 = \beta_{53},\ \ \beta_{14} + \beta_{47} = 1, \\ & \beta_{41} = 1,\ \ \beta_{42} = 0. \end{aligned} \tag{2.24}$$

Developing the relation $S_4\varphi_{13} = \varphi_{11}$, we get after cancellation:

$$\begin{aligned} & u_1S_4c_{25} + u_2c_8(c_{12} + c_4c_8 + c_4c_2^4) \\ & \quad + (u_1c_{11} + u_3c_8 + u_3c_2^4)c_2c_8 + u_2S_4c_{24} \\ & \quad + \varphi_3c_2^4c_4^2 + u_2(\beta_{23}c_2^2c_4^4 + \beta_{24}c_2^{10} + \beta_{29}c_2^6c_4^2 + \beta_{34}c_5^4) \\ & \quad + \varphi_3(c_8^2 + \beta_{47}c_4^4 + \beta_{48}c_2^8 + \beta_{53}c_2^4c_4^2) + u_4c_2(c_{12} + c_4c_2^4) \\ & = u_1c_{21} + u_3((c_{10} + c_5^2 + c_2^2c_6)c_2^4 + c_6c_{12}) + u_4c_2^4c_6 + \varphi_5c_{12} + \varphi_7c_2^4. \end{aligned}$$

After regrouping we have:

$$\begin{aligned} & u_1(S_4c_{25} + c_{21} + c_2c_8c_{11}) + u_2c_4c_8^2 \\ & \quad + u_3c_8^2c_2 + \varphi_3c_8^2 + u_2c_8c_{12} + u_3c_6c_{12} + u_4c_2c_{12} \\ & \quad + \varphi_5c_{12} + u_2c_8c_2^4 + u_3c_2c_8c_2^4 + u_4c_2c_4c_2^4 + u_3(c_{10} + c_5^2 + c_2^2c_6)c_2^4 \\ & \quad + \varphi_3(c_4^2 + c_2^4\beta_{48})c_2^4 + u_4c_6c_2^4 + \varphi_7c_2^4 + u_2(S_4c_{24} + \beta_{23}c_2^2c_4^4 + \beta_{24}c_2^{10} \\ & \quad + \beta_{29}c_2^6c_4^2 + \beta_{34}c_5^4) + \varphi_3(\beta_{47}c_4^4 + \beta_{53}c_2^4c_4^2) = 0. \end{aligned}$$

Hence,

$$\begin{aligned}
&S_4c_{24} = \beta_{23}c_2^2c_4^4 + \beta_{24}c_2^{10} + \beta_{29}c_2^6c_4^2 + \beta_{34}c_5^4 + c_2^4c_{12},\\
&S_4c_{25} = c_{21} + c_2c_8c_{11} + c_5c_8^2 + c_9c_{12} + c_{11}c_2^4,\\
&\beta_{47} = 0,\ \beta_{53} = 0,\ \beta_{48} = 1.
\end{aligned}$$

From these relations and from (2.24), it follows that $\beta_9 = 0$, $\beta_8 = 0$, $\beta_{14} = 1$.

Using the relation $S_2\varphi_{13} = \varphi_{12}$, we get after the cancellation:

$$\begin{aligned}
&u_1(S_2c_{25} + c_{23} + c_2c_5c_8(c_4^2 + c_2^4)\\
&\quad + c_2c_5c_{16} + c_2c_8c_{13} + c_{13}(c_{10} + c_5^2 + c_2^2c_6)\\
&\quad + c_9c_{14} + c_6c_{17} + (c_{19} + c_9c_{10} + c_{11}(c_4^2 + c_2^4))c_2^2) + u_2(S_2c_{24} + c_6c_8^2 + c_4^2c_{14}\\
&\quad + c_{11}^2 + c_6c_4^4 + c_6^2c_{10} + c_{12}(c_5^2 + c_2^2c_6) + c_2^2c_{18}) + u_3(c_5^4(1 + \beta_{34}) + \beta_{24}c_2^{10}\\
&\quad + \beta_{23}c_2^2c_4^4 + \beta_{29}c_2^6c_4^2) = 0.
\end{aligned}$$

Hence,

$$\begin{aligned}
&S_2c_{24} = c_6c_8^2 + c_{11}^2 + c_6c_4^4 + c_6^2c_{10} + c_{12}(c_5^2 + c_2^2c_6) + c_2^2c_{18} + c_4^2c_{14},\\
&S_2c_{25} = c_{23} + c_2c_5c_8(c_4^2 + c_2^4) + c_2c_5c_{16} + c_2c_8c_{13} + c_{13}(c_{10} + c_5^2 + c_2^2c_6)\\
&\qquad\qquad + c_9c_{14} + c_6c_{17} + (c_{19} + c_9c_{10} + c_{11}(c_4^2 + c_2^4))c_2^2,\\
&\beta_{34} = 1,\ \beta_{23} = 0,\ \beta_{24} = 0,\ \beta_{29} = 0.
\end{aligned}$$

The final form of the projection of Φ_{13} is the following:

$$\begin{aligned}
\varphi_{13} &= u_1c_{25} + u_2c_8c_{16} + u_4c_2c_{16} + u_5c_2c_8 + u_2c_{24} + u_3(c_{22} + c_{11}^2 + c_2^6c_{10}\\
&\quad + c_2^2c_4^2c_{10} + c_6c_4^4 + c_6^2c_{10} + c_4^2c_{14} + c_{10}c_{12}) + \varphi_3(c_{20} + c_5^4) + u_4(c_{18}\\
&\quad + c_9^2 + c_2^2c_{14} + c_2^2c_4^2c_6 + c_6^3) + \varphi_5(c_8^2 + c_2^8) + \varphi_6(c_{14} + c_4^2c_6) + u_5(c_{10}\\
&\quad + c_5^2 + c_2^2c_6) + \varphi_{10}c_6.
\end{aligned}$$

Using this decomposition let us calculate the result of action of the operations S_ω on the elements c_{25} and c_{24}.

10. The projection of the element Φ_{14} has the following form:

$$\begin{aligned}
\varphi_{14} &= u_1c_{27} + u_3c_8c_{16} + u_4c_4c_{16} + u_5c_4c_8 + u_2c_{26} + u_3(c_{24} + \beta_1c_{12}^2\\
&\quad + \beta_2c_5^2c_{14} + \beta_3c_{10}c_{14} + \beta_4c_4^6 + \beta_5c_4^2c_8^2 + \beta_6c_4^2c_5^2c_6 + \beta_7c_4^2c_6c_{10} + \beta_8c_6^4\\
&\quad + \beta_9c_6^2c_{12} + \beta_{10}c_6c_9^2 + \beta_{11}c_6c_{18} + \beta_{12}c_2^2c_{20} + \beta_{13}c_2^2c_{10}^2 + \beta_{14}c_2^2c_5^2c_{10}\\
&\quad + \beta_{15}c_2^2c_5^4 + \beta_{16}c_2^4c_8^2 + \beta_{17}c_2^4c_4^4 + \beta_{18}c_2^6c_6^2 + \beta_{19}c_2^4c_6c_{10} + \beta_{20}c_2^6c_{12}\\
&\quad + \beta_{21}c_2^8c_4^2 + \beta_{22}c_2^4c_5^2c_6 + \beta_{23}c_2^2c_6c_{14} + \beta_{24}c_2^2c_4^2c_6^2 + \beta_{25}c_2^2c_4^2c_{12}\\
&\quad + \beta_{26}c_2^{12}) + \varphi_3(\beta_{27}c_{22} + \beta_{28}c_{11}^2 + \beta_{29}c_5^2c_{12} + \beta_{30}c_{10}c_{12} + \beta_{31}c_4^2c_{14}
\end{aligned}$$

$$+\beta_{32}c_5^2c_6^2 + \beta_{33}c_6^2c_{10} + \beta_{34}c_6c_4^4 + \beta_{35}c_6c_8^2 + \beta_{36}c_2^2c_{18} + \beta_{37}c_2^2c_9^2$$
$$+\beta_{38}c_2^4c_{14} + \beta_{39}c_2^2c_6c_{12} + \beta_{40}c_2^6c_{10} + \beta_{41}c_2^6c_5^2 + \beta_{42}c_2^4c_4^2c_6 + \beta_{43}c_2^8c_6$$
$$+\beta_{44}c_2^2c_6^3 + \beta_{45}c_2^2c_4^2c_{10} + \beta_{46}c_2^2c_4^2c_5^2) + u_4(\beta_{47}c_{20} + \beta_{48}c_{10}^2 + \beta_{49}c_5^2c_{10}$$
$$+\beta_{50}c_5^4 + \beta_{51}c_4^2c_{12} + \beta_{52}c_4^2c_6^2 + \beta_{53}c_6c_{14} + \beta_{54}c_2^2c_8^2 + \beta_{55}c_2^2c_4^4 + \beta_{56}c_2^{10}$$
$$+\beta_{57}c_2^2c_6c_{10} + \beta_{58}c_2^4c_6^2 + \beta_{59}c_2^4c_{12} + \beta_{60}c_2^2c_5^2c_6 + \beta_{61}c_2^6c_4^2) + \varphi_5(\beta_{62}c_{18}$$
$$+\beta_{63}c_9^2 + \beta_{64}c_2^2c_{14} + \beta_{65}c_6c_{12} + \beta_{66}c_2^4c_{10} + \beta_{67}c_2^4c_5^2 + \beta_{68}c_2^2c_4^2c_6$$
$$+\beta_{69}c_2^6c_6 + \beta_{70}c_6^3 + \beta_{71}c_4^2c_{10} + \beta_{72}c_4^2c_5^2) + \varphi_6(\beta_{73}c_8^2 + \beta_{74}c_4^4 + \beta_{75}c_2^8$$
$$+\beta_{76}c_6c_{10} + \beta_{77}c_2^2c_6^2 + \beta_{78}c_2^2c_{12} + \beta_{79}c_5^2c_6 + \beta_{80}c_2^4c_4^2) + \varphi_7(\beta_{81}c_{14}$$
$$+\beta_{82}c_2^2c_{10} + \beta_{83}c_2^2c_5^2 + \beta_{84}c_4^2c_6 + \beta_{85}c_2^4c_6) + u_5(\beta_{86}c_{12} + \beta_{87}c_6^2$$
$$+\beta_{88}c_2^2c_4^2 + \beta_{89}c_2^6) + \varphi_9(\beta_{90}c_{10} + \beta_{91}c_5^2 + \beta_{92}c_2^2c_6) + \varphi_{10}(\beta_{93}c_4^2$$
$$+\beta_{94}c_2^4) + \varphi_{11}c_6\beta_{95} + \varphi_{12}c_2^2\beta_{96}.$$

Using the operation $S_{12,12}$, we get:

$$S_{12,12}\varphi_{14} = 0 = u_3(1 + \beta_1 + \beta_{86}),$$

so,

$$\beta_1 + \beta_{86} = 1. \tag{2.25}$$

If we use the operation S_{22}, we get:

$$S_{22}\varphi_{14} = \varphi_3 = u_1S_{22}c_{27} + u_2(S_{22}c_{26} + \beta_{96}c_2^2) + \varphi_3\beta_{27}.$$

i.e.

$$\beta_{27} = 1,\ S_{22}c_{27} = 0,\ S_{22}c_{26} = \beta_{96}c_2^2.$$

Using the operation $S_{11,11}$, we get the equality:

$$S_{11,11}\varphi_{14} = \varphi_3 = u_1S_{11,11}c_{27} + u_2(S_{11,11}c_{26} + \beta_{96}c_2^2) + \varphi_3(1 + \beta_{28}),$$

or,

$$\beta_{28} = 0, \quad S_{11,11}c_{27} = 0, \qquad S_{11,11}c_{26} = \beta_{96}c_2^2.$$

Application of the operation S_{20}, $S_{20}\varphi_{14} = u_4$, gives us:

$$u_4 = u_1S_{20}c_{27} + u_2(S_{20}c_{26} + \beta_{95}c_6) + u_3c_2^2(\beta_{12} + \beta_{96}) + u_4\beta_{47},$$

or

$$S_{20}c_{27}=0,\quad S_{20}c_{26}=\beta_{95}c_6,\quad \beta_{47}=1,\quad \beta_{12}=\beta_{96}. \tag{2.26}$$

From the condition $S_{10,10}\varphi_{14}=0$, it follows that

$$\begin{aligned}0=&u_1S_{10,10}c_{27}+u_2(S_{10,10}c_{26}+c_6(\beta_{76}+\beta_{95}))\\&+u_3c_2^2(\beta_{82}+\beta_{13}+\beta_3+\beta_{96}+1)+u_4(\beta_{48}+\beta_{90}).\end{aligned}$$

Hence,

$$\begin{aligned}S_{10,10}c_{27}=0,\ S_{10,10}c_{26}=c_6(\beta_{76}+\beta_{95}),\\ \beta_{48}=\beta_{90},\ \beta_{82}+\beta_{13}+\beta_3+\beta_{96}=1.\end{aligned} \tag{2.27}$$

Applying the operation $S_{6,6,6,6}$ we get:

$$S_{6,6,6,6}\varphi_{14}=u_3=u_1S_{6,6,6,6}c_{27}+u_2S_{6,6,6,6}c_{26}+u_3(\beta_8+\beta_{70}+\beta_{95})$$

So,

$$S_{6,6,6,6}c_{27}=0,\quad S_{6,6,6,6}c_{26}=0,\quad \beta_8+\beta_{70}+\beta_{95}=1. \tag{2.28}$$

From the condition $S_{4,4,4,4,4,4}\varphi_{14}=0$, we get the equality:

$$\beta_1+\beta_4+\beta_5+\beta_{51}+\beta_{93}=0. \tag{2.29}$$

Let us act by the operation $S_{5,5,5,5}$ on the element φ_{14}. Then we have:

$$\begin{aligned}S_{5,5,5,5}\varphi_{14}=0=&u_1S_{5,5,5,5}c_{27}+u_2(S_{5,5,5,5}c_{26}+c_6(\beta_{76}+\beta_{79}))\\&+u_3c_2^2(1+\beta_{13}+\beta_{14}+\beta_{15}+\beta_2+\beta_3+\beta_{96})\\&+u_4(\beta_{48}+\beta_{49}+\beta_{50}).\end{aligned}$$

This gives

$$\begin{aligned}S_{5,5,5,5}c_{27}=0,\ S_{5,5,5,5}c_{26}=c_6(\beta_{76}+\beta_{79}),\ \beta_{48}+\beta_{49}=\beta_{50},\\ \beta_{13}+\beta_{14}+\beta_{15}+\beta_2+\beta_3+\beta_{96}=1.\end{aligned} \tag{2.30}$$

If we use the operation S_{18}we get:

$$\begin{aligned}S_{18}\varphi_{14}=\varphi_5=&u_1S_{18}c_{27}+u_2(S_{18}c_{26}+\beta_{93}c_4^2+\beta_{94}c_2^4)\\&+u_3c_6(1+\beta_{11}+\beta_{95})+\varphi_3c_2^2(\beta_{36}+\beta_{96})+\varphi_5\beta_{62}.\end{aligned}$$

Hence

$$S_{18}c_{27} = 0,\ S_{18}c_{26} = \beta_{93}c_4^2 + \beta_{94}c_2^4,$$
$$\beta_{62} = 1,\ \beta_{36} = \beta_{96},\ \beta_{11} + \beta_{95} = 1. \tag{2.31}$$

Let us apply now the operation $S_{9,9}$:

$$S_{9,9}\varphi_{14} = \varphi_5 = u_1 S_{9,9}c_{27} + u_2(S_{9,9}c_{26} + \beta_{93}c_4^2 + \beta_{94}c_2^4)$$
$$+u_3c_6(1 + \beta_{11} + \beta_{10}) + \varphi_3c_2^2(\beta_{36} + \beta_{96} + \beta_{37}) + \varphi_5(\beta_{63} + 1).$$

This gives

$$S_{9,9}c_{27} = 0,\ S_{9,9}c_{26} = \beta_{93}c_4^2 + \beta_{94}c_2^4,$$
$$\beta_{63} = 0,\ \beta_{11} + \beta_{10} = 1,\ \beta_{36} + \beta_{96} = \beta_{37}. \tag{2.32}$$

It follows from (2.31) and (2.32) that $\beta_{37} = 0$.

Let us apply the operation S_{16}, this gives:

$$S_{16}\varphi_{14} = \varphi_6 = u_1 S_{16}c_{27} + u_2 S_{16}c_{26} + u_3c_8 + u_4c_4 + u_3c_2^4\beta_{94}$$
$$+u_2(\beta_{90}c_{10} + \beta_{91}c_5^2 + \beta_{92}c_2^2c_6) + u_3c_4^2\beta_{93} + \varphi_3c_6\beta_{95} + u_4c_2^2\beta_{96}.$$

This means that

$$S_{16}c_{27} = c_{11},\ S_{16}c_{26} = (\beta_{90} + 1)c_{10} + \beta_{91}c_5^2 + \beta_{92}c_2^2c_6,$$
$$\beta_{94} = 1,\ \beta_{93} = 0,\ \beta_{95} = 0,\ \beta_{96} = 0.$$

From these relations and from (2.31) it follows that $\beta_{36} = 0$, $\beta_{11} = 1$; from (2.32) it follows that $\beta_{10} = 0$; from (2.26) it follows that $\beta_{12} = 0$.

Let us use the operation $S_{8,8}$, this gives:

$$S_{8,8}\varphi_{14} = 0 = u_1 S_{8,8}c_{27} + u_2(S_{8,8}c_{26} + \beta_{90}c_{10} + \beta_{91}c_5^2 + \beta_{92}c_2^2c_6)$$
$$+u_3(\beta_{16}c_2^4 + \beta_5c_4^2) + \varphi_6\beta_{73} + \varphi_3c_6\beta_{35} + u_4c_2^2\beta_{54}.$$

Hence

$$S_{8,8}c_{27} = 0,\ S_{8,8}c_{26} = \beta_{90}c_{10} + \beta_{91}c_5^2 + \beta_{92}c_2^2c_6,$$
$$\beta_{16} = 0,\ \beta_5 = 0,\ \beta_{73} = 0,\ \beta_{35} = 0,\ \beta_{54} = 0.$$

Using the operation S_{14} we obtain:

$$S_{14}\varphi_{14} = \varphi_7 \ = u_1 S_{14}c_{27} + u_2 S_{14}c_{26} + u_2c_4c_8 + u_3c_2c_8 + u_4c_2c_4$$
$$+u_4c_6 + u_3(c_{10} + c_5^2 + c_2^2c_6) + \varphi_3(c_4^2 + c_2^4) + u_2(\beta_{86}c_{12}$$
$$+\beta_{87}c_6^2 + \beta_{88}c_2^2c_4^2 + \beta_{89}c_2^6) + u_3(c_{10}(\beta_3 + \beta_{90}) + c_5^2(\beta_2 + \beta_{91})$$

$$+c_2^2c_6(\beta_{23}+\beta_{92}))+\varphi_3(\beta_{31}c_4^2+\beta_{38}c_2^4)+u_4c_6\beta_{53}$$
$$+\varphi_5c_2^2\beta_{64}+\varphi_7\beta_{81}.$$

It follows from the relation

$$S_{14}c_{27}=c_{13},\quad S_{14}c_{26}=(\beta_{86}+1)c_{12}+\beta_{87}c_6^2+\beta_{88}c_2^2c_4^2+\beta_{89}c_2^6;$$
$$\beta_3=\beta_{90},\ \beta_2=\beta_{91},\ \beta_{23}=\beta_{92},\ \beta_{31},\ \beta_{38},\ \beta_{53},\ \beta_{64},\ \beta_{81}=0; \tag{2.33}$$

Let us apply the operation S_{12}, we have:

$$S_{12}\varphi_{14}=u_5=u_1S_{12}c_{27}+u_2(S_{12}c_{26}+\beta_{82}c_2^2c_{10}+\beta_{83}c_2^2c_5^2+\beta_{84}c_4^2c_6$$
$$+\beta_{85}c_2^4c_6)+u_3(c_{12}(1+\beta_{86})+c_6^2(\beta_9+\beta_{87})+c_2^2c_4^2(\beta_{25}+\beta_{88})$$
$$+c_2^6(\beta_{20}+\beta_{89}))+\varphi_3(c_{10}(1+\beta_{30}+\beta_{90})+c_5^2(\beta_{29}+\beta_{91})+(\beta_{79}$$
$$+\beta_{92})c_2^2c_6)+u_4(c_4^2\beta_{51}+c_2^4\beta_{59})+\varphi_5c_6\beta_{65}+\varphi_6c_2^2\beta_{78}+u_5\beta_{86}.$$

This gives:

$$S_{12}c_{27}=0,\ S_{12}c_{26}=\beta_{82}c_2^2c_{10}+\beta_{83}c_2^2c_5^2+\beta_{84}c_4^2c_6+\beta_{85}c_2^4c_6;$$
$$\beta_9=\beta_{87},\ \beta_{25}=\beta_{88},\ \beta_{20}=\beta_{89},\ \beta_{29}=\beta_{91},\ \beta_{86}=1,$$
$$\beta_{78},\ \beta_{65},\ \beta_{51},\ \beta_{59}=0;\ \beta_{79}=\beta_{92},\ \beta_{30}+\beta_{90}=1. \tag{2.34}$$

From these relations and from (2.25) it follows that $\beta_1=0$, from (2.29) it follows that $\beta_4=0$.

Calculating the action of the operation $S_{6,6}$ on φ_{14} we obtain:

$$S_{6,6}\varphi_{14}=0=u_1S_{6,6}c_{27}+u_2c_4(c_{10}+c_5^2+c_2^2c_6)+u_3c_2(c_{10}+c_5^2+c_2^2c_6)$$
$$+\varphi_3(c_{10}+c_5^2+c_2^2c_6)+u_2c_2c_4c_8+u_3c_2c_4c_6+u_4c_2^2c_4+\varphi_5c_2c_4$$
$$+u_2(S_{6,6}c_{26}+c_{14}+c_2^4c_6(1+\beta_{85})+c_4^2c_6\beta_{84}+c_2^2c_{10}(\beta_{57}+\beta_{82})$$
$$+c_2^2c_5^2(\beta_{60}+\beta_{83}))+u_3(c_6^2\beta_{70}+c_{12}\beta_9+c_2^6(\beta_{18}+\beta_{69})+c_2^2c_4^2(\beta_{23}$$
$$+\beta_{24}+\beta_{68}))+\varphi_3(c_{10}(\beta_{33}+\beta_{76}+\beta_{90})+c_5^2(\beta_{32}+\beta_{79}+\beta_{91})$$
$$+c_2^2c_6(\beta_{39}+\beta_{44}+\beta_{92}))+u_4(c_4^2(\beta_{52}+\beta_{84})+c_2^4(\beta_{58}+\beta_{85}))$$
$$+u_5\beta_{87}+\varphi_5c_6\beta_{70}+\varphi_6c_2^2(\beta_{92}+\beta_{77}).$$

Hence,

$$S_{6,6}c_{26}=c_{14}+c_2^4c_6(1+\beta_{85})+c_4^2c_6\beta_{84}+c_2^2c_{10}(\beta_{57}+\beta_{82})+c_2^2c_5^2(\beta_{60}+\beta_{83});$$
$$S_{6,6}c_{27}=c_5(c_{10}+c_5^2+c_2^2c_6)+c_2c_4c_8,\beta_{18}=\beta_{69},\ \beta_{23}+\beta_{24}=\beta_{68},$$
$$\beta_{33}+\beta_{76}=\beta_{90},\ \beta_{32}+\beta_{79}=\beta_{91},\ \beta_{39}+\beta_{44}=\beta_{92},$$
$$\beta_{52}=\beta_{84},\ \beta_{58}=\beta_{85},\ \beta_{92}=\beta_{77},\ \beta_{70},\ \beta_9,\ \beta_{87}=0. \tag{2.35}$$

Applying the operation S_{10} we obtain:

$$\begin{aligned}S_{10}\varphi_{14}=\varphi_9 = {} & u_1S_{10}c_{27}+u_2c_6c_{10}\beta_{76}+u_3c_2c_4c_8+u_3c_6c_8+u_4c_2c_4^2\\ & +u_4c_4c_6+\varphi_3c_4c_8+u_3c_6c_4^2+u_3c_6c_2^4+\varphi_5c_4^2\beta_{71}+\varphi_6c_6\beta_{76}\\ & +u_2(S_{10}c_{26}+\beta_{74}c_4^4+\beta_{75}c_2^8+\beta_{77}c_2^2c_6^2+\beta_{79}c_5^2c_6+\beta_{80}c_2^4c_4^2)\\ & +u_3(c_{14}(1+\beta_3)+c_2^2c_{10}(1+\beta_3+\beta_{82})+c_2^2c_5^2(\beta_2+\beta_{14}+\beta_{83})\\ & +c_4^2c_6(\beta_7+\beta_{84})+c_2^4c_6(\beta_{19}+\beta_{23}+\beta_{85}))+\varphi_3(c_{12}\beta_{30}\\ & +c_6^2\beta_{33}+c_2^6(\beta_{40}+\beta_{89})+c_2^2c_4^2(\beta_{45}+\beta_{88}))+u_4(c_{10}(1+\beta_{90})\\ & +c_5^2(1+\beta_{49}+\beta_{91})+c_2^2c_6(\beta_{57}+\beta_{92}+1))+\varphi_7c_2^2\beta_{82}+\varphi_9\beta_{90}\\ & +\varphi_5c_2^4\beta_{66}.\end{aligned}$$

This means that:

$$\begin{aligned}&S_{10}c_{26}=\beta_{74}c_4^4+\beta_{75}c_2^8+\beta_{77}c_2^2c_6^2+\beta_{79}c_5^2c_6+\beta_{80}c_2^4c_4^2,\\ &S_{10}c_{27}=c_4c_5c_8+c_9c_4^2+c_{11}c_6;\ \beta_2+\beta_{14}=\beta_{83},\ \beta_7=\beta_{84},\ \beta_{19}+\beta_{23}=\beta_{85},\\ &\beta_3,\ \beta_{71},\ \beta_{76},\ \beta_{90}=1,\ \beta_{30},\ \beta_{33},\ \beta_{82},\ \beta_{66}=0,\ \beta_{40}=\beta_{89}, \beta_{45}=\beta_{88},\\ &\beta_{49}+\beta_{91}=1,\ \beta_{57}+\beta_{92}=1.\end{aligned} \tag{2.36}$$

It follows from (2.36) and (2.27) that $\beta_{48}=1, \beta_{13}=0$, from (2.28) it follows that $\beta_8=1$.

From the equality $S_8\varphi_{14}=\varphi_{10}$, we obtain after cancellations:

$$\begin{aligned}u_1c_{19}+u_3(c_4^4+c_2^4c_4^2)+\varphi_6c_4^2 = {} & u_1S_8c_{27}+u_3c_4^2c_8+u_4c_4^3+u_2c_4^2c_{10}\\ & +u_2(S_8c_{26}+c_2^4c_5^2\beta_{67}+\beta_{68}c_2^2c_4^2c_6+\beta_{69}c_2^6c_6\\ & +\beta_{72}c_4^2c_5^2)+u_3(c_2^8(1+\beta_{75})+c_4^4\beta_{74}\\ & +c_2^2c_6^2\beta_{77}+c_5^2c_6\beta_{79}+c_2^4c_4^2\beta_{80})+\varphi_3(c_2^2c_5^2\beta_{83}\\ & +c_4^2c_6\beta_{84}+c_2^4c_6\beta_{85})+u_4(c_2^2c_4^2\beta_{88}+c_2^6\beta_{89})\\ & +\varphi_5(c_5^2\beta_{91}+\beta_{92}c_2^2c_6).\end{aligned}$$

Hence,

$$\begin{aligned}&S_8c_{26}=c_2^4c_5^2\beta_{67}+\beta_{68}c_2^2c_4^2c_6+\beta_{69}c_2^6c_6+\beta_{72}c_4^2c_5^2,\\ &\beta_{77},\ \beta_{79},\ \beta_{80},\ \beta_{83},\ \beta_{84},\ \beta_{85},\ \beta_{88},\ \beta_{89},\ \beta_{91},\ \beta_{92}=0,\\ &\beta_{74},\ \beta_{75}=1,\ S_8c_{27}=c_{19}+c_{11}c_4^2.\end{aligned}$$

From these relations and from (2.36) we get $\beta_{57}, \beta_{49}=1, \beta_{45}, \beta_{40}, \beta_7=0$; from (2.35) it follows that $\beta_{58}, \beta_{52}, \beta_{32}=0$; from (2.34) it follows that $\beta_{29}, \beta_{20}, \beta_{25}=0$; from (2.33) it follows that $\beta_2, \beta_{23}=0$; from (2.36) it follows that $\beta_{14}, \beta_{19}=0$; from (2.30) it follows that $\beta_{15}=0$; from (2.30) it follows also that $\beta_{50}=0$.

Using the equality $S_6\varphi_{14} = \varphi_{11}$, we obtain after the cancellation:

$$\begin{aligned}
&u_1c_{21} + u_3c_2^4(c_{10} + c_5^2 + c_2^2c_6) + \varphi_6(c_{10} + c_5^2 + c_2^2c_6) \\
&\quad = u_1S_6c_{27} + u_3c_2c_8^2 + u_3c_8(c_{10} + c_5^2 + c_2^2c_6) + u_4c_4(c_{10} + c_5^2 + c_2^2c_6) \\
&\qquad + u_2c_{10}(c_{10} + c_5^2 + c_2^2c_6) + u_3c_4c_6c_8 + u_4c_2c_4c_8 + \varphi_5c_4c_8 + \varphi_3c_8^2 \\
&\qquad + u_2(S_6c_{26} + c_2^4c_{12} + c_2^4c_{12} + c_2^{10}\beta_{56} + c_2^2c_4^4\beta_{55} + c_2^2c_5^2c_6\beta_{60} + \beta_{61}c_2^6c_4^2) \\
&\qquad + u_3(c_4^2c_5^2(\beta_6 + \beta_{72}) + c_2^4c_5^2(\beta_{22} + \beta_{67}) + c_2^2c_4^2c_6\beta_{68} + c_2^6c_6\beta_{69}) \\
&\qquad + \varphi_3(c_2^4c_4^2\beta_{42} + c_2^8\beta_{43} + c_4^4\beta_{34} + c_2^2c_6^2\beta_{44}) + u_4c_2^2c_5^2\beta_{60} \\
&\qquad + \varphi_5(c_2^2c_4^2\beta_{68} + c_2^6\beta_{69}).
\end{aligned}$$

Hence,

$$\begin{aligned}
&S_6c_{26} = c_2^2c_4^2\beta_{55} + c_2^{10}\beta_{56} + c_2^6c_4^2\beta_{61};\ \beta_6 = \beta_{72},\ \beta_{22} = \beta_{67}. \\
&S_6c_{27} = c_{21} + c_5c_8^2 + c_{11}(c_{10} + c_5^2 + c_2^2c_6) + c_9c_4c_8, \\
&\qquad \beta_{69},\ \beta_{68},\ \beta_{60},\ \beta_{43},\ \beta_{42},\ \beta_{44},\ \beta_{34} = 0.
\end{aligned} \tag{2.37}$$

From these relations and from (2.35) it follows that $\beta_{18}, \beta_{24}, \beta_{39} = 0$.

From the equality $S_4\varphi_{14} = \varphi_{12}$ we get after the cancellation:

$$\begin{aligned}
&u_1S_4c_{27} + u_3c_8(c_{12} + c_2c_{10} + c_4c_8 + c_4c_2^4) + u_4c_4(c_{12} + c_2c_{10} + c_4c_8) \\
&\quad + \varphi_6c_4c_8 + u_3c_2^4c_{12} + \varphi_3c_2^4c_{10} + u_3c_{10}(c_{10} + c_5^2 + c_2^2c_6) + u_4c_6c_{10} \\
&\quad + \varphi_6c_{12} + \varphi_7c_{10} + \varphi_3c_4^2c_{10} + u_2(S_4c_{26} + \beta_{41}c_2^6c_5^2 + \beta_{46}c_2^2c_4^2c_5^2) \\
&\quad + u_3(\beta_{55}c_2^2c_4^2 + \beta_{56}c_2^{10} + \beta_{61}c_2^6c_4^2) + \varphi_3(\beta_{67}c_2^4c_5^2 + \beta_{72}c_4^2c_5^2) = u_1c_{23}.
\end{aligned}$$

This means that

$$\begin{aligned}
&S_4c_{27} = c_{23} + c_{11}c_{12} + c_{13}c_{10} + c_4c_8c_{11}, \\
&S_4c_{26} = \beta_{41}c_2^6c_5^2 + \beta_{46}c_2^2c_4^2c_5^2;\ \beta_{67},\ \beta_{72} = 0,\ \beta_{55},\ \beta_{56},\ \beta_{61} = 0.
\end{aligned}$$

From these relations and from (2.37) it follows that $\beta_6, \beta_{22} = 0$.

To determine the coefficients β_{41} and β_{46} let us apply the operation $S_{5,5}$.

$$\begin{aligned}
S_{5,5}\varphi_{14} = \varphi_9 = {} & u_1S_{5,5}c_{27} + u_3(c_2c_4c_8 + c_6c_8 + c_4^2c_6 + c_2^4c_6) + \varphi_3c_4c_8 \\
& + u_2c_6c_{10} + u_4(c_2c_4^2 + c_4c_6) + \varphi_5c_4^2 + \varphi_6c_6 + \varphi_9 \\
& + u_2(S_{5,5}c_{26} + c_2^8 + c_4^4) + \varphi_3(\beta_{41}c_2^6 + \beta_{46}c_2^2c_4^2).
\end{aligned}$$

Hence,

$$\beta_{41},\ \beta_{46} = 0; \quad S_{5,5}c_{27} = c_4c_5c_8 + c_9c_4^2 + c_{11}c_6, \quad S_{5,5}c_{26} = c_2^8 + c_4^4.$$

To determine the coefficient β_{21} we apply the operation $S_{4,4}$:

$$\begin{aligned} S_{4,4}\varphi_{14} = 0 = {} & u_1 S_{4,4}c_{27} + u_2(c_4^2 + c_2^4)c_{10} + u_3(c_4^2 c_8 + c_8^2 + c_2^8 + c_6 c_{10} \\ & + c_2^4 c_4^2) + u_4(c_4^3 + c_2 c_{10} + c_4 c_8 + c_4 c_2^4) + \varphi_6(c_8 + c_2^4 + c_4^2) \\ & + \varphi_5 c_{10} + u_3 c_2^8 \beta_{21} + u_2(S_{4,4}c_{26} + c_{18}). \end{aligned}$$

It follows from these relations that:

$$\beta_{21} = 0;\ \ S_{4,4}c_{27} = c_{11}(c_8 + c_4^2 + c_2^4) + c_9 c_{10},\ \ S_{4,4}c_{26} = c_{18}.$$

To determine β_{17} let us use the operation $S_{4,4,4,4}$:

$$S_{4,4,4,4}\varphi_{14} = \varphi_6 = u_1 S_{4,4,4,4}c_{27} + u_2 S_{4,4,4,4}c_{26} + u_3 c_2^4(1 + \beta_{17}) + \varphi_6.$$

So,

$$\beta_{17} = 1;\ \ S_{4,4,4,4}c_{27} = S_{4,4,4,4}c_{26} = 0.$$

To determine the coefficient β_{26} we apply the operation $S_{2,2,2,2,2,2,2,2,2,2,2,2,2}$:

$$S_{2,2,2,2,2,2,2,2,2,2,2,2,2}\varphi_{14} = 0 = u_3(1 + \beta_{26}),$$

hence, $\beta_{26} = 1$. The final form of the projection of the element Φ_{14} is the following:

$$\begin{aligned} \varphi_{14} = {} & u_1 c_{27} + u_3 c_8 c_{16} + u_4 c_4 c_{16} + u_5 c_4 c_8 + u_2 c_{26} + u_3(c_{24} + c_{10} c_{14} \\ & + c_6^4 + c_6 c_{18} + c_2^4 c_4^4 + c_2^{12}) + \varphi_3 c_{22} + u_4(c_{20} + c_{10}^2 + c_5^2 c_{10} + c_2^2 c_6 c_{10}) \\ & + \varphi_5(c_{18} + c_4^2 c_{10}) + \varphi_6(c_4^4 + c_2^8 + c_6 c_{10}) + u_5 c_{12} + \varphi_9 c_{10} + \varphi_{10} c_2^4. \end{aligned}$$

Using this decomposition we calculate the action of the operation S_ω on the elements c_{27}, and c_{26}.

2.3 Matrix Massey products in MASS

If elements ξ, η, ζ belong to $E_2^{0,1,t}$, for $t < 106$ then the following triple Massey products are defined and almost all of them contain zero: $\langle \xi, h_0, \eta \rangle$, $\langle \xi, h_0, \zeta \rangle$, $\langle \zeta, h_0, \eta \rangle$. Hence for all such triples the matrix Massey products of the following two types are defined: $\langle \xi, h_0, \eta, h_0 \rangle$, $\langle \xi, h_0, (\eta, \zeta), \begin{pmatrix} \zeta \\ \eta \end{pmatrix} \rangle$. Really, let $c_{\zeta,\eta}$ be the element belonging to $E_1^{0,0,t}$ which is defined uniquely up to cycles of the differential d_1, by the property: $d_1(c_{\zeta,\eta}) \in \langle \zeta, h_0, \eta \rangle$. Let also h_ζ denotes the element belonging to $E_1^{1,0,t}$, and having the property $d_1(h_\zeta) = h_0 \zeta$. Then we have:

$$\langle \xi, h_0, \eta, h_0\rangle = \begin{pmatrix} 0 & \xi & h_\xi & c_{\xi,\eta} & * \\ 0 & 0 & h_0 & h_\eta & 0 \\ 0 & 0 & 0 & \eta & h_\eta \\ 0 & 0 & 0 & 0 & h_0 \end{pmatrix} = h_0 c_{\xi,\eta} + h_\xi h_\eta,$$

$$\left\langle \xi, h_0, (\eta, \zeta), \begin{pmatrix} \zeta \\ \eta \end{pmatrix} \right\rangle = \begin{pmatrix} 0 & \xi & h_\xi & (c_{\xi,\eta}, c_{\xi,\zeta}) & * \\ 0 & 0 & h_0 & (h_\eta, h_\zeta) & c_{\eta,\zeta} \\ 0 & 0 & 0 & (\eta,\zeta) & 0 \\ 0 & 0 & 0 & 0 & \begin{pmatrix} \zeta \\ \eta \end{pmatrix} \end{pmatrix}$$
$$= \xi c_{\eta,\zeta} + \zeta c_{\xi,\eta} + \eta c_{\xi,\zeta}.$$

Let us denote the first product by $\mathcal{A}_{\xi,\eta}$, and the second by $\mathcal{F}_{\xi,\zeta,\eta}$. There is the equality: $\varphi_3 = \mathcal{F}_{u_1,u_2,u_3}$, and in the notations of the work [41] we have: $c_1 = \mathcal{A}_{u_1,u_1}$, $a_6 = \mathcal{A}_{u_1,\varphi_3}$.

As a canonical representative for $\mathcal{F}_{u_1,u_i,u_j}$ we choose the element $\tilde{\varphi}_{i,j}$. As a canonical representative for c_{u_1,u_j} we take $c_{1,j}$. If $\xi \in E_2^{0,1,t}$ has in $E_1^{0,1,t}$ the following decomposition: $\xi = \sum_i u_i \tilde{c}_i$, where $\tilde{c}_i \in E_1^{0,0,t}$, then as h_ξ we take the following element: $h_\xi = \sum_i h_i \tilde{c}_i$. As a canonical representative for $\mathcal{A}_{\xi,\xi}$ we take h_ξ^2. Under these conditions the elements $\mathcal{F}_{\xi,\zeta,\eta}$ and $\mathcal{A}_{\xi,\zeta}$ are defined uniquely for $t < 108$. For the simplicity let us introduce the following new notations:

old	*notation new*
$\mathcal{F}_{u_1,u_i,u_j}$	$\tilde{\varphi}_{i,j}$
$\mathcal{F}_{u_k,u_i,u_j}$	$\omega_{k,i,j}$
$\mathcal{F}_{u_1,u_i,\omega_{i,j,k}}$	$\psi_{\hat{i},j,k}$
$\mathcal{F}_{u_1,u_{i,j},\omega_{i,j,k}}$	$\psi_{\hat{i},\hat{j},k}$

2.4 The cell $E_2^{0,1,t}$ for $t < 108$

Generators are given in the Table 3. The following brief notations are used:

complete	$\omega_{2,3,4}$	$\omega_{2,3,5}$	$\omega_{2,4,5}$	$\omega_{3,4,5}$	$\psi_{\hat{2},3,4}$	$\psi_{2,\hat{3},4}$	$\psi_{\hat{2},\hat{3},4}$
brief	ω_1	ω_2	ω_3	ω_4	ψ_1	ψ_2	ψ_3

complete	$\psi_{2,3,\hat{4}}$	$\psi_{\hat{2},3,\hat{4}}$	$\psi_{\hat{2},3,5}$	$\psi_{2,\hat{3},\hat{4}}$	$\psi_{2,\hat{3},5}$	$\psi_{\hat{2},\hat{3},5}$
brief	ψ_4	ψ_5	ψ_6	ψ_7	ψ_8	ψ_9

Lemma 2.2 *Let all the elements given below are taken from the cell $E_2^{0,1,t}$ for $t < 108$. Let also the following conditions hold: $i \neq j \neq k \neq i$; $i, j, k \in \{2,3,4,5\}$. Let $\xi \neq \zeta \neq \eta \neq \theta \neq \xi \neq \eta, \theta \neq \zeta$. Then the following conditions hold:*

(1) $u_i\tilde{\varphi}_{j,k} + u_j\tilde{\varphi}_{i,k} + u_k\tilde{\varphi}_{i,j} = u_1\omega_{i,j,k}$.
(2) $u_i\tilde{\varphi}_{i,j,k} + \tilde{\varphi}_{i,j}\tilde{\varphi}_{j,k} = u_1\psi_{\hat{i},j,k} + u_j u_k c_{1,i}^2$.
(3) $\tilde{\varphi}_{i,j}\tilde{\varphi}_{i,j,k} = u_1\psi_{\hat{i},\hat{j},k} + u_i\tilde{\varphi}_{i,k}c_{1,j}^2 + u_j\tilde{\varphi}_{j,k}c_{1,i}^2$.
(4) $u_i\psi_{i,\hat{j},k} + u_j\psi_{\hat{i},j,k} = \tilde{\varphi}_{i,j}\omega_{i,j,k}$.

(5) $\tilde{\varphi}_{i,j}^2 = u_1^2 c_{i,j}^2 + u_i c_{1,j}^2 + u_j c_{1,i}^2$.
(6) $u_i \psi_{\hat{i},\hat{j},k} + \tilde{\varphi}_{i,j} \psi_{\hat{i},j,k} = u_1 \tilde{\varphi}_{i,k} c_{i,j}^2 + u_j \omega_{i,j,k} c_{1,i}^2$.
(7) $u_i \psi_{i,\hat{j},\hat{k}} + \tilde{\varphi}_{i,j} \psi_{i,j,\hat{k}} + \tilde{\varphi}_{i,k} \psi_{i,\hat{j},k} = \tilde{\varphi}_{i,j,k} \omega_{i,j,k}$.
(8) $\omega_{i,j,k}^2 = u_i^2 c_{j,k}^2 + u_j^2 c_{i,k}^2 + u_k^2 c_{i,j}^2$.
(9) $\tilde{\varphi}_{i,j,k}^2 = u_1^2 c_{i,j,k}^2 + u_i^2 c_{1,j}^2 c_{1,k}^2 + u_j^2 c_{1,i}^2 c_{1,k}^2 + u_k^2 c_{1,j}^2 c_{1,i}^2$.
(10) $\xi \mathcal{F}_{\zeta,\eta,\theta} + \zeta \mathcal{F}_{\xi,\eta,\theta} + \eta \mathcal{F}_{\xi,\zeta,\theta} + \theta \mathcal{F}_{\xi,\zeta,\eta} = 0$.

Proof It consists in direct check writing the decompositions of participating elements. □

This gives all the relations in $E_2^{0,1,t}$, $t < 108$.

2.5 The cell $E_2^{2,0,t}$, for $t < 108$

Lemma 2.3 (i) *If $\xi, \zeta, \eta \in E_2^{0,1,t}$, and $\theta \in E_2^{*,*,*}$ is an arbitrary element, and expression $\mathcal{A}_{\xi,\zeta}$ is defined, then the expressions $\mathcal{A}_{\theta\xi,\zeta}$ and $\mathcal{A}_{\xi,\theta\zeta}$ are also defined and we have the equality $\theta \mathcal{A}_{\xi,\zeta} = \mathcal{A}_{\theta\xi,\zeta} = \mathcal{A}_{\xi,\theta\zeta}$. ii) If $\mathcal{A}_{\xi,\zeta}$ and $\mathcal{A}_{\xi,\eta}$ are defined, then the relation $\mathcal{A}_{\xi,\zeta+\eta}$ is also defined, and the following equality holds: $\mathcal{A}_{\xi,\zeta} + \mathcal{A}_{\xi,\eta} = \mathcal{A}_{\xi,\zeta+\eta}$.*

Proof (i) If $\mathcal{A}_{\xi,\zeta}$ is defined by the expression:

$$\langle \xi, h_0, \eta, h_0 \rangle = \begin{pmatrix} 0 & \xi & h_\xi & c_{\xi,\eta} & * \\ 0 & 0 & h_0 & h_\eta & 0 \\ 0 & 0 & 0 & \eta & h_\eta \\ 0 & 0 & 0 & 0 & h_0 \end{pmatrix} = h_0 c_{\xi,\eta} + h_\xi h_\eta,$$

then $\mathcal{A}_{\theta\xi,\zeta}$ can be given by the formula:

$$\langle \theta\xi, h_0, \eta, h_0 \rangle = \begin{pmatrix} 0 & \theta\xi & \theta h_\xi & \theta c_{\xi,\eta} & * \\ 0 & 0 & h_0 & h_\eta & 0 \\ 0 & 0 & 0 & \eta & h_\eta \\ 0 & 0 & 0 & 0 & h_0 \end{pmatrix} = \theta(h_0 c_{\xi,\eta} + h_\xi h_\eta),$$

(ii) is proved analogously. □

Lemma 2.4 *Let $\sum_i \xi_i \zeta_i = 0$ be one of the relations of Lemma 1, where the elements $\xi_i, \zeta_i \in E_2^{0,1,t}$ are such that the sum of their t-gradings is less than 108, and the elements $\mathcal{A}_{\xi_i,\zeta_i}$ are defined for all i. Then $\sum_i \mathcal{A}_{\xi_i,\zeta_i} = 0$.*

Proof Let us consider for example the first of relations:

$$u_i \tilde{\varphi}_{j,k} + u_j \tilde{\varphi}_{i,k} + u_k \tilde{\varphi}_{i,j} = u_1 \omega_{i,j,k}.$$

Because of the equality $\tilde{\varphi}_{j,k} = u_1 c_{j,k} + u_k c_{1,j} + u_j c_{1,k}$, we have $h\tilde{\varphi}_{j,k} = h_1 c_{j,k} + h_k c_{1,j} + h_j c_{1,k}$ and $c_{u_i,\tilde{\varphi}_{j,k}} = c_{i,j,k} + c_{1,i} c_{j,k}$. Because of equality $\omega_{i,j,k} = u_i c_{j,k} +$

$u_j c_{i,k} + u_k c_{i,j}$, we have $h_{\omega_{i,j,k}} = h_i c_{j,k} + h_j c_{i,k} + h_k c_{i,j}$, and $c_{u_1,\omega_{i,j,k}} = c_{i,j,k} + c_{1,i} c_{j,k} + c_{1,j} c_{i,k} + c_{1,k} c_{i,j}$. Hence,

$$\begin{aligned}
\mathcal{A}_{u_i,\tilde{\varphi}_{j,k}} &= h_0(c_{i,j,k} + c_{1,i} c_{j,k}) + h_i(h_1 c_{j,k} + h_k c_{1,j} + h_j c_{1,k}),\\
\mathcal{A}_{u_j,\tilde{\varphi}_{i,k}} &= h_0(c_{i,j,k} + c_{1,j} c_{i,k}) + h_j(h_1 c_{i,k} + h_k c_{1,i} + h_i c_{1,k}),\\
\mathcal{A}_{u_k,\tilde{\varphi}_{i,j}} &= h_0(c_{i,j,k} + c_{1,k} c_{i,j}) + h_k(h_1 c_{i,j} + h_i c_{1,j} + h_j c_{1,i}),\\
\mathcal{A}_{u_1,\omega_{i,j,k}} &= h_0(c_{i,j,k} + c_{1,i} c_{j,k} + c_{1,j} c_{i,k} + c_{1,k} c_{i,j})\\
&\quad + h_1(h_i c_{j,k} + h_j c_{i,k} + h_k c_{i,j}).
\end{aligned}$$

Adding these equalities we get the assertion of Lemma. The other equalities are proved analogously. □

Now we can describe the generators of the cell $E_2^{2,0,t}$, $t < 108$. As it will be shown in Sect. 2.4, there are the following equalities for the Massey products $\langle\tilde{\varphi}_7, h_0, \omega_1\rangle = \langle\psi_1, h_0, \tilde{\varphi}_6\rangle = \langle\psi_2, h_0, \tilde{\varphi}_5\rangle = \langle\psi_3, h_0, u_4\rangle = \langle\psi_4, h_0, \tilde{\varphi}_3\rangle = \langle\psi_5, h_0, u_3\rangle = \langle\psi_7, h_0, u_2\rangle$, and they are not equal to zero in $E_2^{*,*,*}$. Equalities are understood in $E_2^{*,*,*}$, indeterminacy of each product is equal to zero in $E_2^{*,*,*}$. Let us call the pairs $(\tilde{\varphi}_7, \omega_1)$, $(\tilde{\varphi}_6, \psi_1)$, $(\tilde{\varphi}_5, \psi_2)$, (u_4, ψ_3), $(\tilde{\varphi}_3, \psi_4)$, (u_3, ψ_5), (u_2, ψ_7) *forbidden*. For each forbidden pair ξ, ζ the element $\mathcal{A}_{\xi,\zeta}$ is not defined.

Let us take for each non-forbidden pair $\xi, \zeta \in E_2^{0,1,*}$, such that the sum of t-gradings of ξ and ζ is less than 108, the element $\mathcal{A}_{\xi,\zeta}$. Let us delete from this set the elements $\mathcal{A}_{\xi,\zeta}$ which according to Lemma 2.3.2 can be expressed by the others. Let us add one element which we shall denote by $\mathcal{A}_{\langle u_2,\psi_7\rangle+\langle\tilde{\varphi}_7,\omega_1\rangle}$, and having the following decomposition:

$$\begin{aligned}
\mathcal{A}_{\langle u_2,\psi_7\rangle+\langle\tilde{\varphi}_7,\omega_1\rangle} &= h_0(c_5c_8c_{13} + c_4c_9c_{13}) + h_1h_3c_9c_{13} + h_1h_4c_5c_{13}\\
&\quad + h_2h_3c_8(c_{13} + c_8c_5 + c_4c_9) + h_3^2c_2c_8c_9 + h_4^2c_2c_4c_5\\
&\quad + h_2h_4c_4(c_{13} + c_8c_5 + c_4c_9) + h_3h_4c_2(c_8c_5 + c_4c_9).
\end{aligned}$$

This gives the final system of generators of the cell $E_2^{2,0,t}$, for $t < 108$. Let us describe relations.

Lemma 2.5 *There are the following relations:*

(1) $\xi\mathcal{A}_{\xi,\eta} = \eta\mathcal{A}_{\xi,\xi}$;

(2) $\xi\mathcal{A}_{\zeta,\eta} = \zeta\mathcal{A}_{\xi,\eta} = \eta\mathcal{A}_{\xi,\zeta}$;

(3) $\mathcal{A}_{\xi,\eta}^2 = h_0c_{\xi,\eta}^2 + h_\xi^2h_\eta^2$;

(4) $\mathcal{A}_{\xi,\eta}\mathcal{A}_{\xi,\zeta} = h_\xi^2\mathcal{A}_{\eta,\zeta} + h_0\mathcal{A}_{\xi,\mathcal{F}_{\xi,\eta,\zeta}}$, $\xi, \eta, \zeta \in E_2^{0,1,t}$;

under the condition that all the expressions appearing in the formulas above are defined.

Proof (1) We have the formula for the first differential: $d_1(h_\xi c_{\xi,\eta}) = \xi\mathcal{A}_{\xi,\eta} + \eta\mathcal{A}_{\xi,\xi}$.
(2) Also we have: $d_1(h_\xi c_{\zeta,\eta} + h_\zeta c_{\xi,\eta}) = \xi\mathcal{A}_{\zeta,\eta} + \zeta\mathcal{A}_{\xi,\eta}$.
Relations (3) and (4) can be checked directly. □

2.6 The cell $E_2^{1,1,t}$ for $t < 108$

Lemma 2.6 (i) *The element $\kappa \in \langle \tilde{\varphi}_7, h_0, \omega_1 \rangle$ is defined and not equal to zero in $E_2^{1,1,t}$;*

(ii) *There are the equalities:*

$$\langle \tilde{\varphi}_7, h_0, \omega_1 \rangle = \langle \tilde{\varphi}_6, h_0, \psi_1 \rangle = \langle \tilde{\varphi}_5, h_0, \psi_2 \rangle = \langle u_4, h_0, \psi_3 \rangle$$
$$= \langle \tilde{\varphi}_3, h_0, \psi_4 \rangle = \langle u_3, h_0, \psi_5 \rangle = \langle u_2, h_0, \psi_7 \rangle,$$

which are understood without an ambiguity because the indeterminacy in the given dimension is equal to zero.

(iii) *There is the equality $h_0\kappa = 0$ in $E_2^{*,*,*}$.*

Proof (i) The element κ has the following decomposition:

$$\kappa = (\alpha_{1,2}c_{11} + \alpha_{1,3}c_9 + \alpha_{1,4}c_5)c_{13} + (\alpha_{2,3}c_2c_8 + \alpha_{2,4}c_2c_4)c_{11}$$
$$+(\alpha_{2,3}c_4c_8 + \alpha_{3,4}c_2c_4)c_9 + (\alpha_{2,4}c_4c_8 + \alpha_{3,4}c_2c_8)c_5.$$

Here $\alpha_{i,j} = u_ih_j + u_jh_i$. Because of

$$d_1[(c_2c_{11} + c_4c_9 + c_5c_8)c_{13}] = \kappa + \alpha_{3,4}c_{11}c_2^2 + \alpha_{2,4}c_9c_4^2 + \alpha_{2,3}c_5c_8^2,$$

and

$$d_1(c_{11}) = \alpha_{3,4}, d_1(c_9) = \alpha_{2,4}, d_1(c_5) = \alpha_{2,3},$$

the fact is proved.

(ii)

$$d_1((c_5c_8 + c_4c_9)c_{13}) = \kappa + \langle u_2, h_0, \psi_7 \rangle,$$
$$d_1((c_5c_8 + c_2c_{11})c_{13}) = \kappa + \langle u_3, h_0, \psi_5 \rangle,$$
$$d_1((c_4c_9 + c_2c_{11})c_{13}) = \kappa + \langle u_4, h_0, \psi_3 \rangle,$$
$$\langle \tilde{\varphi}_6, h_0, \psi_1 \rangle = \langle u_2, h_0, \psi_7 \rangle,$$
$$\langle \tilde{\varphi}_5, h_0, \psi_2 \rangle = \langle u_3, h_0, \psi_5 \rangle,$$
$$\langle u_4, h_0, \psi_3 \rangle = \langle \tilde{\varphi}_3, h_0, \psi_4 \rangle,$$

all the equalities are fulfilled strictly for the decompositions of the given elements in $E_1^{*,*,*}$).

(iii) It follows from the formula:

$$d_1(h_1h_2c_{11}c_{13} + h_2h_3c_2c_8c_{11} + h_2h_4c_2c_4c_{11} + h_1h_3c_9c_{13}$$
$$+h_2h_3c_4c_8c_9 + h_3h_4c_2c_4c_9 + h_1h_4c_5c_{13} + h_2h_4c_4c_5c_8$$
$$+h_3h_4c_2c_5c_8 + h_2^2c_4c_8c_{11} + h_3^2c_2c_8c_9 + h_4^2c_2c_4c_5) = h_0\kappa.$$

□

Remark 2.1 Described elements give a complete set of generators of $E_2^{*,*,t}$ for $t < 108$.

2.7 On the action of differentials $d_r (r > 1)$

Because of multiplicative properties it is sufficient to describe an action of d_r on $E_r^{2,0,t}$ and $E_r^{0,1,t}$. Remind that the cell $E_2^{0,0,t}$ consists of cycles of all the differentials d_r. If $t < 104$ in all the cells $E_2^{q,1,t}$ nonzero elements have t-grading equal to $4m + 2$, and in the cells $E_2^{q,2,t}$ nonzero elements have t-grading equal to $4m$. First "irregular" elements are $\kappa, u_1\kappa h_1^2$ and $(h_1^2)^2\kappa$; $\deg(u_1\kappa h_1^2) = (3, 2, 110)$, $\deg[(h_1^2)^2\kappa] = (2, 5, 112)$. Hence first elements of the cells $E_r^{2,0,t}$ and $E_r^{0,1,t}$ for which a differential d_r can be non-equal to zero for $r > 1$ must have t-grading not less than 110.

Theorem 2.1 *There is an isomorphism of the terms $E_2^{*,*,t}$ and $E_\infty^{*,*,t}$ of MASS up to dimension* 108*:*

Proof We need to prove only the fact that the element κ is an infinite cycle. In Sect. 3 it will be shown that there exists an element $\Omega_1 \in MSp_{49}$, having the order 2 and whose projection to the term E_2 of the Adams–Novikov spectral sequence is an element (having the same notation) $\Omega_1 \in E_2^{1,50}$. This last element has the order 2 and projects into the element $\omega_1 \in E_\infty^{0,1,50}$ in MASS. Hence in the term E_2 of the Adams–Novikov spectral sequence the following Massey product $\langle \Omega_1, 2, \Phi_7 \rangle$ is defined and is associated to the element $\kappa \in \langle \omega_1, h_0, \phi_7 \rangle$ defined in MASS. All the elements in $E_2^{0,0,*}$ are infinite cycles. Hence for the last product all the conditions of Theorem 3 of the work [20] about the convergence of Massey products in spectral sequences are fulfilled. Hence the element κ is an infinite cycle. □

3 The Adams–Novikov spectral sequence for $t - s < 56$

3.1 Algebraic structure

In the work [41] of the second author the ring MSp_* was calculated up to dimensions $* < 32$. We continue these calculations and compute this ring in dimensions $* < 56$ (up to some integer relations). On the base of these calculations we construct the element $\Omega_{49} \in MSp_{49}$ of the order 2 and with the following properties: $\theta_1^2\Omega_1 = 0$ and $0 \in \langle \theta_1, 2, \Omega_1 \rangle$ in MSp_*. Let us determine the term E_2 of the Adams–Novikov spectral sequence, which is associated to the term E_∞ of the MASS which is calculated up to $t < 106$ in the Sect. 2. We also calculate the action of the differentials in the Adams–Novikov spectral sequence for $32 < t - s < 52$. These results are given in Tables 6 and 7.

1. We follow the work [41] and denote by $\pi_i^1(x)$ the projection of the element $x \in MSp_*$ (if x is in the i-th module of filtration) into the term E_2 of the Adams–Novikov spectral sequence. Let us introduce the following notations for the projections

of the elements $\theta_1 \in MSp_1$, $\Phi_i \in MSp_{8i-3}$:

$$U_1 = \pi_1^1(\theta_1) \in E_2^{1,2}, \quad U_{i+2} = \pi_1^1(\Phi_2 i) \in E_2^{1,2(2^i-1)},$$
$$\Phi_j = \pi_1^1(\Phi_j) \in E_2^{1,8j-2}, \quad \text{if} \;\; j \neq 2^i.$$

We have: $2U_i = 0,\ 2\Phi_j = 0$ and $U_i,\ \Phi_j$ are the infinite cycles of the Adams–Novikov spectral sequence.

2. We follow the work [41] and denote by $\pi_i^2(x)$ the projection of the element $x \in E_2 \cong \mathrm{Ext}_A(BP^*(MSp), BP^*)$, (here we put $A = A^{BP}$) into the i-th line of the term E_∞ of MASS if x is in the i-th module of filtration corresponding to MASS. For the shortening let us introduce new notations of some generators of $E_\infty^{2,0,t}$, $t < 56$, results are given in Table 5. Let us consider the elements z_i, $y_j (1 \leq i, j \leq 8)$, having the following projections into the term E_∞ of MASS (notations for the elements of MASS are taken from Table 5):

$$\pi_2^2(z_1) = a_1,\ \pi_2^2(z_2) = a_2,\ \pi_2^2(z_3) = a_3,\ \pi_2^2(z_4) = a_4,$$
$$\pi_0^2(y_4) = e_4,\ \pi_2^2(z_5) = a_5,\ \pi_2^2(z_6) = a_6,\ \pi_0^2(y_6) = c_6,$$
$$\pi_2^2(z_7) = a_7,\ \pi_2^2(y_7) = b_7,\ \pi_2^2(z_8) = a_8,\ \pi_0^2(y_8) = e_8.$$

These notations differ from notations of the work [41] by the transposition of the elements y_6 and z_6. The action of the differentials on these elements are described in the work [41]. Also in the work [41] the following elements are introduced:

$$\tau_1 = U_1 y_4 + U_2 z_3 \in E_2^{1,18},\ \tau_2 = U_1 y_6 + U_2 z_5 \in E_2^{1,26},$$
$$\tau_3 = U_2 y_6 + U_3 y_4 \in E_2^{1,30}.$$

These elements are the infinite cycles in the Adams–Novikov spectral sequence and define the elements $\tau_1 \in MSp_{17}$, $\tau_2 \in MSp_{25}$, $\tau_3 \in MSp_{29}$. Let us denote by $F^i E_2^{*,*}$ the set of elements from $E_2^{*,*} \cong \mathrm{Ext}_A^{*,*}(BP^*(MSp), BP^*)$ having the filtration corresponding MASS not less than i.

3. It follows from the relation $h_0^2 e_8 = a_1 a_7 + a_4^2$, which is fulfilled in the term E_∞ of MASS that in the term E_2 of the Adams–Novikov spectral sequence we have the relation:

$$\begin{aligned}4y_8 = {} & z_1 z_7 + z_4^2 + \beta_1 4 z_8 + \beta_2 (z_1^2 z_2^3 + z_1^4 y_4) + \beta_3 (z_1 z_3 z_4 + z_1^2 y_6) \\ & + \beta_4 2 z_1 y_7 + \beta_5 2 z_2 z_6 + \beta_6 2 z_1 z_7 + \beta_7 2 z_3 z_5 + \beta_8 z_1^3 z_5 + \beta_9 2 z_4^2 \\ & + \beta_{10} z_1^2 z_2 z_4 + \beta_{11} z_1^2 z_3^2 + \beta_{12} z_1 z_2^2 z_3 + \beta_{13} z_1^5 z_3 + \beta_{14} z_2^4 + \beta_{15} z_1^4 z_2^2 \\ & + \beta_{16} z_1^8 + \beta_{17} 2 z_1^2 z_2 y_4 + \beta_{18} 2 z_1^4 z_4 + \beta_{19} 2 z_1^3 z_2 z_3 + \beta_{20} 2 z_1^6 z_2 \\ & + \beta_{21} 2 z_1^2 z_6 + \beta_{22} 2 z_1 z_3 y_4 + \beta_{23} 2 z_1 z_2 z_5 + \beta_{24} 2 z_2^2 z_4 + \beta_{25} 2 z_2 z_3^2 \\ & + \beta_{26} 2 z_2^2 y_4 + \beta_{27} 4 y_4 z_4 + \beta_{28} 8 y_4^2 + \beta_{29} 4 z_2 y_6.\end{aligned}$$

Let us choose the element y_8 in such a way that $\beta_{28} = 0$ (changing y_8 for $y_8^* = y_8 + 2\beta_{28}y_4^2$). Let us multiply by U_1 both parts of studying equality, then we have:

$$\beta_2 z_1^4\tau_1 + \beta_3 z_1^2\tau_2 + (\beta_8 + \beta_{10})z_1^3 z_5 U_1$$
$$+(\beta_{11} + \beta_{12} + \beta_{14})z_2^4 U_1 + (\beta_{13} + \beta_{15})z_1^5 z_3 U_1 + \beta_{16} z_1^8 U_1 = 0.$$

we obtain that $\beta_2, \beta_3, \beta_{16} \equiv 0 \mod 2$; $\beta_8 \equiv \beta_{10}, \beta_{13} \equiv \beta_{15} \mod 2$; $\beta_{11} + \beta_{12} + \beta_{14} \equiv 0 \mod 2$. Not loosing the generality in the last relation we can have $\beta_{11} \equiv \beta_{12} \mod 2$, then $\beta_{14} \equiv 0 \mod 2$. We can choose the element z_7 in such a way that the action of the differential $d_3(z_7) = U_1U_3^2$ and the relation $U_1z_7 = U_3z_4$ are conserved and our equation takes the form:

$$4y_8 = z_1z_7 + z_4^2 + \beta_1 4z_8 + \beta_5 2z_2z_6 + \beta_7 2z_3z_5 + \beta_8(z_1^3z_5 + z_1^2z_2z_4)$$
$$+\beta_9 2z_4^2 + \beta_{11}(z_1^2z_3^2 + z_1z_2^2z_3) + \beta_{13}(z_1^5z_3 + z_1^4z_2^2) + \beta_{14}2z_2^4$$
$$+\beta_{24}2z_2^2z_4 + \beta_{25}2z_2z_3^2 + \beta_{26}2z_2^2y_4 + \beta_{27}4y_4z_4 + \beta_{29}4z_2y_6.$$

Let us apply the operation $S_{4,4}$ to the both parts of the last equality

$$S_{4,4}y_8 \equiv ((S_2z_4)/2)^2(1 + 2\beta_9) \mod 4.$$

We can suppose (changing y_8 if necessary), that $S_{4,4}y_8 \equiv ((S_2z_4)/2)^2 \mod 4$ Hence $\beta_9 \equiv 0 \mod 2$. Then by the choice of y_8 we can achieve that our relation take the form:

$$4y_8 = z_1z_7 + z_4^2 + \beta_1 4z_8 + \beta_5 2z_2z_6 + \beta_7 2z_3z_5 + \beta_8(z_1^3z_5 + z_1^2z_2z_4)$$
$$+\beta_{11}(z_1^2z_3^2 + z_1z_2^2z_3) + \beta_{13}(z_1^5z_3 + z_1^4z_2^2) + \beta_{14}2z_2^4$$
$$+\beta_{24}2z_2^2z_4 + \beta_{25}2z_2z_3^2 + \beta_{26}2z_2^2y_4 + \beta_{27}4y_4z_4 + \beta_{29}4z_2y_6.$$

Because of the fact $S_7(U_2U_3^2) = 0$, it follows that $S_7y_8 \equiv 0 \mod (2z_1)$, and because of the fact $S_7U_4 = U_1$, it follows that $S_7z_8 \equiv z_1 \mod (2z_1)$. Applying S_7 to our equality we obtain $4z_1 \equiv 4\beta_1z_1 \mod (8z_1)$, hence $\beta_1 \equiv 1 \mod 2$, and so one can choose the element y_8 in such a way that our equality takes the form:

$$4y_8 + 4z_8 = z_1z_7 + z_4^2 + \beta_5 2z_2z_6 + \beta_7 2z_3z_5 + \beta_8(z_1^3z_5 + z_1^2z_2z_4)$$
$$+\beta_{11}(z_1^2z_3^2 + z_1z_2^2z_3) + \beta_{13}(z_1^5z_3 + z_1^4z_2^2) + \beta_{14}2z_2^4$$
$$+\beta_{24}2z_2^2z_4 + \beta_{25}2z_2z_3^2 + \beta_{26}2z_2^2y_4 + \beta_{27}4y_4z_4 + \beta_{29}4z_2y_6.$$

From the relations $2z_6 = z_1z_5 + z_2z_4$ and $4z_4 + 4y_4 = z_1z_3 + z_2^2$ it follows that $\beta_8(z_1^3z_5 + z_1^2z_2z_4) = \beta_8 2z_1^2z_6$, $\beta_{11}(z_1^2z_3^2 + z_1z_2^2z_3) = \beta_{11}z_1z_3 4(z_4 + y_4)$, $\beta_{13}(z_1^5z_3 + z_1^4z_2^2) = \beta_{13}z_1^4 4(z_4 + y_4)$. Hence one can choose z_7 (not changing the relation $U_1z_7 = U_3z_4$) in such a way that our expression takes the form:

$$4y_8 + 4z_8 = z_1z_7 + z_4^2 + \beta_5 2z_2z_6 + \beta_7 2z_3z_5 + \beta_{14}2z_2^4 + \beta_{24}2z_2^2z_4$$
$$+\beta_{25}2z_2z_3^2 + \beta_{26}2z_2^2y_4 + \beta_{27}4y_4z_4 + \beta_{29}4z_2y_6.$$

From the relation $4z_4+4y_4 = z_1z_3+z_2^2$ it follows that $2\beta_{24}z_2^2z_4 = 2\beta_{24}z_4(4z_4+4y_4-z_1z_3)$, $\beta_{26}2z_2^2y_4 = 2\beta_{26}y_4(4z_4+4z_4-z_1z_3)$, $\beta_{14}2z_2^4 = \beta_{14}2z_2^2(4z_4+4y_4-z_1z_3)$. Hence we can choose y_8 and z_8 that our equality takes the form:

$$4y_8+4z_8 = z_1z_7+z_4^2+\beta_5 2z_2z_6+\beta_7 2z_3z_5+\beta_{25}2z_2z_3^2 \\ +\beta_{27}4y_4z_4+\beta_{29}4z_2y_6.$$

From the conditions $S_6c_6 = 1$ and $S_6c_8 = c_2$ it follows that $S_6y_6 \equiv 1 \mod 2$; $S_6z_8 \equiv z_2 \mod (2z_2, z_1^2)$; $S_6y_8 \equiv 0 \mod (2z_2, z_1^2)$; $S_6z_7 \equiv z_1 \mod (2z_1)$. It follows from these facts that $4z_2 \equiv \beta_{29}4z_4 \mod (8z_4)$, hence $\beta_{29} \equiv 1 \mod 2$, and one can choose y_8 that our equality takes the form:

$$4y_8+4z_8+4z_2y_6 = z_1z_7+z_4^2+\beta_7 2z_3z_5+\beta_{25}2z_2z_3^2+\beta_{27}4y_4z_4+\beta_5 2z_2z_6.$$

From the relations $S_4c_4 = 1$, $S_4c_6 = 0$, $S_4c_8 = c_4$ we obtain

$$S_4z_6 \equiv z_2 \mod (2z_2, z_1^2);\ S_4z_4 \equiv 0 \mod 2;\ S_4z_7 \equiv 0 \mod (2z_3, z_1z_2, 2z_1^3); \\ S_4y_6 \equiv 0 \mod (2z_2, z_1^2);\ S_4y_8 \equiv 0 \mod (2z_4, 2y_4, 2z_1^2z_2, z_2^2, z_1z_3, z_1^4); \\ S_4z_8 \equiv z_4 \mod (2z_4, 2y_4, 2z_1^2z_2, z_2^2, z_1z_3, z_1^4).$$

If we apply the operation S_4 to our equation then we get the relation: $2\beta_5z_2^2 \equiv 0 \mod (2z_2^2)$, hence it follows that $\beta_5 \equiv 0 \mod 2$, hence one can choose y_8 in such a way that the relation holds:

$$4y_8+4z_8+4z_2y_6 = z_1z_7+z_4^2+\beta_7 2z_3z_5+\beta_{25}2z_2z_3^2+\beta_{27}4y_4z_4.$$

From the following relations in MASS: $S_{2,2}c_8 = 0$, $S_2c_6 = c_2^2$, $S_2c_4=c_2$, $S_{2,2}c_4=0$, and the condition $S_{2,2}U_4 = 0$ we get the relations:

$$S_{2,2}y_8 \equiv y_4 \mod (2y_4, 2z_4, z_1z_3, z_2^2, 2z_1^2z_2, z_1^4);\ S_{2,2}y_6 \equiv 0 \mod (2z_2, z_1^2); \\ S_{2,2}z_7 \equiv z_3 \mod (2z_3, z_1z_2, z_1^3);\ S_2z_5 \equiv z_3 \mod (2z_3, z_1z_2, z_1^3); \\ S_{2,2}z_8 \equiv 0 \mod (2y_4, 2z_4, z_1z_3, z_2^2, 2z_1^2z_2, z_1^4);\ S_{2,2}z_5 \equiv 0 \mod (2z_1); \\ S_2y_6 \equiv y_4 \mod (2y_4, 2z_4, z_1z_3, z_2^2, 2z_1^2z_2, z_1^4);\ S_2z_4 \equiv z_2 \mod (2z_2, z_1^2); \\ S_{2,2}z_4 \equiv 0 \mod 4;\ S_2z_3 \equiv 0 \mod (2z_1).$$

Hence if we apply $S_{2,2}$ to our equation we get a relation:

$$4\beta_{27}z_4+4y_4 \equiv z_1z_3+z_2^2 \mod (8y_4, 8z_4, 2z_1z_3, 2z_2^2).$$

Hence, $\beta_{27} \equiv 1 \mod 2$, and we can choose y_8, in such a way that our equation takes the form:

$$4y_8+4z_8+4z_2y_6+4y_4z_4 = z_1z_7+z_4^2+\beta_7 2z_3z_5+\beta_{25}2z_2z_3^2.$$

From the conditions $S_3(U_1U_2U_3) = U_1^2U_2$ and $S_3(U_1U_3^2) = 0$ it follows that

$$S_3z_5 \equiv z_2 \mod (2z_2, z_1^2),\ S_3z_4 \equiv z_1 \mod (2z_1),$$
$$S_3z_7 \equiv 0 \mod (2z_4, 2y_4, z_2^2, z_1z_3, 2z_1^2z_2, z_1^4),$$

so the application of the operation S_3 to our equation gives the relation:

$$2\beta_7z_2z_3 \equiv 0 \mod (4z_2z_3).$$

Hence $\beta_7 \equiv 0 \mod 2$. Let us choose y_8 in such a way that our equation takes the form:

$$4y_8 + 4z_8 + 4z_2y_6 + 4y_4z_4 = z_1z_7 + z_4^2 + \beta_{25}2z_2z_3^2.$$

Multiplying this equation by z_1, we obtain:

$$(4y_8 + 4z_8 + 4z_2y_6 + 4y_4z_4)z_1 = z_1^2z_7 + z_4^2z_1 + \beta_{25}2z_2z_3^2z_1.$$

From the relation 4 of the work [41] it follows that $z_1z_3 = 4y_4 + 4z_4 - z_2^2$, hence our equation can be rewritten in the form:

$$4(y_8 + z_8 + z_2y_6 + y_4z_4)z_1 = z_1^1z_7 + z_4^2z_1 + \beta_{25}2z_2z_3(4z_4 + 4y_4 - z_2^2).$$

From the relations $S_{1,1}U_3^2 = 0$; $U_1z_7 = U_3z_4$; $U_1z_4 = U_3z_1$ it follows that

$$S_{1,1}z_4 \equiv z_2 \mod (2z_2, z_1^2);\ S_1z_4 \equiv 0 \mod (2z_3, z_1z_2, 2z_1^3);$$
$$S_{1,1}z_7 \equiv 0 \mod (2z_5, 2z_1z_4, 2z_2z_3, 2z_1^2z_3, 2z_1z_2^2, 2z_1^3z_2, 2z_1^5);$$
$$S_{1,1}z_3 \equiv z_1 \mod (2z_1).$$

Applying to the studying equality the operation $S_{1,1}$, we have $0 \equiv \beta_{25}2z_1z_2^3 \mod (4z_1z_2^3)$, or $0 \equiv \beta_{25} \mod 2$. Choosing y_6 transform our equality to the form:

$$4(y_8 + z_8 + z_2y_6 + y_4z_4) = z_1z_7 + z_4^2.$$

4. Consider the expression U_1z_8. Because of the equality $d_3(U_1z_8) = U_1^3U_4$, and the identity $u_1a_8 = u_4a_1$, fulfilled in the term E_∞ of MASS we have the following equality in the term E_2 of the Adams–Novikov spectral sequence:

$$U_1z_8 = U_4z_1 + \beta_1z_1^4\tau_1 + \beta_2z_2^2\tau_1 + \beta_3z_1^2\tau_2 + \beta_4z_1y_7U_1 + \beta_5z_1z_7U_1$$
$$+\beta_6z_3z_5U_1 + \beta_7z_1^3z_5U_1 + \beta_8z_2^4U_1 + \beta_9z_1^8U_1 + \beta_{10}z_1^5z_3U_1.$$

Changing z_8 and z_7 in such a way that the equality of the item 3 and the relation $U_1z_7 = U_3z_4$ are conserved, it is possible to transform our equality to the form:

$$U_1z_8 = U_4z_1 + \beta_6z_3z_5U_1 + \beta_8z_2^4U_1.$$

Changing z_8 and z_4 in such a way that the equality of the item 3 and the equalities 3,4,5,6,9 and 11 of the work [15] are conserved, it is possible to transform our equality to the form:

$$U_1 z_8 = U_4 z_1 + \beta_6 z_3 z_5 U_1.$$

Changing z_8 simultaneously with the changing of y_8 in order to conserve the equality of the item 3, it is possible to transform our equality to the form:

$$U_1 z_8 = U_4 z_1.$$

5. Let us consider the expression $U_2 y_7$. Because of the equality $d_3(U_2 y_7) = U_1 U_2^2 \Phi_3$ and the identity $u_2 b_7 = \varphi_3 a_3$, fulfilled for the term E_∞ of the MASS the following equation is valid in the term E_2 of the Adams–Novikov spectral sequence:

$$\begin{aligned} U_2 y_7 = \Phi_3 z_3 &+ \beta_1 z_1^4 \tau_1 + \beta_2 z_2^2 \tau_1 + \beta_3 z_1^2 \tau_2 + \beta_4 z_1 y_7 U_1 + \beta_5 z_1 z_7 U_1 \\ &+ \beta_6 z_3 z_5 U_1 + \beta_7 z_1^3 z_5 U_1 + \beta_8 z_2^4 U_1 + \beta_9 z_1^8 U_1 + \beta_{10} z_1^5 z_3 U_1. \end{aligned}$$

Let us multiply both parts of this equality by U_1 and use the relations $U_1 z_3 = U_2 z_2$ and $U_1 y_7 = \Phi_3 z_2$ from [15]. We get the relations $\beta_i \equiv 0 \mod 2$ for each i. Hence we have:

$$U_2 y_7 = \Phi_3 z_3.$$

6. Let us consider the expression $U_2 z_7$. Because of the equality $d_3(U_2 z_7) = U_1 U_2 U_3^2$, and the identity $u_2 a_7 = u_3 a_5$, fulfilled for the term E_∞ of the MASS the following equation is valid in the term E_2 of the Adams–Novikov spectral sequence:

$$\begin{aligned} U_2 z_7 = U_3 z_5 &+ \beta_1 z_1^4 \tau_1 + \beta_2 z_2^2 \tau_1 + \beta_3 z_1^2 \tau_2 + \beta_4 z_1 y_7 U_1 + \beta_5 z_1 z_7 U_1 \\ &+ \beta_6 z_3 z_5 U_1 + \beta_7 z_1^3 z_5 U_1 + \beta_8 z_2^4 U_1 + \beta_9 z_1^8 U_1 + \beta_{10} z_1^5 z_3 U_1. \end{aligned}$$

Let us multiply both parts of this equality by U_1 and use the relations $U_1 z_7 = U_3 z_4$ and $U_1 z_5 = U_2 z_4$ from [15]. We obtain the relations $\beta_i \equiv 0 \mod 2$ for each i. Hence we have:

$$U_2 z_7 = U_3 z_5.$$

7. An element $z_9 \in E_2^{0,36}$ we choose in such a way that $\pi_2^2(z_9) = a_9$. From the action of the Landweber–Novikov operation S_1: $S_1 a_9 = a_8$ it follows that

$$S_1(z_9 + \text{any element in this dimension not equal to } z_9) = z_8 + \text{decomposables}.$$

Hence, the following conditions must be fulfilled:

$$d_3(z_9) \neq 0,$$
$$d_3(z_9 + \text{ any element in this dimension not equal to } z_9) \neq 0.$$

So,

$$d_3(z_9) = \beta_1 U_1 U_2 U_4 + \beta_2 U_1 U_3 \Phi_3 + \beta_3 U_2^2 (U_1 y_6 + U_3 z_3) \\ + \beta_4 U_1^2 (U_1 y_8 + U_2 z_7) + \beta_5 U_2 U_3 (U_1 y_4 + U_2 z_3) + \beta_6 U_1^3 y_4^2.$$

Applying the operation $S_{4,4}$ to this equality and because of the relations $S_{4,4} z_9 \equiv 0 \mod 2$, $S_{4,4} y_8 \equiv 1 \mod 2$ we obtain $\beta_4 U_1^3 = 0$, or $\beta_4 \equiv 0 \mod 2$. Let us use the operation S_7, then we have $S_7 z_9 \equiv z_2 \mod (2z_2, z_1^2)$, $S_7 U_4 = U_1$. So $d_3(z_2) = U_1^2 U_2 = \beta_1 U_1^2 U_2$, i.e. $\beta_1 \equiv 1 \mod 2$. Let us use the operation $S_{2,2,2,2}$; it follows from the relations $S_{2,2,2,2} z_9 \equiv 0 \mod (2z_1)$; $S_{2,2} y_4 \equiv 1 \mod 2$, that $\beta_6 U_1^3 = 0$, or $\beta_6 \equiv 0 \mod 2$. Let us use the operation S_6, we have $S_6 z_9 \equiv z_3 \mod (2z_3, z_1 z_2, 2z_1^3)$. Hence, $d_3(z_3) = U_1 U_2^2 = (\beta_3 + 1) U_1 U_2^2$, so $\beta_3 \equiv 0 \mod 2$. Applying the operation S_5 and having in mind $S_5 z_9 \equiv 0 \mod (z_2^2, 2z_4, 2y_4, z_1 z_3, 2z_1^2 z_2, z_1^4)$, and also $S_5 \Phi_3 = U_1$, we obtain $\beta_2 U_1^2 U_3 = 0$, or $\beta_2 \equiv 0 \mod 2$. From the conditions $S_{2,2} U_4 = 0$, $S_{2,2} \Phi_3 = U_2$, $S_{2,2} z_9 \equiv 0 \mod (2z_5, z_1 z_4, z_2 z_3, 2z_1^2 z_3, z_1^3 z_2, 2z_1^5)$ and $S_{2,2} y_4 \equiv 1 \mod 2$, it follows the relation $\beta_5 U_1 U_2 U_3 = 0$, from which it follows that $\beta_5 \equiv 0 \mod 2$. So

$$d_3(z_9) = U_1 U_2 U_4.$$

8. An element $y_9 \in E_2^{0,36}$ we choose so that $\pi_2^2(y_9) = b_9$. From the action of the Landweber–Novikov operation $S_2 : S_2 a_9 = a_7 + b_7$, it follows that

$$S_2(y_9 + \text{ any element in this dimension not equal to } y_9) = z_7 + y_7 + \text{ decomposables.}$$

Hence the following conditions must be fulfilled:

$$d_3(y_9) \neq 0, \\ d_3(y_9 + \text{ any element in this dimension not equal to } y_9) \neq 0.$$

So,

$$d_3(y_9) = U_1 U_3 \Phi_3 + \beta_1 U_1 U_2 U_4 + \beta_2 U_2^2 (U_1 y_6 + U_3 z_3) \\ + \beta_3 U_1^2 (U_1 y_8 + U_2 z_7) + \beta_4 U_2 U_3 (U_1 y_4 + U_2 z_3) + \beta_5 U_1^3 y_4^2.$$

Applying the operation $S_{4,4}$ we obtain $\beta_3 U_1^3 = 0$, so $\beta_3 \equiv 0 \mod 2$. Applying the operation $S_{2,2,2,2}$ we obtain the relation $\beta_5 U_1^3 = 0$, that means that $\beta_5 \equiv 0 \mod 2$. Acting by the operation S_6 we arrive to the expression $0 = \beta_2 U_1 U_2^2$, i.e. $\beta_2 \equiv 0 \mod 2$. From the condition $S_{2,2} b_9 = 0$, fulfilled in MASS the equality follows $S_{2,2} y_9 \equiv 0 \mod (2z_5, z_1 z_4, z_2 z_3, 2z_1^5)$. Hence applying the operation $S_{2,2}$ to the equality that we are studying we obtain $\beta_4 U_1 U_2 U_3 = 0$, hence, $\beta_4 \equiv 0 \mod 2$. Because of the equality $S_7 b_9 = 0$, it follows that $S_7 y_9 \equiv 0 \mod (2z_2, z_1^2)$, this gives $\beta_1 U_1^2 U_2 = 0$, or $\beta_1 \equiv 0 \mod 2$. We get the following final form of the action of the differential d_3 on y_9:

$$d_3(y_9) = U_1 U_3 \Phi_3.$$

9. From the condition $h_0b_9 = a_2a_7 + a_4a_5$, fulfilled in the term E_∞ of the MASS it follows that one can choose an element y_9 in such a way that the following equality is fulfilled:

$$\begin{aligned}2y_9 = {} & z_2z_7 + z_4z_5 + \beta_1 2z_4z_5 + \beta_2 2z_3^3 + \beta_3 2z_3(z_2y_4 + z_3^2) + \beta_4 2z_1^5y_4 \\ & +\beta_5 2z_1^2z_3y_4 + \beta_6 2z_1z_2^2y_4 + \beta_7 2z_1^2y_7 + \beta_8 z_1^3(z_2y_4 + z_3^2) \\ & +\beta_9 2z_1z_2z_6 + \beta_{10} 2z_1^2z_7 + \beta_{11} 2z_1z_4^2 + \beta_{12} 2z_1z_3z_5 \\ & +\beta_{13} 2z_2^2z_5 + \beta_{14} 2z_1^3z_2z_4 + \beta_{15} 2z_1^4z_5 + \beta_{16} 2z_1z_2^4 \\ & +\beta_{17} 2z_1^9 + \beta_{18} 2z_1^3y_6 + \beta_{19} 2z_1^6z_3 + \beta_{20} 2z_1^5z_2^2 \\ & +\beta_{21} 2z_1^2z_2^2z_3 + \beta_{22} 2z_2z_3z_4 + \beta_{23} z_2(z_1z_6 + z_2z_5) \\ & +\beta_{24} z_4(z_1y_4 + z_2z_3) + \beta_{25} z_2^3z_3 + \beta_{26} z_1^2z_2z_5 + \beta_{27} z_1^2z_3z_4 \\ & +\beta_{28} z_1^3z_6 + \beta_{29} z_1z_2z_3^2 + \beta_{30} z_1z_2^2z_4 + \beta_{31} z_1^5z_4 \\ & +\beta_{32} z_1^3z_2^3 + \beta_{33} z_1^4z_2z_3 + \beta_{34} z_1^7z_2 + \beta_{35} 2z_1^3z_3^2.\end{aligned}$$

Let us multiply both parts of the equality by U_1. Then we have the following identity:

$$\begin{aligned}0 = {} & \beta_8 z_1^3z_2\tau_1 + \beta_{18} z_1^3\tau_2 + \beta_{23} z_1z_2\tau_2 + \beta_{24} z_1z_4\tau_1 + \beta_{28} U_1z_1^3z_6 \\ & +(\beta_{25} + \beta_{29})U_1z_2^3z_3 + (\beta_{26} + \beta_{27} + \beta_{30})U_1z_1^2z_2z_5 + \beta_{31} U_1z_1^5z_4 \\ & +(\beta_{32} + \beta_{33})U_1z_1^3z_2^3 + \beta_{34} U_1z_1^7z_2.\end{aligned}$$

It follows from this identity that $\beta_8, \beta_{18}, \beta_{23}, \beta_{24}, \beta_{28}, \beta_{31}, \beta_{34} \equiv 0 \mod 2$; $\beta_{25} \equiv \beta_{29} \mod 2$; $\beta_{32} \equiv \beta_{33} \mod 2$; $\beta_{26} + \beta_{30} + \beta_{27} \equiv 0 \mod 2$. Without loss of generality we can consider that $\beta_{30} \equiv 0 \mod 2$, hence $\beta_{26} \equiv \beta_{27} \mod 2$, and we can choose y_9 in such a way that the considering equality takes the form:

$$\begin{aligned}2y_9 = {} & z_2z_7 + z_4z_5 + \beta_1 2z_4z_5 + \beta_2 2z_3^3 + \beta_4 2z_1^5y_4 \\ & +\beta_5 2z_1^2z_3y_4 + \beta_6 2z_1z_2^2y_4 + \beta_7 2z_1^2y_7 + \beta_9 2z_1z_2z_6 \\ & +\beta_{10} 2z_1^2z_7 + \beta_{11} 2z_1z_4^2 + \beta_{12} 2z_1z_3z_5 + \beta_{13} 2z_2^2z_5 \\ & +\beta_{14} 2z_1^3z_2z_4 + \beta_{15} 2z_1^4z_5 + \beta_{16} 2z_1z_2^4 + \beta_{17} 2z_1^9 \\ & +\beta_{18} 2z_1^3y_6 + \beta_{19} 2z_1^6z_3 + \beta_{20} 2z_1^5z_2^2 + \beta_{21} 2z_1^2z_2^2z_3 \\ & +\beta_{22} 2z_2z_3z_4 + \beta_{25}(z_2^3z_3 + z_1z_2z_3^2) + \beta_{35} 2z_1^3z_3^2 \\ & +\beta_{26}(z_1^2z_2z_5 + z_1^2z_3z_4) + \beta_{32}(z_1^3z_2^3 + z_1^4z_2z_3).\end{aligned}$$

The following relations arrive from Table 2 of the work [41]:

$$\begin{aligned}\beta_{25}(z_2^3z_3 + z_1z_2z_3^2) &= \beta_{25} z_2z_3(4z_4 + 4y_4), \\ \beta_{26}(z_1^2z_2z_5 + z_1^2z_3z_4) &= \beta_{26} 2z_1z_2z_6, \\ \beta_{32}(z_1^3z_2^3 + z_1^4z_2z_3) &= \beta_{32} z_1^3z_3(4z_4 + 4y_4), \\ \beta_5 2z_1^2z_3y_4 + \beta_6 2z_1z_2^2y_4 &= \beta_6 2z_1y_4(4z_4 + 4y_4) + (\beta_5 - \beta_6)2z_1^2z_3y_4, \\ \beta_{10} 2z_1^2z_7 + \beta_{11} 2z_1z_4^2 &= \beta_{10} 2z_1(4z_8 + 4y_8 + 4z_2y_6 + 4z_4y_4) \\ &\quad +(\beta_{10} - \beta_{11})2z_1z_4^2, \\ \beta_{12} 2z_1z_3z_5 + \beta_{22} 2z_2z_3z_4 &= 2z_3y_6\beta_{12} + (\beta_{22} - \beta_{12})2z_2z_3z_4,\end{aligned}$$

$$
\begin{aligned}
\beta_{14} 2z_1^3 z_2 z_4 + \beta_{15} 2z_1^4 z_5 &= \beta_{14} 2z_1^3 2y_6 + (\beta_{15} - \beta_{14}) 2z_1^4 z_5,\\
\beta_{16} 2z_1 z_2^4 + \beta_{21} 2z_1^2 z_2^2 z_3 + \beta_{35} 2z_1^3 z_3^2 &= \beta_{16} 2z_1 z_2^2 (4y_4 + 4z_4)\\
&\quad + (\beta_{21} - \beta_{16}) 2z_1^2 z_3 (4y_4 + 4z_4)\\
&\quad + (\beta_{35} - \beta_{21} + \beta_{16}) 2z_1^3 z_3^2;\\
\beta_{19} 2z_1^6 z_3 + \beta_{20} 2z_1^5 z_2^2 &= \beta_{19} 2z_1^5 (4y_4 + 4z_4) + (\beta_{20} - \beta_{19}) 2z_1^5 z_2^2.
\end{aligned}
$$

Hence one can choose the element y_9 in order to get the relation (changing the notations of the elements β_i for convenience):

$$
\begin{aligned}
2y_9 = {} & z_2 z_7 + z_4 z_5 + \beta_1 2z_4 z_5 + \beta_2 2z_3^3 + \beta_4 2z_1^5 y_4\\
& + \beta_5 2z_1^2 z_3 y_4 + \beta_7 2z_1^2 y_7 + \beta_9 2z_1 z_2 z_6 + \beta_{11} 2z_1 z_4^2\\
& + \beta_{13} 2z_2^2 z_5 + \beta_{15} 2z_1^4 z_5 + \beta_{17} 2z_1^9 + \beta_{18} 2z_1^3 y_6\\
& + \beta_{19} 2z_1^6 z_3 + \beta_{22} 2z_2 z_3 z_4 + \beta_{35} 2z_1^3 z_3^2.
\end{aligned}
$$

Using the relation $2y_7 = z_2 z_5 + 3z_3 z_4$ we can choose the element y_9 so that $\beta_{22} = 0$. Let us apply the operation $S_{2,2}$ to our equality. From the conditions

$$
\begin{aligned}
S_{2,2} y_9 \equiv 0 \mod (2z_1^5, 2z_1^2 z_3),\ & S_2 z_5 \equiv 0 \mod (2z_1^3),\\
S_2 z_4 \equiv 0 \mod (2z_1^2),\ & S_2 z_7 \equiv 0 \mod (2z_1^5),\\
& S_2 z_2 \equiv 0 \mod 2,
\end{aligned}
$$

we obtain that $\beta_4 2z_1^5 \equiv 0 \mod (4z_1^5)$, or $\beta_4 \equiv 0 \mod 2$. By the choice of y_9 we transform our equality to the form:

$$
\begin{aligned}
2y_9 = {} & z_2 z_7 + z_4 z_5 + \beta_1 2z_4 z_5 + \beta_2 2z_3^3 + \beta_5 2z_1^2 z_3 y_4\\
& + \beta_7 2z_1^2 y_7 + \beta_9 2z_1 z_2 z_6 + \beta_{11} 2z_1 z_4^2 + \beta_{13} 2z_2^2 z_5\\
& + \beta_{15} 2z_1^4 z_5 + \beta_{17} 2z_1^9 + \beta_{18} 2z_1^3 y_6 + \beta_{19} 2z_1^6 z_3\\
& + \beta_{35} 2z_1^3 z_3^2.
\end{aligned}
$$

From the condition $S_6 y_9 \equiv 0 \mod (2z_1^3)$ we obtain that $\beta_{18} 2z_1^3 \equiv 0 \mod (4z_1^3)$, hence, $\beta_{18} \equiv 0 \mod 2$, and we can choose the element y_9, in such a way that the corresponding summand will be equal to zero. So, we get:

$$
\begin{aligned}
2y_9 = {} & z_2 z_7 + z_4 z_5 + \beta_1 2z_4 z_5 + \beta_2 2z_3^3 + \beta_5 2z_1^2 z_3 y_4\\
& + \beta_7 2z_1^2 y_7 + \beta_9 2z_1 z_2 z_6 + \beta_{11} 2z_1 z_4^2 + \beta_{13} 2z_2^2 z_5\\
& + \beta_{15} 2z_1^4 z_5 + \beta_{17} 2z_1^9 + \beta_{19} 2z_1^6 z_3 + \beta_{35} 2z_1^3 z_3^2.
\end{aligned}
$$

Multiplying our expression by z_1, and changing (according to the formula in the item 3) the expression $z_1 z_2 z_7$ by

$$
z_1 z_2 z_7 = z_2 (4y_8 + 4z_8 + 4z_2 y_6 + 4z_4 y_4) - z_2 z_4^2,
$$

and changing (according to the relation $2y_6 = z_1z_5 + z_2z_4$ from the work [41]) the expression $z_1z_4z_5$ by:

$$z_1z_4z_5 = z_42z_6 - z_2z_4^2.$$

Now let us divide all coefficients by 2, we obtain:

$$\begin{aligned}&z_1y_9 + z_4y_6 + 2(y_8 + z_8 + z_2y_6 + z_4y_4)z_2 = z_2z_4^2 + \beta_1z_1z_4z_5\\ &\quad+\beta_2z_1z_3^3 + \beta_5z_1^3z_3y_4 + \beta_7z_1^3y_7 + \beta_9z_1^2z_2z_6 + \beta_{11}z_1^2z_4^2\\ &\quad+\beta_{13}z_1z_2^2z_5 + \beta_{15}z_1^5z_5 + \beta_{17}z_1^{10} + \beta_{19}z_1^7z_3 + \beta_{35}z_1^4z_3^2.\end{aligned}$$

The rest of coefficients we determine later.

10. Let us consider the expression U_1y_9. From the action of the differential d_3: $d_3(U_1y_9) = U_1^2U_3\Phi_3$ and the relation in the MASS: $u_1b_9 = \varphi_3a_4$ it follows that one can choose the element y_9 (not changing the relation of the previous item) in such a way that the equality is fulfilled:

$$\begin{aligned}U_1y_9 = \Phi_3z_4 &+ U_1[\gamma_1z_3(z_3^2 + z_2y_4) + \gamma_2z_2(z_3z_4 + z_1y_6)\\ &+\gamma_3z_4(z_1y_4 + z_2y_3) + \gamma_4z_1^2z_2(z_1y_4 + z_2z_3) + \gamma_5z_2y_7\\ &+\gamma_6z_2z_7 + \gamma_7z_2^3z_3 + \gamma_8z_1z_8 + \gamma_9z_1^2z_2z_5 + \gamma_{10}z_1^3y_6\\ &+\gamma_{11}z_1^5z_4 + \gamma_{12}z_1^4z_2^3 + \gamma_{13}z_1^8z_2] + \gamma_{14}U_2z_3z_5.\end{aligned}$$

Let us multiply this equality by z_1, and the last equality of the item 9 we multiply by U_1 and add them up. Then we get an equality:

$$\begin{aligned}&(\beta_1 + \gamma_6 + 1)U_1z_2z_4^2 + \beta_2U_1z_1z_3^3 + \beta_5U_1z_1^3z_3y_4 + (\beta_7\\ &\quad+\beta_9)U_1z_1^3y_7 + \beta_{11}U_1z_1^2z_4^2 + \beta_{13}U_1z_1z_2^2z_5 + \beta_{15}U_1z_1^5z_5\\ &\quad+\beta_{17}U_1z_1^{10} + \beta_{19}U_1z_1^7z_3 + \beta_{35}U_1z_1^4z_3^2 + \gamma_1U_1z_1z_3(z_3^2\\ &\quad+z_2y_4) + \gamma_2U_1z_1z_2(z_3z_4 + z_1y_6) + \gamma_3U_1z_1z_4(z_1y_4\\ &\quad+z_2y_3) + \gamma_4U_1z_1^3z_2(z_1y_4 + z_2z_3) + \gamma_5U_1z_1z_2y_7\\ &\quad+\gamma_7U_1z_1z_2^3z_3 + \gamma_8U_1z_1^2z_8 + \gamma_9U_1z_1^3z_2z_5 + \gamma_{10}U_1z_1^4y_6\\ &\quad+\gamma_{11}U_1z_1^6z_4 + \gamma_{12}U_1z_1^5z_2^3 + \gamma_{13}U_1z_1^8z_2 + \gamma_{14}U_1z_2z_3z_5 = 0.\end{aligned}$$

from this we obtain that $\beta_2, \beta_5, \beta_{11}, \beta_{13}, \beta_{15}, \beta_{17}, \beta_{19}, \beta_{35}, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_7, \gamma_8, \gamma_9, \gamma_{10}, \gamma_{11}, \gamma_{12}, \gamma_{13}, \gamma_{14} \equiv 0 \mod 2$, $\beta_1 + \gamma_1 + 1 \equiv 0 \mod 2$, $\beta_7 \equiv \beta_9 \mod 2$. Hence we can change the choice of y_9, in order to obtain the relations:

$$\begin{aligned}2y_9 &= z_2z_7 + z_4z_5 + 2(\beta + 1)z_4z_5 + \gamma(z_1^2y_7 + z_1z_2z_6),\\ U_1y_9 &= \Phi_3z_4 + \beta U_1z_2z_7.\end{aligned}$$

We can choose y_9 so that $\beta = 0$, then we obtain:

$$U_1y_9 = \Phi_3z_4, 2y_9 = z_2z_7 + 3z_4z_5 + \gamma(z_1^2y_7 + z_1z_2z_6).$$

Applying the operation S_4 to this equality and having in mind the equality $S_4 y_7 \equiv z_3 \mod (2z_3)$, we obtain $\gamma z_1^2 z_3 \equiv 0 \mod (2z_1^2 z_3, 2z_1^3 z_2, 2z_1^5)$, hence it follows that $\gamma \equiv 0 \mod 2$, so one can choose the element y_9 to obtain:

$$2y_9 = z_2 z_7 + 3z_4 z_5.$$

11. Let us consider the expression $U_3 y_6$. From the action of the differential d_3: $d_3(U_3 y_6) = U_1^2 U_3 \Phi_3$ and the relation in the MASS: $u_3 b_3 = \varphi_3 a_4$, we obtain the following equality:

$$\begin{aligned} U_3 y_6 = \Phi_3 z_4 + U_1[&\beta_1 z_3(z_3^2 + z_2 y_4) + \beta_2 z_2(z_1 y_6 + z_3 z_4) \\ &+\beta_3 z_4(z_1 y_4 + z_2 z_3) + \beta_4 z_1^2 z_2(z_1 y_4 + z_2 z_3) + \beta_5 z_2 y_7 \\ &+\beta_6 z_2 z_7 + \beta_7 z_2^3 z_3 + \beta_8 z_1 z_8 + \beta_9 z_1^2 z_2 z_5 + \beta_{10} z_1^3 z_6 \\ &+\beta_{11} z_1^5 z_4 + \beta_{12} z_1^3 z_2^3 + \beta_{13} z_1^7 z_2] + U_2 \beta_{14} z_3 z_5. \end{aligned}$$

Let us multiply this equation by U_1 and use the relations:

$$U_1 y_6 = \Phi_3 z_1, U_3 z_1 = U_1 z_3,$$

we obtain $\beta_i \equiv 0 \mod 2$. Hence we obtain:

$$U_3 y_6 = \Phi_3 z_4.$$

12. Let us consider the expression $U_1 z_9$. From the condition $d_3(U_1 z_9) = U_1^2 U_2 U_4$ and the relation $u_1 a_9 = u_4 a_2$ fulfilled in the MASS it follows that one can choose the element z_9, in such a way that the equality is fulfilled:

$$U_1 z_9 = U_4 z_2 + \beta U_2 z_2 z_5.$$

Let us apply the operation S_2 to this equality. From the relations

$$\begin{aligned} &S_2 z_9 \equiv y_7 \mod (2y_7, 2z_7, \ldots), S_2 U_4 = \Phi_3, S_2 z_5 \equiv z_3 \\ &\quad \mod (2z_3, z_1 z_2, 2z_1^3) \end{aligned}$$

we obtain that $\beta U_2 z_3^2 \equiv 0 \mod 2$, hence, $\beta \equiv 0 \mod 2$. So our equality takes the form:

$$U_1 z_9 = U_4 z_2.$$

13. Let us consider the expression $U_2 z_8$. Because of relation $d_3(U_2 z_8) = U_1 U_2 U_4$, of the term E_3 of the Adams–Novikov spectral sequence and the relation $u_2 a_8 = u_4 a_2$, fulfilled in the MASS we obtain the following:

$$\begin{aligned} U_2 z_8 = U_4 z_2 + \beta_1 U_1[&\beta_1 z_3(z_3^2 + z_2 y_4) + \beta_2 z_2(z_1 y_6 + z_3 z_4) \\ &+\beta_3 z_4(z_1 y_4 + z_2 z_3) + \beta_4 z_1^2 z_2(z_1 y_4 + z_2 z_3) + \beta_5 z_2 y_7 \end{aligned}$$

$$+\beta_6 z_2 z_7 + \beta_7 z_2^3 z_3 + \beta_8 z_1 z_8 + \beta_9 z_1^2 z_2 z_5 + \beta_{10} z_1^3 z_6$$
$$+\beta_{11} z_1^5 z_4 + \beta_{12} z_1^3 z_2^3 + \beta_{13} z_1^7 z_2] + U_2 \beta_{14} z_3 z_5.$$

If we multiply this equality by U_1, and then use the relation: $U_1 z_8 = U_4 z_1$ and $U_2 z_1 = U_1 z_2$, then we get the equality:

$$\begin{aligned}
&U_1^2[\beta_1 z_3(z_3^2 + z_2 y_4) + \beta_2 z_2(z_1 y_6 + z_3 z_4) + \beta_3 z_4(z_1 y_4 + z_2 z_3)\\
&\quad + \beta_4 z_1^2 z_2(z_1 y_4 + z_2 z_3) + \beta_5 z_2 y_7 + \beta_6 z_2 z_7 + \beta_7 z_2^3 z_3 + \beta_8 z_1 z_8\\
&\quad + \beta_9 z_1^2 z_2 z_5 + \beta_{10} z_1^3 z_6 + \beta_{11} z_1^5 z_4 + \beta_{12} z_1^3 z_2^3 + \beta_{13} z_1^7 z_2]\\
&\quad + U_1 U_2 \beta_{14} z_3 z_5 = 0.
\end{aligned}$$

From the last equality we conclude that all the elements β_i are equal to $0 \mod 2$ and so the initial equality can be transformed to the form:

$$U_2 z_8 = U_4 z_2.$$

14. Let us choose an element $z_{10} \in E_2^{0,40}$ so that the following equality is fulfilled: $\pi_2^2(z_{10}) = b_{10} + a_4 c_6$. From the action of the Landweber–Novikov operation S_2 in the MASS: $S_2 b_{10} = b_8$, we conclude that in the Adams–Novikov spectral sequence the following relation need to be fulfilled:

$$S_2(z_{10} + \text{ any element from } \quad F^2 E_2^{0,40}, \text{not equal to } z_{10}) = z_8 + \text{decomposables.}$$

Hence

$$d_3(z_{10}) \neq 0,$$
$$d_3(z_{10} + \text{ decomposable}) \neq 0.$$

Hence we choose the element z_{10} so that the following equality will be fulfilled:

$$\begin{aligned}
d_3(z_{10}) = {} & U_1^2 \Phi_5 + \beta_1 U_3^3 + \beta_2 U_2 U_3 \Phi_3 + \beta_3 U_2^2 U_4\\
& + \beta_4 U_1 U_2 (U_1 y_8 + U_3 z_7) + \beta_5 U_1^2 U_2 y_4^2\\
& + \beta_6 U_2^2 (U_2 y_6 + U_3 y_4) + \beta_7 U_1 \Phi_3 (U_1 y_4 + U_2 z_3)\\
& + \beta_8 U_1 U_3 (U_1 y_6 + U_3 z_3).
\end{aligned}$$

Let us apply the operation $S_{4,4}$ to this equality. From the conditions: $S_{4,4}\Phi_5 = \Phi_1$, $S_{4,4} z_{10} \equiv z_2 \mod (2z_2, z_1^2)$, $S_{4,4} y_8 \equiv 1 \mod 2$, we get the relation $\beta_4 \equiv 0 \mod 2$. From the relations fulfilled for the operation $S_{2,2,2,2}$: $S_{2,2,2,2}\Phi_5 = \Phi_1$, $S_{2,2,2,2} z_{10} \equiv z_2 \mod (2z_2, z_1^2)$, and the operation $S_{2,2}$: $S_{2,2} y_4 \equiv 1 \mod 2$, $S_{2,2}$ $\Phi_3 = \Phi_1$, we obtain the equality

$$d_3(z_2) = (1 + \beta_5 + \beta_7 + \beta_8) U_1^2 U_2,$$

from which it follows that $\beta_5+\beta_7+\beta_8\equiv 0 \mod 2$. Using the operation S_7, we arrive to the relation $\beta_3 U_1 U_2^2$, from which we conclude, that $\beta_3\equiv 0 \mod 2$. Acting by the operation S_6 on our equality and having in mind the relation: $S_6 z_{10}\equiv z_4 \mod (2z_4, 2y_4, z_1z_3, 2z_1^2z_2, z_2^2, z_1^4)$, we get the equality $d_3(z_4)=(1+\beta_6)U_1^2U_3+\beta_8U_2^3$. Hence, $\beta_6, \beta_8\equiv 0 \mod 2$. Using the conditions

$$S_{2,2}z_{10}\equiv z_6 \mod (2z_6, 2y_6, z_1z_5, z_2z_4, z_3^2, 2z_1^2z_4, z_1^3z_3, 2z_1^4z_2, z_1^2z_2^2, z_1^6),$$
$$S_{2,2}\Phi_5=\Phi_3,$$

and applying the operation $S_{2,2}$ to the considering equality, we obtain:

$$d_3(y_6)=\beta_7U_1U_2(U_1y_4+U_2z_3)+\beta_1U_2^2U_3+(1+\beta_7)U_1^2\Phi_3.$$

This gives the relations: $\beta_1, \beta_7\equiv 0 \mod 2$. From the relation $\beta_5+\beta_7+\beta_8\equiv 0 \mod 2$ we conclude $\beta_5\equiv 0 \mod 2$. Applying the operation S_2 to our equality. From the equality $S_2(U_2U_3\Phi_3)=U_2^2\Phi_3+U_2U_3^2$ it follows that $\beta_2\equiv 0 \mod 2$. So, finally we get:

$$d_3(z_{10})=U_1^2\Phi_5.$$

15. Choose the element $y_{10}\in E_2^{0,40}$ so that $\pi_0^2(y_{10})=c_{10}$. From the condition:

$$S_2(y_{10}+\text{any element in this dimension not equal to } y_{10})$$
$$\equiv y_8 \mod (\text{elements from} F^2E_2^{0,32})$$

the relations follow:

$$d_3(y_{10})\neq 0,$$
$$d_3(y_{10}+\text{any element in this dimension not equal to } y_{10})\neq 0.$$

Hence we can choose the element y_{10} so that the equality will be fulfilled:

$$d_3(y_{10})=\beta_1U_3^3+\beta_2U_2U_3\Phi_4+\beta_3U_2^2U_4+\beta_4U_2^2(U_2y_6+U_3y_4).$$

Apply the operation S_6 to this equality and notice that $S_6y_{10}\equiv 0 \mod (2y_4)$, so we get the relation $(\beta_3+\beta_4)U_2^3=0$, hence $\beta_3\equiv\beta_4 \mod 2$. Because of the fact $S_{2,2}y_{10}\equiv y_6 \mod (\text{elements from } F^2E_2^{0,24})$, after the application of the operation $S_{2,2}$ to our equality we obtain $d_3(y_6)=(\beta_1+\beta_4)U_2^2U_3$. This gives the relation $\beta_1+\beta_4\equiv 1 \mod 2$. Using the operation S_2 we arrive to the equality

$$d_3(y_8)=(\beta_1+\beta_2)U_2U_3^2+(\beta_2+\beta_3)U_2^2\Phi_3.$$

This gives us the relations: $\beta_2\equiv\beta_3 \mod 2\beta_1+\beta_2\equiv 1 \mod 2$. Finally if we use the operation $S_{1,1}$, and have in mind the fact that $S_{1,1}y_{10}\equiv y_8+y_4^2 \mod$ (elements from

$F^2E_2^{0,32}$), the we get the following expression: $d_3(y_8) = \beta_1 U_2 U_3^2 + (\beta_2+\beta_3)U_2^2\Phi_3 + \beta_4 U_1^2(U_2 y_6 + U_3 y_4)$. Hence $\beta_1 \equiv 1 \mod 2$, $\beta_2, \beta_3, \beta_4 \equiv 0 \mod 2$. Finally we have the formula for the action of d_3:

$$d_3(y_{10}) = U_3^3.$$

16. Choose the element $y_{10}^* \in E_2^{0,40}$ so that $\pi_0^2(y_{10}^*) = c_5^2 + c_2^2 c_6$. From the condition:

$$S_{3,3}(y_{10}^* + \text{ any element from this dimension not equal to } y_{10}^*)$$
$$\equiv y_4 \mod (\text{elements form } F^2E_2^{0,16})$$

the relations follow:

$$d_3(y_{10}) \neq 0,$$
$$d_3(y_{10} + \text{ any element from his dimension not equal to } y_{10}^*) \neq 0.$$

Hence it is possible to choose the element y_{10}^* so that the following equality is fulfilled:

$$d_3(y_{10}^*) = \beta_1 U_3^3 + \beta_2 U_2 U_3 \Phi_4 + \beta_3 U_2^2 U_4 + \beta_4 U_2^2(U_2 y_6 + U_3 y_4).$$

Applying the operation $S_{3,3}$ to this equality and noticing that $S_{3,3}y_{10} \equiv y_4 \mod (2y_4)$,
$S_{3,3}y_6 \equiv 0 \mod 2$, $S_{3,3}U_4 = U_2$, we obtain the relation $d_3(y_4) = \beta_3 U_2^3$, hence, $\beta_3 \equiv 1 \mod 2$. If we apply the operation S_6 to our equality and notice that $S_6 y_{10}^* \equiv y_4 \mod (2y_4)$, we get the relation $d_3(y_4) = (1+\beta_4)U_2^3$, hence $\beta_4 \equiv 0 \mod 2$. Using the operation S_2, and having in mind the equality

$$S_2 y_{10}^* \equiv y_4^2 \mod (\text{elements from } F^2E_2^{0,32} \text{ and decomposables}),$$

we come to the expression

$$0 = (\beta_1 + \beta_2)U_2 U_3^2 + (\beta_2 + 1)U_2^2 \Phi_3,$$

hence, $\beta_1 \equiv 1 \mod 2$, $\beta_2 \equiv 1 \mod 2$. Finally our equality has the form:

$$d_3(y_{10}^*) = U_2^2 U_4 + U_2 U_3 \Phi_3 + U_3^3.$$

17. The is an equality: $h_0^2 e_{10} = a_3 a_7 + a_5^2$ in the MASS. Hence, one can choose the element y_{10}^* so that the following equality will be fulfilled:

$$4(y_{10}^* + y_4 y_6) = (1+2\beta_1)z_3 z_7 + (1+2\beta_2)z_5^2 + X_{40}^1(\widehat{y_{10}^*}, \widehat{y_4 y_6}, \widehat{z_3 z_7}, \widehat{z_5^2}).$$

where $X_{40}^1 \in F^6(E_2^{0,40})$, $d_3(X_{40}^1) = 0$, and the sign $(\widehat{\ })$ over an element means that the given element does not appear in the expression for X_{40}^1. Let us apply the operation

$S_{5,5}$ to our equality, we get the relation $S_{5,5}y^*_{10} \equiv ((S_5z_5)/2)^2(1+2\beta_2) \mod 4$; we may suppose that $S_{5,5}y^*_{10} \equiv ((S_5z_5)/2)^2 \mod 4$. Hence, $\beta_2 \equiv 0 \mod 2$, so, by the choice of y_{10} we can transform our equality to the form

$$4(y^*_{10} + y_4y_6) = (1+2\beta_1)z_3z_7 + z_5^2 + X^1_{40}(\widehat{y^*_{10}}, \widehat{y_4y_6}, \widehat{z_3z_7}, \widehat{z_5^2}).$$

Let us apply the operation S_7 to this expression, we get the relation: $S_7y^*_{10} \equiv ((S_7z_7)/4)(1+2\beta_1)z_3 \mod (8z_3)$. We may suppose that $S_7y^*_{10} \equiv ((S_7z_7)/4)z_3 \mod (8z_3)$. Hence, $\beta_1 \equiv 0 \mod 2$ and by the choice of y^*_{10} it is possible to transform our expression to the form:

$$4(y^*_{10} + y_4y_6) = z_3z_7 + z_5^2 + X^1_{40}(\widehat{y^*_{10}}, \widehat{y_4y_6}, \widehat{z_3z_7}, \widehat{z_5^2}).$$

18. There is the equality: $h_0b_{10} = a_2a_8 + a_1a_9$ in the MASS. Hence it is possible to choose the element z_{10} so that the following equality is fulfilled:

$$2(z_{10} + z_4y_6) = z_2z_8 + (1+2\beta)z_1z_9 + X^2_{40}(\widehat{z_{10}}, \widehat{z_4y_6}, \widehat{z_2z_8}, \widehat{z_1z_9}).$$

In this expression we have: $X^2_{40} \in F^6(E_2^{0,40})$, $d_3(X^2_{40}) = 0$. Applying the operation S_9 to this expression we get the following relation: $S_9z_{10} \equiv ((S_9z_9)/2)(1+2\beta)z_1 \mod (4z_1)$. We may suppose that $S_9z_{10} \equiv ((S_9z_9)/2)3z_1 \mod (4z_1)$. Hence it is possible to choose z_{10} so that the following equality is fulfilled:

$$2(z_{10} + z_4y_6) = z_2z_8 + 3z_1z_9 + X^2_{40}(\widehat{z_{10}}, \widehat{z_4y_6}, \widehat{z_2z_8}, \widehat{z_1z_9}).$$

Applying the operation S_7 to this relation we obtain

$$S_7z_{10} \equiv 0 \mod (2z_1z_2, 2z_1^3, 4z_3).$$

19. From the relation $u_1(b_{10} + a_4c_6) = a_1\varphi_6$ in the MASS and the action of the differential: $d_3(U_1z_{10}) = U_1^3\Phi_5$ it follows that we may choose the element z_{10} so that the equality is fulfilled:

$$U_1z_{10} = \Phi_5z_1 + \beta U_1z_2z_8.$$

Applying the operation S_7 to this equality we obtain the following relation: $\beta U_1z_1z_2 \equiv 0 \mod (2z_1z_2)$, from where it follows that one may choose the element z_{10} so that the equality is fulfilled:

$$U_1z_{10} = \Phi_5z_1.$$

20. Let us consider the expression: U_2z_9. From the condition: $d_3(U_2z_9) = U_1^2U_2U_4$ and $u_2a_9 = u_4a_3$ in the MASS it follows that one can choose the element z_9 (not violating the previous relations) that the following relation is fulfilled: $U_2z_9 = U_4z_2 + Y[\beta_i]$ (*), where $Y[\beta_i]$ has the following form:

$$\begin{aligned}Y[\beta_i] = {} & [\beta_1 z_3(z_3 y_4 + z_3 z_4) + \beta_2 z_1(z_1 y_8 + z_2 z_7) + \beta_3 z_1 z_2^2(z_1 y_4 + z_2 z_3) \\ & + \beta_4 z_1^3(z_1 y_6 + z_2 z_5) + \beta_5 z_1^4(z_1^2 y_4 + z_2^3) + \beta_6 z_3(z_1 y_6 + z_2 z_5) \\ & + \beta_7 z_5(z_1 y_4 + z_2 z_3) + \beta_8 z_1 z_9 + \beta_9 z_3 z_7 + \beta_{10} z_1^3 z_7 + \beta_{11} z_2^3 z_4 \\ & + \beta_{12} z_1^5 z_5 + \beta_{13} z_2^2 z_3^2 + \beta_{14} z_1^2 z_2^4 + \beta_{15} z_1^6 z_2^2 + \beta_{16} z_1^{10} + \beta_{17} z_4 y_6 \\ & + \beta_{18} z_3 y_7 + \beta_{19} z_1^3 y_7 + \beta_{20} z_1^2 y_4^2] U_1 + \beta_{21} U_2 z_3(z_2 y_4 + z_3^2)\end{aligned}$$

If we multiply now the expression (*) by U_1 and use the relations $U_1 U_2 z_9 = U_2 U_4 z_1 = U_1 U_4 z_2$, then we get $U_1 Y[\beta_i] = 0$ Hence, it follows for all $\beta_i \equiv 0 \mod 2$, and so:

$$U_2 z_9 = U_4 z_2.$$

21. From the action of the differential d_3 and the relation $u_2 a_9 = u_3 a_7 = \varphi_3 a_5$ in the MASS it follows that one can choose the element y_9 (not violating the previous relations) that in the Adams–Novikov spectral sequence the following relations will be fulfilled:

$$U_2 y_9 = U_3 y_7 + Y[\beta_i], \quad U_2 y_9 = \Phi_3 z_5 + Y[\beta_i].$$

Here $Y[\beta_i]$ denotes the expression from the item 20. Multiplying both relations (term by term) by U_1, we obtain that in both cases all coefficients $\beta_i \equiv 0 \mod 2$. So, we have:

$$U_2 y_9 = U_3 y_7, \quad U_2 y_9 = \Phi_3 z_5.$$

22. There is the relation in the MASS: $\varphi_3^2 = u_1^2 c_5^2 + u_2^2 c_4^4 + u_3^2 c_2^2$. Hence it is possible to choose the element y_{10}^* (not violating his properties) so that the following equality is fulfilled in the Adams–Novikov spectral sequence:

$$\begin{aligned}\Phi_3^2 = {} & U_1[U_1 y_{10}^* + U_2(y_9 + z_9) + U_3 z_7] + U_2^2 y_8 + U_3^2 y_4 \\ & + U_1[U_1 y_4 y_6 + U_2(y_4 z_5 + y_6 z_3)] + \beta U_2^2 z_3 z_5.\end{aligned}$$

Let us apply the operation S_2 to our equality. From the relation $U_1 z_8 = U_4 z_1$ it follows that $S_2 z_8 \equiv 0 \mod (2z_3^2)$, and because of the relation from the item 3 it follows that $S_2 y_8 \equiv 0 \mod (2z_3^2)$. So, we obtain that $\beta U_2^2 z_3^2 \equiv 0 \mod (2U_2^2 z_3^2)$, Hence, $\beta \equiv 0 \mod 2$. Finally:

$$\begin{aligned}\Phi_3^2 = {} & U_1[U_1 y_{10}^* + U_2(y_9 + z_9) + U_3 z_7] + U_2^2 y_8 + U_3^2 y_4 \\ & + U_1[U_1 y_4 y_6 + U_2(y_4 z_5 + y_6 z_3)].\end{aligned}$$

23. An element $z_{11} \in E_2^{0,44}$ we choose so that $\pi_2^2(z_{11}) = a_{11}$. Because of the relation $S_3 a_{11} = a_8$ we obtain that

$S_3(z_{11} +$ any element from this dimension not equal to $z_{11}) = z_8 +$ decomposables.

Hence,

$$d_3(z_{11}) \neq 0,$$
$$d_3(z_{11} + \text{ any element from this dimension not equal to } z_{11}) \neq 0.$$

Hence it is possible to choose the element z_{11} so that for the action of the differential d_3 the following equality will be fulfilled:

$$\begin{aligned} d_3(z_{11}) = U_1U_3U_4 + \beta_1 U_1\Phi_3^2 + \beta_2 U_1U_2\Phi_5 + \beta_3 U_2\Phi_3(U_1y_4 \\ +U_2z_3) + \beta_4 U_2^2(U_1y_8 + U_2z_7) + \beta_5 U_2U_3(U_2y_6 + U_3y_4) \\ +\beta_6 U_3^2(U_1y_4 + U_2z_3) + \beta_7 U_1U_2^2y_4^2. \end{aligned}$$

Let us apply the operation S_9 to this expression. From the relation $S_9z_{11} \equiv 0 \mod (2z_2)$, we obtain the equality $\beta_2 U_1^2 U_2 = 0$. Hence, $\beta_2 \equiv 0 \mod 2$. Acting on the equality under consideration by the operation $S_{5,5}$ and using the relation $S_{5,5}z_{11} \equiv 0 \mod (2z_1)$, we obtain $\beta_1 \equiv 0 \mod 2$. Let us use now the operation $S_{4,4}$. Because of the equality $S_{4,4}z_{11} \equiv 0 \mod (2z_3)$, it follows the relation: $\beta_4 \equiv 0 \mod 2$. Analogous use of the operation $S_{2,2,2,2}$ together with the relations $S_{2,2,2,2}z_{11} \equiv 0 \mod (2z_3)$, $S_{2,2,2}y_6 \equiv 1 \mod 2$, $S_{2,2}\Phi_3 = U_2$, gives us the condition $(\beta_3 + \beta_5 + \beta_6 + \beta_7)U_1U_2^2 = 0$. Hence, $\beta_3 + \beta_5 + \beta_6 + \beta_7 \equiv 0 \mod 2$. Using the operation S_6 we get the following: $S_6z_{11} \equiv z_5 \mod (2z_5)$, $S_6y_6 \equiv 1 \mod 2$, and so, $d_3(z_5) = U_1U_2U_3 = U_1U_2U_3 + \beta_5 U_1U_2U_3$. Hence, $\beta_5 \equiv 0 \mod 2$. From the following property of the operation S_5: $S_5z_{11} \equiv 0 \mod (2y_6, 2z_6)$, it follows that $\beta_3 \equiv 0 \mod 2$. Finally using the operation $S_{2,2}$ we have: $S_{2,2}z_{11} \equiv y_7 \mod (2y_7, 2z_7)$, that gives:

$$d_3(y_7) = U_1U_2\Phi_3 = U_1U_2\Phi_3 + \beta_6 U_1U_3^2 + \beta_6 U_2^2(U_1y_4 + U_2z_3).$$

Hence, $\beta_6 \equiv 0 \mod 2$. From the previous relations we obtain now that $\beta_7 \equiv 0 \mod 2$. Summarizing all relations we get finally:

$$d_3(z_{11}) = U_1U_3U_4.$$

24. An element $y_{11} \in E_2^{0,44}$ we choose so that $\pi_2^2(y_{11}) = b_{11} + a_5c_6$. Because of the relation $S_9b_{11} = a_2$, we obtain that

$$S_9(y_{11} + \text{ any element of this dimension not equal to } y_{11}) \equiv z_2 \mod (2z_2, z_1^2).$$

Hence,

$$d_3(y_{11}) \neq 0,$$
$$d_3(y_{11} + \text{ any element of this dimension not equal to } y_{11}) \neq 0.$$

Hence it is possible to choose the element y_{11}, so that for the action of the differential d_3 the following equality is fulfilled:

$$\begin{aligned} d_3(y_{11}) = {} & U_1U_2\Phi_5 + \beta_1 U_1U_3U_4 + \beta_2 U_1\Phi_3^2 + \beta_3 U_2\Phi_3(U_1y_4 + U_2z_3) \\ & + \beta_4 U_2^2(U_1y_8 + U_2z_7) + \beta_5 U_2U_3(U_2y_6 + U_3y_4) \\ & + \beta_6 U_3^2(U_1y_4 + U_2z_3) + \beta_7 U_1U_2^2y_4^2. \end{aligned}$$

Acting by the operation $S_{5,5}$ on our equality an using the relation $S_{5,5}y_{11} \equiv 0 \mod (2z_1)$, we obtain that $\beta_2 \equiv 0 \mod 2$. Analogous use of the operation $S_{2,2,2,2}$, together with the relations

$$S_{2,2,2,2}y_{11} \equiv z_3 \mod (2z_3),\ S_{2,2,2}y_6 \equiv 1 \mod 2,\ S_{2,2,2,2}\Phi_5 = U_2,$$

gives the formula

$$d_3(z_3) = U_1U_2^2 = U_1U_2^2 + (\beta_3 + \beta_5 + \beta_6 + \beta_7)U_1U_2^2.$$

Hence, $\beta_3 + \beta_5 + \beta_6 + \beta_7 \equiv 0 \mod 2$. Let us use the operation $S_{4,4}$. From the relations: $S_{4,4}y_{11} \equiv z_3 \mod (2z_3)$ and $S_{4,4}\Phi_5 = U_2$ we obtain the equality: $d_3(z_3) = U_1U_2^2 = U_1U_2^2 + \beta_4 U_1U_2^2$, hence, it follows that $\beta_4 \equiv 0 \mod 2$. Let us apply the operation S_7 to our equality. Because of the relation $S_7y_{11} \equiv 0 \mod (2z_4, 2y_4)$, we obtain: $\beta_1 \equiv 0 \mod 2$. From the following property of the operation S_6: $S_6y_{11} \equiv z_5 \mod (2z_5)$, we obtain that

$$d_3(z_5) = U_1U_2U_3 = U_1U_2U_3 + \beta_5 U_1U_2U_3.$$

Hence, $\beta_5 \equiv 0 \mod 2$. From the following property of the operation S_5: $S_5y_{11} \equiv 0 \mod (2y_6, 2z_6)$ we obtain that $\beta_3 U_1U_2(U_1y_4 + U_2z_3) = 0$. Hence, $\beta_3 \equiv 0 \mod 2$. Finally using the operation $S_{2,2}$, we obtain: $S_{2,2}y_{11} \equiv y_7 \mod (2y_7, 2z_7)$, this gives:

$$d_3(y_7) = U_1U_2\Phi_3 = U_1U_2\Phi_3 + \beta_6 U_1U_3^2 + \beta_6 U_2^2(U_1y_4 + U_2z_3).$$

Hence, $\beta_6 \equiv 0 \mod 2$. From the previous relations we obtain now $\beta_7 \equiv 0 \mod 2$. So, finally we get:

$$d_3(y_{11}) = U_1U_2\Phi_5.$$

25. There are the following equalities in the MASS: $h_0e_4a_7 + h_0e_8a_3 = a_2b_9 + a_4b_7$, and $h_0e_8a_3 + h_0e_{10}a_1 = a_4b_7 + a_5b_6$. Using the relations 4, 6, 9 from the work [41] and relations of the items 3, 9, 16 of the present work, we obtain that there is the

equality in the Adams–Novikov spectral sequence:

$$\begin{aligned}2y_8z_3+2(y_{10}^*+y_6y_4)z_1 &= z_4y_7+z_5z_6\\ &\quad +X_{44}^2\left(\widehat{y_6z_5},\widehat{y_8z_3},z_1(\widehat{y_{10}^*+y_6y_4}),\widehat{z_4y_7}\right).\\ 2y_4z_7+2y_8z_3 &= z_2y_9+z_4y_7\\ &\quad +X_{44}^1\left(\widehat{y_4z_7},\widehat{y_8z_3},\widehat{z_2y_9},\widehat{z_4y_7}\right).\end{aligned}$$

Here we have: $X_{44}^i\in F^6(E_2^{0,44})$, $d_3(X_{44}^i)=0$, and the symbol $\widehat{\ }$ over an element means that this element does not appear in the given expression.

26. From the relation $h_0b_{11}=a_3a_8+a_2a_9$ in the MASS, the action of the operation and the relation 9 from the work [15] it follows that we can choose the element y_{11} so that the relation is fulfilled:

$$2y_{11}=z_2z_9+3z_3z_8+X_{44}\left(\widehat{y_{11}},\widehat{z_2z_9},\widehat{z_3z_8}\right).$$

Here: $X_{44}\in F^6(E_2^{0,44})$, $d_3(X_{44})=0$, and the symbol $\widehat{\ }$ has the same meaning as earlier.

27. From the relation: $u_1(a_1e_{10}+a_3e_8+a_7e_4)=\varphi_3b_6$, valid in the MASS, the equality follows:

$$\begin{aligned}&U_1(z_1y_{10}^*+z_1y_4y_6+z_3y_8+z_7y_4)=\Phi_3y_6+U_1[\beta_1z_1z_{10}\\ &\quad+\beta_2z_2z_9+\beta_3z_2y_9+\beta_4z_1^2z_2z_7+\beta_5z_4z_7+\beta_6z_2(z_1y_8\\ &\quad+z_2z_7)+\beta_7z_4(z_1y_6+z_3z_4)+\beta_8z_1^2z_4(z_1y_4+z_2z_3)\\ &\quad+\beta_9z_1^2z_2(z_1y_6+z_3z_4)+\beta_{10}z_2^3(z_1y_4+z_2z_3)+\beta_{11}z_1^4\\ &\quad\times z_2(z_1y_4+z_2z_3)+\beta_{12}z_6(z_1y_4+z_2z_3)+\beta_{13}z_3(z_2y_6\\ &\quad+z_4y_4)+\beta_{14}z_3(z_2y_6+z_3z_5)+\beta_{15}z_1^2z_2y_7+\beta_{16}z_1^3z_8\\ &\quad+\beta_{17}z_1^3z_2z_3^2+\beta_{18}z_1^3z_2^2z_4+\beta_{19}z_1^7z_4+\beta_{20}z_1^5z_6+\beta_{21}\\ &\quad\times z_1^5z_2^3+\beta_{22}z_1^9z_2+\beta_{23}z_1z_2z_3z_5+\beta_{24}z_1z_2y_4^2+\beta_{25}z_4z_7\\ &\quad+\beta_{26}z_2z_3^3]+\beta_{27}U_2z_3z_7+\beta_{28}U_2z_3y_7.\end{aligned}$$

Multiplying this equality by U_1, and the equality from the item 22 by z_1 and add them. Using the relations: $U_1^2z_3=U_2^2z_1$, $U_1^2z_7=U_3^2z_1$, $U_1y_6=\Phi_3z_1$, $U_1z_2=U_2z_1$, we obtain, that $\beta_2,\beta_3,\beta_5,\beta_{13}\equiv 1\mod 2$, and for all the other i the relation is valid: $\beta_i\equiv 0\mod 2$. So, finally we get:

$$\begin{aligned}&U_1(z_1y_{10}^*+z_1y_4y_6+z_3y_8+z_7y_4)+U_1z_4z_7+U_1z_2z_9\\ &\quad+U_1z_2y_9+U_1z_3(z_2y_6+z_4y_4)=\Phi_3y_6.\end{aligned}$$

28. Let us consider the product U_1y_{11}. From the action of the differential d_3: $d_3(U_1y_{11})=U_1^2U_2\Phi_5$ and the relation valid in the MASS: $u_1(b_{11}+a_5c_6)=\varphi_5a_2$,

it follows that one can choose y_{11} so that the following formula holds:

$$U_1 y_{11} = \Phi_5 z_2 + \beta_1 U_2 z_3 z_7 + \beta_2 U_2 z_3 y_7 + \beta_3 U_3 y_4^2 + \beta_4 U_1 z_3 z_8.$$

From the relation of the item 25 it follows that $S_6 y_{11} \equiv 0 \mod (2z_2z_3)$. Hence, if we apply the operation S_6 to our equality we obtain: $\beta_4 U_1 z_2 z_3 = 0$, hence, $\beta_4 \equiv 0 \mod 2$. Let us apply the operation $S_{2,2,2,2}$ to our equality, then we have: $U_1 z_3 = U_2 z_2 + \beta_3 U_3$. Hence, $\beta_3 \equiv 0 \mod 2$. Now let us act by the operation $S_{3,3}$ on our equality. We shall have: $\beta_1 U_2 z_1 z_3 = 0$, hence, $\beta_1 \equiv 0 \mod 2$. Finally, if we apply the operation S_5, then we get the relation $\beta_2 U_2 z_2 z_3 = 0$, in other words: $\beta_2 \equiv 0 \mod 2$. Hence, it is possible to choose the element y_{11} in such a way that the relation holds:

$$U_1 y_{11} = \Phi_5 z_2.$$

29. Let us consider the product $U_2 z_{10}$. The action of the differential d_3 on this element in the following: $d_3(U_2 z_{10}) = U_1^2 U_2 \Phi_5$. Also we have the formula in the MASS: $u_2(b_{10} + a_4 c_6) = \varphi_5 a_2$. Hence by the choice of the element z_{10} we can obtain the following equality: $U_2 z_{10} = \Phi_5 z_2 + X(\beta_i)$, where $X(\beta_i)$ has the decomposition:

$$\begin{aligned} X(\beta_i) = U_1[&\beta_1 z_1^2 z_2 z_7 + \beta_2 z_1 z_{10} + \beta_3 z_2(z_{111} y_8 + z_2 z_7) + \beta_4 z_4(z_1 y_6 + z_3 z_4) \\ &+ \beta_5 z_1^2 z_4(z_1 y_4 + z_2 z_3) + \beta_6 z_1^2 z_2(z_1 y_6 + z_2 z_3) + \beta_7 z_2^3(z_1 y_4 + z_2 z_3) \\ &+ \beta_8 z_1^4 z_2(z_1 y_4 + z_2 z_3) + \beta_9 z_6(z_1 y_4 + z_2 z_3) + \beta_{10} z_3(z_2 y_6 + z_4 y_4) \\ &+ \beta_{11} z_3(z_2 y_6 + z_3 z_5) + \beta_{12} z_1^2 z_2 y_7 + \beta_{13} z_1^3 z_2 z_3^2 + \beta_{14} z_1^3 z_8 \\ &+ \beta_{15} z_1^7 z_4 + \beta_{16} z_1^3 z_2^2 z_4 + \beta_{17} z_1^5 z_6 + \beta_{18} z_1^5 z_2^3 + \beta_{19} z_1^9 z_2 \\ &+ \beta_{20} z_4 z_7 + \beta_{21} z_1 z_2 z_3 z_5 + \beta_{22} z_1 z_2 y_4^2 + \beta_{23} z_2 y_9 + \beta_{24} z_2 z_9 \\ &+ \beta_{25} z_2 z_3^3] + \beta_{26} U_2 z_3 z_7 + \beta_{27} U_2 z_3 y_7. \end{aligned}$$

Multiplying the expression $U_2 z_{10} = \Phi_5 z_2 + X(\beta_i)$ by U_1, and using the relations: $U_1 z_{10} = \Phi_5 z_1$ and $U_1 z_2 = U_2 z_1$ we get the following: $U_1 X(\beta_i) = 0$. Hence it follows that for all $i = 1, 2, \ldots, 27$, $\beta_i \equiv 0 \mod 2$. So, finally we obtain:

$$U_2 z_{10} = \Phi_5 z_2.$$

30. Let us consider the expression: $U_1 z_{11}$. Having in mind the action of the differential d_3: $d_3(U_1 z_{11}) = U_1^2 U_3 U_4$ and the relation in the MASS: $u_1 a_{11} = u_3 a_8$, we get the equality:

$$U_1 z_{11} = U_3 z_8 + \beta_1 U_2 z_3 z_7 + \beta_2 U_2 z_3 y_7 + \beta_3 U_3 y_4^2.$$

Let us act by the operation $S_{2,2,2,2}$ on this relation, we get $\beta_3 U_3 = 0$, hence, $\beta_3 \equiv 0 \mod 2$. Acting by the operation $S_{3,3}$, we get: $U_1 z_5 = U_3 z_2 + \beta_1 U_2 z_1 z_3$, so, $\beta_1 \equiv 0 \mod 2$. Finally, using the operation S_5, we get the expression: $\beta_2 \equiv 0 \mod 2$. The final form of the equality is the following:

$$U_1 z_{11} = U_3 z_8.$$

31. Let us consider the product: U_3z_8. Because of the fact that $d_3(U_3z_8) = U_1^2U_3U_4$ and in the MASS we have the relation: $u_3a_8 = u_4a_4$, it follows that it is possible to choose the element z_8 to fulfill the equality:

$$U_3z_8 = U_4z_4 + X(\beta_i),$$

where the expression $X(\beta_i)$ is defined in the item 29. Let us multiply this expression by U_1 and use the relations: $U_1z_8 = U_4z_1$ and $U_1z_4 = U_3z_1$. Then we get: $U_1X(\beta_i) = 0$. Hence, $\beta_i \equiv 0 \mod 2$. So, we have:

$$U_3z_8 = U_4z_4.$$

32. An element $z_{12} \in E_2^{0,40}$ we choose to fulfill the condition: $\pi_2^2(z_{12}) = b_{12} + a_2c_{10} + a_4c_4^2$. From the condition:

$$S_2(z_{12} + \text{any element of this dimension not equal to } z_{12}) = z_{10} + \cdots$$

it follows that the conditions are fulfilled: $d_3(z_{12}) \neq 0$, $d_3(z_{12} + \cdots) \neq 0$. Hence, we can choose z_{12} to satisfy the conditions:

$$\begin{aligned} d_3(z_{12}) = {} & U_1^2\Phi_6 + \beta_1U_2^2\Phi_5 + \beta_2U_2\Phi_3^2 + \beta_3U_2U_3U_4 + \beta_4U_3^2\Phi_3 + \beta_5U_2^3y_4^2 \\ & +\beta_6U_2^2(U_2y_8 + U_3y_6) + \beta_7U_2U_3(U_2y_6 + U_3y_4) + \beta_8U_1^2U_3y_4^2 \\ & +\beta_{12}U_1U_4(U_1y_4 + U_2z_3) + \beta_9U_1U_3(U_1y_8 + U_2z_7) + \beta_{10}U_1\Phi_3(U_1y_6 \\ & +U_3z_3) + \beta_{11}U_1U_2(U_1(y_{10} + y_{10}^*) + U_2(z_9 + y_9)). \end{aligned}$$

Applying the operation S_{10} to our equality and having in mind the relation $S_{10}(z_{12}) \equiv z_2 \mod (2z_2, z_1^2)$, we obtain: $d_3(z_2) = \beta_{11}U_1^2U_2 + U_1^2U_2$. Hence, $\beta_{11} \equiv 0 \mod 2$. Let us use the operation $S_{5,5}$. We have the relations: $S_{5,5}z_{12} \equiv z_2 \mod (2z_2, z_1^2)$ and $S_{5,5}\Phi_6 = U_2$. So, we get the equality: $\beta_2 \equiv 0 \mod 2$. From the conditions: $S_9z_{12} \equiv 0 \mod (2z_3)$ and $S_9\Phi_5 = U_1$ it follows that if we apply the operation S_9 to our equality we obtain the relation: $\beta_1 \equiv 0 \mod 2$. Using the operation $S_{4,4}$ and having in mind the facts: $S_{4,4}z_{12} \equiv 0 \mod (2z_4, 2y_4)$ and $S_{4,4}\Phi_6 = 0$, we obtain the relation: $0 = \beta_6U_2^3 + \beta_9U_1^2U_3$. Hence, $\beta_6, \ \beta_9 \equiv 0 \mod 2$. Let us the operation $S_{2,2,2,2}$: $S_{2,2,2,2}z_{12} \equiv z_4 \mod (2z_4, 2y_4)$, $S_{2,2,2,2}\Phi_6 = U_3$. Hence,

$$d_3(z_4) = U_1^2U_3 = U_1^2U_3 + (\beta_4 + \beta_5)U_2^3 + (\beta_8 + \beta_{10})U_1^2U_3.$$

So, we obtain: $\beta_4 \equiv \beta_5 \mod 2$, $\beta_8 \equiv \beta_{10} \mod 2$. Let us act by the operation S_7 on our equality, we obtain: $0 = \beta_3U_1U_2U_3 + \beta_{12}U_1(U_1y_4 + U_2z_3)$. Hence, $\beta_3, \ \beta_{12} \equiv 0 \mod 2$. Using the operation S_6 and the relation $S_6z_{12} \equiv z_6 \mod (2z_6, 2y_6)$, gives us the relation:

$$d_3(z_6) = U_1^2\Phi_3 = \beta_7U_2^2U_3 + (1 + \beta_{10})U_1^2\Phi_3,$$

from which it follows that $\beta_7,\ \beta_{10} \equiv 0 \mod 2$, and from the previous relations we get: $\beta_8 \equiv 0 \mod 2$. Let us use the operation S_5, we obtain that $0 = \beta_4 U_1 U_3^2$, i.e. $\beta_4 \equiv 0 \mod 2$, and so $\beta_5 \equiv 0 \mod 2$. Finally we obtain:

$$d_3(z_{12}) = U_1^2 \Phi_6.$$

33. An element $y_{12} \in E_2^{0,48}$ we choose so that $\pi_0^2(y_{12}) = c_{12}$. From the relation in the MASS: $S_2 c_{12} = c_5^2 + c_2^2 c_6$, we obtain:

$$S_2(y_{12} + \text{any element of this dimension not equal to } y_{12}) = y_{10}^* + \cdots .$$

Hence, $d_3(y_{12}) \neq 0$, $d_3(y_{12} + \cdots) \neq 0$. Choose the element y_{12} so that:

$$d_3(y_{12}) = \beta_1 U_2 \Phi_3^2 + \beta_2 U_2^2 \Phi_5 + \beta_3 U_2 U_3 U_4 + \beta_4 U_3^2 \Phi_3 + \beta_5 U_2^3 y_4^2$$
$$+\beta_6 U_2^2 (U_2 y_8 + U_3 y_6) + \beta_7 U_2 U_3 (U_2 y_6 + U_3 y_4).$$

Let us apply the operation S_8 to this equality, we have: $S_8 y_{12} \equiv 0 \mod (2y_4)$, so $0 = \beta_2 U_2^3$. Hence: $\beta_2 \equiv 0 \mod 2$. Let us apply the operation $S_{4,4}$. From the condition $S_{4,4} y_{12} \equiv 0 \mod (2y_4)$, we obtain $0 = \beta_1 U_2^3 + \beta_6 U_2^3$. Hence, $\beta_1 \equiv \beta_6 \mod 2$. Let us act by the operation $S_{2,2,2,2}$ on our equality, we have: $S_{2,2,2,2} y_{12} \equiv y_4 \mod (2y_4)$. Hence, we have: $d_3(y_4) = U_2^3 = \beta_1 U_2^3 + \beta_4 U_2^3 + \beta_5 U_2^3 + \beta_6 U_2^3$. So: $\beta_1 + \beta_4 + \beta_5 + \beta_6 \equiv 1 \mod 2$. Let us apply the operation S_6, then we have: $S_6 y_{12} \equiv y_6 \mod (2y_6)$. So it will be: $d_3(y_6) = U_2^2 U_3 = (\beta_3 + \beta_6 + \beta_7) U_2^2 U_3$, hence, $\beta_3 + \beta_6 + \beta_7 \equiv 1 \mod 2$. Let us apply the operation $S_{3,3}$, then we obtain: $S_{3,3} y_{12} \equiv y_6 \mod (2y_6)$. Hence, we have: $d_3(y_6) = U_2^2 U_3 = \beta_3 U_2^2 U_3 + \beta_4 U_1^2 \Phi_3$. So, $\beta_3 \equiv 1 \mod 2$. Let us apply the operation $S_{2,2,2}$, then we obtain: $S_{2,2,2} y_{12} \equiv y_6 \mod (2y_6)$, and so $d_3(y_6) = U_2^2 U_3 = \beta_4 U_2^2 U_3 + \beta_6 U_2^2 U_3$. Hence, $\beta_4 + \beta_6 \equiv 1 \mod 2$. Let us act by the operation $S_{2,2}$ on our equality. From the relation $S_{2,2} y_{12} \equiv y_8 \mod (2y_8)$ we get:

$$d_3(y_8) = U_2 U_3^2 = \beta_1 U_2 U_3^2 + U_2^2 \Phi_3 + \beta_4 U_2^2 \Phi_3 + \beta_4 U_2 U_3^2 + \beta_7 U_2 U_3^2.$$

Hence, $\beta_4 \equiv 1 \mod 2$, $\beta_1 \equiv \beta_7 \mod 2$. So, $\beta_1, \beta_5, \beta_6, \beta_7 \equiv 0 \mod 2$. Finally we obtain:

$$d_3(y_{12}) = U_2 U_3 U_4 + U_3^2 \Phi_3.$$

34. There is a formula in the MASS: $b_6^2 = a_2^2 e_8 + a_1 a_7 e_4 + a_1^2 e_{10}$. Hence in the Adams–Novikov spectral sequence we have a relation:

$$z_6^2 = (1 + 2\beta_1) z_2^2 y_8 + (1 + 2\beta_2) z_1 z_7 y_4 + (1 + 2\beta_3) z_1^2 (y_{10}^* + y_4 y_6) + X_{48}^1.$$

Here $X_{48}^1 \in F^6(E_2^{0,48})$, $d_3(X_{48}^1) = 0$.

35. In the MASS there is the relation: $h_0 b_{12} = a_1 a_{11} + a_4 a_8$. Hence, we can change the choice of z_{12}, not changing the properties, to satisfy the relation:

$$2z_{12} = z_1 z_{11} + 3 z_4 z_8 + X_{48}^2(\widehat{y_{12}}, \widehat{z_1 z_{11}}, \widehat{z_4 z_8}),$$

where $X_{48}^2 \in F^6(E_2^{0,48})$, $d_3(X_{48}^2) = 0$, and the meaning of the symbol $\widehat{\ }$ is the same as in the item 25.

36. Consider the expression $U_1 z_{12}$. Because of the formula $d_3(U_1 z_{12}) = U_1^3 \Phi_6$ and the equality in the MASS: $u_1(b_{12} + a_2 c_{10} + a_4 e_4^2)$, it is possible to change the choice of z_{12}, conserving its properties to satisfy the relation:

$$U_1 z_{12} = \Phi_6 z_1 + \beta_1 U_1 z_4 z_8.$$

From the relation of the item 35 it follows that $S_7 z_{12} \equiv 0 \mod (2 z_1 z_4)$, hence applying the operation S_7, we arrive to relation: $\beta_1 \equiv 0 \mod 2$. So,

$$U_1 z_{12} = \Phi_6 z_1.$$

37. Let us consider the expression $U_2 y_{11}$. From the action of d_3: $d_3(U_2 y_{11}) = U_1 U_2^2 \Phi_5$ and the relation in the MASS: $u_2(b_{11} + a_5 c_6) = \varphi_5 z_3$, it follows that the equality is fulfilled:

$$U_2 y_{11} = \Phi_5 z_3 + Y_{49}(\beta_i),$$

where $Y_{49} \in F^4(E_2^{1,49})$ and $d_3(Y_{49}) = 0$. Multiplying this expression by U_1, and using the relations: $U_1 y_{11} = \Phi_5 z_2$ and $U_1 z_3 = U_2 z_2$, we obtain: $Y_{49} U_1 = 0$, that gives after the consideration the relations $\beta_i \equiv 0 \mod 2$ for all i. Hence,

$$U_2 y_{11} = \Phi_5 z_3.$$

38. Let us consider the expression $U_2 z_{11}$. From the condition $d_3(U_2 z_{11}) = U_1 U_2 U_3 U_4$ and relation in the MASS: $u_2 a_{11} = u_3 a_9 = u_4 a_5$ it follows that:

$$U_2 z_{11} = U_3 z_9 + Y_{49}(\beta_i), \quad U_3 z_9 = U_4 z_5 + Y(\beta_i).$$

where $Y(\beta_i)$ has the same sense as in the item 36. Multiplying both last relations by U_1 and using the equalities:

$$U_1 z_{11} = U_3 z_8 = U_4 z_4, \quad U_1 z_9 = U_2 z_8, \quad U_1 z_5 = U_2 z_4,$$

we obtain that in both relations coefficients $\beta_i \equiv 0 \mod 2$. Hence,

$$U_2 z_{11} = U_3 z_9 = U_4 z_5.$$

39. Let us consider the expression $U_3 y_9$. Because of the formula $d_3(U_3 y_9) = U_1 U_3^2 \Phi_3$ and the equality in the MASS: $u_3 b_9 = \varphi_3 a_7$ we get the relation: $U_3 y_9 = \Phi_3 z_7 + Y_{49}(\beta_i)$, where $Y(\beta_i)$ has the same sense as in the item 37. Multiply this expression by U_1 and use the relations $U_1 y_9 = \Phi_3 z_4$ and $U_1 z_7 = U_3 z_4$. We have for all β_i: $\beta_i \equiv 0 \mod 2$. The final relation is the following:

$$U_3 y_9 = \Phi_3 z_7.$$

40. From the relation in the MASS $u_2(a_1e_{10} + a_3e_8 + a_7e_4) = \varphi_3 b_7$ the relation in the Adams–Novikov spectral sequence follows:

$$U_2(z_1 y_{10}^* + z_3 y_8 + z_1 y_4 y_6 + z_7 y_4) = \Phi_3 y_7 + \beta_1 U_2 z_1 z_{10} + \beta_2 U_2 z_2 z_9 \\ +\beta_3 U_2 z_2 y_9 + \beta_4 U_2 z_3 (z_2 y_6 + z_4 y_4) + Y_{49}^*(\beta_i).$$

In this expression $Y(\beta_i) \in F^4(E_2^{1,49})$ denotes a linear combination of various generators of the given cell of the Adams–Novikov spectral sequence such that the filtration of these generators is not less than 4 and there do not participate the monomials that are already mentioned. Here we consider all coefficients β_i with $5 \le i \le 37$. Let us multiply this expression by U_1 and use the relation of the item 26. We obtain: $\beta_1, \beta_2, \beta_3, \beta_4 \equiv 1 \mod 2$, $\beta_i \equiv 0 \mod 2$ for all $5 \le i \le 37$. So, we get the relation:

$$\Phi_3 y_7 = U_2(z_1(y_{10}^* + z_{10} + y_4 y_6) + z_2(y_9 + z_9) + z_3(y_8 + y_6 z_2 + y_4 z_4) + z_7 y_4).$$

41. Let us choose the element $\Omega_1 \in E_2^{1,49}$ so that $\pi_0^2(\Omega_1) = \omega_1$. For any cycle x of the differential d_3, lying in $E_2^{4,48} = E_3^{4,48}$ of the Adams–Novikov spectral sequence, there exists an element $\tilde{x} \in F^2(E_2^{1,49})$ such that: $d_3(\tilde{x}) = x$. Hence, the element Ω_1 can be chosen in such a way that $d_3(\Omega_1) = 0$. Because in the term $E_4^{*,*}$ of the Adams–Novikov spectral sequence for all $s > 4$ the cells $E_4^{s,48}$ consist of zeros all higher differentials map the element Ω_1 to zero. Hence, Ω_1 lives to infinity and define an indecomposable element $\Omega_1 \in MSp_{49}$. All the cells $E_4^{s,49}$ of the Adams–Novikov spectral sequence for $s > 1$ consist of zeros, so there is no extension problem for the cell $E_\infty^{1,49}$. Hence, the order of the element $\Omega_1 \in MSp_{49}$ is equal to two.

42. There is the relation in the MASS

$$u_1\omega_1 = u_2(\varphi_6 + u_2 c_{10} + u_4 e_4^2) + u_3(\varphi_5 + u_3 c_6) + u_4 \varphi_3.$$

Hence it is possible to choose Ω_1 in such a way that the equality is satisfied:

$$U_1\Omega_1 = U_2\Phi_6 + U_3\Phi_5 + U_4\Phi_3 + U_2^2 y_{10} + U_3^2 y_6 \\ +\beta_1 U_2^2 z_5^2 + \beta_2 U_2^2 z_3 y_7 + U_2 U_3 y_4^2.$$

Let us apply the operation $S_{3,3}$ to this equality. Then we get the relation: $U_1 S_{3,3}\Omega_1 = U_2^2 S_{3,3} y_{10} + U_1^2 y_6 + \beta_1 U_2^2 z_3^2$ (because of relation $U_1 z_5 = U_3 z_2$ we have $S_3 z_5 \equiv z_2 \mod (2z_2, 2z_1^2)$, and because of relation $U_1 y_7 = \Phi_3 z_2$ we have $S_{3,3} y_7 \equiv 0 \mod (2z_1)$). So, $S_{3,3} y_{10} \equiv \sigma z_2^2 + z_4 \mod (2z_2^2, 2z_4, 2z_1^2 z_2, z_1 z_3, z_1^4)$. Let us choose Ω_1 so that $S_{3,3}\Omega_1 \equiv \sigma U_1 z_3^2 \mod (U_1 z_1^2 z_2^2, U_1 z_1^6)$. Then we have $\beta_1 \equiv 0 \mod 2$. Let us apply the operation S_5: $S_5 y_7 \equiv z_2 \mod (2z_2, z_1^2)$ (corollary of the condition $U_1 y_7 = \Phi_2 z_2$), $S_5 y_{10} \equiv \gamma z_2 z_3 \mod (2z_2 z_3)$, $S_5 y_6 \equiv 0 \mod (2z_1)$. We obtain: $U_1 S_5 \Omega_1 = U_1 U_4 + U_1 U_2(\gamma + \beta_2) z_3^2$. Choose Ω_1 so that $S_5\Omega_1 \equiv U_4 + \gamma U_2 z_3^2 \mod (U_2 2 z_3^2)$. Then $\beta_2 \equiv 0 \mod 2$. We have:

$$U_1\Omega_1 = U_2\Phi_6 + U_3\Phi_5 + U_4\Phi_3 + U_2^2 y_{10} + U_3^2 y_6 + U_2 U_3 y_4^2.$$

43. We choose an element $z_{13} \in E_2^{0,52}$ to satisfy the relation: $\pi_2^2(z_{13}) = a_{13}$. From the condition

$$S_9(z_{13} + \text{any element of this dimension not equal to } z_{13}) = z_4 + \text{ decomposables}$$

we obtain: $d_3(z_{13}) \neq 0$, $d_3(z_{13} + \cdots) \neq 0$. It is possible to choose z_{13} so that

$$\begin{aligned} d_3(z_{13}) = {} & U_1^2\Omega_1 + \beta_1 U_1 U_2 \Phi_6 + \beta_2 U_1 U_3 \Phi_5 + \beta_4 U_1 U_4 \Phi_3 + \beta_5 U_1 U_2 U_3 y_4^2 \\ & + \beta_6 U_1^2 y_6^2 + \beta_7 U_1 U_3 (U_2 y_8 + U_3 y_6) + \beta_8 U_1 U_2 (U_2 y_{10} + U_3 y_8) \\ & + \beta_{19} U_1^2 (U_1 y_{12} + U_2 z_{11} + U_3 y_8) + \beta_{20} U_1^2 y_4^2 (U_1 y_4 + U_2 z_3) \\ & + \beta_9 U_2^2 (U_1 (y_{10}^* + y_{10}) + U_2 (y_9 + z_9)) + \beta_{10} U_2^2 (U_1 y_{10} + U_3 z_7) \\ & + \beta_{11} U_2 U_3 (U_2 y_8 + U_2 z_7) + \beta_{12} U_3^2 (U_1 y_6 + U_3 z_3) \\ & + \beta_{13} U_3 \Phi_3 (U_1 y_4 + U_2 z_3) + \beta_{14} U_2 \Phi_3 (U_1 y_6 + U_3 z_3) \\ & + \beta_{15} U_2 U_4 (U_1 y_4 + U_2 z_3) + \beta_{16} U_1 \Phi_3 (U_2 y_6 + U_3 y_4) \\ & + \beta_{17} U_1 U_2 (U_2 (y_{10}^* + y_{10}) + \Phi_3 y_6 + U_4 y_4) \\ & + \beta_{18} U_2^2 (U_1 y_6 y_4 + U_2 (y_6 z_3 + y_4 z_5)) \\ & + \beta_{21} U_1^2 (U_1 y_8 y_4 + U_2 y_8 z_3 + U_3 y_4 z_5). \end{aligned}$$

Applying the operation S_{12} and having in mind that $S_{12} z_{13} \equiv 0 \mod (2z_1)$, $S_{12}\Omega_1 = 0$, we obtain the relation: $\beta_{19} \equiv 0 \mod 2$. Using the operation $S_{6,6}$ and relations $S_{6,6} z_{13} \equiv 0 \mod (2z_1)$, $S_{6,6}\Omega_1 = 0$, we get: $\beta_6 \equiv 0 \mod 2$. Let us consider the action of the operation $S_{2,2,2,2,2,2}$. There are relations: $S_{2,2,2,2,2,2} z_{13} \equiv 0 \mod (2z_1)$,
$S_{2,2,2,2,2,2}\Omega_1 = 0$, hence, $\beta_{20} \equiv 0 \mod 2$. Now we apply the operation S_{11}. Because of the formulae $S_{11} z_{13} \equiv z_2 \mod (2z_2, z_1^2)$, $S_{11}\Omega_1 \equiv U_1$, we have the relation: $d_3(z_2) = (1 + \beta_1) U_1^2 U_2 = U_1^2 U_2$, or, $\beta_1 \equiv 0 \mod 2$. Let us use the operation S_{10}. Because of relations: $S_{10} z_{13} \equiv 0 \mod (2z_3, z_1 z_2, 2z_1^3)$, $S_{10}\Omega_1 \equiv 0 \mod (U_1 z_1^2)$, we obtain that $\beta_8 + \beta_9 + \beta_{10} + \beta_{17} \equiv 0 \mod 2$. Let us consider the action of the operation $S_{5,5}$ on our equality. We have $S_{5,5} z_{13} \equiv 0 \mod (2z_3, z_1 z_2, 2z_1^2)$ and $S_{5,5}\Omega_1 \equiv 0 \mod (U_1 z_1^2)$. This gives the relation: $\beta_8 + \beta_{10} \equiv 0 \mod 2$. Using the relations: $S_9 z_{13} \equiv z_4 \mod (2z_4, z_1 z_3, 2y_4, z_2^2, 2z_1^2 z_2, z_1^4)$ and $S_9\Omega_1 \equiv U_3 \mod (U_2 z_1^2)$, and applying the operation S_9 to the equality under the consideration we get the following: $d_3(z_4) = (1 + \beta_2) U_1^2 U_3 = U_1^2 U_3$, so, $\beta_2 \equiv 0 \mod 2$. Let us consider the action of the operation $S_{4,4}$, we have:

$$\begin{aligned} S_{4,4} z_{13} &\equiv 0 \mod (2z_5, z_1 z_4, z_2 z_3, 2z_1^2 z_3, 2z_1 z_2^2, 2z_1^5, 2z_1 y_4, 2z_1^3 z_2), \\ S_{4,4}\Omega_1 &\equiv 0 \mod (U_1 z_1^4, U_1 z_2^2). \end{aligned}$$

It follows from these relations that:

$$\beta_4 U_1 U_2 U_3 + \beta_7 U_1 U_2 U_3 + \beta_8 U_1 U_2 U_3 + \beta_{11} U_1 U_2 U_3 + \beta_{21} U_1^2 (U_1 y_4 + U_2 z_3) \equiv 0.$$

Hence, $\beta_{21} \equiv 0 \mod 2$, $\beta_4 + \beta_7 + \beta_8 + \beta_{11} \equiv 0 \mod 2$. Let us apply the operation $S_{2,2,2,2}$. We obtain the following equalities:

$$S_{2,2,2,2}z_{13} \equiv 0 \mod (2z_5, z_1z_4, z_2z_3, 2z_1^2z_3, 2z_1z_2^2, 2z_1^5, 2z_1y_4, 2z_1^3z_2),$$
$$S_{2,2,2,2}\Omega_1 \equiv 0 \mod (U_1z_2^2, U_2z_1^4).$$

From these conditions it follows that: $\beta_5U_1U_2U_3 + \beta_7U_1U_2U_3 + \beta_{14}U_1U_2U_3 \equiv 0$. Hence, $\beta_5 + \beta_7 + \beta_{14} \equiv 0 \mod 2$. Let us consider the action of the operation S_7, we have:

$$S_7z_{13} \equiv z_6 \mod (2z_6, 2y_6, z_1z_5, z_2z_4, z_3^2, 2z_1^2z_4, z_1^2z_2^2, z_1^3z_3, z_1^6, 2z_1^2y_4, 2z_1z_2z_3, 2z_2y_4, 2z_1^4z_2),$$
$$S_7\Omega_1 \equiv \Phi_3 \mod (U_3z_1^2, U_2z_2^2, U_2z_1^4).$$

Hence, $d_3(z_6) = U_1^2\Phi_3 \equiv (1 + \beta_4)U_1^2\phi_3 + (\beta_{15} + \beta_{17})U_1U_2(U_1y_4 + U_2z_3)$. So, $\beta_4 \equiv 0 \mod 2$, $\beta_{15} + \beta_{17} \equiv 0 \mod 2$. Let us apply the operation S_6. There are the following relations: $S_6z_{13} \equiv 0 \mod (2z_7, 2y_7, \ldots.)$, $S_6\Omega_1 \equiv 0 \mod (U_1z_1^6, \ldots.)$. It follows from these conditions:

$$\beta_7U_1U_3^2 + \beta_9U_2^2(U_1y_4 + U_2z_3) + \beta_{12}U_1U_3^2 + \beta_{14}U_1U_2\Phi_3$$
$$+\beta_{15}U_2^2(U_1y_4 + U_2z_3) + \beta_{16}U_1U_2\Phi_3 + \beta_{17}U_1U_2\Phi_3$$
$$+\beta_{18}U_2^2(U_1y_4 + U_2z_3) \equiv 0 \mod (U_1z_1^6, \ldots.).$$

So, we have the following: $\beta_7 + \beta_{12} \equiv 0 \mod 2$, $\beta_{14} + \beta_{16} + \beta_{17} \equiv 0 \mod 2$, $\beta_9 + \beta_{15} + \beta_{18} \equiv 0 \mod 2$. From these and previous relations it follows that $\beta_{18} \equiv 0 \mod 2$. Applying $S_{2,2,2}$ and using the relations: $S_{2,2,2}z_{13} \equiv 0 \mod (2z_7, 2y_7, \ldots)$, $S_{2,2,2}\Omega_1 \equiv 0 \mod ((U_1z_1^6, \ldots)$, we obtain:

$$0 \equiv (\beta_7 + \beta_{12} + \beta_{13})U_1U_3^2 + (\beta_{13} + \beta_{14} + \beta_{15} + \beta_{16})U_1U_2\Phi_3$$
$$+(\beta_9 + \beta_{10} + \beta_{11} + \beta_{12} + \beta_{13} + \beta_{14})U_2^2(U_1y_4 + U_2z_3).$$

Hence, $\beta_9 + \beta_{10} + \beta_{11} + \beta_{12} + \beta_{13} + \beta_{14} \equiv 0 \mod 2$, $\beta_{13} + \beta_{14} + \beta_{15} + \beta_{16} \equiv 0 \mod 2$, $\beta_7+\beta_{12}+\beta_{13} \equiv 0 \mod 2$. From these and previous conditions it follows that: $\beta_{13} \equiv 0 \mod 2$, $\beta_{16} \equiv 0 \mod 2$. From the action of the operation $S_{2,2}$: $S_{2,2}z_{13} \equiv 0 \mod (F^2E_2^{0,36})$, $S_{2,2}\Omega_1 \equiv 0 \mod (F^2E_2^{1,35})$, we have:

$$(\beta_8 + \beta_{17})U_1U_2(U_2y_6 + U_3y_4) + (\beta_{10} + \beta_{12} + \beta_{14})U_2^2(U_1y_6 + U_3z_3)$$
$$+(\beta_{11} + \beta_{14})U_2U_3(U_1y_4 + U_2z_3) + (\beta_{15} + \beta_{17})U_1U_2U_4 \equiv 0$$
$$\mod ((U_1z_1^8, \ldots).$$

Hence, $\beta_8 + \beta_{17} \equiv \beta_{10} + \beta_{12} + \beta_{14} \equiv \beta_{11} + \beta_{14} \mod 2$, $\beta_{15} \equiv \beta_{17} \mod 2$. From this relation and from previous ones we obtain: $\beta_{12} \equiv \beta_7 \equiv 0 \mod 2$, $\beta_{10} \equiv \beta_{11} \mod 2$, $\beta_5 \equiv \beta_{14} \mod 2$. Finally using the operation $S_{1,1,1,1}$ and relations:

$S_{1,1,1,1}z_{13} \equiv z_1 y_4^2 \mod (F^4 E_2^{0,36})$, $S_{1,1,1,1}\Omega_1 \equiv U_1 y_4^2 \mod (F^2 E_2^{1,35})$ we obtain:

$$\begin{aligned} d_3(z_1 y_4^2) = U_1^3 y_4^2 &\equiv (1 + \beta_5 + \beta_8 + \beta_9 + \beta_{10} + \beta_{17}) U_1^3 y_4^2 \\ &+ (\beta_8 + \beta_{17}) U_1 U_2 (U_2 y_6 + U_3 y_4) + (\beta_9 + \beta_{10}) U_2^2 (U_1 y_6 + U_3 z_3) \\ &+ (\beta_{11} + \beta_{15}) U_2 U_3 (U_1 y_4 + U_2 z_3) + (\beta_{10} + \beta_{11}) U_1^2 (U_1 y_8 \\ &+ U_2 z_7) \mod (F^2 E_2^{1,35}). \end{aligned}$$

Hence, $\beta_5 + \beta_8 + \beta_9 + \beta_{10} + \beta_{17} \equiv 0 \mod 2$, $\beta_8 + \beta_{17} \equiv \beta_9 + \beta_{10} \equiv \beta_{11} + \beta_{15} \mod 2$, $\beta_{10} \equiv \beta_{11} \mod 2$. From the previous equalities we determine step by step the meanings of the rest β_i. All of them are nonzero. So, we have:

$$d_3(z_{13}) = U_1^2 \Omega_1.$$

44. An element $y_{13} \in E_2^{0,52}$ we choose so that $\pi_2^2(y_{13}) = b_{13}$. From the condition:

$$S_6(y_{13} + \text{ any element of this dimension non-equal to } y_{13}) = y_9 + \cdots$$

it follows that $d_3(y_{13}) \neq 0$, $d_3(y_{13} + \cdots) \neq 0$. Hence we can choose y_{13} so that:

$$d_3(y_{13}) = U_1 U_4 \Phi_3 + \beta_1 U_1 U_2 \Phi_6 + \beta_2 U_1 U_3 \Phi_5 + \beta_3 U_1^2 \Omega_1 + \cdots,$$

where the dots denote summands starting with the coefficient β_5 as in the item 43. Using the same operations as in the item 42, we get analogous relations for the coefficients β_i excluding the following relations. Using the operation S_{11} gives the relation: $\beta_1 \equiv \beta_3 \mod 2$, the operation S_{10} gives in addition to the equality of the item 43: $\beta_1 \equiv \beta_2 \mod 2$. As a result of the action of the rest of the operations there will be the same values a in the item 43. The same way we get:

$$d_3(y_{13}) = U_1 U_4 \Phi_3.$$

45. An element $y_{13}^* \in E_2^{0,52}$ we choose so that $\pi_2^2(y_{13}^*) = f_{13} + a_5 e_8 + a_3 c_{10}$. From the condition:

$$S_8(y_{13}^* + \text{any element of this dimension non-equal to } y_{13}^*) = z_5 + \cdots$$

we get, that $d_3(y_{13}^*) \neq 0$, $d_3(y_{13}^* + \cdots) \neq 0$. Considerations analogous to those given in the items 42 and 43, show that it is possible to choose the element y_{13}^*, so that the equality is fulfilled:

$$d_3(y_{13}^*) = U_1 U_2 \Phi_6.$$

The results of the calculations are given in Tables 15, 16 and 17. In Table 17 the action of the differential d_3 is given for the generators of dimension not bigger than 52 in the term $E_3 \simeq E_2 \simeq \mathrm{Ext}_A(BP^*(MSp), BP^*)$ of the Adams–Novikov spectral sequence for the spectrum MSp, and in Table 16 there are given relation between generators in the given dimensions (part of the "integer" are given only modulo filtration corresponding to the MASS), which continue the analogous Table from the work [41]. Vertical lines in Table 15 denote the multiplication by the element U_1 in the term E_2. Because of the fact $\pi_*(MSp) \otimes \mathbb{Z}_{(p)} \cong \mathbb{Z}_{(p)}[z_1, \ldots, z_k, \ldots]$ for all $p > 2$, Tables 15, 16 and 17 describe the structure of the initial term and the action of the differential d_3 in the integer case of the Adams–Novikov spectral sequence. In this case we mast consider that the zero line consists of free abelian groups and not of free $\mathbb{Z}_{(2)}$-modules. From Tables 15 and 17 it is possible to see that in the considering dimensions $E_4 \simeq E_\infty$ and the extension problem from E_∞ to $\pi_*(MSp)$ is trivial. So, we get a description of the symplectic cobordism ring (not complete because not all the relation among "free" generators are known) up to dimension 52. Table 18 describes the ring $\pi_*(MSp)$ as a subring of $\mathrm{Ext}_A U(MU^*(MSp), MU^*)$. To diminish the volume there is a familiarity in description which we hope does not lead to ambiguity, for example, many evident relations are not shown.

3.2 Computations of the action of the differential d_3 of the Adams–Novikov spectral sequence

The aim of the present section is a calculation of the differential d_3 on the generators $y_{26}, y_{26}^* \in E_2^{0,104}$ of the Adams–Novikov spectral sequence modulo θ_1 and elements having F-filtration (corresponding to MASS) strictly greater than zero, $E_2^{*,*} \cong \mathrm{Ext}_A(BP^*(MSp), BP^*)$, $A = A^{BP}$. These spectral sequences were described in particular in the works of the second author [40–43]. The term E_∞ of the MASS up to dimension 106 is described in Sect. 2, and the action of Landweber–Novikov operations on the generators c_j of the MASS is described in Sect. 1.

1. All subsequent calculations are done modulo 2, and images of the differential d_3 are expressions considered modulo triple products which either contain θ_1, or have F-filtration (corresponding to MASS) strictly greater than zero. The scheme of the calculations will be the same for all dimensions: we write down an image of the differential d_3 on a given element as a linear combination of generators in given dimension with unknown coefficients. Applying subsequently various Landweber–Novikov operations we define all the coefficients. All notations dealing with dimensions less than 32 coincide with the notations of mentioned works of the second author. The only exclusions are the changes: V_i by U_i and c_{2i-1} by h_i^2. This is true for notations in MASS, in the Adams–Novikov spectral sequence as well as in the symplectic cobordism ring MSp_*. By $\pi_0^2(x)$ we denote the projection of an element $x \in E_2^{0,*}$ of the Adams–Novikov spectral sequence into the term E_∞ of the MASS.

Remark 3.1 Our notation y_6 corresponds to z_6 from the works of the second author.

2. Choose elements $y_{10}, y_{10}^* \in E_2^{0,40}$ so that $\pi_0^2(y_{10}) = c_{10}$, $\pi_0^2(y_{10}^*) = c_{10} + c_5^2 + c_2^2 c_6$ and also that:

$$d_3(y_{10}) = \beta_1 U_2^2 U_4 + \beta_2 U_2 U_3 \Phi_3 + \beta_3 U_3^3 + \beta_4 U_2^2 \tau_3. \quad (3.1)$$

Table 1 Calculation of the action of differential d_3 on y_{10}, y_{10}^*

Operation S_ω	y_j	S_ω on y_j	S_ω on the right part of the formula (3.1)	Relation among the elements β_i.
S_6	y_{10}	0	$(\beta_1+\beta_4)U_2^3$	$\beta_1+\beta_4=0$
S_6	y_{10}^*	y_4	$U_2^3(\beta_1^*+\beta_4^*)$	$\beta_1^*+\beta_4^*=1$
$S_{3,3}$	y_{10}	0	$\beta_1 U_2^3$	$\beta_1=0$
$S_{3,3}$	y_{10}^*	y_4	$\beta_1^* U_2^3$	$\beta_1^*=1$
$S_{2,2,2}$	y_{10}	y_4	$\beta_3 U_2^3$	$\beta_3=1$
$S_{2,2,2}$	y_{10}^*	y_4	$\beta_3^* U_2^3$	$\beta_3^*=1$
S_4	y_{10}	0	$\beta_2 U_2^2 U_3$	$\beta_2=0$
S_4	y_{10}^*	0	$(1+\beta_2^*)U_2^2 U_3$	$\beta_2^*=1$

Table 2 Calculation of the action of the differential d_3 on y_{12}

Operation S_ω	y_j	S_ω on y_j	S_ω on the right part of the formula (3.2)	Relation among the elements β_i
S_8	y_{12}	0	$\beta_1 U_2^3$	$\beta_1=0$
$S_{4,4}$	y_{12}	0	$(\beta_4+\beta_5)U_2^3$	$\beta_4=\beta_5$
S_6	y_{12}	y_6	$(\beta_2+\beta_5)U_2^2 U_3$	$\beta_2+\beta_5=1$
$S_{3,3}$	y_{12}	y_6	$\beta_2 U_2^2 U_3$	$\beta_2=1$
$S_{2,2,2,2}$	y_{12}	y_4	$(\beta_3+\beta_6)U_2^3$	$\beta_3+\beta_6=1$
S_4	y_{12}	y_4^2	$(\beta_3+1)U_2 U_3^2$	$\beta_3=1$

In the analogous equality for y_{10}^*, β_i are changed by β_i^*. To determine coefficients β_i let us use Landweber–Novikov operations S_ω (Table 1).

It follows from this table that

$$d_3(y_{10}^*) = U_2^2 U_4 + U_2 U_3 \Phi_3 + U_3^3,$$
$$d_3(y_{10}) = U_3^3.$$

3. Choose an element $y_{12} \in E_2^{0,48}$ so that $\pi_0^2(y_{12}) = c_{12}$ and also that:

$$d_3(y_{12}) = \beta_1 U_2^2 \Phi_5 + \beta_2 U_2 U_3 U_4 + \beta_3 U_3^2 \Phi_3 + \beta_4 U_2 \Phi_3^2 + \beta_5 U_2^2 \tau_5 + \beta_6 U_2^3 y_4^2. \tag{3.2}$$

To determine the coefficients β_i we use operations S_ω (Table 2).

Solving the system of equations on the coefficients β_i we come to the formula:

$$d_3(y_{12}) = U_2 U_3 U_4 + U_3^2 \Phi_3.$$

Table 3 Calculation of the action of the differential d_3 on y_{14}

Operation S_ω	y_j	S_ω on y_j	S_ω on of the right part of the formula (3.3)	Relation among the elements β_i
S_{10}	y_{14}	y_4	$(\beta_1+\beta_6)U_2^3$	$\beta_1+\beta_6=1$
$S_{5,5}$	y_{14}	y_4	$\beta_1 U_2^3$	$\beta_1=1$
S_8	y_{14}	0	$(\beta_1+\beta_2)U_2^2U_3$	$\beta_2=1$
$S_{4,4}$	y_{14}	0	$(\beta_2+\beta_3+\beta_5)U_2^2U_3$	$\beta_3+\beta_5=1$
S_6	y_{14}	y_8	$(\beta_3+\beta_8+1)U_2^2\Phi_3$	$\beta_3+\beta_8=1$
			$+(1+\beta_4+\beta_7)U_2U_3^2$	$\beta_4\beta_7=0$
$S_{3,3}$	y_{14}	0	$(1+\beta_3)U_2^2\Phi_3$	$\beta_3=1, \beta_4=0$
			$+\beta_4U_2U_3^2$	
S_2	y_{14}	y_4y_8	$U_2\Phi_3^2+\beta_9U_2^3y_4^2$	$\beta_9=0$

4. Choose an element $y_{14}\in E_2^{0,56}$ so that $\pi_0^2(y_{14})=c_{14}$ and also that:

$$\begin{aligned} d_3(y_{14}) = {} & \beta_1U_2^2\Phi_6+\beta_2U_2U_3\Phi_5+\beta_3U_2\Phi_3U_4+\beta_4U_3^2U_4 \\ & +\beta_5U_3\Phi_3^2+\beta_6U_2^2\tau_7^*+\beta_7U_3^2\tau_3+\beta_8U_2\Phi_3\tau_3+\beta_9U_2^2U_3y_4^2. \end{aligned} \tag{3.3}$$

To determine the coefficients β_i we again use operations S_ω (Table 3).
Solving the system of equations on the coefficients β_i we come to the formula:

$$d_3(y_{14})=U_2^2\Phi_6+U_2U_3\Phi_5+U_2\Phi_3U_4.$$

5. Let us choose $y_{16}\in E_2^{0,64}$ so that $\pi_0^2(y_{16})=c_8^2$ and also that:

$$\begin{aligned} d_3(y_{16}) = {} & \beta_1U_2^2\Phi_7+\beta_2U_2U_3\Phi_6+\beta_3U_2\Phi_3\Phi_5+\beta_4U_3^2\Phi_5+\beta_5U_2U_4^2 \\ & +\beta_{11}U_2U_4(U_2y_6+U_3y_4)+\beta_{12}U_2U_3^2y_4^2+\beta_{13}U_2^3y_6^2 \\ & +\beta_{14}U_2^2\Phi_3y_4^2+\beta_6U_3\Phi_3U_4+\beta_7\Phi_3^3+\beta_8U_2U_3(U_2(y_{10}+y_{10}^*) \\ & +\Phi_3y_6+U_4y_4)+\beta_9U_3\Phi_3(U_2y_6+U_3y_4) \\ & +\beta_{10}U_3^2(U_2y_6+U_3y_4). \end{aligned} \tag{3.4}$$

To determine the coefficients β_i we again use operations S_ω (Table 4).
Solving the linear system we obtain:

$$d_3(y_{16})=U_2U_4^2.$$

Table 4 Calculation of the action of the differential d_3 on y_{16}

Operation S_ω	y_j	S_ω on y_j	S_ω on of the right part of the formula (3.4)	Relation among the elements β_i
S_{12}	y_{16}	0	$\beta_1 U_2^3$	$\beta_1 = 0$
S_{10}	y_{16}	0	$(\beta_2 + \beta_8) U_2^2 U_3$	$\beta_2 = \beta_8$
$S_{5,5}$	y_{16}	0	$\beta_2 U_2^2 U_3$	$\beta_2 = 0$
$S_{6,6}$	y_{16}	y_4	$(\beta_5 + \beta_{11} + \beta_{13}) U_2^3$	$\beta_5 + \beta_{11} + \beta_{13} = 1$
$S_{3,3,3,3}$	y_{16}	y_4	$\beta_5 U_2^3$	$\beta_5 = 1$
$S_{4,4,4}$	y_{16}	0	$(\beta_3 + \beta_7) U_2^3$	$\beta_3 = \beta_7$
S_8	y_{16}	0	$\beta_3 U_2^2 \Phi_3 + \beta_4 U_2 U_3^2$	$\beta_3 = 0, \beta_4 = 0$
$S_{4,4}$	y_{16}	y_8	$(\beta_5 + \beta_6 + \beta_{10}) U_2 U_3^2$	$\beta_5 + \beta_6 + \beta_{10} = 1$
S_6	y_{16}	0	$(\beta_6 + \beta_9) U_2 U_3 \Phi_3$	$\beta_6 = \beta_9, \beta_{10} = 0$
			$+\beta_{10} U_3^3 + \beta_{11} U_2^2 U_4$	$\beta_{11} = 0$
S_4	y_{16}	0	$\beta_{14} U_2^3 y_4$	$\beta_{14} = 0$
$S_{2,2}$	y_{16}	$y_4 y_8$	$U_2 \Phi_3^2 + \beta_{12} U_2^3 y_4^2$	$\beta_{12} = 0$

6. Elements $y_{18}, y_{18}^* \in E_2^{0,72}$ we choose so that $\pi_0^2(y_{18}) = c_{18}, \pi_0^2(y_{18}^*) = c_9^2 + c_2^2 c_{14} + c_4^2(c_{10} + c_5^2) + c_6^3$, and also that:

$$\begin{aligned} d_3(y_{18}) = {} & \beta_1 U_2^2 U_5 + \beta_2 U_2 U_3 \Phi_7 + \beta_3 U_2 \Phi_3 \Phi_6 + \beta_4 U_2 U_4 \Phi_5 + \beta_5 U_3^2 \Phi_6 \\ & +\beta_7 U_3 U_4^2 + \beta_8 \Phi_3^2 U_4 + \beta_9 U_2^2 (U_2 y_{14} + U_3 y_4 y_8 + \Phi_3 (y_{10} + y_{10}^*) \\ & +\Phi_5 y_6 + \Phi_6 y_4) + \beta_{10} U_2 \Phi_3 (U_2 (y_{10} + y_{10}^*) + \Phi_3 y_6 + U_4 y_4) \\ & +\beta_{11} U_3^2 (U_2 y_6 + U_3 y_4) + \beta_{12} U_3^2 (U_2 (y_{10} + y_{10}^*) + \Phi_3 y_6 + U_4 y_4) \\ & +\beta_{13} U_3 \Phi_3 (U_2 y_8 + U_3 y_6) + \beta_{14} U_2 \Phi_5 (U_2 y_6 + U_3 y_4) \\ & +\beta_{15} U_3 U_4 (U_2 y_6 + U_3 y_4) + \beta_{16} \Phi_3^2 (U_2 y_6 + U_3 y_4) \\ & +\beta_{17} U_2^2 U_3 y_6^2 + \beta_{18} U_2^2 U_4 y_4^2 + \beta_{19} U_2 U_3 \Phi_3 y_4^2 + \beta_{20} U_3^3 y_4^2 \\ & +\beta_{21} U_2^2 (U_2 y_6 + U_3 y_4) y_4^2 + \beta_6 U_3 \Phi_3 \Phi_5. \end{aligned} \tag{3.5}$$

In analogous equality for y_{18}^*, coefficients β_i are changed by β_i^*).

To determine the coefficients β_i we use operations S_ω (Table 5).

Solving the linear system we get the following relations:

$$\begin{aligned} d_3(y_{18}^*) &= U_3 U_4^2 + U_2^2 U_5 + U_2 U_3 \Phi_7 + U_2 \Phi_3 \Phi_6 + U_2 U_4 \Phi_5, \\ d_3(y_{18}) &= U_3 U_4^2. \end{aligned}$$

7. Choose the element y_{20} so that $\pi_0^2(y_{20}) = c_{20}$ and so that:

$$\begin{aligned} d_3(y_{20}) = {} & \beta_1 U_2^2 \Phi_9 + \beta_2 U_2 U_3 U_5 + \beta_3 U_2 \Phi_3 \Phi_7 + \beta_4 U_2 U_4 \Phi_6 + \beta_5 U_2 \Phi_5^2 \\ & +\beta_6 U_3^2 \Phi_7 + \beta_7 U_3 \Phi_3 \Phi_6 + \beta_8 U_3 U_4 \Phi_5 + \beta_9 \Phi_3^2 \Phi_5 \end{aligned}$$

Table 5 Calculation of the action of the differential d_3 on $y_{18},\ y^*_{18}$

Operation S_ω	y_j	S_ω on y_j	S_ω on of the right part of the formula (3.5)	Relation among the elements β_i
S_{14}	y_{18}	0	$(\beta_1+\beta_9)U_2^3$	$\beta_1=\beta_9$
S_{14}	y^*_{18}	y_4	$(\beta^*_1+\beta^*_9)U_2^3$	$\beta^*_1+\beta^*_9=1$
$S_{7,7}$	y_{18}	0	$\beta_1 U_2^3$	$\beta_1=0$
$S_{7,7}$	y^*_{18}	y_4	$\beta_1 U_2^3$	$\beta_1=1$
S_{12}	y_{18}	0	$\beta_2 U_2^3$	$\beta_2=0$
S_{12}	y^*_{18}	0	$(\beta^*_1+\beta^*_2)U_2^2U_3$	$\beta^*_2=1$
$S_{6,6}$	y_{18}	y_6	$(\beta_4+\beta_7+\beta_{14}+\beta_{15}$ $+\beta_{17})U_2^2U_3$	$\beta_4+\beta_7+\beta_{14}+\beta_{15}$ $+\beta_{17}=1$
$S_{6,6}$	y^*_{18}	y_6	$(\beta^*_4+\beta^*_7+\beta^*_{14}+\beta^*_{15}$ $+\beta_{17}U_2^2U_3$	$\beta^*_4+\beta^*_7+\beta^*_{14}+\beta^*_{15}$ $+\beta^*_{17}=0$
$S_{3,3,3,3}$	y_{18}	y_6	$\beta_7 U_2^2U_3$	$\beta_7=1$
$S_{3,3,3,3}$	y^*_{18}	0	$(\beta^*_7+1)U_2^2U_3$	$\beta^*_7=1$
S_{10}	y_{18}	0	$(\beta_5+\beta_{11}+\beta_{12})U_2U_3^2$ $+(\beta_3+\beta_{10})U_2^2\Phi_3$	$\beta_5+\beta_{11}+\beta_{12}=0$ $\beta_3=\beta_{10}$
S_{10}	y^*_{18}	y_8	$(1+\beta^*_5+\beta^*_{11}+\beta^*_{12})U_2U_3^2$ $+(1+\beta^*_3+\beta^*_{10})U_2^2\Phi_3$	$\beta^*_5+\beta^*_{11}+\beta^*_{12}=1$ $\beta^*_3+\beta^*_{10}=1$
$S_{5,5}$	y_{18}	0	$(\beta_5+\beta_{11})U_2U_3^2$ $+\beta_3U_2^2\Phi_3$	$\beta_5=\beta_{11}$ $\beta_3=0$
$S_{5,5}$	y^*_{18}	y_8	$(1+\beta^*_5+\beta^*_{11})U_2U_3^2$ $+(1+\beta^*_3)U_2^2\Phi_3$	$\beta^*_5=\beta^*_{11}$ $\beta^*_3=1$
S_8	y_{18}	0	$\beta_4U_2^2U_4+\beta_5U_3^3$ $+\beta_6U_2U_3\Phi_3$	$\beta_4,\beta_5,\beta_6=0$
S_8	y^*_{18}	0	$(\beta^*_4+1)U_2^2U_4+\beta^*_5U_3^3$ $+\beta^*_6U_2U_3\Phi_3$	$\beta^*_5,\beta^*_6=0$ $\beta^*_4=1$
$S_{4,4}$	y_{18}	y_{10}	$(1+\beta_{11})U_3^3+\beta_8U_2^2U_4$ $+\beta_{13}U_2U_3\Phi_3+\beta_{14}U_2^2\tau_3$	$\beta_{11},\beta_8,\beta_{13},\beta_{14}=0$
$S_{4,4}$	y^*_{18}	y_{10}	$(1+\beta^*_{11})U_3^3+\beta^*_8U_2^2U_4$ $+\beta^*_{13}U_2U_3\Phi_3+\beta^*_{14}U_2^2\tau_3$	$\beta_{11},\beta_8,\beta_{13},\beta_{14}=0$
S_6	y_{18}	0	$\beta_{15}U_2U_3U_4+\beta_{16}U_2\Phi_3^2$ $+\beta_{18}U_2^3y_4^2+\beta_{21}U_2^3y_4^2$ $+\beta_{15}U_2U_3\tau_3$	$\beta_{15},\beta_{16}=0$ $\beta_{18}=\beta_{21}$
S_6	y^*_{18}	y_4y_8	$\beta^*_{15}U_2U_3U_4+\beta^*_{16}U_2\Phi_3^2$ $+\beta^*_{18}U_2^3y_4^2+\beta^*_{21}U_2^3y_4^2$ $+\beta^*_{15}U_2U_3\tau_3+U_2\Phi_3^2$	$\beta^*_{15},\beta^*_{16}=0$ $\beta^*_{18}=\beta^*_{21}$
$S_{3,3}$	y_{18}	0	$\beta_{18}U_2^3y_4^2$	$\beta_{18}=0$
$S_{3,3}$	y^*_{18}	y_4y_8	$\beta^*_{18}U_2^3y_4^2+U_2\Phi_3^2$	$\beta^*_{18}=0$

Table 5 continued

Operation S_ω	y_j	S_ω on y_j	S_ω on of the right part of the formula (3.5)	Relation among the elements β_i
S_4	y_{18}	0	$\beta_{19}U_2^2U_3y_4^2$	$\beta_{19}=0$
S_4	y_{18}^*	0	$\beta_{19}^*U_2^2U_3y_4^2$	$\beta_{19}^*=0$
$S_{2,2,2}$	y_{18}	y_4y_8	$U_2\Phi_3^2+\beta_{20}U_2^3y_4^2$	$\beta_{20}=0$
$S_{2,2,2}$	y_{18}^*	0	$\beta_{20}^*U_2^3y_4^2$	$\beta_{20}^*=0$

$$+\beta_{10}\Phi_3U_4^2+\beta_{11}U_2U_3[U_2y_{14}+U_3y_4y_8+\Phi_3(y_{10}+y_{10}^*)$$
$$+\Phi_5y_6+\Phi_6y_4]+\beta_{12}U_3^2[U_2y_{12}+\Phi_3y_8+U_4y_6]$$
$$+\beta_{13}U_2U_4[U_2(y_{10}+y_{10}^*)+\Phi_3y_6+U_4y_4]$$
$$+\beta_{14}U_3\Phi_3[U_2(y_{10}+y_{10}^*)+\Phi_3y_6+U_4y_4]$$
$$+\beta_{15}U_3\Phi_3[U_2y_{10}+U_3y_8]+\beta_{16}U_3U_4[U_2y_8$$
$$+U_3y_6]+\beta_{17}\Phi_3^2[U_2y_8+U_3y_6]+\beta_{18}U_2\Phi_6[U_2y_6$$
$$+U_3y_4]+\beta_{19}U_3\Phi_5[U_2y_6+U_3y_4]+\beta_{20}U_4\Phi_3[U_2y_6+U_3y_4]$$
$$+\beta_{21}U_2[U_2y_6+U_3y_4][U_2(y_{10}+y_{10}^*)+\Phi_3y_6+U_4y_4]$$
$$+\beta_{22}U_2^3y_8^2+\beta_{23}U_2^2\Phi_5y_4^2+\beta_{24}U_2^2\Phi_3y_6^2$$
$$+\beta_{25}U_2U_3U_4y_4^2+\beta_{26}U_2\Phi_3^2y_4^2+\beta_{27}U_3^2\Phi_3y_4^2$$
$$+\beta_{28}U_2^2[U_2y_8+U_3y_6]y_4^2+\beta_{29}U_2U_3^2y_6^2+\beta_{30}U_2^3y_4^4. \tag{3.6}$$

To determine the coefficients β_i we use operations S_ω (Table 6).

Finally applying the operation $S_{2,2,2,2,2,2,2,2,2}$ to the equality (vi) we obtain $0=\beta_{30}U_2^3$, from where it follows $\beta_{30}=0$. Solving the linear system on the coefficients β_i we obtain:

$$d_3(y_{20})=U_2U_3U_5+U_2U_4\Phi_6+\Phi_3U_4^2+U_2U_3U_4y_4^2.$$

8. Choose elements $y_{22}, y_{22}^*\in E_2^{0,88}$ so that $\pi_0^2(y_{22})=c_{22}$,

$$\pi_0^2(y_{22}^*)=c_{11}^2+c_{14}(c_4^2+c_2^4)+c_{10}(c_{12}+c_2^2c_4^2+c_2^6)+c_6(c_2^4c_4^2+c_8^2+c_2^8),$$

and so that:

$$d_3(y_{22})=\beta_1U_2^2\Phi_{10}+\beta_2U_2U_3\Phi_9+\beta_3U_2\Phi_3U_5+\beta_4U_2U_4\Phi_7+\beta_5U_2\Phi_5\Phi_6$$
$$+\beta_6U_3^2U_5+\beta_7U_3\Phi_3\Phi_7+\beta_8U_3U_4\Phi_6+\beta_9U_3\Phi_5^2+\beta_{10}\Phi_3^2\Phi_6$$
$$+\beta_{11}\Phi_3U_4\Phi_5+\beta_{12}U_4^3+\beta_{13}U_2^2[U_2(y_{18}+y_{18}^*)+\Phi_3y_{14}+U_4y_4y_8$$
$$+\Phi_5(y_{10}+y_{10}^*)+\Phi_7y_6+U_5y_4]+\beta_{14}U_2\Phi_3[U_2y_{14}+U_3y_4y_8$$
$$+\Phi_3(y_{10}+y_{10}^*)+\Phi_5y_6+\Phi_6y_4]+\beta_{15}U_3^2[U_3y_{12}+\Phi_3y_{10}+U_4y_8]$$
$$+\beta_{16}U_2U_4[U_2y_{12}+U_3(y_{10}+y_{10}^*)]+\beta_{17}U_3\Phi_3[U_2y_{12}+U_3(y_{10}+y_{10}^*)]$$

Table 6 Calculation of the action of the differential d_3 on y_{20}

Operation S_ω	y_j	S_ω on y_j	S_ω on of the right part of the formula (3.6)	Relation among the elements β_i
S_{16}	y_{20}	0	$\beta_1 U_2^3$	$\beta_1 = 0$
S_{14}	y_{20}	y_6	$(\beta_2 + \beta_{11})U_2^2 U_3$	$1 + \beta_2 = \beta_{11}$
$S_{7,7}$	y_{20}	y_6	$\beta_2 U_2^2 U_3$	$\beta_2 = 1$
S_{12}	y_{20}	y_8	$(1 + \beta_6 + \beta_{12})U_2 U_3^2$ $+\beta_3 U_2^2 \Phi_3$	$\beta_6 = \beta_{12}, \beta_3 = 0$
$S_{4,4,4}$	y_{20}	y_8	$(\beta_6 + \beta_8 + \beta_{10} + \beta_{15}$ $+\beta_{16})U_2 U_3^2 + \beta_9 U_2^2 \Phi_3$	$(\beta_6 + \beta_8 + \beta_{10} + \beta_{15}$ $+\beta_{16}) = 1, \beta_9 = 0$
$S_{8,8}$	y_{20}	0	$\beta_5 U_2^3$	$\beta_5 = 0$
$S_{4,4,4,4}$	y_{20}	0	$(\beta_9 + \beta_{17} + \beta_{22})U_2^3$	$\beta_9 + \beta_{17} + \beta_{22} = 0$
$S_{3,3,3,3}$	y_{20}	y_8	$U_2 U_3^2 + (\beta_4$ $+ \beta_{10})U_2^2 \Phi_3$	$\beta_4 = \beta_{10}$
$S_{6,6}$	y_{20}	0	$(\beta_6 + \beta_8 + \beta_{12} + \beta_{16}$ $+ \beta_{19} + \beta_{29})U_2 U_3^2$ $+ (\beta_4 + \beta_{10} + \beta_{13}$ $+ \beta_{18} + \beta_{20} + \beta_{21}$ $\beta_{24})U_2^2 \Phi_3$	$\beta_6 + \beta_8 + \beta_{16} + \beta_{19}$ $+\beta_{12} + \beta_{29} = 0$ $\beta_4 + \beta_{10} + \beta_{13} + \beta_{18}$ $+\beta_{20} + \beta_{21} = \beta_{24}$
S_{10}	y_{20}	y_{10} $+y_{10}^*$	$(1 + \beta_7 + \beta_{14} + \beta_{15})$ $\times U_2 U_3 \Phi_3 + (\beta_4$ $+\beta_{13})U_2^2 U_4 + \beta_6 U_3^3$ $+(\beta_{21} + \beta_{18})U_2^2 \tau_3$	$\beta_7 + \beta_{14} + \beta_{15} = 0$ $\beta_4 + \beta_{13} = 1$ $\beta_6 = 0$ $\beta_{21} = \beta_{18}$
$S_{5,5}$	y_{20}	y_{10} $+y_{10}^*$	$(1 + \beta_7 + \beta_{15})U_2 U_3$ $\times \Phi_3 + \beta_4 U_2^2 U_4$ $+\beta_{18} U_2^2 \tau_3$	$\beta_7 + \beta_{15} = 0$ $\beta_4 = 1$ $\beta_{18} = 0$
S_8	y_{20}	0	$\beta_7 U_3^2 \Phi_3 + \beta_{23} U_2^2$ $\times U_3 y_4^2 + \beta_{19} U_2 U_3 \tau_3$ $+\beta_8 U_2 U_3 U_4$	$\beta_7 = 0 \quad \beta_{23} = 0$ $\beta_{19} = 0 \quad \beta_8 = 0$
$S_{4,4}$	y_{20}	y_{12}	$U_2 U_3 U_4 + \beta_{17} U_2 \Phi_3^2$ $+U_3^2 \Phi_3 + \beta_{20} U_2 U_3 \tau_3$ $+\beta_{17} U_2^2 \tau_5 + \beta_{26} U_2^3 y_4^2$	$\beta_{17} = \beta_{20}$ $\beta_{26} = 0$
S_6	y_{20}	y_{14} $+y_6 y_4^2$	$U_2^2 \Phi_6 + \beta_{20} U_2 \Phi_3 \tau_3$ $+U_2 U_3 \Phi_5 + \beta_{17} U_3 \Phi_3^2$ $+(\beta_{25} + \beta_{28})U_2^2 U_3 y_4^2$ $+(1 + \beta_{20})U_2 \Phi_3 U_4$	$\beta_{17}, \beta_{20} = 0$ $\beta_{25} + \beta_{28} = 1$
$S_{3,3}$	y_{20}	y_{14} $+y_6 y_4^2$	$U_2^2 \Phi_6 + \beta_{25} U_2^2 U_3 y_4^2$ $+U_2 U_3 \Phi_5 + U_2 \Phi_3 U_4$	$\beta_{25} = 1$
S_4	y_{20}	$y_8 y_4^2$	$(1 + \beta_{27})U_2 U_3^2 y_4^2$	$\beta_{27} = 0$

$$
\begin{aligned}
&+\beta_{18}U_2\Phi_5[U_2(y_{10}+y_{10}^*)+\Phi_3y_6+U_4y_4]+\beta_{19}U_3U_4[U_2y_{10}+U_3y_8]\\
&+\beta_{20}U_3U_4[U_2(y_{10}+y_{10}^*)+\Phi_3y_6+U_4y_4]+\beta_{21}\Phi_3^2[U_2y_{10}+U_3y_8]\\
&+\beta_{22}\Phi_3^2[U_2(y_{10}+y_{10}^*)+\Phi_3y_6+U_4y_4]+\beta_{23}\Phi_3U_4[U_2y_{10}+U_3y_8]\\
&+\beta_{24}U_2\Phi_7[U_2y_6+U_3y_4]+\beta_{25}U_3\Phi_6[U_2y_6+U_3y_4]+\beta_{26}\Phi_3\Phi_5[U_2y_6\\
&+U_3y_4]+\beta_{27}U_4^2[U_2y_6+U_3y_4]+\beta_{28}U_2[U_2y_6+U_3y_4][U_2y_{12}+\Phi_3y_8\\
&+U_4y_6]+\beta_{29}U_3[U_2y_6+U_3y_4][U_2y_{10}+U_3y_8]+\beta_{30}U_3[U_2y_6+U_3y_4]\\
&\times[U_2(y_{10}+y_{10}^*)+\Phi_3y_6+U_4y_4]+\beta_{31}\Phi_3[U_2y_6+U_3y_4][U_2y_8+U_3y_6]\\
&+\beta_{32}U_2^2U_3y_8^2+\beta_{33}U_2^2U_4y_6^2+\beta_{34}U_2U_3\Phi_3y_6^2+\beta_{35}U_3^3y_6^2+\beta_{36}U_2^2[U_2y_6\\
&+U_3y_4]+\beta_{37}U_2^2\Phi_6y_4^2+\beta_{38}U_2U_3\Phi_5y_4^2+\beta_{39}U_2\Phi_3U_4y_4^2+\beta_{40}U_3^2U_4y_4^2\\
&+\beta_{41}U_3\Phi_3^2y_4^2+\beta_{42}U_2^2[U_2(y_{10}+y_{10}^*)+\Phi_3y_6+U_4y_4]y_4^2\\
&+\beta_{43}U_2^2U_3y_4^4+\beta_{44}U_3^2[U_2y_6+U_3y_4]y_4^2. \qquad (3.7)
\end{aligned}
$$

For the image of the differential d_3 on the element y_{22}^* there is analogous equality with the change of β_i on β_i^*). Let us determine the unknown coefficients applying operations S_ω to equality (vii). For almost all operations except $S_{8,8}$, $S_{4,4,4,4}$, $S_{6,6}$, $S_{4,4}$ and $S_{2,2,2,2,2,2,2,2}$ their action on y_{22} and y_{22}^* is the same. Analogously the action of almost all operations except mentioned on the right part of the formula (vii). Because of this fact only the action of these operations on the element y_{22} will be given in the Table and for the relations we give only for β_i, except the mentioned operations where two cases are given (Table 7).

Table 7 Calculation of the action of the differential d_3 on y_{22} and y_{22}^*

Operation S_ω	y_j	S_ω on y_j	S_ω on of the right part of the formula (3.7)	Relation among the elements β_i
S_{18}	y_{22}	0	$(\beta_1+\beta_{13})U_2^3$	$\beta_1=\beta_{13}$
$S_{9,9}$	y_{22}	0	$\beta_1U_2^3$	$\beta_1=0$
$S_{6,6,6}$	y_{22}	y_4	$(\beta_4+\beta_{12}+\beta_{16}$ $+\beta_{27}+\beta_{28}+\beta_{33}$ $+\beta_{24}+\beta_{36})U_2^3$	$\beta_4+\beta_{12}+\beta_{16}$ $+\beta_{27}+\beta_{28}+\beta_{33}$ $+\beta_{24}+\beta_{36}=1$
S_{16}	y_{22}	0	$\beta_2U_2^3$	$\beta_2=0$
$S_{8,8}$	y_{22}	0	$(\beta_2+\beta_5+\beta_9)U_2^2U_3$	$\beta_2+\beta_5+\beta_9=0$
$S_{8,8}$	y_{22}^*	y_6	$(\beta_2^*+\beta_5^*+\beta_9^*)U_2^2U_3$	$\beta_2^*+\beta_5^*+\beta_9^*=1$
$S_{7,7}$	y_{22}	y_8	$U_2U_3^2+\beta_3U_2^2\Phi_3$	$\beta_3=0$
$S_{4,4,4,4}$	y_{22}	0	$(\beta_2+\beta_4+\beta_7+\beta_9$ $+\beta_{11}+\beta_{16}+\beta_{17}$ $+\beta_{21}+\beta_{23}+\beta_{32})U_2^2U_3$	$\beta_2+\beta_4+\beta_7+\beta_9$ $+\beta_{11}+\beta_{16}+\beta_{17}$ $+\beta_{21}+\beta_{23}+\beta_{32}=0$

Table 7 continued

Operation S_ω	y_j	S_ω on y_j	S_ω on of the right part of the formula (3.7)	Relation among the elements β_i
$S_{4,4,4,4}$	y_{22}^*	y_6	$(\beta_2^*+\beta_4^*+\beta_7^*+\beta_9^*$ $+\beta_{11}^*+\beta_{16}^*+\beta_{17}^*$ $+\beta_{21}^*+\beta_{23}^*+\beta_{32}^*)U_2^2U_3$	$(\beta_2^*+\beta_4^*+\beta_7^*+\beta_9^*$ $+\beta_{11}^*+\beta_{16}^*+\beta_{17}^*$ $+\beta_{21}^*+\beta_{23}^*+\beta_{32}^*=1$
S_{14}	y_{22}	y_8	$(\beta_2+\beta_6)U_2U_3^2$ $+(\beta_3+\beta_{14})U_2^2\Phi_3$	$\beta_2=\beta_6$ $\beta_3=\beta_{14}$
S_{12}	y_{22}	y_{10}	$(\beta_4+\beta_{16})U_2^2U_4$ $+(\beta_2+\beta_3+\beta_7$ $+\beta_{17})U_2U_3\Phi_3$ $+(\beta_6+\beta_{15})U_3^3$ $+(\beta_{24}+\beta_{28})U_2^2\tau_3$	$\beta_4=\beta_{16}$ $\beta_2+\beta_3$ $=\beta_7+\beta_{17}$ $\beta_6=\beta_{15}$ $\beta_{24}=\beta_{28}$
$S_{3,3,3,3}$	y_{22}	y_{10}^*	$U_2^2U_4+\beta_8U_2U_3\Phi_3$ $+(\beta_{16}+\beta_{27})U_2^2\tau_3+U_3^3$	$\beta_8=1$ $\beta_{16}=\beta_{27}$
$S_{6,6}$	y_{22}	y_{10} $+y_{10}^*$	$(\beta_{12}+\beta_{24}+\beta_{28}$ $+\beta_{33})U_2^2U_4+(\beta_5$ $+\beta_7+\beta_8+\beta_{11}$ $+\beta_{18}+\beta_{20}+\beta_{23}$ $+\beta_{25}+\beta_{26}+\beta_{30}$ $+\beta_{31}+\beta_{34})U_2U_3\Phi_3$ $+(\beta_9+\beta_{35})U_3^3$ $+(\beta_{16}+\beta_{24}+\beta_{27}$ $+\beta_{28}+\beta_{36})U_2^2\tau_3$	$\beta_{12}+\beta_{24}+\beta_{28}$ $+\beta_{33}=1,\ \beta_5$ $+\beta_7+\beta_8+\beta_{11}$ $+\beta_{18}+\beta_{20}+\beta_{23}$ $+\beta_{25}+\beta_{26}+\beta_{30}$ $+\beta_{31}+\beta_{34}=1$ $\beta_9=\beta_{35}$ $\beta_{16}+\beta_{24}+\beta_{27}$ $+\beta_{28}+\beta_{36}=0$
$S_{6,6}$	y_{22}^*	y_{10} $+y_{10}^*$	$(\beta_{12}^*+\beta_{24}^*+\beta_{28}^*$ $+\beta_{33}^*)U_2^2U_4+(\beta_5^*$ $+\beta_7^*+\beta_8^*+\beta_{11}^*$ $+\beta_{18}^*+\beta_{20}^*+\beta_{23}^*$ $+\beta_{25}^*+\beta_{26}^*+\beta_{30}^*$ $+\beta_{31}^*+\beta_{34}^*)U_2U_3\Phi_3$ $+(\beta_9^*+\beta_{35}^*+1)U_3^3$ $+(\beta_{16}^*+\beta_{24}^*+\beta_{27}^*$ $+\beta_{28}^*+\beta_{36}^*)U_2^2\tau_3$	$\beta_{12}^*+\beta_{24}^*+\beta_{28}^*$ $+\beta_{33}^*=1,\ \beta_5^*$ $+\beta_7^*+\beta_8^*+\beta_{11}^*$ $+\beta_{18}^*+\beta_{20}^*+\beta_{23}^*$ $+\beta_{25}^*+\beta_{26}^*+\beta_{30}^*$ $+\beta_{31}^*+\beta_{34}^*=1$ $\beta_9^*=\beta_{35}^*+1$ $\beta_{16}^*+\beta_{24}^*+\beta_{27}^*$ $+\beta_{28}^*+\beta_{36}^*=0$
$S_{4,4,4}$	y_{22}	y_{10}	$(\beta_4+\beta_5+\beta_{10}$ $+\beta_{11}+\beta_{16}+\beta_{23})$ $\times U_2^2U_4+(\beta_4+\beta_{17}$ $+\beta_{23})U_2U_3\Phi_3$ $+(\beta_{19}+1)U_3^3+(\beta_{18}$ $+\beta_{22}+\beta_{24}+\beta_{26}$ $+\beta_{37})U_2^2\tau_3$	$\beta_4+\beta_5+\beta_{10}$ $+\beta_{11}+\beta_{16}+\beta_{23}=0$ $\beta_{19}=0$ $\beta_4+\beta_{17}+\beta_{23}=0$ $\beta_{18}+\beta_{22}+\beta_{24}$ $+\beta_{26}+\beta_{37}=0$

Table 7 continued

Operation S_ω	y_j	S_ω on y_j	S_ω on of the right part of the formula (3.7)	Relation among the elements β_i
S_{10}	y_{22}	y_{12}	$(\beta_4+\beta_{16}+\beta_{19}$ $+\beta_{20}+1)U_2U_3U_4$ $+(\beta_7+1+\beta_{17})U_3^2\Phi_3$ $+(\beta_5+\beta_{18})U_2^2\Phi_5$ $+U_2\Phi_3^2(\beta_{10}+\beta_{21}$ $+\beta_{22})+(\beta_{24}+\beta_{25}$ $+\beta_{29}+\beta_{30})U_2U_3\tau_3$ $+(\beta_{37}+\beta_{42})U_2^3y_4^2$	$\beta_4+\beta_{16}+\beta_{19}$ $+\beta_{20}=0$ $\beta_7=\beta_{17}$ $\beta_5=\beta_{18}$ $\beta_{10}+\beta_{21}+\beta_{22}=0$ $\beta_{24}+\beta_{25}+\beta_{29}$ $+\beta_{30}=0$ $\beta_{37}=\beta_{42}$
$S_{5,5}$	y_{22}	y_{12}	$(1+\beta_{19})U_2U_3U_4$ $+U_3^2\Phi_3+\beta_5U_2^2\Phi_5$ $+(\beta_{10}+\beta_{21})U_2\Phi_3^2$ $+(\beta_{25}+\beta_{29})U_2U_3\tau_3$ $+\beta_{37}U_2^3y_4^2$	$\beta_5=0$ $\beta_{19}=0$ $\beta_{10}=\beta_{21}$ $\beta_{37}=0$ $\beta_{25}=\beta_{29}$
S_8	y_{22}	0	$\beta_5U_2^2\Phi_6+(\beta_{11}$ $+\beta_4)U_2U_4\Phi_3+(\beta_7$ $+\beta_{10})U_3\Phi_3^2+\beta_6$ $\times U_2U_3\Phi_5+\beta_{18}U_2^2\tau_7^*$ $+(\beta_{24}+\beta_{26})U_2\Phi_3\tau_3$ $+(\beta_{37}+\beta_{38})U_2^2U_3y_4^2$ $+\beta_{25}U_3^2\tau_3$	$\beta_{11}=\beta_4$ $\beta_{18}=0$ $\beta_{10}=\beta_7$ $\beta_{24}=\beta_{26}$ $\beta_{37}=\beta_{38}$ $\beta_5=0$ $\beta_6=0\beta_{25}=0$
$S_{4,4}$	y_{22}	0	$\beta_{10}U_2^2\Phi_6+\beta_{21}U_2^2\tau_7$ $(\beta_7+\beta_{21})U_3\Phi_3^2$ $+\beta_{23}U_2U_4\Phi_3$ $+\beta_7U_2U_3\Phi_5$ $+\beta_{23}U_2U_3\tau_5$ $+\beta_{41}U_2^2U_3y_4^2$ $+\beta_{31}U_2\Phi_3\tau_3$	$\beta_7=\beta_{41}$ $\beta_{31}=0$
S_6	y_{22}	y_{16} $+y_6y_{10}$	$(\beta_4+\beta_{24})U_2^2\Phi_7$ $+(\beta_{25}+1)U_2U_3\Phi_6+$ $+U_2\Phi_3\Phi_5(\beta_{11}+\beta_{26})$ $+(1+\beta_{27}+\beta_4)U_2U_4^2$ $+U_3\Phi_3U_4(1+\beta_7$ $+\beta_{11}+\beta_{23})+U_3^2\Phi_5$ $+\beta_{10}\Phi_3^3+\beta_{30}U_2U_3\tau_7^*$ $+\beta_{16}U_2^2\tau_9^*+\beta_{28}U_2^2\tau_9$	$\beta_4+\beta_{24}$ $=\beta_{11}+\beta_{26}$ $=\beta_{16}=\beta_{39}$ $\beta_{40}+\beta_{44}=1$ $\beta_{17}+\beta_{26}$ $+\beta_{30}=\beta_{10}$ $\beta_{33}=0$

Table 7 continued

Operation S_ω	y_j	S_ω on y_j	S_ω on of the right part of the formula (3.7)	Relation among the elements β_i
			$+\beta_{33}U_2^3y_6^2+U_2U_4\tau_3\times$ $\times(\beta_{16}+\beta_{24}+\beta_{28})$ $+\beta_{23}U_2\Phi_3\tau_5+(\beta_{17}$ $+\beta_{26}+\beta_{30})U_3\Phi_3\tau_3$ $+(\beta_{40}+\beta_{44})U_2U_3^2y_4^2$ $+\beta_{39}U_2^2\Phi_3y_4^2$ $+\beta_{29}U_2U_3\tau_7$	$\beta_{28}=0$
$S_{3,3}$	y_{22}	y_{16} $+y_6y_{10}$	$U_2U_3\Phi_6+U_3^2\Phi_5$ $+U_3\Phi_3U_4+\beta_{10}\Phi_3^3$ $+\beta_{23}U_2\Phi_3\tau_5$ $+\beta_{17}U_3\Phi_3\tau_3$ $+\beta_{40}U_2U_3^2y_4^2$ $+U_2U_4^2$	$\beta_{40}=1$
S_4	y_{22}	$y_{10}y_4^2$	$\beta_7(U_2U_3\Phi_7$ $+U_3\Phi_3\Phi_5)+U_3^3y_4^2$ $+\beta_{10}\Phi_3^2U_4$ $+\beta_{23}(U_2U_4$ $+U_3\Phi_3)\tau_5$ $\beta_{17}U_2U_3\tau_9^*$	$\beta_7=0$ $\beta_{17}=0$ $\beta_{10}=0$ $\beta_{23}=0$

Applying the operation $S_{2,2,2,2,2,2,2,2}$ to the considering equality we get the relation: $\beta_{43}U_2^2U_3=0$, or $\beta_{43}=0$. Applying the operation $S_{3,3,3,3,3,3}$ we get the relation: $d_3y_4=\beta_{12}U_2^3$, or $\beta_{12}=1$. Solving the corresponding linear system we have:

$$d_3(y_{22})=U_3^2U_5+U_3U_4\Phi_6+U_4^3+U_3^2U_4y_4^2,$$
$$d_3(y_{22}^*)=U_3^2U_5+U_3U_4\Phi_6+U_4^3+U_3^2U_4y_4^2+U_3\Phi_5^2.$$

9. Choose elements $y_{26}, y_{26}^*\in E_2^{0,104}$ so that $\pi_0^2(y_{26})=c_{26}+c_{10}c_2^8$, $\pi_0^2(y_{26}^*)=c_{13}^2+c_2^2c_{11}^2+c_4^2c_9^2+c_5^2c_8^2$ and also such that:

$$\begin{aligned}d_3(y_{26})=&\ \beta_1U_2^2\Phi_{12}+\beta_2U_2U_3\Phi_{11}+\beta_3U_2\Phi_3\Phi_{10}+\beta_4U_3^2\Phi_{10}+\beta_5U_2U_4\Phi_9\\&+\beta_6U_3\Phi_3\Phi_9+\beta_7U_2\Phi_5U_5+\beta_8U_3U_4U_5+\beta_9\Phi_3^2U_5+\beta_{10}U_2\Phi_6\Phi_7\\&+\beta_{11}U_3\Phi_5\Phi_7+\beta_{12}\Phi_3U_4\Phi_7+\beta_{13}U_3\Phi_6^2+\beta_{14}\Phi_3\Phi_5\Phi_6+\beta_{15}U_4^2\Phi_6\\&+\beta_{16}U_4\Phi_5^2+\beta_{17}U_2\Phi_3[U_2(y_{18}+y_{18}^*)+\Phi_3y_{14}+U_4y_4y_8+\Phi_5(y_{10}\\&+y_{10}^*)+\Phi_7y_6+U_5y_4]+\beta_{19}U_3^2[U_2(y_{18}+y_{18}^*)+\Phi_3y_{14}+U_4y_4y_8\\&+\Phi_5(y_{10}+y_{10}^*)+\Phi_7y_6+U_5y_4]+\beta_{18}U_3^2[U_2y_{18}+U_3y_{16}]\end{aligned}$$

$$
\begin{aligned}
&+\beta_{20}U_2\Phi_5[U_2y_{14}+U_3y_4y_8+\Phi_3(y_{10}+y_{10}^*)+\Phi_5y_6+\Phi_6y_4]\\
&+\beta_{21}U_3\Phi_3[U_2(y_{10}y_6+y_8y_4^2)+U_3y_{14}]+\beta_{22}U_3U_4[U_3y_{12}+\Phi_3y_{10}\\
&+U_4y_8]+\beta_{23}U_3U_4[U_2y_{14}+U_3y_4y_8+\Phi_3(y_{10}+y_{10}^*)+\Phi_5y_6+\Phi_6y_4]\\
&+\beta_{24}U_3\Phi_6[U_2y_{10}+U_3y_8]+\beta_{25}U_3\Phi_6[U_2(y_{10}+y_{10}^*)+\Phi_3y_6\\
&+U_4y_4]+\beta_{26}\Phi_3\Phi_5[U_2y_{10}+U_3y_8]+\beta_{27}\Phi_3\Phi_5[U_2(y_{10}+y_{10}^*)\\
&+\Phi_3y_6+U_4y_4]+\beta_{28}U_4^2[U_2y_{10}+U_3y_8]+\beta_{29}U_4^2[U_2(y_{10}+y_{10}^*)\\
&+\Phi_3y_6+U_4y_4]+\beta_{30}\Phi_3\Phi_6[U_2y_8+U_3y_6]+\beta_{31}U_4\Phi_5[U_2y_8+U_3y_6]\\
&+\beta_{32}U_2\Phi_9[U_2y_6+U_3y_4]+\beta_{33}U_3U_5[U_2y_6+U_3y_4]+\beta_{34}U_4\Phi_6[U_2y_6\\
&+U_3y_4]+\beta_{35}U_2[U_2y_6+U_3y_4][U_2y_{16}+U_3(y_{14}+y_4y_{10}+y_6(y_8+y_4^2))\\
&+\Phi_3y_{12}+U_4(y_{10}+y_{10}^*)]+\beta_{36}U_2(U_2y_8+U_3y_6)[U_2y_{14}+U_3y_4y_8\\
&+\Phi_3(y_{10}+y_{10}^*)+\Phi_5y_6+\Phi_6y_4]+\beta_{37}U_2[U_2y_{10}+U_3y_8][U_2y_{12}\\
&+U_3(y_{10}+y_{10}^*)]+\beta_{38}U_2[U_2(y_{10}+y_{10}^*)+\Phi_3y_6+U_4y_4][U_2y_{12}\\
&+U_3(y_{10}+y_{10}^*)]+\beta_{39}U_3[U_2y_6+U_3y_4][U_3y_{12}+\Phi_3y_{10}+U_4y_8]\\
&+\beta_{40}U_3[U_2y_6+U_3y_4][U_2y_{14}+U_3y_4y_8+\Phi_3(y_{10}+y_{10}^*)+\Phi_5y_6\\
&+\Phi_6y_4]+\beta_{41}U_3[U_2y_8+U_3y_6][U_2y_{12}+\Phi_3y_8+U_4y_6]\\
&+\beta_{42}U_3[U_2y_8+U_3y_6][U_2y_{12}+U_3(y_{10}+y_{10}^*)]+\beta_{43}\Phi_3[U_2y_6+U_3y_4]\\
&\times[U_2y_{12}+\Phi_3y_8+U_4y_6]+\beta_{44}\Phi_3[U_2y_6+U_3y_4][U_2y_{12}+U_3(y_{10}\\
&+y_{10}^*)]+\beta_{45}\Phi_3[U_2y_8+U_3y_6][U_2y_{10}+U_3y_8]+\beta_{46}\Phi_3[U_2(y_{10}+y_{10}^*)\\
&+\Phi_3y_6+U_4y_4][U_2y_8+U_3y_6]+\beta_{47}U_4[U_2y_6+U_3y_4][U_2y_{10}+U_3y_8]\\
&+\beta_{48}U_4[U_2y_6+U_3y_4][U_2(y_{10}+y_{10}^*)+\Phi_3y_6+U_4y_4]+\beta_{49}\Phi_5[U_2y_6\\
&+U_3y_4][U_3y_8+U_4y_6]+\beta_{50}U_2^2U_3y_{10}^2+\beta_{51}U_2^2U_3(y_{10}^*)^2+\beta_{52}U_2^2U_4y_8^2\\
&+\beta_{53}U_2U_3\Phi_3y_8^2+\beta_{54}U_3^3y_8^2+\beta_{55}U_2^2[U_2y_6+U_3y_4]y_8^2+\beta_{56}U_2U_3\Phi_5y_6^2\\
&+\beta_{57}U_2\Phi_3U_4y_6^2+\beta_{58}U_3^2U_4y_6^2+\beta_{59}U_2^2[U_2(y_{10}+y_{10}^*)+\Phi_3y_6+U_4y_4]\\
&\times y_6^2+\beta_{60}U_2^2y_6^6[U_2y_{10}+U_3y_8]+\beta_{61}U_2U_3[U_2y_8+U_3y_6]+\beta_{62}U_2\Phi_3\\
&\times[U_2y_6+U_3y_4]y_6^2+\beta_{63}U_3^2[U_2y_6+U_3y_4]+\beta_{64}U_2^2U_3y_4^2y_6^2+\beta_{65}U_2^2U_5\\
&\times y_4^2+\beta_{66}U_2U_4\Phi_5y_4^2+\beta_{67}U_3^2\Phi_6y_4^2+\beta_{68}U_3\Phi_3\Phi_5+\beta_{69}U_3U_4^2y_4^2\\
&+\beta_{70}U_2^2[U_2y_{14}+U_3y_4y_8+\Phi_3(y_{10}+y_{10}^*)+\Phi_5y_6+\Phi_6y_4]y_4^2\\
&+\beta_{71}U_2U_3[U_2y_{12}+U_3(y_{10}+y_{10}^*)]y_4^2+\beta_{72}U_2\Phi_3[U_2y_{10}+U_3y_8]y_4^2\\
&+\beta_{73}U_2\Phi_3[U_2(y_{10}+y_{10}^*)+\Phi_3y_6+U_4y_4]y_4^2+\beta_{74}U_3^2[U_2y_{10}+U_3y_8]y_4^2\\
&+\beta_{75}U_3^2[U_2(y_{10}+y_{10}^*)+\Phi_3y_6+U_4y_4]+\beta_{76}U_2U_4[U_2y_8+U_3y_6]y_4^2
\end{aligned}
$$

$$+\beta_{77}U_3\Phi_3[U_2y_8+U_3y_6]y_4^2+\beta_{78}U_2\Phi_5[U_2y_6+U_3y_4]y_4^2+\beta_{79}U_3U_4 \times[U_2y_6+U_3y_8]y_4^2+\beta_{80}U_2^2U_4y_4^4+\beta_{81}U_2U_3\Phi_3y_4^4+\beta_{82}U_3^3y_4^4+\beta_{83}U_2^2 \times[U_2y_6+U_3y_4]y_4^4+\beta_{84}U_2[U_2y_6+U_3y_4][U_2y_8+U_3y_6]y_4^2+\beta_{85}U_2\Omega_1^2. \tag{3.8}$$

To determine the coefficients β_i we use operations S_ω. When the actions of the operation S_ω on y_{26} and y_{26}^* coincide then in the table this action is described only for y_{26}.

9. Choose elements $y_{26}, y_{26}^* \in E_2^{0,104}$ so that $\pi_0^2(y_{26}) = c_{26}+c_{10}c_2^8, \pi_0^2(y_{26}^*) = c_{13}^2+c_2^2c_{11}^2+c_4^2c_9^2+c_5^2c_8^2$ and also such that:

$$d_3(y_{26}) = \beta_1U_2^2\Phi_{12}+\beta_2U_2U_3\Phi_{11}+\beta_3U_2\Phi_3\Phi_{10}+\beta_4U_3^2\Phi_{10}+\beta_5U_2U_4\Phi_9 +\beta_6U_3\Phi_3\Phi_9+\beta_7U_2\Phi_5U_5+\beta_8U_3U_4U_5+\beta_9\Phi_3^2U_5+\beta_{10}U_2\Phi_6\Phi_7 +\beta_{11}U_3\Phi_5\Phi_7+\beta_{12}\Phi_3U_4\Phi_7+\beta_{13}U_3\Phi_6^2+\beta_{14}\Phi_3\Phi_5\Phi_6+\beta_{15}U_4^2\Phi_6 +\beta_{16}U_4\Phi_5^2+\beta_{17}U_2\Phi_3[U_2(y_{18}+y_{18}^*)+\Phi_3y_{14}+U_4y_4y_8+\Phi_5(y_{10} +y_{10}^*)+\Phi_7y_6+U_5y_4]+\beta_{19}U_3^2[U_2(y_{18}+y_{18}^*)+\Phi_3y_{14}+U_4y_4y_8 +\Phi_5(y_{10}+y_{10}^*)+\Phi_7y_6+U_5y_4]+\beta_{18}U_3^2[U_2y_{18}+U_3y_{16}] +\beta_{20}U_2\Phi_5[U_2y_{14}+U_3y_4y_8+\Phi_3(y_{10}+y_{10}^*)+\Phi_5y_6+\Phi_6y_4] +\beta_{21}U_3\Phi_3[U_2(y_{10}y_6+y_8y_4^2)+U_3y_{14}]+\beta_{22}U_3U_4[U_3y_{12}+\Phi_3y_{10} +U_4y_8]+\beta_{23}U_3U_4[U_2y_{14}+U_3y_4y_8+\Phi_3(y_{10}+y_{10}^*)+\Phi_5y_6+\Phi_6y_4] +\beta_{24}U_3\Phi_6[U_2y_{10}+U_3y_8]+\beta_{25}U_3\Phi_6[U_2(y_{10}+y_{10}^*)+\Phi_3y_6 +U_4y_4]+\beta_{26}\Phi_3\Phi_5[U_2y_{10}+U_3y_8]+\beta_{27}\Phi_3\Phi_5[U_2(y_{10}+y_{10}^*) +\Phi_3y_6+U_4y_4]+\beta_{28}U_4^2[U_2y_{10}+U_3y_8]+\beta_{29}U_4^2[U_2(y_{10}+y_{10}^*) +\Phi_3y_6+U_4y_4]+\beta_{30}\Phi_3\Phi_6[U_2y_8+U_3y_6]+\beta_{31}U_4\Phi_5[U_2y_8+U_3y_6] +\beta_{32}U_2\Phi_9[U_2y_6+U_3y_4]+\beta_{33}U_3U_5[U_2y_6+U_3y_4]+\beta_{34}U_4\Phi_6[U_2y_6 +U_3y_4]+\beta_{35}U_2[U_2y_6+U_3y_4][U_2y_{16}+U_3(y_{14}+y_4y_{10}+y_6(y_8+y_4^2)) +\Phi_3y_{12}+U_4(y_{10}+y_{10}^*)]+\beta_{36}U_2(U_2y_8+U_3y_6)[U_2y_{14}+U_3y_4y_8 +\Phi_3(y_{10}+y_{10}^*)+\Phi_5y_6+\Phi_6y_4]+\beta_{37}U_2[U_2y_{10}+U_3y_8][U_2y_{12} +U_3(y_{10}+y_{10}^*)]+\beta_{38}U_2[U_2(y_{10}+y_{10}^*)+\Phi_3y_6+U_4y_4][U_2y_{12} +U_3(y_{10}+y_{10}^*)]+\beta_{39}U_3[U_2y_6+U_3y_4][U_3y_{12}+\Phi_3y_{10}+U_4y_8] +\beta_{40}U_3[U_2y_6+U_3y_4][U_2y_{14}+U_3y_4y_8+\Phi_3(y_{10}+y_{10}^*)+\Phi_5y_6 +\Phi_6y_4]+\beta_{41}U_3[U_2y_8+U_3y_6][U_2y_{12}+\Phi_3y_8+U_4y_6] +\beta_{42}U_3[U_2y_8+U_3y_6][U_2y_{12}+U_3(y_{10}+y_{10}^*)]+\beta_{43}\Phi_3[U_2y_6+U_3y_4] \times[U_2y_{12}+\Phi_3y_8+U_4y_6]+\beta_{44}\Phi_3[U_2y_6+U_3y_4][U_2y_{12}+U_3(y_{10}$$

$$
\begin{aligned}
&+y_{10}^*)] + \beta_{45}\Phi_3[U_2y_8 + U_3y_6][U_2y_{10} + U_3y_8] + \beta_{46}\Phi_3[U_2(y_{10} + y_{10}^*) \\
&+\Phi_3y_6 + U_4y_4][U_2y_8 + U_3y_6] + \beta_{47}U_4[U_2y_6 + U_3y_4][U_2y_{10} + U_3y_8] \\
&+\beta_{48}U_4[U_2y_6 + U_3y_4][U_2(y_{10} + y_{10}^*) + \Phi_3y_6 + U_4y_4] + \beta_{49}\Phi_5[U_2y_6 \\
&+U_3y_4][U_3y_8 + U_4y_6] + \beta_{50}U_2^2U_3y_{10}^2 + \beta_{51}U_2^2U_3(y_{10}^*)^2 + \beta_{52}U_2^2U_4y_8^2 \\
&+\beta_{53}U_2U_3\Phi_3y_8^2 + \beta_{54}U_3^3y_8^2 + \beta_{55}U_2^2[U_2y_6 + U_3y_4]y_8^2 + \beta_{56}U_2U_3\Phi_5y_6^2 \\
&+\beta_{57}U_2\Phi_3U_4y_6^2 + \beta_{58}U_3^2U_4y_6^2 + \beta_{59}U_2^2[U_2(y_{10} + y_{10}^*) + \Phi_3y_6 + U_4y_4] \\
&\times y_6^2 + \beta_{60}U_2^2y_6^6[U_2y_{10} + U_3y_8] + \beta_{61}U_2U_3[U_2y_8 + U_3y_6] + \beta_{62}U_2\Phi_3 \\
&\times[U_2y_6 + U_3y_4]y_6^2 + \beta_{63}U_3^2[U_2y_6 + U_3y_4] + \beta_{64}U_2^2U_3y_4^2y_6^2 + \beta_{65}U_2^2U_5 \\
&\times y_4^2 + \beta_{66}U_2U_4\Phi_5y_4^2 + \beta_{67}U_3^2\Phi_6y_4^2 + \beta_{68}U_3\Phi_3\Phi_5 + \beta_{69}U_3U_4^2y_4^2 \\
&+\beta_{70}U_2^2[U_2y_{14} + U_3y_4y_8 + \Phi_3(y_{10} + y_{10}^*) + \Phi_5y_6 + \Phi_6y_4]y_4^2 \\
&+\beta_{71}U_2U_3[U_2y_{12} + U_3(y_{10} + y_{10}^*)]y_4^2 + \beta_{72}U_2\Phi_3[U_2y_{10} + U_3y_8]y_4^2 \\
&+\beta_{73}U_2\Phi_3[U_2(y_{10}+y_{10}^*)+\Phi_3y_6+U_4y_4]y_4^2+\beta_{74}U_3^2[U_2y_{10}+U_3y_8]y_4^2 \\
&+\beta_{75}U_3^2[U_2(y_{10} + y_{10}^*) + \Phi_3y_6 + U_4y_4] + \beta_{76}U_2U_4[U_2y_8 + U_3y_6]y_4^2 \\
&+\beta_{77}U_3\Phi_3[U_2y_8 + U_3y_6]y_4^2 + \beta_{78}U_2\Phi_5[U_2y_6 + U_3y_4]y_4^2 + \beta_{79}U_3U_4 \\
&\times[U_2y_6 + U_3y_8]y_4^2 + \beta_{80}U_2^2U_4y_4^4 + \beta_{81}U_2U_3\Phi_3y_4^4 + \beta_{82}U_3^3y_4^4 + \beta_{83}U_2^2 \\
&\times[U_2y_6 + U_3y_4]y_4^4 + \beta_{84}U_2[U_2y_6 + U_3y_4][U_2y_8 + U_3y_6]y_4^2 + \beta_{85}U_2\Omega_1^2.
\end{aligned} \tag{3.9}
$$

To determine the coefficients β_i we use operations S_ω. When the actions of the operation S_ω on y_{26} and y_{26}^* coincide then in the table this action is described only for y_{26}.

Remark 3.2 The following notation is used in the table $\widehat{y_{14}} = y_{14}+y_{10}y_4+y_6(y_8+y_4^2)$ (Table 8).

Let us apply the operation $S_{3,3,3,3,3,3,3,3,3}$ to the equality under consideration, we obtain the relation: $S_{3,3,3,3,3,3,3,3,3}(y_{26}) = 0 = \beta_8U_2U_3^2 + \beta_{15}U_2^2\Phi_3$, this gives: $\beta_8 = 0$, $\beta_{15} = 0$.

If we apply the operation $S_{2,2,2,2,2,2,2,2,2,2,2,2}$, then we have

$$
\begin{aligned}
S_{2,2,2,2,2,2,2,2,2,2,2,2}(y_{26}) = y_4 = (&\beta_{14} + \beta_{20} + \beta_{30} + \beta_{42} + \beta_{51} \\
&+\beta_{56} + \beta_{61} + \beta_{64} + \beta_{67} + \beta_{68} + \beta_{70} \\
&+\beta_{71} + \beta_{77} + \beta_{81} + \beta_{82} + 1)U_2^3.
\end{aligned}
$$

This gives the relation:

$$
\begin{aligned}
&\beta_{14} + \beta_{20} + \beta_{30} + \beta_{42} + \beta_{51} + \beta_{56} + \beta_{61} + \beta_{64} + \beta_{67} \\
&\quad+\beta_{68} + \beta_{70} + \beta_{71} + \beta_{77} + \beta_{81} + \beta_{82} = 0
\end{aligned}
$$

Table 8 Calculation of the action of the differential d_3 on y_{26}, y_{26}^*

Operation S_ω	y_j	S_ω on y_j	S_ω on of the right part of the formula (3.9)	Relation among the elements β_i
S_{22}	y_{26}	0	$\beta_1 U_2^3$	$\beta_1 = 0$
$S_{11,11}$	y_{26}	0	$\beta_{85} U_2^3$	$\beta_{85} = 0$
$S_{11,11}$	y_{26}^*	y_4	$\beta_{85}^* U_2^3$	$\beta_{85}^* = 1$
S_{20}	y_{26}	0	$\beta_2 U_2^3$	$\beta_2 = 0$
$S_{10,10}$	y_{26}	y_6	$(\beta_{10} + \beta_{13} + \beta_{24}$ $+\beta_{25} + \beta_{36} + \beta_{38}$ $+\beta_{50} + \beta_{51})U_2^2 U_3$	$\beta_{10} + \beta_{13} + \beta_{24}$ $+\beta_{25} + \beta_{36} + \beta_{38}$ $+\beta_{50} + \beta_{51} = 1$
$S_{10,10}$	y_{26}^*	0	$(\beta_{10}^* + \beta_{13}^* + \beta_{24}^*$ $+\beta_{25}^* + \beta_{36}^* + \beta_{38}^*$ $+\beta_{50}^* + \beta_{51}^*)U_2^2 U_3$	$\beta_{10}^* + \beta_{13}^* + \beta_{24}^*$ $+\beta_{25}^* + \beta_{36}^* + \beta_{38}^*$ $+\beta_{50}^* + \beta_{51}^* = 0$
$S_{5,5,5,5}$	y_{26}	y_6	$(\beta_{13} + \beta_{24} + \beta_{50})$ $\times U_2^2 U_3$	$\beta_{13} + \beta_{24} + \beta_{50} = 1$
$S_{5,5,5,5}$	y_{26}^*	0	$(\beta_{13}^* + \beta_{24}^* + \beta_{50}^*)$ $\times U_2^2 U_3$	$\beta_{13}^* + \beta_{24}^* + \beta_{50}^* = 0$
S_{18}	y_{26}	0	$(\beta_3 + \beta_{17})U_2^2 \Phi_3 + (\beta_4$ $+\beta_{18} + \beta_{19})U_2 U_3^2$	$\beta_4 + \beta_{18} + \beta_{19} = 0$ $\beta_3 = \beta_{17}$
$S_{9,9}$	y_{26}	0	$(\beta_4 + \beta_{18})U_2 U_3^2$ $+\beta_3 U_2^2 \Phi_3$	$\beta_4 = \beta_{18}$ $\beta_3 = 0$
S_{16}	y_{26}	0	$\beta_{32} U_2^2 \tau_3 + \beta_5 U_2^2 U_4$ $+\beta_6 U_2 \Phi_3 U_3 + \beta_4 U_3^3$	$\beta_4 = 0 \beta_{32} = 0$ $\beta_5 = 0 \beta_6 = 0$
$S_{8,8}$	y_{26}	y_{10}	$(\beta_7 + \beta_{16})U_2^2 U_4$ $+(\beta_{10} + \beta_{11} + \beta_{14})$ $\times U_2 U_3 \Phi_3 + \beta_{13} U_3^3$	$\beta_{10} + \beta_{11} + \beta_{14} = 0$ $\beta_7 = \beta_{16}$ $\beta_{13} = 1$
$S_{8,8}$	y_{26}^*	0	$(\beta_7^* + \beta_{16}^*)U_2^2 U_4$ $+(\beta_{10}^* + \beta_{11}^* + \beta_{14}^*)$ $\times U_2 U_3 \Phi_3 + \beta_{13}^* U_3^3$	$\beta_{10}^* + \beta_{11}^* + \beta_{14}^* = 0$ $\beta_7^* = \beta_{16}^*$ $\beta_{13}^* = 0$
$S_{6,6,6}$	y_{26}	0	$(\beta_{10} + \beta_{12} + \beta_{15}$ $+\beta_{29} + \beta_{34} + \beta_{43}$ $+\beta_{48} + \beta_{57} + \beta_{59}$ $+\beta_{62})U_2^2 \Phi_3$ $+(\beta_{11} + \beta_{16} + \beta_{20}$ $+\beta_{23} + \beta_{31} + \beta_{36}$ $+\beta_{40} + \beta_{41} + \beta_{49}$ $+\beta_{56} + \beta_{58} + \beta_{61}$ $\beta_{63})U_2 U_3^2$	$\beta_{10} + \beta_{12} + \beta_{15}$ $+\beta_{29} + \beta_{34} + \beta_{43}$ $+\beta_{48} + \beta_{57} + \beta_{59}$ $+\beta_{62} = 0$ $\beta_{11} + \beta_{16} + \beta_{20}$ $+\beta_{23} + \beta_{31} + \beta_{36}$ $+\beta_{40} + \beta_{41} + \beta_{49}$ $+\beta_{56} + \beta_{58} + \beta_{61}$ $\beta_{63} = 0$
$S_{7,7}$	y_{26}	0	$\beta_7 U_2^2 \Phi_5 + \beta_{33} U_2 U_3 \tau_3$ $+\beta_9 U_2 \Phi_3^2 + \beta_{65} U_2^3 y_4^2$	$\beta_7 = 0 \beta_{33} = 0$ $\beta_9 = 0 \beta_{65} = 0$
$S_{4,4,4,4,4}$	y_{26}	0	$(\beta_{11} + \beta_{12} + \beta_{26}$ $+\beta_{31} + \beta_{37} + \beta_{41}$ $+\beta_{42} + \beta_{45} + \beta_{52}$ $+\beta_{53})U_2^2 U_3$	$\beta_{11} + \beta_{12} + \beta_{26}$ $+\beta_{31} + \beta_{37} + \beta_{41}$ $+\beta_{42} + \beta_{45} + \beta_{52}$ $+\beta_{53} = 0$
$S_{4,4,4,4}$	y_{26}	0	$U_2^2 U_4(\beta_{10} + \beta_{12} + \beta_{14}$ $+\beta_{30} + \beta_{31} + \beta_{52})$ $+U_2 U_3 \Phi_3(\beta_{31} + \beta_{45}$ $+\beta_{53}) + U_3^3(\beta_{28} + \beta_{54})$ $+U_2^2 \tau_3(\beta_{20} + \beta_{36}$ $+\beta_{38} + \beta_{44} + \beta_{46}$ $+\beta_{49} + \beta_{52} + \beta_{55})$	$0 = \beta_{10} + \beta_{12} + \beta_{14}$ $+\beta_{30} + \beta_{31} + \beta_{52}$ $\beta_{53} + \beta_{31} + \beta_{45} = 0$ $\beta_{28} = \beta_{54}$ $0 = \beta_{20} + \beta_{36}$ $+\beta_{38} + \beta_{44} + \beta_{46}$ $+\beta_{49} + \beta_{52} + \beta_{55}$
S_{14}	y_{26}	0	$\beta_{20} U_2^2 \Phi_5 + \beta_{36} U_2^2 \tau_5$ $+\beta_{21} U_3^2 \Phi_3 + \beta_{70} U_2^3 y_4^2$	$\beta_{20} = 0 \beta_{36} = 0$ $\beta_{21} = 0 \beta_{70} = 0$

Table 8 continued

Operation S_ω	y_j	S_ω on y_j	S_ω on of the right part of the formula (3.9)	Relation among the elements β_i
			$+(\beta_{35}+\beta_{40})U_2U_3\tau_3$	$\beta_{35}=\beta_{40}$
			$+\beta_{23}U_2U_3U_4$	$\beta_{23}=0$
S_{12}	y_{26}	0	$\beta_{10}U_2^2\Phi_6+\beta_{11}U_2U_3\Phi_5$	$\beta_{10}=\beta_{11}$
			$+\beta_{12}U_2\Phi_3U_4+\beta_{22}U_3^2U_4$	$=\beta_{12}=\beta_{71}$
			$+\beta_{37}U_2^2\tau_7+\beta_{38}U_2^2\tau_7^*$	$=\beta_{41}+\beta_{42}$
			$+(\beta_{41}+\beta_{42})U_2U_3\tau_5$	$\beta_{22}=0\beta_{39}=0$
			$+(\beta_{43}+\beta_{44})U_2\Phi_3\tau_3$	$\beta_{43}=\beta_{44}$
			$+\beta_{39}U_3^2\tau_3+\beta_{71}U_2^2U_3y_4^2$	
$S_{3,3,3,3}$	y_{26}	$\widehat{y_{14}}$	$U_3\Phi_3^2+\beta_{28}U_2^2\tau_7$	$\beta_{28}=0\beta_{29}=0$
			$+(\beta_{51}+\beta_{69})U_2^2U_3y_4^2$	$\beta_{51}=\beta_{69}$
			$+\beta_{29}U_2^2\tau_7^*$	
$S_{6,6}$	y_{26}	$\widehat{y_{14}}$	$U_3\Phi_3^2+U_2U_3\Phi_5(\beta_{11}$	$\beta_{10}+\beta_{34}=\beta_{11}$
			$+\beta_{31}+\beta_{40}+\beta_{49}+\beta_{56})$	$+\beta_{31}+\beta_{40}+\beta_{49}$
			$+U_2^2\Phi_6(\beta_{10}+\beta_{34})$	$+\beta_{56}=\beta_{10}+\beta_{34}$
			$+U_3\Phi_3U_4(\beta_{10}+\beta_{34}$	$+\beta_{43}+\beta_{48}+\beta_{57}$
			$+\beta_{43}+\beta_{48}+\beta_{57})$	$=\beta_{28}+\beta_{60}=\beta_{35}$
			$+U_3^2U_4(\beta_{11}+\beta_{31}$	$+\beta_{51}+\beta_{64}+\beta_{66}$
			$+\beta_{41}+\beta_{58})+U_2^2\tau_7$	$+\beta_{69}+\beta_{76}+\beta_{78}$
			$\times(\beta_{28}+\beta_{60})+U_2^2\tau_7^*$	$+\beta_{79}+\beta_{84}$
			$\times\beta_{29}+\beta_{35}+\beta_{48}$	$\beta_{25}+\beta_{27}+\beta_{30}$
			$+\beta_{59})+U_3\Phi_3^2(\beta_{25}$	$+\beta_{46}=\beta_{42}+\beta_{49}$
			$+\beta_{27}+\beta_{30}+\beta_{46})$	$+\beta_{63}=\beta_{10}+\beta_{34}$
			$+U_2U_3\tau_5(\beta_{40}+\beta_{49}$	$+\beta_{40}+\beta_{49}+\beta_{61}$
			$+\beta_{61})+U_3^2\tau_3(\beta_{42}$	$\beta_{29}+\beta_{35}+\beta_{48}$
			$+\beta_{49}+\beta_{63})+U_3\Phi_3\tau_3$	$=\beta_{59}$
			$\times(\beta_{38}+\beta_{44}+\beta_{62})$	$\beta_{11}+\beta_{31}+\beta_{41}$
			$+U_2^2U_3y_4^2(\beta_{35}+\beta_{51}$	$=\beta_{58}$
			$+\beta_{64}+\beta_{66}+\beta_{69}$	$\beta_{38}+\beta_{44}=\beta_{62}$
			$+\beta_{76}+\beta_{78}+\beta_{79}$	
			$+\beta_{84})$	
S_{10}	y_{26}	0	$\beta_{10}U_2^2\Phi_7+U_2U_3\Phi_6$	$\beta_{10}=\beta_{26}+\beta_{27}$
			$\times(\beta_{10}+\beta_{24}+\beta_{25})$	$=\beta_{37}+\beta_{38}=\beta_{30}$
			$+U_2\Phi_3\Phi_5(\beta_{26}+\beta_{27})$	$+\beta_{45}+\beta_{46}=\beta_{34}$
			$+\beta_{11}U_3^2\Phi_5+\beta_{12}U_3\Phi_3U_4$	$+\beta_{35}+\beta_{47}+\beta_{48}$
			$+U_2^2\tau_9^*(\beta_{37}+\beta_{38})$	$=\beta_{72}+\beta_{73}$
			$+(\beta_{24}+\beta_{37})U_2U_3\tau_7$	$\beta_{10}+\beta_{24}+\beta_{25}$
			$+(\beta_{25}+\beta_{38})U_2U_3\tau_7^*$	$=\beta_{11}=\beta_{12}=\beta_{24}$
			$+U_2\Phi_3\tau_5(\beta_{30}+\beta_{45}$	$+\beta_{37}=\beta_{42}=\beta_{67}$
			$+\beta_{46})+\beta_{42}U_3^2\tau_5$	$+\beta_{71}+\beta_{74}+\beta_{75}$
			$+(\beta_{34}+\beta_{35}+\beta_{47}$	$\beta_{25}=\beta_{38}$
			$+\beta_{48})U_2U_4\tau_3+(\beta_{67}$	$\beta_{40}=\beta_{44}$
			$+\beta_{71}+\beta_{74}+\beta_{75})$	$\beta_{59}=\beta_{60}$
			$\times U_2U_3^2y_4^2+(\beta_{40}+\beta_{44})$	
			$\times U_3\Phi_3\tau_3+(\beta_{59}+\beta_{60})$	
			$\times U_2^3y_6^2+(\beta_{72}+\beta_{73})$	
			$\times U_2^2\Phi_3y_4^2$	
$S_{5,5}$	y_{26}	0	$\beta_{24}U_2U_3\Phi_6+U_2\Phi_3\tau_5$	$\beta_{10}=\beta_{26}=\beta_{37}$
			$\times(\beta_{30}+\beta_{45})+U_2^2\Phi_7$	$=\beta_{30}+\beta_{45}$

Table 8 continued

Operation S_ω	y_j	S_ω on y_j	S_ω on of the right part of the formula (3.9)	Relation among the elements β_i
			$\times\beta_{10} + \beta_{26}U_2\Phi_3\Phi_5$	$= \beta_{34} + \beta_{47}$
			$+\beta_{24}U_2U_3\tau_7 + U_2^2\tau_9^*$	$= \beta_{72}$
			$\times\beta_{37} + \beta_{25}U_2U_3\tau_7^*$	$\beta_{67} = \beta_{74}$
			$+(\beta_{34} + \beta_{47})U_2U_4\tau_3$	$\beta_{24} = 0$
			$+(\beta_{67} + \beta_{74})U_2U_3^2y_4^2$	$\beta_{25} = 0$
			$+\beta_{60}U_2^3y_6^2 + \beta_{72}U_2^2\Phi_3y_4^2$	$\beta_{60} = 0$
S_8	y_{26}	0	$\beta_{30}U_3\Phi_3\tau_5 + U_3U_4\tau_5$	$\beta_{40} = \beta_{56} = \beta_{67}$
			$\times\beta_{31} + \beta_{49}U_2\tau_3\tau_5$	$\beta_{30} = 0\beta_{31} = 0$
			$+\beta_{40}U_3\tau_3^2 + U_2^2U_3y_6^2$	$\beta_{49} = 0\beta_{66} = 0$
			$\times\beta_{56} + \beta_{66}U_2^2U_4y_4^2$	$\beta_{68} = 0\beta_{78} = 0$
			$+\beta_{67}U_3^3y_4^2 + U_2^2\tau_3y_4^2$	
			$\times\beta_{78} + \beta_{68}U_2U_3\Phi_3y_4^2$	
S_6	y_{26}		$(\beta_{76} + \beta_{79})U_2U_3U_4y_4^2$	$\beta_{75} = \beta_{77}$
			$+(\beta_{75} + \beta_{77})U_3^2\Phi_3y_4^2$	$\beta_{76} = \beta_{79}$
			$+(\beta_{80} + \beta_{83})U_2^3y_4^4$	$\beta_{80} = \beta_{83}$
			$+\beta_{84}U_2^2\tau_5y_4^2$	$\beta_{84} = 0$
			$+(\beta_{79} + \beta_{84})U_2U_3\tau_3y_4^2$	

Finally using the operation $S_{2,2,2,2,2,2,2,2}$, we obtain:

$$S_{2,2,2,2,2,2,2,2}(y_{26}) = y_{10} = \beta_{80}U_2^2U_4 + \beta_{83}U_2^2\tau_3 \\ +(1 + \beta_{82})U_3^3 + \beta_{81}U_2U_3\Phi_3.$$

This gives: $\beta_{80} = 0$, $\beta_{81} = 0$, $\beta_{82} = 0$, $\beta_{83} = 0$.

Solving the linear system for β_i and β_j^*, we get:

$$d_3(y_{26}) = U_3\Phi_6^2 \quad d_3(y_{26}^*) = U_2\Omega_1^2$$

4 Construction of the elements of order four in MSp_*

Let us show that that the product $U_1\Phi_7\Omega_1 \in E_2^{3,103} \simeq E_3^{3,103}$ lives to infinity and not equal to zero in E_∞ of the Adams–Novikov spectral sequence. Each factor in this product is a nonzero infinite cycle. So, it is sufficient to prove that there is no element in $E_2^{0,104} \simeq E_3^{0,104}$ that kills $U_1\Phi_7\Omega_1$ by the action of the differential d_3, because all higher differentials are equal to zero by the dimension arguments.

From the conditions: $S_{13}(U_1\Phi_7\Omega_1) = U_1^2\Omega_1$ and $d_3(z_{13}) = U_1^2\Omega_1$, z_{13} has the F-filtration, corresponding to MASS equal to 2, operations S_ω conserve this filtration, it follows that the killing element can't have the F-filtration greater than 2. If there exists an element X such that: $X \in F^2(E_2^{0,104})$, $X \notin F^3(E_2^{0,104})$ and $d_3(X) = U_1\Phi_7\Omega_1$, then for the projection x of this element in MASS there must be the relation $S_{13}x = u_1\varphi_7\omega_1$ and $x \in E_\infty^{0,1,106}$ in MASS. However from the description of the E_∞ it follows that there is no such an element in the given dimension. Hence the only possible candidate for killing of this product is a multiplicatively nondecomposable element of zero filtration in $E_2^{0,104}$ of the Adams–Novikov spectral sequence. There are two such elements: y_{26} and y_{26}^* and we proved two following relations for them:

$$d_3(y_{26}) \equiv U_3\Phi_6^2 \mod (F^2E_3^{3,103} \cup (\text{ elements containing } U_1)),$$
$$d_3(y_{26}^*) \equiv U_2\Omega_1^2 \mod (F^2E_3^{3,103} \cup (\text{ elements containing } U_1)).$$

Hence, if the element y_{26}^* (for y_{26} this is analogous) kills $U_1\Phi_7\Omega_1$, then there must exist another element η, different from y_{26}^*, and such that: $d_3(\eta) = U_2\Omega_1^2$. Then from the action of the operation $S_{11,11}$: $S_{11,11}(U_2\Omega_1^2) = U_2^3$ and the action of the differential d_3, we conclude that $\eta = y_4y_{22}^* + \cdots$. However because of multiplicativity of the action of d_3 and the formula $d_3(y_{22}^*) = U_3^2U_5 + \cdots$, it follows the necessity of eliminating of the summand $y_4U_3^2U_5$, which is deleted either with the help of $d_3(y_4y_{22})$, (and then there appears $U_2^3y_{22}$ which is also undeletable), or again by $d_3(y_4y_{22}^*)$. Hence such η does not exist. So, $U_1\Phi_7\Omega_1$ not equal to zero in E_∞ of the Adams–Novikov spectral sequence and hence in the ring MSp_*.

Let us see the connection between the elements $\langle\Phi_7, 2, \Omega_1\rangle$ and $\theta_1\Phi_7\Omega_1$ of the ring MSp_*.

Theorem 4.1 *Let* $m_1, m_2 \in \operatorname{Tors} MSp_*$ *are elements of the order two. Then we have a relation:*

$$\theta_1 m_1 m_2 \in 2\langle m_1, 2, m_2\rangle$$

where $\langle m_1, 2, m_2\rangle$ *denotes a triple Massey products in the theory of symplectic cobordisms and* $\theta_1 \in MSp_2$ *is the generator.*

Proof Let us consider inclusions of manifolds $i_j : M_j^{n_j} \to \mathbb{R}^{n_j+r_j}$, $j = 1, 2$ which together with the normal bundles of these inclusions define on manifolds M_1 and M_2 some (B, f)-structures which represent the elements m_1 and m_2 respectively. Let MSp-manifolds X_j, $(j = 1, 2)$ are included in $\mathbb{R}_+^{n_j+r_j+1}$ so that $\partial X_j \cong 2M_j$ and a collar neighborhood of the boundary in X_j has the form $\partial X_j \times [0, 1)$, $\partial X_j \subset \mathbb{R}^{n_j+r_j} \times 0$ and X_j meets the boundary transversally. Let us consider the products together with injections $M_1 \times X_2 \subset \mathbb{R}_+^{n_1+r_1+n_2+r_2+1}$ and $M_2 \times X_1 \subset \mathbb{R}_-^{n_1+r_1+n_2+r_2+1}$ such that under restrictions of both injections on the neighborhoods of each copy of $M_1 \times M_2$ their images coincide. Let us glue the products $M_1 \times X_2$ and $X_1 \times M_2$ by the given identification $\partial(M_1 \times X_2)$ with $\partial(X_1 \times M_2)$. A manifold that we have obtained $Y \cong (M_1 \times X_2 \cup X_1 \times M_2)/(M_1 \times \partial X_2 \cong \partial X_1 \times M_2)$ with the MSp-structure induced by this inclusion $Y \subset \mathbb{R}^{n_1+r_1+n_2+r_2+1}$ depicts an element from $\langle m_1, 2, m_2\rangle$ in the ring MSp_*. Let an interval $I = [0, 1]$ be included in $\mathbb{R}^{1+r}$ with the trivial framing. Let us take two copies of the manifold with a boundary $Y \times I \subset \mathbb{R}^{n_1+r_1+n_2+r_2+r+2}$. Let us glue the upper boundary $Y \times \{1\}$ of the given manifolds by the following rule. We choose in the manifold Y one of the copies of $M_1 \times M_2$ by the gluing along which the manifold Y was constructed. A normal boundary of $M_1 \times M_2$ in Y has the form: $M_1 \times M_2 \times (-1, 1)$. Let us identify two copies of $Y \times \{1\}$ by the identical map out of the boundary $M_1 \times M_2 \times (-1/2, 1/2)$. We obtained a manifold whose lower boundary consists of two copies of Y, upper boundary has the form: $M_1 \times M_2 \times S^1$, where S^1 is included in $\mathbb{R}^{2+r}$ in the form of the figure "eight". With the help of such inclusion the framing induced on S^1 gives the element θ_1 in MSp_1. So, we constructed a cobordism between the elements representing the elements $\theta_1 m_1 m_2$ and $2\langle m_1, 2, m_2\rangle$ respectively. The codimensions that we have chosen are sufficiently large, so the result does not depend from the choice of concrete inclusions. □

Let Γ_1 be an element belonging to the Massey product $\langle\Phi_7, 2, \Omega_1\rangle$. From Theorem 4.1 it follows that $2\Gamma_1 = \theta_1\Phi_7\Omega_1 \neq 0$, hence, the element Γ_1 has the order four.

We denote by Γ_i, $(i = 3, 4, \ldots)$ a series of elements belonging (for each i respectively) to the following Massey product: $\langle\Phi_{6+i}, 2, \Omega_1\rangle$, $i = 3, 4, \ldots$. We have S_4 $\Omega_1 = 0$ and $S_6\Omega_1 = 0$, because in the MASS there are the formulas $S_4\omega_1 = 0$ and $S_6\omega_1 = 0$, and MSp_{41} and MSp_{37} consist of elements having projections in the term E_2 with filtration of the MASS equal to zero. For $i = 5, 6, \ldots$, $S_{2(i-1)}\Omega_1 = 0$ by the dimensional reasons. Also there are the formulas $S_{2(i-1)}\Phi_{6+i} = \Phi_7$, so we obtain: $2\Gamma_i = \theta_1\Phi_{6+i}\Omega_1 \neq 0$, $i = 3, 4, \ldots$, and hence the elements of the series Γ_i have the order four in MSp_*.

5 Tables

See Tables 9, 10, 11, 12, 13, 14, 15, 16, 17 and 18.

Table 9 The action of Landweber–Novikov operations on the Ray's elements Φ_i (nonzero values)

$S_k\Phi_m = \Phi_{m-k}, m > k$; $S_{2m-1}\Phi_m = \theta_1$; $S_{k,k}\Phi_m = (m-k)\Phi_{m-k}, m > k$;
$S_{k,k,k}\Phi_m = (m-k)\Phi_{m-3s}, k = 2s, m > 3s$.
$S_{2,2,2,2}\Phi_5 = \Phi_1$.
$S_{2,2,2,2}\Phi_6 = \Phi_2$, $S_{2,2,2,2,2,2}\Phi_6 = \Phi_1$.
$S_{3,3,3,3}\Phi_8 = \Phi_2$.
$S_{2,2,2,2}\Phi_9 = \Phi_5$, $S_{2,2,2,2,2,2}\Phi_9 = \Phi_4$, $S_{2,2,2,2,2,2,2}\Phi_9 = \Phi_3$, $S_{3,3,3,3}\Phi_9 = \Phi_3$,
$S_{2,2,2,2,2,2,2,2}\Phi_9 = \Phi_2$, $S_{4,4,4,4}\Phi_9 = \Phi_1$, $S_{2,2,2,2,2,2,2,2,2}\Phi_9 = \Phi_1$.
$S_{2,2,2,2}\Phi_{10} = \Phi_6$, $S_{2,2,2,2,2,2}\Phi_{10} = \Phi_5$, $S_{4,4,4,4}\Phi_{10} = \Phi_2$, $S_{2,2,2,2,2,2,2,2}\Phi_{10} = \Phi_2$,
$S_{2,2,2,2,2,2,2,2,2,2}\Phi_{10} = \Phi_1$.
$S_{2,2,2,2,2,2,2,2}\Phi_{11} = \Phi_3$, $S_{2,2,2,2,2,2,2,2,2,2}\Phi_{11} = \Phi_2$.
$S_{3,3,3,3}\Phi_{12} = \Phi_6$, $S_{2,2,2,2,2,2,2,2}\Phi_{12} = \Phi_4$, $S_{5,5,5,5}\Phi_{12} = \Phi_2$, $S_{3,3,3,3,3,3}\Phi_{12} = \Phi_3$.
$S_{2,2,2,2,2,2,2,2,2}\Phi_{12} = \Phi_3$,
$S_{2,2,2,2}\Phi_{13} = \Phi_9$, $S_{2,2,2,2,2,2}\Phi_{13} = \Phi_8$, $S_{2,2,2,2,2,2,2}\Phi_{13} = \Phi_7$, $S_{3,3,3,3}\Phi_{13} = \Phi_7$,
$S_{2,2,2,2,2,2,2,2}\Phi_{13} = \Phi_6$, $S_{4,4,4,4}\Phi_{13} = \Phi_5$, $S_{5,5,5,5}\Phi_{13} = \Phi_3$, $S_{4,4,4,4,4,4}\Phi_{13} = \Phi_3$,
$S_{6,6,6,6}\Phi_{13} = \Phi_1$, $S_{4,4,4,4,4,4,4,4}\Phi_{13} = \Phi_1$.
$S_{2,2,2,2}\Phi_{14} = \Phi_2$, $S_{2,2,2,2,2,2}\Phi_{14} = \Phi_9$, $S_{4,4,4,4}\Phi_{14} = \Phi_6$, $S_{4,4,4,4,4,4}\Phi_{14} = \Phi_4$,
$S_{6,6,6,6}\Phi_{14} = \Phi_2$.

Table 10 The action of Landweber–Novikov operations on the elements c_i of the MASS

$S_2c_2 = 1$, $S_{1,1}c_2 = 0$.
$S_2c_4 = c_2$, $S_{1,1}c_4 = c_2$, $S_4c_4 = 1$, $S_{2,2}c_4 = 0$.
$S_1c_5 = c_4$, $S_3c_5 = c_2$, $S_5c_5 = 1$.
$S_2c_6 = c_2^2$, $S_{1,1}c_6 = c_2^2$, $S_6c_6 = 1$, $S_{3,3}c_6 = 0$, $S_{2,2,2}c_6 = 1$.
$S_2c_8 = c_2c_4 + c_6$, $S_{1,1}c_8 = c_2c_4 + c_6$, $S_4c_8 = c_4$, $S_{2,2}c_8 = c_2^2$, $S_6c_8 = c_2$,
$S_{3,3}c_8 = c_2$, $S_{2,2,2}c_8 = 0$, $S_8c_8 = 1$, $S_{4,4}c_8 = 1$, $S_{2,2,2,2}c_8 = 0$.
$S_1c_9 = c_8$, $S_2c_9 = c_2c_5$, $S_{1,1}c_9 = c_2c_5$, $S_3c_9 = c_6$, $S_4c_9 = c_5$, $S_5c_9 = 0$, $S_{2,2}c_9 = 0$,
$S_7c_9 = c_2$, $S_9c_9 = 1$, $S_{3,3,3}c_9 = 0$.
$S_2c_{10} = c_4^2 + c_2^4$, $S_{1,1}c_{10} = c_4^2 + c_2^4$, $S_4c_{10} = 0$, $S_{2,2}c_{10} = c_6$, $S_{3,3}c_{10} = 0$,
$S_{2,2,2}c_{10} = c_2^2$, $S_{10}c_{10} = 1$, $S_{5,5}c_{10} = 1$, $S_{2,2,2,2,2}c_{10} = 0$, $S_6c_{10} = 0$.
$S_1c_{11} = c_{10}$, $S_2c_{11} = c_9 + c_4c_5$, $S_{1,1}c_{11} = c_9 + c_4c_5$, $S_3c_{11} = c_8 + c_2^4$,
$S_{2,2}c_{11} = c_2c_5$, $S_4c_{11} = 0$, $S_5c_{11} = 0$, $S_6c_{11} = c_5$, $S_{3,3}c_{11} = c_5$, $S_7c_{11} = c_4$,
$S_{3,3,3}c_{11} = c_2$, $S_{11}c_{11} = 1$.
$S_2c_{12} = c_5^2 + c_2^2c_6$, $S_{1,1}c_{12} = c_{10} + c_5^2 + c_2^2c_6$, $S_4c_{12} = c_2^4$, $S_{2,2}c_{12} = c_4^2 + c_2^4$,
$S_6c_{12} = c_6$, $S_{3,3}c_{12} = c_6$, $S_{2,2,2}c_{12} = c_6$, $S_8c_{12} = 0$, $S_{4,4}c_{12} = 0$, $S_{2,2,2,2}c_{12} = c_2^2$,
$S_{6,6}c_{12} = 0$, $S_{12}c_{12} = 1$, $S_{4,4,4}c_{12} = 1$, $S_{3,3,3,3}c_{12} = 0$, $S_{2,2,2,2,2,2}c_{12} = 0$.
$S_1c_{13} = c_{12} + c_4c_8$, $S_2c_{13} = c_{11} + c_2c_4c_5 + c_5c_6$, $S_{1,1}c_{13} = c_2c_4c_5 + c_5c_6$,
$S_3c_{13} = c_2c_8 + c_{10} + c_5^2 + c_2^2c_6$, $S_4c_{13} = c_9$, $S_{2,2}c_{13} = c_9 + c_4c_5 + c_2^2c_5$,
$S_5c_{13} = c_2^4 + c_4^2$, $S_6c_{13} = 0$, $S_{3,3}c_{13} = 0$, $S_{2,2,2}c_{13} = c_2c_5$, $S_7c_{13} = c_2c_4 + c_6$,
$S_8c_{13} = c_5$, $S_{4,4}c_{13} = 0$, $S_{3,3,3}c_{13} = 0$, $S_{2,2,2,2}c_{13} = 0$, $S_{13}c_{13} = 1$.
$S_2c_{14} = c_6^2 + c_2^2c_4^2$, $S_{1,1}c_{14} = c_6^2 + c_2^2c_4^2$, $S_4c_{14} = 0$, $S_{2,2}c_{14} = 0$, $S_6c_{14} = c_4^2$,
$S_{3,3}c_{14} = 0$, $S_{2,2,2}c_{14} = c_4^2$, $S_8c_{14} = 0$, $S_{4,4}c_{14} = 0$, $S_{2,2,2,2}c_{14} = c_6$, $S_{10}c_{14} = c_2^2$,
$S_{5,5}c_{14} = c_2^2$, $S_{2,2,2,2,2}c_{14} = 0$, $S_{14}c_{14} = 1$, $S_{7,7}c_{14} = 0$, $S_{2,2,2,2,2,2,2}c_{14} = 0$.
$S_2c_{16} = c_{14} + c_2c_{12} + c_4c_{10} + c_6c_8 + c_2c_4c_8 + c_2^2c_{10} + c_4c_5^2 + c_2^2c_4c_6 + c_2^5c_4 + c_2c_4^3$,
$S_{1,1}c_{16} = c_{14} + c_2c_{12} + c_4c_{10} + c_6c_8 + c_2c_4c_8 + c_2^2c_{10} + c_4c_5^2 + c_2^2c_4c_6 + c_2^5c_4 + c_2c_4^3$,
$S_4c_{16} = c_{12} + c_2c_{10} + c_4c_8 + c_4c_2^4$, $S_{2,2}c_{16} = c_6^2 + c_2^2c_4^2$, $S_{2,2,2}c_{16} = 0$,
$S_6c_{16} = c_2c_8 + c_{10} + c_5^2 + c_2^2c_6 + c_4c_6$, $S_{3,3}c_{16} = c_2c_8 + c_{10} + c_5^2 + c_2^2c_6 + c_4c_6$,
$S_8c_{16} = c_4^2 + c_8$, $S_{4,4}c_{16} = c_4^2$, $S_{2,2,2,2}c_{16} = 0$, $S_{3,3,3}c_{16} = 0$, $S_{10}c_{16} = c_2c_4 + c_6$,
$S_{5,5}c_{16} = c_2c_4 + c_6$, $S_{2,2,2,2,2}c_{16} = 0$, $S_{12}c_{16} = c_4$, $S_{6,6}c_{16} = c_2^2$, $S_{4,4,4}c_{16} = 0$,
$S_{3,3,3,3}c_{16} = c_4 + c_2^2$, $S_{2,2,2,2,2,2}c_{16} = 0$, $S_{14}c_{16} = c_2$, $S_{7,7}c_{16} = c_2$,
$S_{2,2,2,2,2,2,2}c_{16} = 0$, $S_{16}c_{16} = 1$, $S_{8,8}c_{16} = 1$, $S_{4,4,4,4}c_{16} = 0$,
$S_{2,2,2,2,2,2,2,2}c_{16} = 0$.
$S_2c_{17} = c_2c_{13} + c_5c_{10} + c_6c_9 + c_2^2c_{11} + c_5^3 + c_2^2c_5c_6 + c_2c_5c_4^2 + c_5c_2^5$,
$S_{1,1}c_{17} = c_2c_{13} + c_5c_{10} + c_6c_9 + c_2^2c_{11} + c_5^3 + c_2^2c_5c_6 + c_2c_5c_4^2 + c_5c_2^5$,
$S_1c_{17} = c_{16}$, $S_3c_{17} = c_{14}$, $S_4c_{17} = c_{13} + c_2c_{11} + c_5c_2^4$, $S_{2,2}c_{17} = 0$, $S_5c_{17} = 0$,
$S_6c_{17} = c_2c_9 + c_5c_6$, $S_{3,3}c_{17} = c_2c_9 + c_5c_6$, $S_{2,2,2}c_{17} = 0$, $S_7c_{17} = c_{10} + c_5^2 + c_2^2c_6$
$S_8c_{17} = c_9$, $S_{4,4}c_{17} = 0$, $S_{2,2,2,2}c_{17} = 0$, $S_9c_{17} = c_4^2$, $S_{3,3,3}c_{17} = 0$, $S_{10}c_{17} = c_2c_5$,

Table 10 continued

$S_{5,5}c_{17} = c_2c_5$, $S_{2,2,2,2,2}c_{17} = 0$, $S_{11}c_{17} = c_6$, $S_{12}c_{17} = c_5$, $S_{6,6}c_{17} = 0$,
$S_{4,4,4}c_{17} = 0$, $S_{3,3,3,3}c_{17} = c_5$, $S_{2,2,2,2,2,2}c_{17} = 0$, $S_{15}c_{17} = c_2$, $S_{5,5,5}c_{17} = c_2$,
$S_{3,3,3,3,3}c_{17} = c_2$, $S_{17}c_{17} = 1$.
$S_2c_{18} = c_8^2$, $S_{1,1}c_{18} = c_8^2$, $S_4c_{18} = 0$, $S_{2,2}c_{18} = c_6(c_4^2 + c_2^4) + c_{14} + c_2^2c_{10}$,
$S_6c_{18} = 0$, $S_{3,3}c_{18} = 0$, $S_{2,2,2}c_{18} = c_6^2 + c_2^2c_4^2$, $S_8c_{18} = 0$, $S_{4,4}c_{18} = c_{10}$,
$S_{2,2,2,2}c_{18} = 0$, $S_{10}c_{18} = c_2^4$, $S_{5,5}$
$S_{6,6}c_{18} = c_6$, $S_{4,4,4}c_{18} = 0$, $S_{3,3,3,3}c_{18} = c_6$, $S_{14}c_{18} = 0$, $S_{7,7}c_{18} = 0$,
$S_{2,2,2,2,2,2}c_{18} = 0$, $S_{2,2,2,2,2,2,2}c_{18} = 0$, $S_{18}c_{18} = 1$, $S_{9,9}c_{18} = 1$, $S_{6,6,6}c_{18} = 0$,
$S_{3,3,3,3,3,3}c_{18} = 0$, $S_{2,2,2,2,2,2,2,2,2}c_{18} = 0$.
$S_1c_{19} = c_{18}$, $S_2c_{19} = c_{17} + c_9c_4^2 + c_{11}c_6 + c_{13}c_4 + c_5c_4^3 + c_5c_{12} + c_4c_5c_2^4$,
$S_{1,1}c_{19} = c_{17} + c_9c_4^2 + c_{11}c_6 + c_{13}c_4 + c_5c_4^3 + c_5c_{12} + c_4c_5c_2^4$,
$S_3c_{19} = c_{16} + c_8^2 + c_4^4 + c_6c_{10} + c_4^2c_2^4$, $S_4c_{19} = c_5c_{10} + c_4c_{11}$,
$S_{2,2}c_{19} = c_2c_{13} + c_5(c_{10} + c_5^2 + c_2^2c_6) + c_2c_5(c_4^2 + c_2^4) + c_6c_9 + c_2^2c_{11}$, $S_5c_{19} = 0$,
$S_6c_{19} = c_{13} + c_4c_9$, $S_{2,2,2}c_{19} = 0$, $S_{3,3}c_{19} = c_{13} + c_4c_9$, $S_7c_{19} = c_{12}$, $S_8c_{19} = c_{11}$,
$S_{4,4}c_{19} = 0$, $S_{2,2,2,2}c_{19} = 0$, $S_9c_{19} = c_{10}$, $S_{3,3,3}c_{19} = c_{10} + c_5^2 + c_2^2c_6 + c_2c_8 + c_4c_6$
$S_{10}c_{19} = c_4c_5$, $S_{5,5}c_{19} = c_4c_5$, $S_{2,2,2,2,2}c_{19} = 0$, $S_{12}c_{19} = 0$, $S_{11}c_{19} = c_4^2 + c_2^4$,
$S_{6,6}c_{19} = 0$, $S_{3,3,3,3}c_{19} = 0$, $S_{2,2,2,2,2,2}c_{19} = 0$, $S_{13}c_{19} = 0$, $S_{14}c_{19} = c_5$,
$S_{7,7}c_{19} = c_5$, $S_{2,2,2,2,2,2,2}c_{19} = 0$, $S_{15}c_{19} = c_4$, $S_{3,3,3,3,3}c_{19} = c_4$, $S_{5,5,5}c_{19} = c_4$,
$S_{17}c_{19} = 0$, $S_{19}c_{19} = 1$.
$S_2c_{20} = c_9^2 + c_2^2c_{14} + c_4^2(c_{10} + c_5^2) + c_2^4(c_{10} + c_5^2 + c_2^2c_6) + c_6^3$,
$S_{1,1}c_{20} = c_{18} + c_9^2 + c_2^2c_{14} + c_4^2(c_{10} + c_5^2) + c_2^4(c_{10} + c_5^2 + c_2^2c_6) + c_6^3$,
$S_4c_{20} = c_4^4 + c_2^4c_4^2$, $S_{2,2}c_{20} = c_8^2 + c_2^2c_{12} + c_6(c_{10} + c_5^2 + c_2^2c_6)$, $S_6c_{20} = c_{14} + c_2^4c_6$
$S_{3,3}c_{20} = c_{14} + c_2^4c_6$, $S_{2,2,2}c_{20} = c_{14} + c_2^2c_{10} + c_6(c_4^2 + c_2^4)$, $S_8c_{20} = 0$,
$S_{4,4}c_{20} = c_{12}$, $S_{2,2,2,2}c_{20} = c_6^2 + c_2^2c_4^2$, $S_{10}c_{20} = c_{10} + c_5^2 + c_2^2c_6$,
$S_{5,5}c_{20} = c_{10} + c_5^2 + c_2^2c_6$, $S_{6,6}c_{20} = 0$, $S_{2,2,2,2,2}c_{20} = 0$, $S_{12}c_{20} = c_4^2 + c_2^4$,
$S_{4,4,4}c_{20} = c_4^2$, $S_{3,3,3,3}c_{20} = c_4^2$, $S_{2,2,2,2,2,2}c_{20} = 0$, $S_{14}c_{20} = c_6$, $S_{7,7}c_{20} = c_6$,
$S_{2,2,2,2,2,2,2}c_{20} = 0$, $S_{16}c_{20} = 0$, $S_{8,8}c_{20} = 0$, $S_{4,4,4,4}c_{20} = 0$, $S_{2,2,2,2,2,2,2,2}c_{20} = 0$,
$S_{20}c_{20} = 1$, $S_{10,10}c_{20} = 0$,
$S_1c_{21} = c_{20} + c_4c_{16}$,
$S_2c_{21} = c_{19} + c_2c_4c_{13} + c_6c_{13} + c_2c_4c_5(c_2^4 + c_4^2) + c_5(c_{14} + c_2^2c_{10})$,
$S_{1,1}c_{21} = c_2c_4c_{13} + c_6c_{13} + c_2c_4c_5(c_2^4 + c_4^2) + c_5(c_{14} + c_2^2c_{10})$,
$S_3c_{21} = c_2c_{16} + c_{18} + c_9^2 + c_2^2(c_{14} + c_4^2c_6) + c_2^4(c_{10} + c_5^2 + c_2^2c_6) + c_6(c_{12} + c_6^2)$,
$S_4c_{21} = c_{17} + c_2c_4c_{11} + c_4^2c_9 + c_5c_{12}$,
$S_{2,2}c_{21} = c_{17} + c_4^2c_9 + c_6c_{11} + c_4c_{13} + c_4c_5(c_2^4 + c_4^2) + c_5(c_{12} + c_6^2 + c_2^2c_4^2)$,
$S_5c_{21} = c_4^4$, $S_6c_{21} = c_2c_4c_9 + c_2^2c_{11} + c_5(c_{10} + c_5^2 + c_2^2c_6)$,
$S_{3,3}c_{21} = c_2c_4c_9 + c_2^2c_{11} + c_5(c_{10} + c_5^2 + c_2^2c_6)$,
$S_{2,2,2}c_{21} = c_2c_{13} + c_2^2c_{11} + c_6c_9 + c_5(c_{10} + c_5^2 + c_2^2c_6) + c_2c_5(c_2^4 + c_4^2)$,
$S_8c_{21} = c_{13} + c_5c_4^2$, $S_{4,4}c_{21} = c_{13} + c_2c_{11} + c_5(c_2^4 + c_4^2)$, $S_{2,2,2,2}c_{21} = 0$,

Table 10 continued

$S_7c_{21} = c_{14} + c_2^2c_{10} + c_6(c_2^4 + c_4^2)$, $S_9c_{21} = c_{12}$,
$S_{3,3,3}c_{21} = c_2(c_{10} + c_5^2 + c_2^2c_6) + c_2^6 + c_2c_4c_6 + c_2^2c_8$, $S_{10}c_{21} = c_2c_4c_5 + c_5c_6$,
$S_{5,5}c_{21} = c_2c_4c_5 + c_5c_6$, $S_{2,2,2,2,2}c_{21} = 0$, $S_{12}c_{21} = 0$, $S_{6,6}c_{21} = 0$, $S_{4,4,4}c_{21} = 0$,
$S_{3,3,3,3}c_{21} = c_5c_2^2$, $S_{2,2,2,2,2,2}c_{21} = 0$, $S_{11}c_{21} = c_{10} + c_5^2 + c_2^2c_6$, $S_{13}c_{21} = c_2^4$,
$S_{14}c_{21} = 0$, $S_{7,7}c_{21} = 0$, $S_{2,2,2,2,2,2,2}c_{21} = 0$, $S_{15}c_{21} = c_2c_4 + c_6$,
$S_{5,5,5}c_{21} = c_2c_4 + c_6$, $S_{3,3,3,3,3}c_{21} = c_2^3$, $S_{16}c_{21} = c_5$, $S_{8,8}c_{21} = c_5$,
$S_{4,4,4,4}c_{21} = 0$, $S_{2,2,2,2,2,2,2,2}c_{21} = 0$.
$S_2c_{22} = c_{12}(c_4^2 + c_2^4) + c_{10}(c_{10} + c_5^2 + c_2^2c_6) + c_5^4$, $S_4c_{22} = c_2^4c_{10}$,
$S_{1,1}c_{22} = c_{12}(c_4^2 + c_2^4) + c_{10}(c_{10} + c_5^2 + c_2^2c_6) + c_5^4$, $S_6c_{22} = c_8^2 + c_2^8 + c_6c_{10}$,
$S_{2,2}c_{22} = c_{18} + c_9^2 + c_2^2c_{14} + c_6(c_{12} + c_6^2 + c_2^2c_4^2) + c_2^4(c_{10} + c_5^2 + c_2^2c_6)$,
$S_{3,3}c_{22} = c_8^2 + c_2^8 + c_6c_{10}$, $S_8c_{22} = 0$, $S_{2,2,2}c_{22} = c_2^2c_{12} + c_6(c_{10} + c_5^2 + c_2^2c_6)$,
$S_{4,4}c_{22} = 0$, $S_{2,2,2,2}c_{22} = 0$, $S_{10}c_{22} = c_{12}$, $S_{5,5}c_{22} = c_{12}$, $S_{2,2,2,2,2}c_{22} = 0$,
$S_{12}c_{22} = c_{10}$, $S_{6,6}c_{22} = c_{10} + c_5^2 + c_2^2c_6$, $S_{4,4,4}c_{22} = c_{10}$, $S_{3,3,3,3}c_{22} = c_5^2 + c_2^2c_6$,
$S_{2,2,2,2,2,2}c_{22} = 0$, $S_{14}c_{22} = c_4^2$, $S_{7,7}c_{22} = c_4^2$, $S_{2,2,2,2,2,2,2}c_{22} = 0$, $S_{16}c_{22} = 0$,
$S_{8,8}c_{22} = 0$, $S_{4,4,4,4}c_{22} = 0$, $S_{2,2,2,2,2,2,2,2}c_{22} = 0$, $S_{18}c_{22} = 0$, $S_{9,9}c_{22} = 0$,
$S_{6,6,6}c_{22} = c_2^2$, $S_{3,3,3,3,3,3}c_{22} = c_2^2$, $S_{2,2,2,2,2,2,2,2,2}c_{22} = 0$, $S_{22}c_{22} = 1$,
$S_{11,11}c_{22} = 1$, $S_{2,2,2,2,2,2,2,2,2,2,2}c_{22} = 0$.
$S_2c_{23} = c_{21} + c_5c_{16} + c_9c_{12} + c_{11}(c_{10} + c_5^2 + c_2^2c_6) + c_8c_{13} + c_5c_8^2 + (c_{13}$
$+c_5c_8)(c_4^2 + c_2^4)$, $S_1c_{23} = c_{22}$, $S_3c_{23} = c_{20} + c_5^4$,
$S_{1,1}c_{23} = c_{21} + c_5c_{16} + c_9c_{12} + c_{11}(c_{10} + c_5^2 + c_2^2c_6) + c_8c_{13} + c_5c_8^2 + (c_{13}$
$+c_5c_8)(c_4^2 + c_2^4)$, $S_4c_{23} = c_{19} + c_8c_{11} + c_9c_{10} + c_{11}c_2^4$, $S_5c_{23} = c_{18}$,
$S_{2,2}c_{23} = c_2c_4(c_{13} + c_5c_8) + c_2c_5c_{12} + c_4c_5(c_{10} + c_5^2 + c_2^2c_6) + c_5c_8c_6$,
$S_6c_{23} = c_{17} + c_8c_9 + c_{11}c_6$, $S_{3,3}c_{23} = c_{17} + c_8c_9 + c_{11}c_6$, $S_{2,2,2}c_{23} = 0$,
$S_7c_{23} = c_2^8$, $S_8c_{23} = 0$, $S_{4,4}c_{23} = c_4c_{11} + c_5c_{10}$, $S_{2,2,2,2}c_{23} = 0$, $S_9c_{23} = 0$,
$S_{3,3,3}c_{23} = c_{14} + c_2^4c_6$, $S_{10}c_{23} = c_{13} + c_5c_8$, $S_{5,5}c_{23} = c_{13} + c_5c_8$, $S_{2,2,2,2,2}c_{23} = 0$
$S_{11}c_{23} = 0$, $S_{12}c_{23} = c_{11}$, $S_{6,6}c_{23} = c_2c_9 + c_5c_6$, $S_{4,4,4}c_{23} = 0$, $S_{2,2,2,2,2,2}c_{23} = 0$,
$S_{3,3,3,3}c_{23} = c_2c_9 + c_5c_6$, $S_{13}c_{23} = 0$, $S_{14}c_{23} = c_9$, $S_{7,7}c_{23} = c_9$, $S_{15}c_{23} = c_8 + c_4^2$,
$S_{2,2,2,2,2,2,2}c_{23} = 0$, $S_{5,5,5}c_{23} = c_8 + c_4^2 + c_2^4$, $S_{3,3,3,3,3}c_{23} = c_8 + c_2^4$, $S_{16}c_{23} = 0$,
$S_{8,8}c_{23} = 0$, $S_{4,4,4,4}c_{23} = 0$, $S_{2,2,2,2,2,2,2,2}c_{23} = 0$, $S_{18}c_{23} = 0$, $S_{9,9}c_{23} = 0$,
$S_{6,6,6}c_{23} = 0$, $S_{3,3,3,3,3,3}c_{23} = c_5$.
$S_2c_{24} = c_{11}^2 + c_2^2c_{18} + c_6(c_8^2 + c_4^4) + c_4^2c_{14} + c_{12}(c_5^2 + c_2^2c_6) + c_6^2c_{10}$,
$S_{1,1}c_{24} = c_{11}^2 + c_2^2c_{18} + c_6(c_8^2 + c_4^4) + c_4^2c_{14} + c_{12}(c_5^2 + c_2^2c_6) + c_6^2c_{10} + c_{22}$,
$S_4c_{24} = c_2^4c_{12} + c_5^4$, $S_{2,2}c_{24} = c_6c_{14} + c_4^2(c_{12} + c_2^6) + c_5^2c_{10} + c_2^4(c_{12} + c_2^6) + c_2^{10}$,
$S_6c_{24} = c_{18} + c_9^2 + c_2^2c_{14} + c_6c_{12} + c_2^2c_4^2c_6 + c_6^3$, $S_{4,4}c_{24} = c_8^2 + c_4^4 + c_2^8$,
$S_{3,3}c_{24} = c_{18} + c_9^2 + c_2^2c_{14} + c_6c_{12} + c_2^2c_4^2c_6 + c_6^3$, $S_8c_{24} = c_2^8$, $S_{2,2,2,2,2}c_{24} = 0$,
$S_{2,2,2}c_{24} = c_{18} + c_9^2 + c_6(c_{12} + c_2^2c_4^2) + c_2^4c_5^2$, $S_{10}c_{24} = c_{14} + c_2^2c_{10} + c_4^2c_6$,
$S_{2,2,2,2}c_{24} = c_2^2(c_{12} + c_2^2c_4^2 + c_6^2) + c_6(c_{10} + c_5^2 + c_2^2c_6)$, $S_{12}c_{24} = c_{12}$,

Table 10 continued

$S_{5,5}c_{24} = c_{14} + c_2^2c_{10} + c_4^2c_6$, $S_{6,6}c_{24} = c_6^2$, $S_{4,4,4}c_{24} = c_{12}$, $S_{3,3,3,3}c_{24} = c_{12} + c_6^2$,
$S_{2,2,2,2,2,2}c_{24} = 0$, $S_{14}c_{24} = c_{10} + c_5^2 + c_2^2c_6$, $S_{7,7}c_{24} = c_{10} + c_5^2 + c_2^2c_6$,
$S_{2,2,2,2,2,2,2}c_{24} = 0$, $S_{16}c_{24} = 0$, $S_{8,8}c_{24} = c_4^2$, $S_{2,2,2,2,2,2,2,2}c_{24} = 0$,
$S_{4,4,4,4}c_{24} = c_4^2$, $S_{18}c_{24} = c_6$, $S_{9,9}c_{24} = c_6$, $S_{6,6,6}c_{24} = 0$, $S_{3,3,3,3,3,3}c_{24} = c_6$,
$S_{2,2,2,2,2,2,2,2,2}c_{24} = 0$, $S_{20}c_{24} = 0$, $S_{10,10}c_{24} = c_2^2$, $S_{2,2,2,2,2,2,2,2,2,2}c_{24} = 0$,
$S_{5,5,5,5}c_{24} = c_2^2$, $S_{4,4,4,4,4}c_{24} = 0$, $S_{12,12}c_{24} = 1$, $S_{8,8,8}c_{24} = 1$, $S_{6,6,6,6}c_{24} = 0$,
$S_{4,4,4,4,4,4}c_{24} = 0$, $S_{3,3,3,3,3,3,3,3}c_{24} = 0$, $S_{2,2,2,2,2,2,2,2,2,2,2,2}c_{24} = 0$.
$S_2c_{25} = c_{23} + c_2c_5c_{16} + c_2c_8c_{13} + c_{19}c_2^2 + c_6c_{17} + c_2c_5c_8(c_4^2 + c_2^4) + c_{13}(c_{10}$
$+c_5^2 + c_2^2c_6) + c_{11}(c_2^2c_4^2 + c_2^6) + c_9(c_{14} + c_2^2c_{10})$, $S_1c_{25} = c_{24} + c_8c_{16}$,
$S_{1,1}c_{25} = c_2c_5c_{16} + c_2c_8c_{13} + c_{19}c_2^2 + c_6c_{17} + c_2c_5c_8(c_4^2 + c_2^4) + c_{13}(c_{10} + c_5^2$
$+c_2^2c_6) + c_{11}(c_2^2c_4^2 + c_2^6) + c_9(c_{14} + c_2^2c_{10})$,
$S_3c_{25} = c_{22} + c_{11}^2 + c_4^2c_{14} + c_{10}(c_{12} + c_6^2 + c_2^2c_4^2 + c_2^6) + c_6c_4^4$,
$S_4c_{25} = c_{21} + c_2c_8c_{11} + c_{13}c_2^4 + c_9c_{12} + c_5c_8^2$,
$S_{2,2}c_{25} = c_{21} + c_5c_{16} + c_8c_{13} + c_2^2(c_{17} + c_4c_{13} + c_4c_5c_8) + (c_2c_{13} + c_2c_5c_8)c_6$
$+(c_{13} + c_5c_8)(c_4^2 + c_2^4) + (c_{11} + c_2c_4c_5)(c_{10} + c_5^2 + c_2^2c_6) + c_9c_{12}$
$+c_2c_5(c_{14} + c_2^2c_{10} + c_6(c_2^4 + c_4^2)) + c_5(c_6(c_{10} + c_5^2 + c_2^2c_6) + c_8^2)$,
$S_5c_{25} = c_{20} + c_5^4$, $S_6c_{25} = c_2c_8c_9 + c_{13}c_6 + c_9(c_{10} + c_5^2 + c_2^2c_6)$,
$S_{3,3}c_{25} = c_2c_8c_9 + c_{13}c_6 + c_9(c_{10} + c_5^2 + c_2^2c_6)$,
$S_{2,2,2}c_{25} = c_2c_4c_{13} + c_2c_4c_5c_8 + c_2c_{13}c_2^2 + (c_{13} + c_5c_8)c_6 + c_{11}c_2^4 + c_9c_2^2c_6$
$+c_4c_5(c_{10} + c_5^2 + c_2^2c_6) + c_5c_2^2(c_{10} + c_5^2 + c_2^2c_6) + c_2c_5(c_{12} + c_6^2 + c_2^6)$,
$S_7c_{25} = c_{18} + c_2c_{16} + c_9^2 + c_2^2c_{14} + c_6(c_2^2c_4^2 + c_6^2)$, $S_8c_{25} = c_{17} + c_9c_4^2$,
$S_{4,4}c_{25} = c_5c_{12} + c_{17} + c_2c_4c_{11}$, $S_{2,2,2,2}c_{25} = 0$, $S_9c_{25} = c_8^2 + c_2^8$, $S_{3,3,3}c_{25} = 0$,
$S_{10}c_{25} = c_2c_5c_8 + c_{11}c_2^2 + c_9c_6$, $S_{5,5}c_{25} = c_2c_5c_8 + c_{11}c_2^2 + c_9c_6$, $S_{2,2,2,2,2}c_{25} = 0$,
$S_{11}c_{25} = c_{14} + c_4^2c_6$, $S_{12}c_{25} = c_{13}$, $S_{6,6}c_{25} = c_9c_2^2$, $S_{3,3,3,3}c_{25} = c_{13} + c_9c_2^2$,
$S_{4,4,4}c_{25} = c_{13} + c_2c_{11} + c_5(c_4^2 + c_2^4)$, $S_{2,2,2,2,2,2}c_{25} = 0$, $S_{13}c_{25} = 0$, $S_{14}c_{25} = 0$,
$S_{7,7}c_{25} = 0$, $S_{2,2,2,2,2,2,2}c_{25} = 0$, $S_{15}c_{25} = c_{10} + c_5^2 + c_2^2c_6$, $S_{5,5,5}c_{25} = 0$,
$S_{3,3,3,3,3}c_{25} = c_{10} + c_5^2 + c_2^2c_6$, $S_{16}c_{25} = c_9$, $S_{8,8}c_{25} = 0$, $S_{4,4,4,4}c_{25} = 0$,
$S_{2,2,2,2,2,2,2,2}c_{25} = 0$, $S_{17}c_{25} = 0$, $S_{18}c_{25} = 0$, $S_{9,9}c_{25} = 0$, $S_{6,6,6}c_{25} = 0$,
$S_{3,3,3,3,3,3}c_{25} = 0$, $S_{19}c_{25} = c_6$, $S_{2,2,2,2,2,2,2,2,2}c_{25} = 0$, $S_{20}c_{25} = 0$, $S_{10,10}c_{25} = 0$,
$S_{21}c_{25} = 0$, $S_{5,5,5,5}c_{25} = 0$, $S_{4,4,4,4,4}c_{25} = 0$, $S_{2,2,2,2,2,2,2,2,2,2}c_{25} = 0$,
$S_{7,7,7}c_{25} = 0$, $S_{3,3,3,3,3,3,3}c_{25} = 0$, $S_{23}c_{25} = 0$, $S_{25}c_{25} = 1$, $S_{5,5,5,5,5}c_{25} = 0$.
$S_2c_{26} = c_4^2c_8^2 + c_2^2c_{10}^2 + c_{12}^2 + c_2^4c_4^4 + c_6^4 + c_2^{12}$,
$S_{1,1}c_{26} = c_4^2c_8^2 + c_2^2c_{10}^2 + c_{12}^2 + c_2^4c_4^4 + c_6^4 + c_2^{12}$, $S_4c_{26} = 0$,
$S_{2,2}c_{26} = c_{22} + c_{11}^2 + c_6c_8^2 + c_{14}(c_4^2 + c_2^4) + c_2^2c_4^2(c_{10} + c_2^2c_6) + c_2^6c_{10} + c_{10}c_{12}$,
$S_6c_{26} = 0$, $S_{3,3}c_{26} = 0$, $S_{2,2,2}c_{26} = c_2^2c_8^2 + c_6^2(c_4^2 + c_2^4) + c_{10}^2 + c_5^4 + c_2^{10}$,
$S_8c_{26} = 0$, $S_{4,4}c_{26} = c_{18}$, $S_{2,2,2,2}c_{26} = 0$, $S_{10}c_{26} = c_4^4 + c_2^8$, $S_{5,5}c_{26} = c_4^4 + c_2^8$,
$S_{2,2,2,2,2}c_{26} = 0$, $S_{12}c_{26} = 0$, $S_{6,6}c_{26} = c_{14} + c_4^2c_6 + c_2^4c_6 + c_2^2c_{10}$, $S_{4,4,4}c_{26} = 0$,

Table 10 continued

$S_{3,3,3,3}c_{26} = c_{14} + c_4^2c_6 + c_2^4c_6 + c_2^2c_{10}$, $S_{2,2,2,2,2,2}c_{26} = 0$, $S_{14}c_{26} = 0$, $S_{7,7}c_{26} = 0$

$S_{2,2,2,2,2,2,2}c_{26} = 0$, $S_{16}c_{26} = 0$, $S_{8,8}c_{26} = c_{10}$, $S_{4,4,4,4}c_{26} = 0$, $S_{18}c_{26} = c_2^4$,

$S_{2,2,2,2,2,2,2,2}c_{26} = 0$, $S_{9,9}c_{26} = c_2^4$, $S_{6,6,6}c_{26} = 0$, $S_{3,3,3,3,3,3}c_{26} = 0$, $S_{10,10}c_{26} = c_6$

$S_{2,2,2,2,2,2,2,2,2}c_{26} = 0$, $S_{20}c_{26} = 0$, $S_{5,5,5,5}c_{26} = c_6$, $S_{2,2,2,2,2,2,2,2,2,2}c_{26} = 0$,

$S_{4,4,4,4,4}c_{26} = 0$, $S_{22}c_{26} = 0$, $S_{11,11}c_{26} = 0$, $S_{2,2,2,2,2,2,2,2,2,2,2}c_{26} = 0$,

$S_{13,13}c_{26} = 1$, $S_{2,2,2,2,2,2,2,2,2,2,2,2,2}c_{26} = 0$.

$S_2c_{27} = c_{25} + c_4c_5c_{16} + c_4c_8c_{13} + c_{19}c_6 + c_{17}c_4^2 + c_4c_5c_8(c_4^2 + c_2^4) + c_{13}c_{12}$
$\quad + c_{11}(c_2^2c_{10} + c_{14}) + c_9(c_2^4c_4^2 + c_4^4)$, $S_1c_{27} = c_{26}$,

$S_{1,1}c_{27} = c_{25} + c_4c_5c_{16} + c_4c_8c_{13} + c_{19}c_6 + c_{17}c_4^2 + c_4c_5c_8(c_4^2 + c_2^4) + c_{13}c_{12}$
$\quad + c_{11}(c_2^2c_{10} + c_{14}) + c_9(c_2^4c_4^2 + c_4^4)$,

$S_3c_{27} = c_8c_{16} + c_{24} + c_6c_{18} + c_{10}c_{14} + c_6^4 + c_2^4c_4^4 + c_2^{12}$,

$S_4c_{27} = c_{23} + c_4c_8c_{11} + c_{13}c_{10} + c_{11}c_{12}$,

$S_{2,2}c_{27} = c_2c_5c_{16} + c_2c_8c_{13} + c_6(c_{17} + c_4c_{13} + c_4c_5c_8) + c_2c_{13}c_4^2 + c_{13}(c_{10}$
$\quad + c_5^2 + c_2^2c_6) + c_{11}(c_2^2c_4^2 + c_2^6) + c_2c_4c_5c_{12} + c_5c_6c_{12} + c_9(c_{14} + c_2^2c_{10})$
$\quad + c_4c_5(c_2^2c_{10} + c_6(c_4^2 + c_2^4))$, $S_5c_{27} = c_{22}$,

$S_6c_{27} = c_{21} + c_4c_8c_9 + c_{11}(c_{10} + c_5^2 + c_2^2c_6) + c_5c_8^2$,

$S_{3,3}c_{27} = c_{21} + c_4c_8c_9 + c_{11}(c_{10} + c_5^2 + c_2^2c_6) + c_5c_8^2$,

$S_{2,2,2}c_{27} = c_{11}c_2^2c_6 + c_2^2(c_{17} + c_4c_5c_8) + c_6(c_2c_{13} + c_2c_5c_8) + c_2c_4c_5(c_{10} + c_5^2$
$\quad + c_2^2c_6) + c_2^2c_4^2c_9 + c_4c_5(c_6^2 + c_2^6) + c_2c_5(c_{14} + c_2^2c_{10} + c_6(c_4^2 + c_2^4))$
$\quad + c_5(c_2^2c_{12} + c_6(c_{10} + c_5^2 + c_2^2c_6))$, $S_{4,4}c_{27} = c_8c_{11} + c_{11}(c_4^2 + c_2^4) + c_9c_{10}$,

$S_7c_{27} = c_4c_{16} + c_{20} + c_{10}(c_{10} + c_5^2 + c_2^2c_6)$, $S_8c_{27} = c_{19} + c_{11}c_4^2$,

$S_{2,2,2,2}c_{27} = c_2^3c_{13} + c_2^4c_{11} + c_2^2c_6c_9 + c_2c_5(c_6^2 + c_2^6) + c_2^2c_5(c_{10} + c_5^2 + c_2^2c_6)$,

$S_{3,3,3}c_{27} = c_2c_{16} + c_4c_6c_8 + c_8(c_{10} + c_5^2 + c_2^2c_6) + c_{18} + c_9^2 + c_2^2c_{14} + c_6(c_{12}$
$\quad + c_2^2c_4^2 + c_6^2)$, $S_9c_{27} = c_{18} + c_4^2c_{10}$,

$S_{10}c_{27} = c_4c_5c_8 + c_6c_{11} + c_4^2c_9$, $S_{5,5}c_{27} = c_4c_5c_8 + c_6c_{11} + c_4^2c_9$, $S_{2,2,2,2,2}c_{27} = 0$,

$S_{11}c_{27} = c_6c_{10} + c_2^8 + c_4^4$, $S_{12}c_{27} = 0$, $S_{6,6}c_{27} = c_2c_4c_9 + c_5(c_{10} + c_5^2 + c_2^2c_6)$,

$S_{4,4,4}c_{27} = 0$, $S_{3,3,3,3}c_{27} = c_2c_4c_9 + c_5(c_{10} + c_5^2 + c_2^2c_6)$, $S_{2,2,2,2,2,2}c_{27} = 0$,

$S_{13}c_{27} = 0$, $S_{14}c_{27} = c_{13}$, $S_{7,7}c_{27} = c_{13}$, $S_{2,2,2,2,2,2,2}c_{27} = 0$, $S_{15}c_{27} = c_4c_8 + c_{12}$,

$S_{5,5,5}c_{27} = c_4c_8$, $S_{3,3,3,3,3}c_{27} = c_2^2c_8 + c_2c_4c_6 + c_2(c_{10} + c_5^2 + c_2^2c_6)$, $S_{16}c_{27} = c_{11}$,

$S_{8,8}c_{27} = 0$, $S_{4,4,4,4}c_{27} = 0$, $S_{2,2,2,2,2,2,2,2}c_{27} = 0$, $S_{17}c_{27} = c_{10}$, $S_{18}c_{27} = 0$,

$S_{9,9}c_{27} = 0$, $S_{6,6,6}c_{27} = c_2^2c_5$, $S_{3,3,3,3,3,3}c_{27} = c_2^2c_5$, $S_{19}c_{27} = 0$, $S_{20}c_{27} = 0$,

$S_{2,2,2,2,2,2,2,2,2}c_{27} = 0$, $S_{10,10}c_{27} = 0$, $S_{5,5,5,5}c_{27} = 0$, $S_{2,2,2,2,2,2,2,2,2,2}c_{27} = 0$,

$S_{4,4,4,4,4}c_{27} = 0$, $S_{21}c_{27} = 0$, $S_{7,7,7}c_{27} = c_2c_4 + c_6$, $S_{3,3,3,3,3,3,3}c_{27} = c_2^3$,

$S_{22}c_{27} = 0$, $S_{11,11}c_{27} = 0$, $S_{2,2,2,2,2,2,2,2,2,2,2}c_{27} = 0$, $S_{23}c_{27} = 0$, $S_{25}c_{27} = 0$,
$S_{5,5,5,5,5}c_{27} = 0$.

Table 11 The cell $E_2^{0,1,t}$ of the MASS for $t < 108$ (generators)

106	$\omega_4(= u_3c_{23} + u_4c_{19} + u_5c_{11})$,
	$\psi_{10}(= u_1c_9c_{17} + u_2(c_{25} + c_8c_{17} + c_9c_{16}) + u_4c_2c_{17} + u_5c_2c_9)$,
	$\psi_9(= u_1c_5c_{21} + u_2c_4(c_{21} + c_4c_{17} + c_5c_{16}) + u_3c_2(c_{21} + c_2c_{19} + c_5c_{16}) + u_5c_2c_4c_5)$.
102	$\tilde{\varphi}_{13}(= u_1c_{25} + u_2c_8c_{16} + u_4c_2c_{16} + u_5c_2c_8)$.
98	$\omega_3(= u_2c_{23} + u_4c_{17} + u_5c_9)$, $\psi_8(= u_1c_5c_{19} + u_2c_4c_{19} + u_3(c_{21} + c_2c_{19} + c_5c_{16})$
	$+u_5c_4c_5)$, $\psi_7(= u_1c_{11}c_{13} + u_2c_4c_8c_{11} + u_3c_8(c_{13} + c_2c_{11} + c_5c_8)$
	$+u_4c_4(c_{13} + c_2c_{11} + c_4c_9))$.
94	$\tilde{\varphi}_{12}(= u_1c_{23} + u_4c_{16} + u_5c_8)$.
90	$\psi_6(= u_1c_5c_{17} + u_2(c_{21} + c_4c_{17} + c_5c_{16}) + u_3c_2c_{17} + u_5c_2c_5)$.
	$\psi_5(= u_1c_9c_{13} + u_2c_8(c_{13} + c_5c_8 + c_4c_9) + u_3c_2c_8c_9 + u_4c_2(c_{13} + c_2c_{11} + c_4c_9))$.
86	$\tilde{\varphi}_{11}(= u_1c_{21} + u_2c_4c_{16} + u_3c_2c_{16} + u_5c_2c_4)$.
82	$\omega_2(= u_2c_{19} + u_3c_{17} + u_5c_5)$,
	$\psi_4(= u_1c_9c_{11} + u_2c_8c_{11} + u_3c_8c_9 + u_4(c_{13} + c_2c_{11} + c_4c_9)$.
78	$\tilde{\varphi}_{10}(= u_1c_{19} + u_3c_{16} + u_5c_4)$.
74	$\psi_3(= u_1c_5c_{13} + u_2c_4(c_{13} + c_4c_9 + c_5c_8) + u_3c_2(c_{13} + c_5c_8 + c_2c_{11}) + u_4c_2c_4c_5$
70	$\tilde{\varphi}_9(= u_1c_{17} + u_2c_{16} + u_5c_2)$.
66	$\psi_2(= u_1c_5c_{11} + u_2c_4c_{11} + u_3(c_{13} + c_2c_{11} + c_5c_8) + u_4c_4c_5)$.
62	u_5.
58	$\psi_1(= u_1c_5c_9 + u_2(c_{13} + c_4c_9 + c_5c_8) + u_3c_2c_9 + u_4c_2c_5)$.
54	$\tilde{\varphi}_7(= u_1c_{13} + u_2c_4c_8 + u_3c_2c_8 + u_4c_2c_4)$.
50	$\omega_1(= u_2c_{11} + u_3c_9 + u_4c_5)$.
46	$\tilde{\varphi}_6(= u_1c_{11} + u_3c_8 + u_4c_4)$.
38	$\tilde{\varphi}_5(= u_1c_9 + u_2c_8 + u_4c_2)$.
30	u_4.
22	$\varphi_3(= u_1c_5 + u_2c_4 + u_3c_2)$.
14	u_3.
6	u_2.
2	u_1.
$\uparrow$	
t	

Table 12 The cell $E_2^{0,0,t}$ of the MASS for $t < 108$ (generators)

104	c_{26}, $e_{26}(= c_{13}^2)$.
96	c_{24}.
88	c_{22}, $e_{22}(= c_{11}^2)$.
80	c_{20}.
72	c_{18}, $e_{18}(= c_9^2)$.
64	$e_{16}(= c_8^2)$.
56	c_{14}.
48	c_{12}.
40	c_{10}, $e_{10}(= c_5^2)$.
32	$e_8(= c_4^2)$.
24	c_6.
16	$e_4(= c_2^2)$.
0	1.
↑	t

Table 13 The generators of the cell $E_\infty^{2,0,t}$ of the MASS for $t < 54$

52	$a_{13}(= h_0c_{13} + (h_0c_2 + h_1h_2)c_{11} + (h_0c_4 + h_1h_3)c_9 + (h_0c_8 + h_1h_4)c_5)$
	$b_{13}(= h_0c_{13} + (h_0c_8 + h_1h_4)c_5 + h_2h_4c_4 + h_3h_4c_2)$
	$f_{13}(= h_0c_{13} + (h_0c_2 + h_1h_2)c_{11} + h_2h_3c_8 + h_2h_4c_4)$
48	$b_{12}(= h_0c_4c_8 + h_1h_3c_8 + h_1h_4c_4 + h_1^2c_{11})$
44	$a_{11}(= h_0c_{11} + h_3h_4)$
	$b_{11}(= h_0c_2c_9 + h_1h_2c_9 + h_2h_4c_2 + h_2^2c_8)$
40	$b_{10}(= h_0c_2c_8 + h_1h_2c_8 + h_1h_4c_2 + h_1^2c_9)$
36	$a_9(= h_0c_9 + h_2h_4)$
	$b_9(= h_0c_4c_5 + h_1h_3c_5 + h_2h_3c_4 + h_3^2c_2)$
32	$a_8(= h_0c_8 + h_1h_4)$
28	$a_7(= h_3^2)$
	$b_7(= h_0c_2c_5 + h_1h_2c_5 + h_2h_3c_2 + h_2^2c_4)$
24	$b_6(= h_0c_2c_4 + h_1h_2c_4 + h_1h_3c_2 + h_1^2c_5)$
20	$a_5(= h_0c_5 + h_2h_3)$
16	$a_4(= h_0c_4 + h_1h_3)$
12	$a_3(= h_2^2)$
8	$a_2(= h_0c_2 + h_1h_2)$
4	$a_1(= h_1^2)$
0	h_0
↑	t

Table 14 Relations in the term $E_\infty^{*,*,t}$ for $t < 54$

$h_i^2 u_j^2 = h_j^2 u_i^2$, $(i, j = 1, 2, 3, 4)$.
$h_0 x = 0$, where x is an arbitrary element with the second grading > 0.
$u_i(h_0 c_{[i,k]} + h_i h_k) = u_k h_i^2$, where $[i, k] = 2^{i-1} + 2^{k-1} - 1$; $i, k \in \{1, 2, 3, 4\}$; $i \neq k$.
$u_i(h_0 c_{[j,k]} + h_j h_k) = u_j(h_0 c_{[i,k]} + h_i h_k) = u_k(h_0 c_{[i,j]} + h_i h_j)$, where
$i \neq j \neq k \neq i \in \{1, 2, 3, 4\}$; $[i, k] = 2^{i-1} + 2^{k-1} - 1$, $[i, j] = 2^{i-1} + 2^{j-1} - 1$;
$[j, k] = 2^{j-1} + 2^{k-1} - 1$.
$h_0^2 e_{2k} = a_k^2 + h_i^2 h_j^2$, where $k = 2^{i-1} + 2^{j-1} - 1$, $i \neq j \in \{1, 2, 3\}$.
$h_0(h_0 c_k c_j + h_i h_j c_l + h_s h_j c_k + h_j^2 c_r) = (h_0 c_k + h_i h_j)(h_0 c_l + h_s h_j)(h_0 c_r + h_s h_i)$
where $k = 2^{i-1} + 2^{j-1} - 1$, $l = 2^{s-1} + 2^{j-1} - 1$, $r = 2^{s-1} + 2^{i-1} - 1$;
$i \neq j \neq k \neq i \in \{1, 2, 3\}$.
$h_0(a_{13} + b_{13}) = a_2 a_{11} + a_4 a_9$;
$u_1 b_6 = \varphi_3 a_1$;
$h_0(a_{13} + f_{13}) = a_5 a_8 + a_4 a_9$;
$u_1 b_7 = \varphi_3 a_2 = u_2 b_6$;
$b_6^2 = a_2^2 e_8 + a_1 a_7 e_4 + a_1^2 e_{10} = a_4^2 e_4 + a_1 a_3 e_8 + a_1^2 e_{10}$;
$u_2 b_7 = \varphi_3 a_3$;
$u_1 b_9 = \varphi_3 a_4 = u_3 b_6$;
$h_0(e_4 a_7 + e_8 a_3) = a_2 b_9 + a_4 b_7$;
$u_1 b_{10} = \varphi_5 a_1 + u_3 c_6 a_1$;
$h_0(e_8 a_3 + a_1 e_{10}) = a_4 b_7 + a_5 b_6$;
$u_2 b_9 = \varphi_3 a_5 = u_3 b_7$;
$u_1 \omega_1 = u_2(\varphi_6 + u_2 c_{10} + u_3 c_2^4) + u_3(\varphi_5 + u_3 c_6) + u_4 \varphi_3$;
$u_1 b_{11} = u_2 b_{10} = \varphi_5 a_2 + u_3 c_6 a_2$;
$u_2 b_{11} = \varphi_5 a_3 + u_3 c_6 a_3$;
$u_1(a_1 e_{10} + a_3 e_8 + a_7 e_4) = b_6 \varphi_3$;
$u_3 b_9 = \varphi_3 a_7$;
$u_2(a_1 e_{10} + a_3 e_8 + a_7 e_4) = b_7 \varphi_3$;
$u_1 b_{12} = \varphi_6 a_1 + u_2 c_{10} a_1 + u_3 c_2^4 a_1$;
$\omega_1 a_1 = u_2 b_{12} + u_3 b_{10} + u_4 b_6$.

Table 15 The term $E_2 = Ext_A(BP^*(MSp), BP^*)$

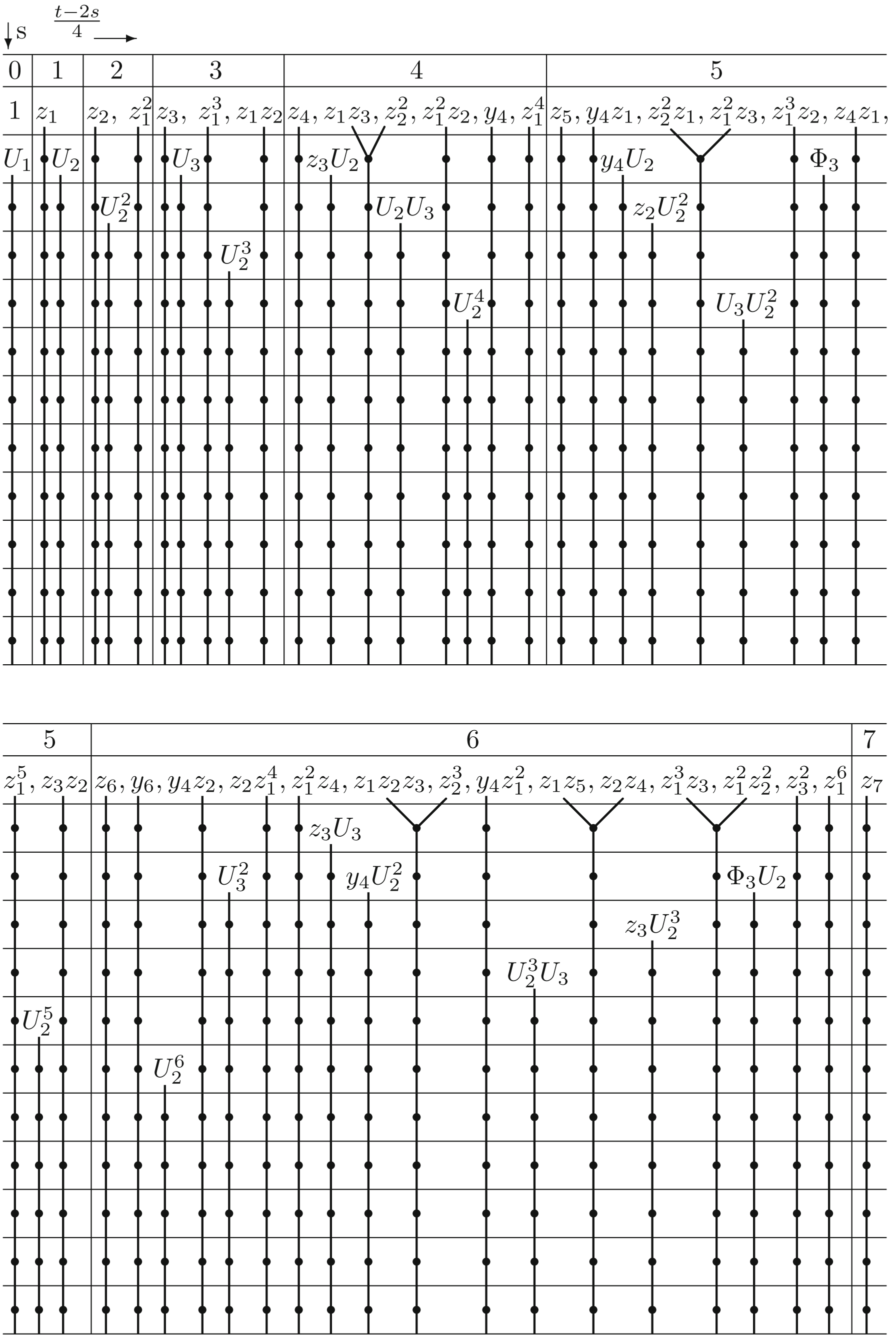


Table 15 continued

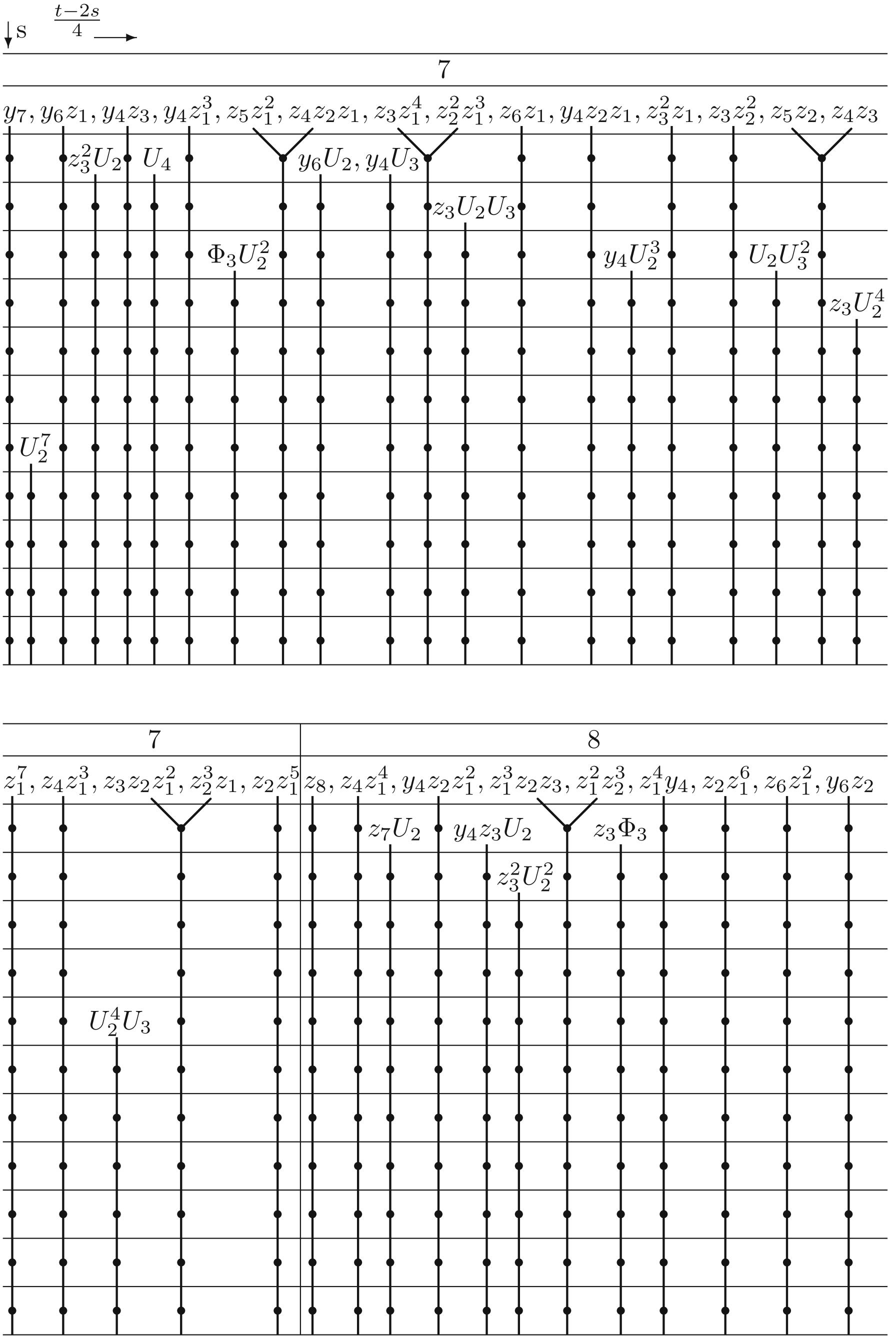
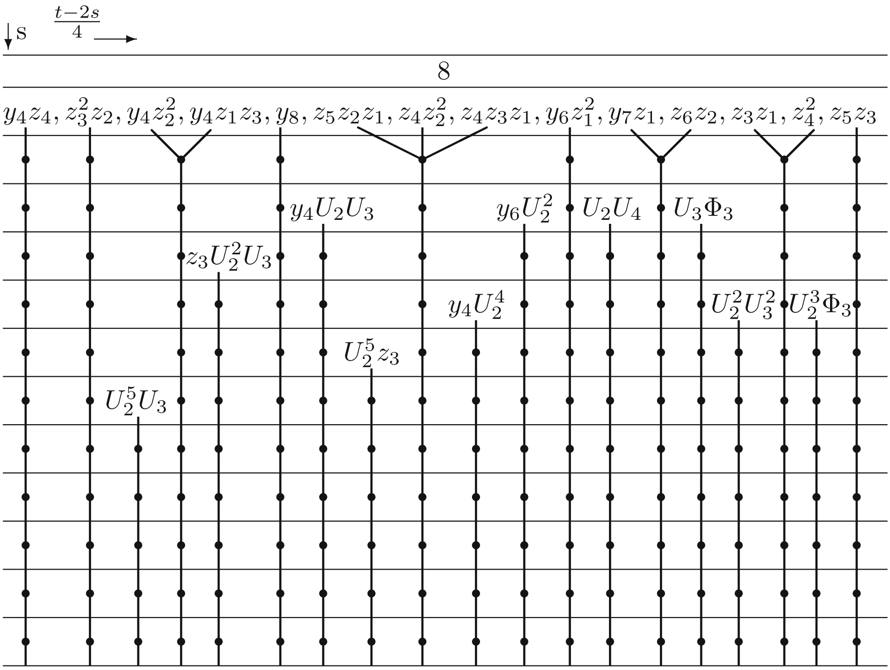


Table 15 continued



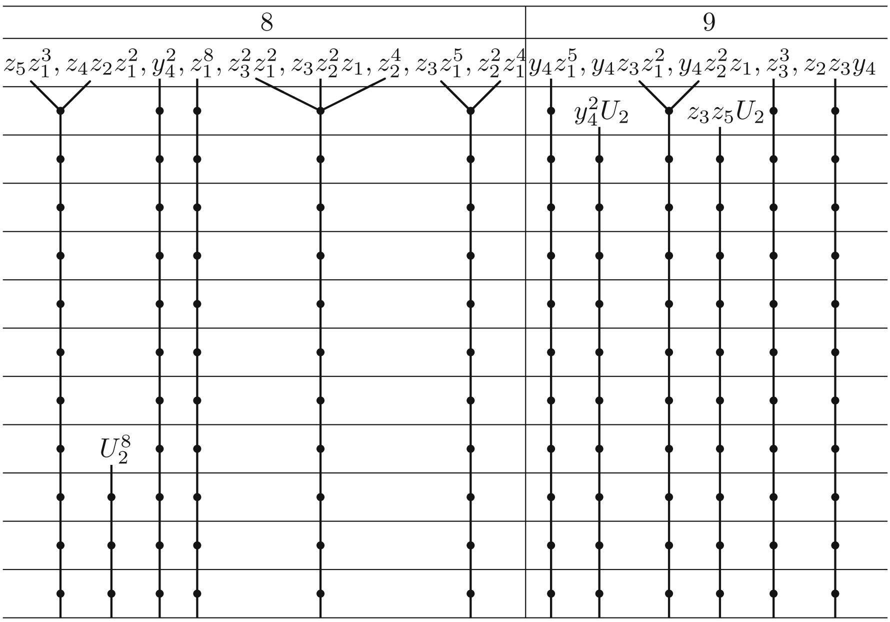


Table 15 continued

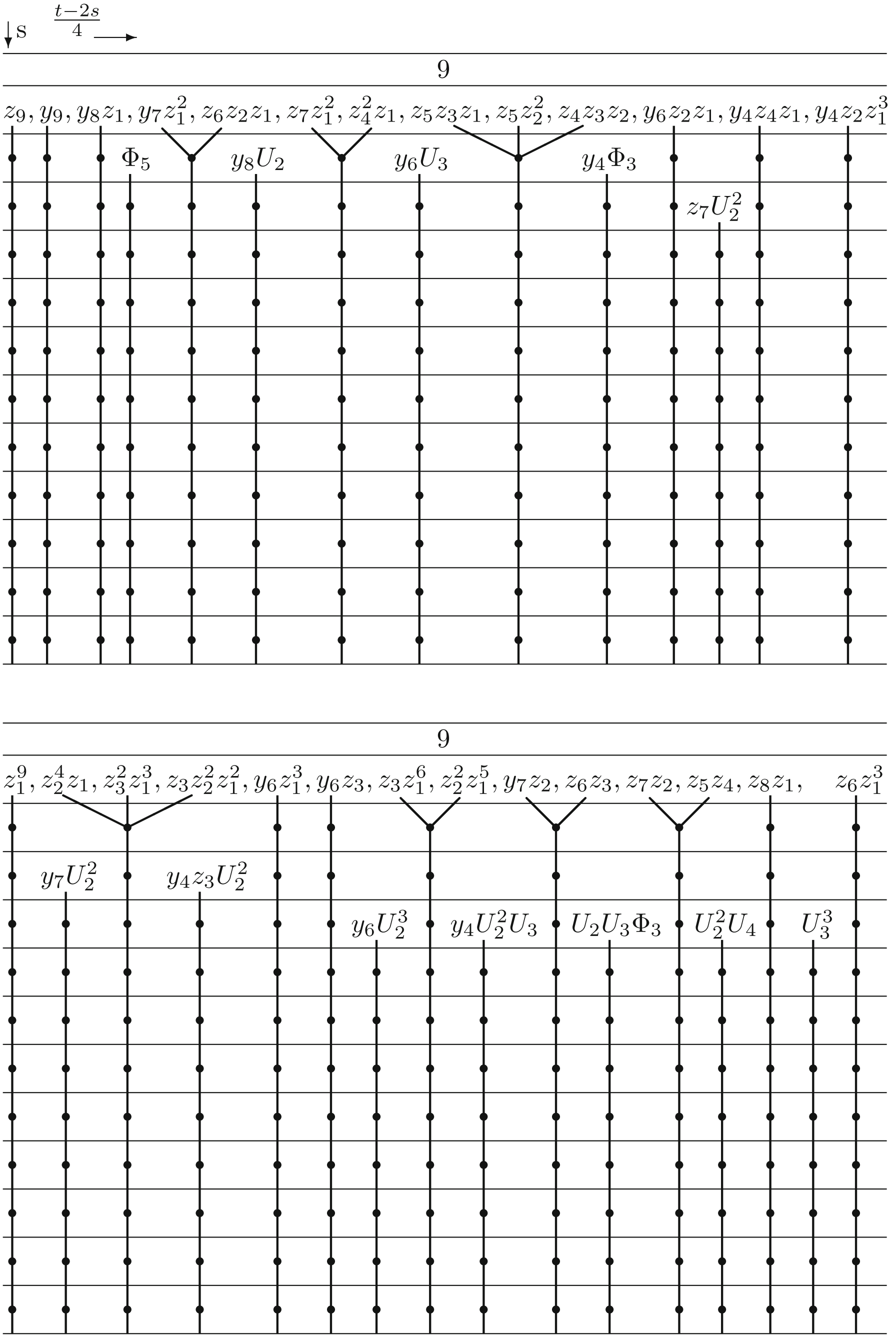


Table 15 continued

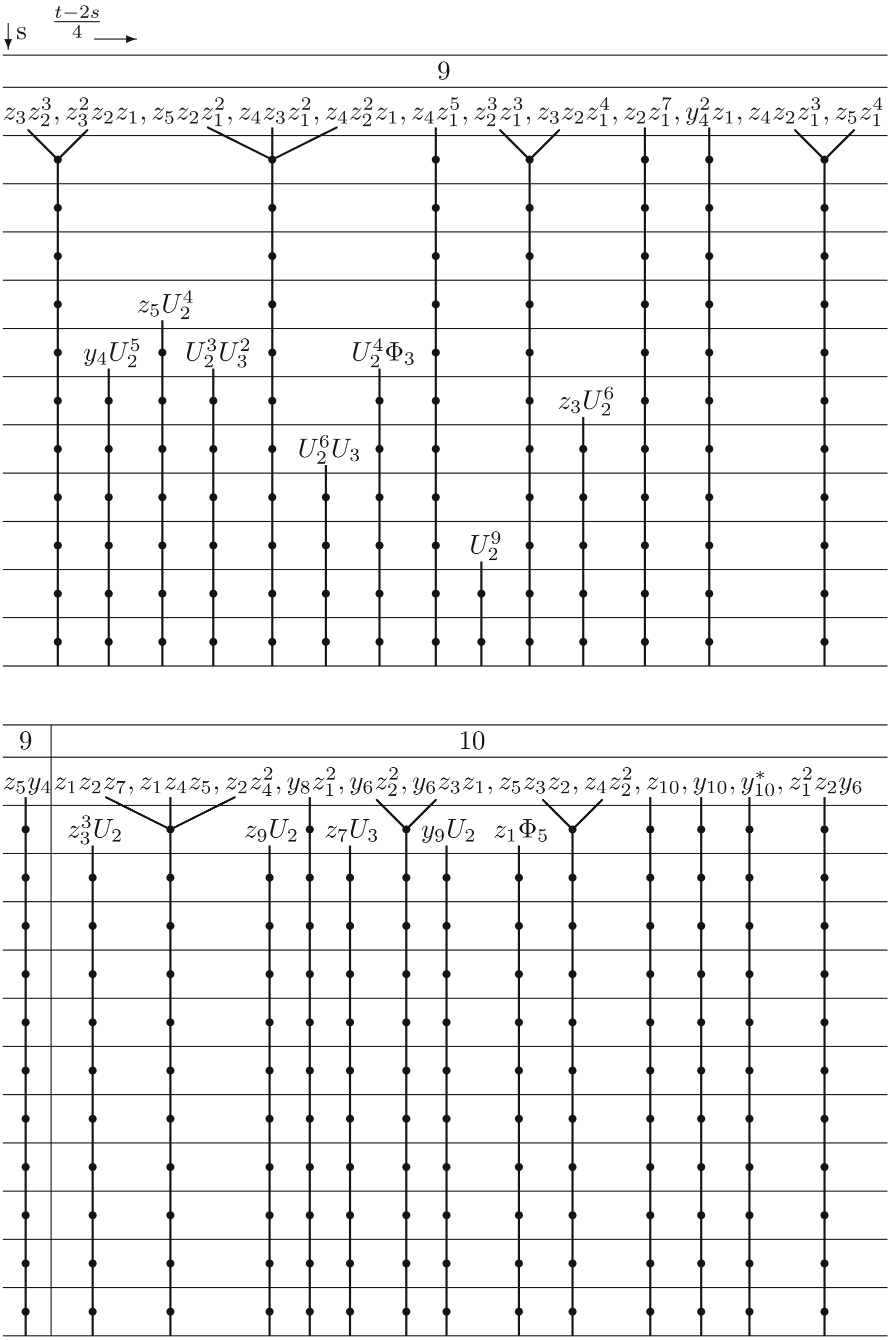
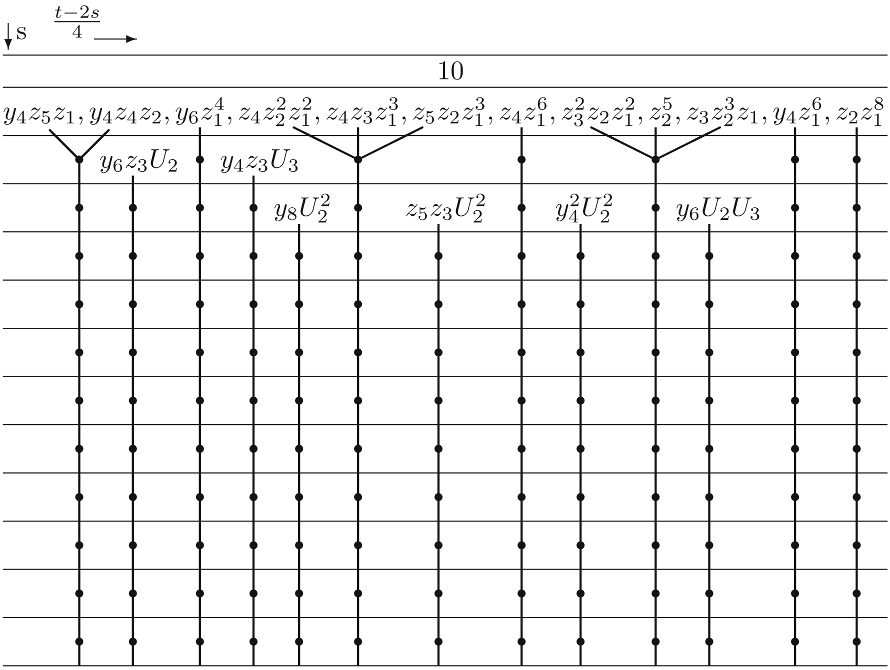


Table 15 continued



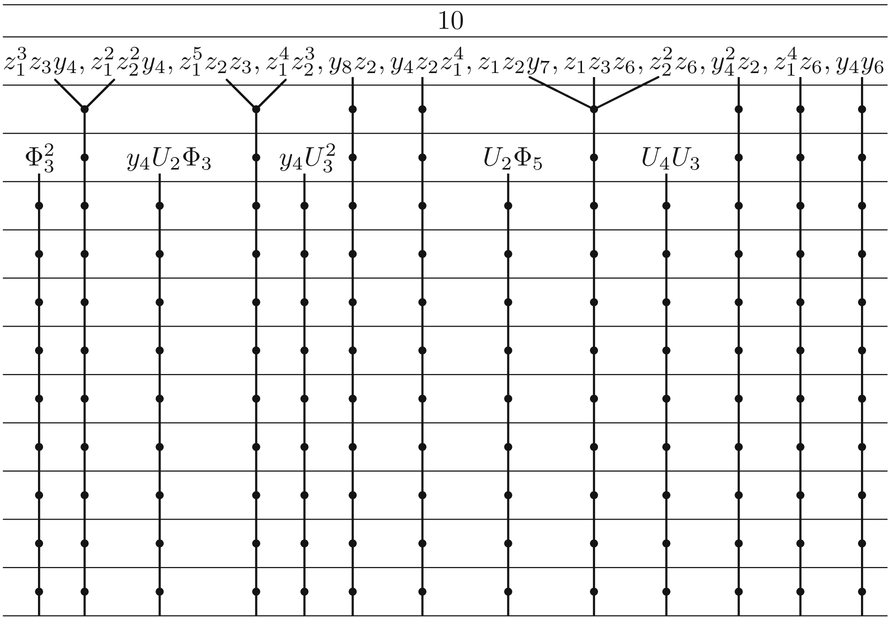
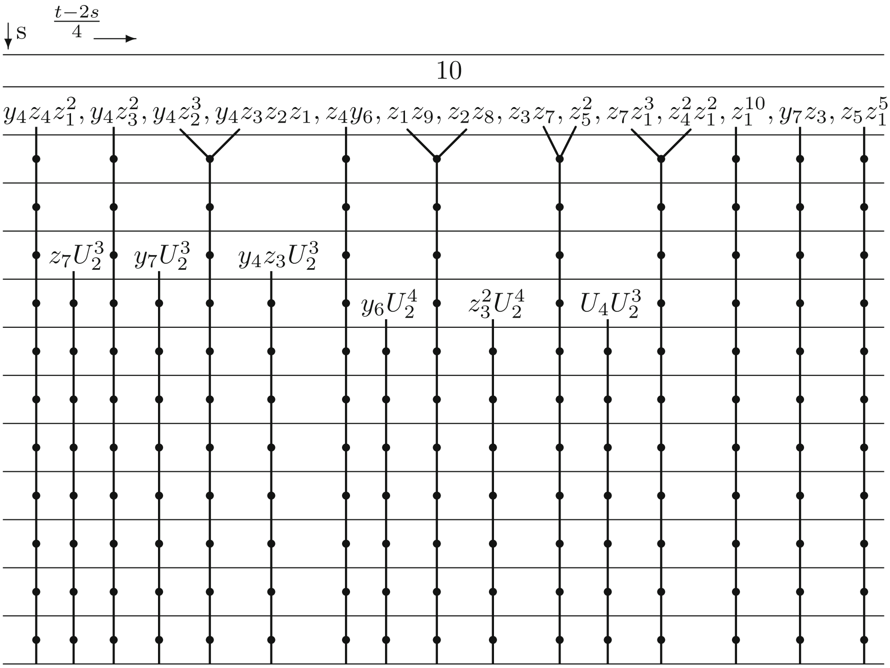


Table 15 continued



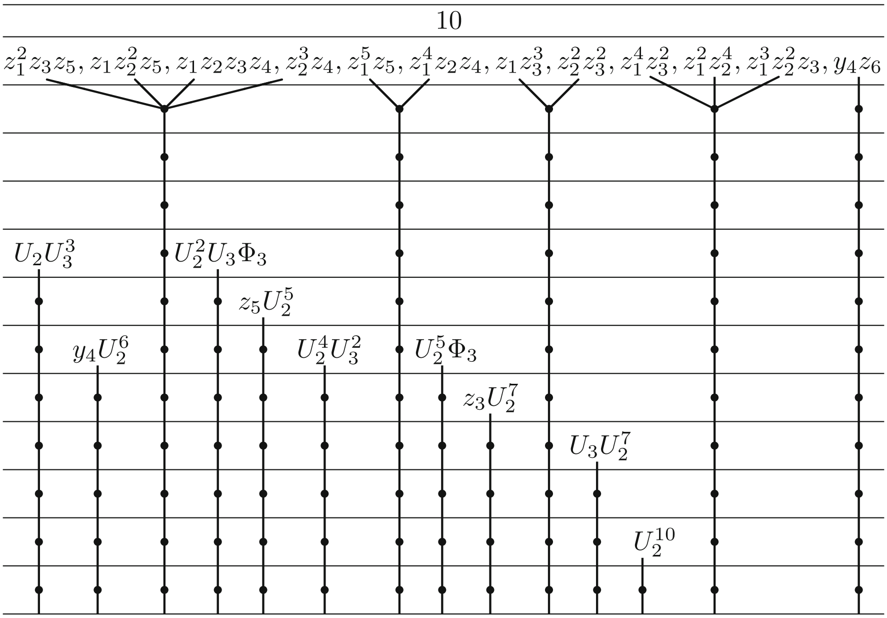


Table 15 continued

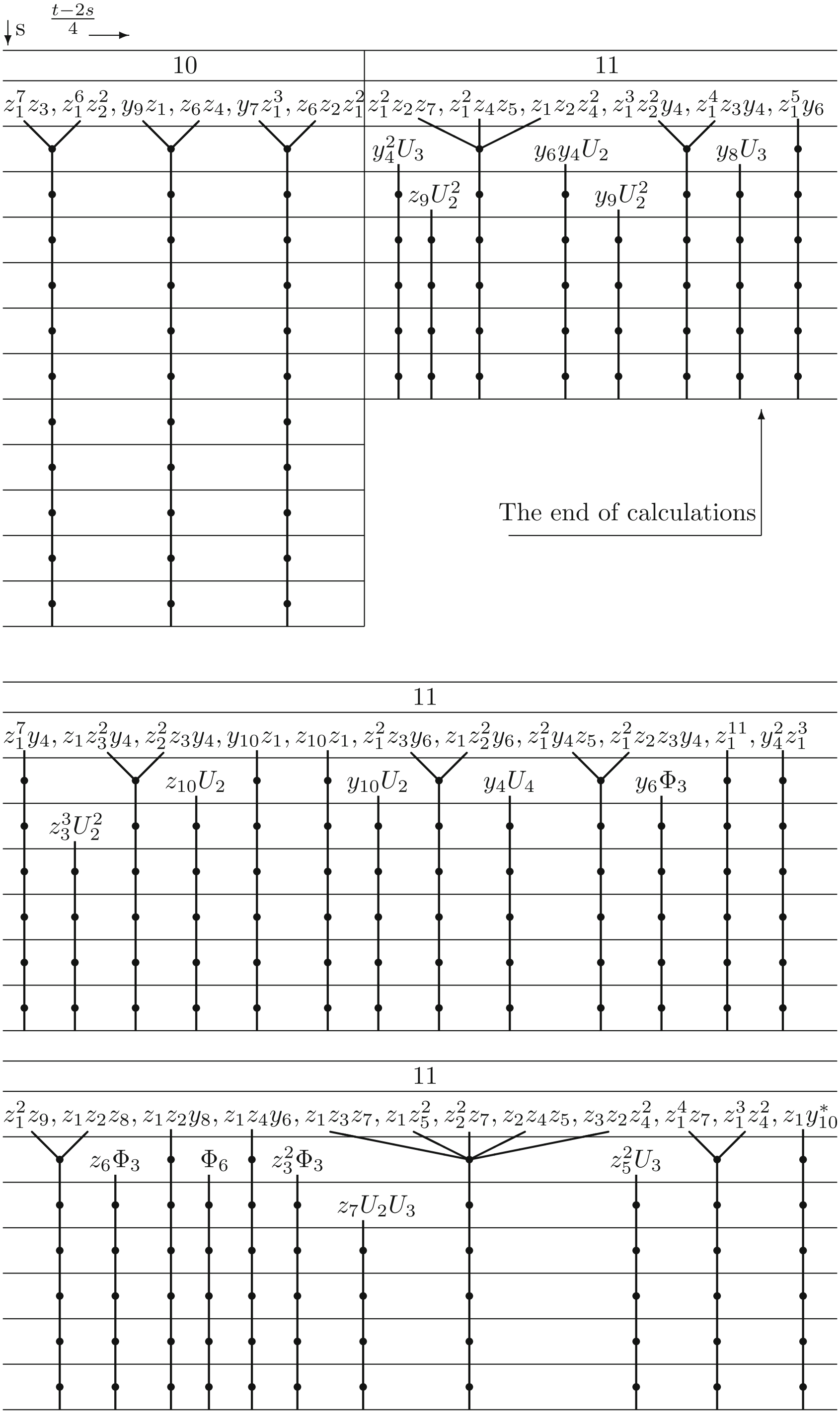


Table 15 continued

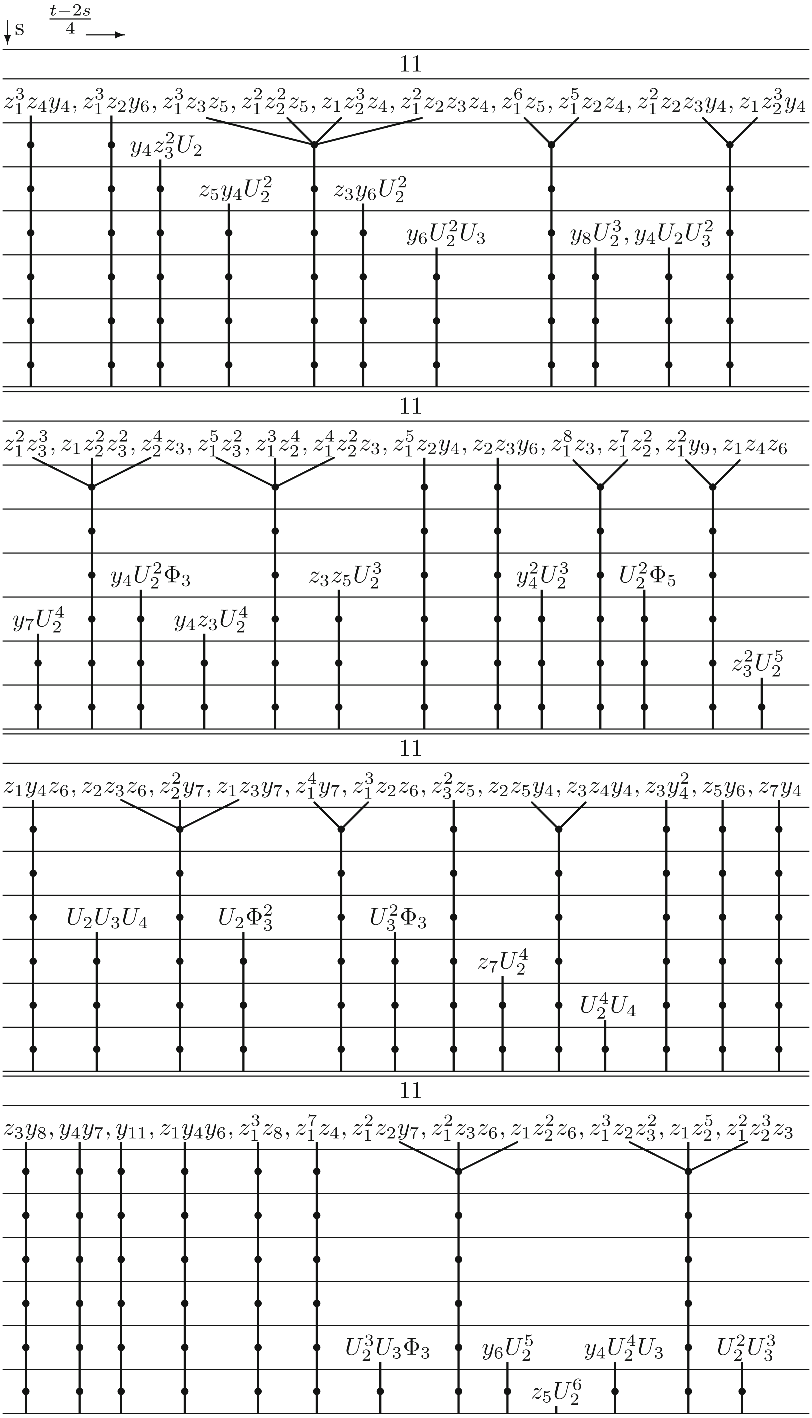


Table 15 continued

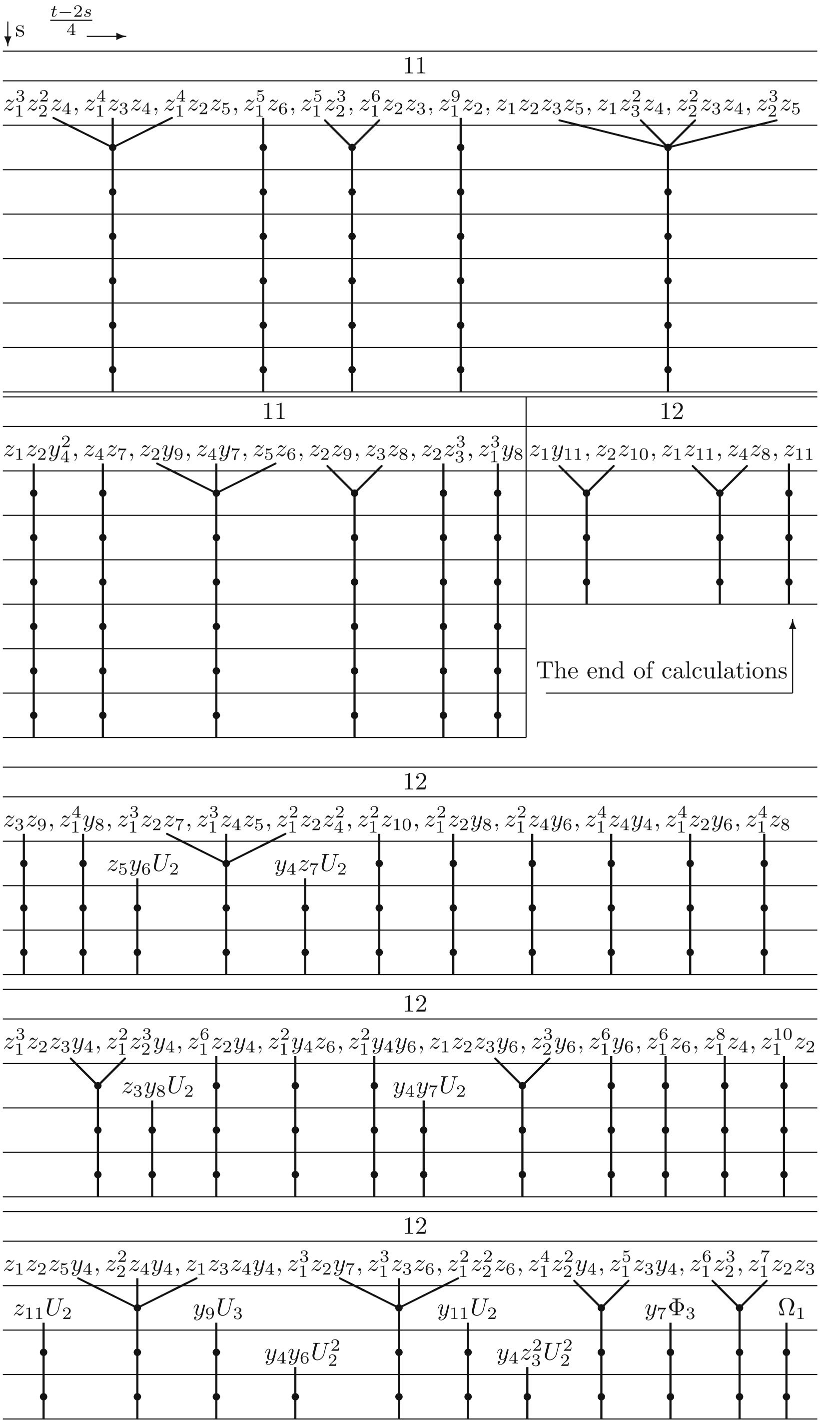


Table 15 continued

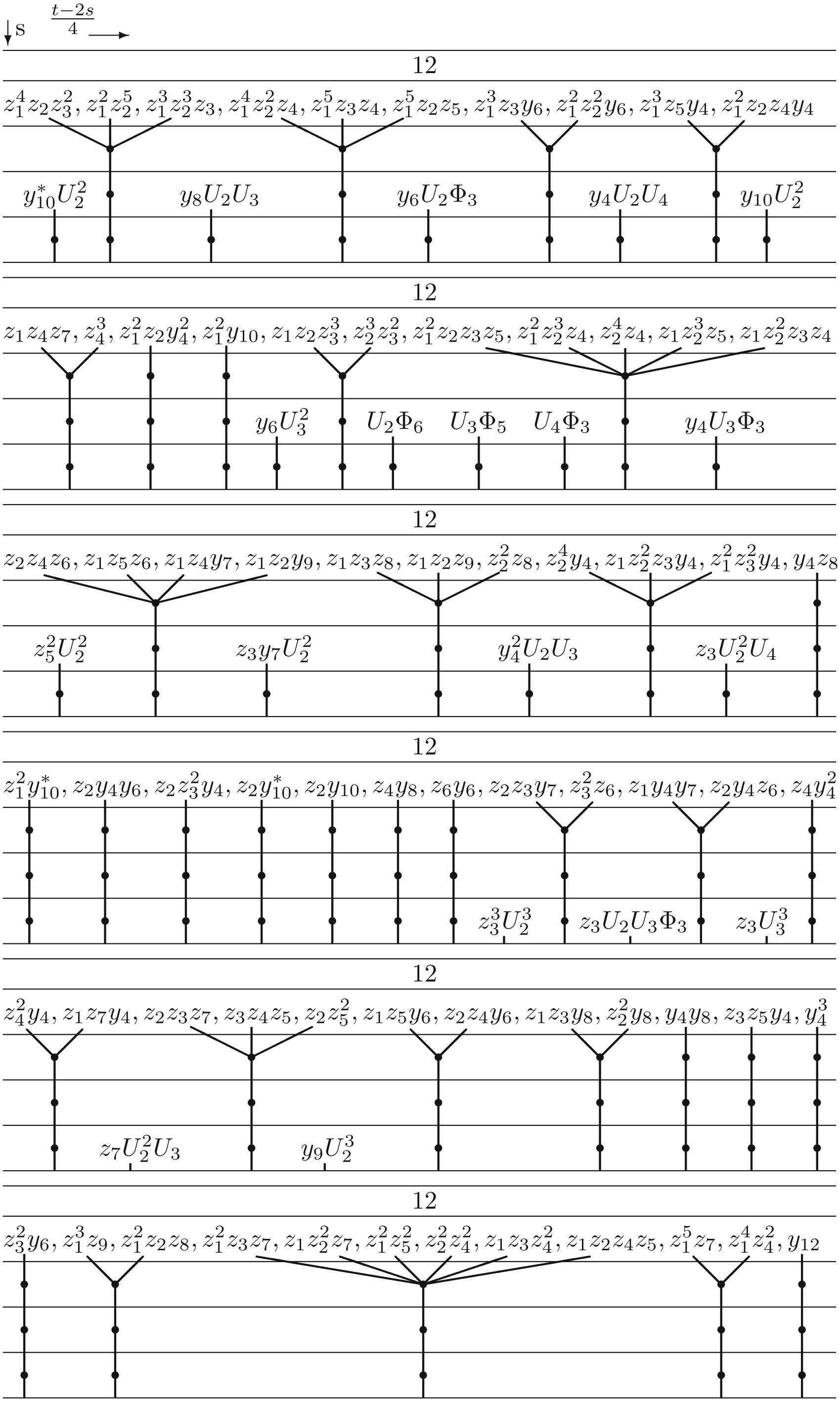


Table 15 continued

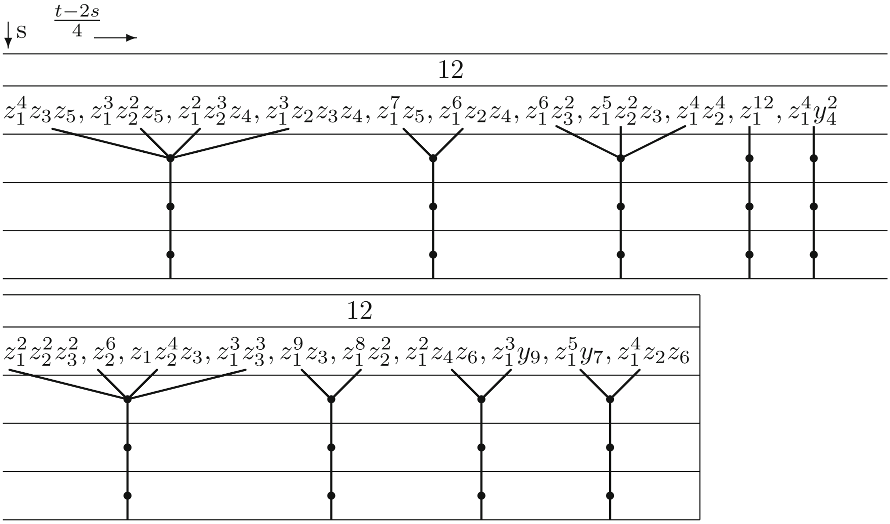


Table 16 The relations in $\mathrm{Ext}_A(BP^*(MSp), BP^*)$

1.	$4(y_8 + z_8 + z_2y_6 + y_4z_4) = z_1z_7 + z_4^2.$
2.	$U_1z_8 = U_4z_1.$
3.	$U_2y_7 = \Phi_3z_3.$
4.	$U_2z_7 = U_3z_5.$
5.	$2y_9 = z_2z_7 + 3z_4z_5$
6.	$U_1y_9 = U_3z_6 = \Phi_3z_4.$
7.	$U_1z_9 = U_2z_8 = U_4z_2.$
8.	$4(y_{10}^* + y_4y_6) = z_3z_7 + z_5^2 + X_{40}^1(\widehat{y_{10}^*}, \widehat{y_4y_6}, \widehat{z_3z_7}, \widehat{z_5^2}).$
9.	$2(z_{10} + z_4y_6) = z_2z_8 + 3z_1z_9 + X_{40}^2(\widehat{z_{10}}, \widehat{z_4y_6}, \widehat{z_2z_8}, \widehat{z_1z_9}).$
10.	$U_1z_{10} = \Phi_5z_1$
11.	$U_2z_9 = U_4z_3.$
12.	$U_2y_9 = U_3y_7 = \Phi_3z_5.$
13.	$\Phi_3^2 = U_1[U_1y_{10}^* + U_2(y_9 + z_9) + U_3z_7] + U_2^2y_8 + U_3^2y_4 + U_1[U_1y_4y_6 + U_2(y_4z_5 + y_6z_3)].$
14.	$2y_4z_7 + 2y_8z_3 = z_2y_9 + z_4y_7 + X_{44}^1(\widehat{y_4z_7}, \widehat{y_8z_3}, \widehat{z_2y_9}, \widehat{z_4y_7}).$
15.	$2y_8z_3 + 2(y_{10}^* + y_6y_4)z_1 = z_4y_7 + z_5z_6 + X_{44}^2(\widehat{y_6z_5}, \widehat{y_8z_3}, z_1(\widehat{y_4y_6 + y_{10}^*}), \widehat{z_4y_7}).$
16.	$2y_{11} = z_2z_9 + 3z_3z_8 + X_{44}^3(\widehat{y_{11}}, \widehat{z_2z_9}, \widehat{z_3z_8}).$
17.	$U_1(z_1y_{10}^* + z_1y_4y_6 + z_3y_8 + z_7y_4) + U_1z_4z_7 + U_1z_2z_9 + U_1z_2y_9$ $+U_1z_3(z_2y_6 + z_4y_4) = \Phi_3y_6.$
18.	$U_1y_{11} = U_2z_{10} = \Phi_5z_2.$
19.	$U_1z_{11} = U_3z_8 = U_4z_4.$
20.	$z_6^2 = (1 + 2\beta_1)z_2^2y_8 + (1 + 2\beta_2)z_1z_7y_4 + (1 + 2\beta_3)z_1^2(y_{10}^* + y_4y_6) + X_{48}^1.$
21.	$2z_{12} = z_1z_{11} + 3z_4z_8 + X_{48}^2(\widehat{y_{12}}, \widehat{z_1z_{11}}, \widehat{z_4z_8}),$
22.	$U_1z_{12} = \Phi_6z_1.$

Table 16 continued

23.	$U_2y_{11}=\Phi_5z_3$.
24.	$U_2z_{11}=U_3z_9=U_4z_5$.
25.	$U_3y_9=\Phi_3z_7$.
26.	$\Phi_3y_7=U_2(z_1(y_{10}^*+z_{10}+y_4y_6)+z_2(y_9+z_9)+z_3(y_8+y_6z_2+y_4z_4)z_7y_4)$
27.	$U_1\Omega_1=U_2\Phi_6+U_3\Phi_5+U_4\Phi_3+U_2^2y_{10}+U_3^2y_6+U_2U_3y_4^2$.

Remark The summand of the form $X_n(\dots,\hat{x},\dots)$ has the following properties:

(i) $X_n\in F^6(E_2^{0,n})$,
(ii) $d_3(X_n)=0$,
(iii) the summand x do not appear in the expression for X_n.

Table 17 The action of the differential d_3 of the Adams–Novikov spectral sequence (continuation of Table 4 of the work [41])

(7)	$d_3(U_1^nU_2^4z_3)=U_1^{n+1}U_2^6$, $d_3(U_1^nU_2^4U_3)=0$, $d_3(U_1^nU_2^7)=0$.
(8)	$d_3(U_1^nU_2z_7)=d_3(U_1^nU_3z_5)=U_1^{n+1}U_2U_3^2$, $d_3(U_1^n\Phi_3z_3)=U_1^{n+1}U_2^2\Phi_3$, $d_3(U_1^nU_2z_3y_4)=U_1^nU_2(U_2^3z_3+U_1U_2^2y_4)$, $d_3(U_1^nU_2^2U_3z_3)=U_1^{n+1}U_2^4U_3$, $d_3(U_1^nU_2U_3y_4)=d_3(U_1^nU_2^2y_6)=U_1^nU_2^4U_3$, $d_3(U_1^nU_2^4y_4)=U_1^nU_2^7$, $d_3(U_1^n\times U_2^8)=0$, $d_3(U_1^nU_2^5z_3)=U_1^{n+1}U_2^7$, $d_3(U_1^nU_3\Phi_3)=0$, $d_3(U_1^nU_2U_4)=0$, $d_3(U_1^nU_2^2z_3^2)=0$, $d_3(U_1^nU_2^2U_3^2)=0$, $d_3(U_1^nU_2^3\Phi_3)=0$, $d_3(U_1^nU_2^5U_3)=0$.
(9)	$d_3(U_1^nz_9)=U_1^{n+1}U_2U_4$, $d_3(U_1^ny_9)=U_1^{n+1}U_3\Phi_3$, $d_3(U_1^nz_1^5y_4)=U_1^{n+3}z_1^2\times(z_1^2y_4+z_2^3)$, $d_3(U_1^nz_3^3)=d_3(U_1^nz_3z_3y_4)=U_1^{n+1}U_2^2z_3^2$, $d_3(U_1^nz_1^2z_3y_4)=d_3(U_1^nz_1z_2^2)=U_1^{n+3}z_2(z_2y_4+z_3^2)$, $d_3(U_1^nz_1^2y_7)=d_3(U_1^nz_1z_2z_6)=U_1^{n+3}z_1y_7$, $d_3(U_1^nz_1^2z_7)=d_3(U_1^nz_1z_4^2)=U_1^{n+3}z_1z_7$, $d_3(U_1^nz_1z_3z_5)=d_3(U_1^nz_1z_2y_6)=d_3(U_1^nz_2z_3z_4)=d_3(U_1^nz_1z_4y_4)=d_3(U_1^nz_2^2z_5)=U_1^{n+3}\times z_3z_5$, $d_3(U_1^nz_4z_2z_1^3)=d_3(U_1^nz_5z_1^4)=U_1^{n+3}z_1^3z_5$, $d_3(U_1^nz_1y_4^2)=U_1^{n+3}y_4^2$, $d_3(U_1^nz_1z_2^4)=d_3(U_1^nz_1^3z_3^2)=d_3(U_1^nz_1^3z_2y_4)=d_3(U_1^nz_1^2z_2^2z_3)=U_1^{n+3}z_2^4$, $d_3(U_1^nz_1y_8)=U_1^{n+2}(U_1y_8+U_2z_7)$, $d_3(U_1^nz_1^3y_6)=U_1^{n+3}z_1(z_1y_6+z_3z_4)$, $d_3(U_1^nz_1^9)=U_1^{n+3}z_1^8$, $d_3(U_1^nz_5y_4)=U_1^nU_2U_3(U_1y_4+U_2z_3)$, $d_3(U_1^nz_3y_6)=U_1^nU_2^2(U_6y_6+U_3z_3)$, $d_3(U_1^nz_1^6z_3)=d_3(U_1^nz_1^5z_2^2)=U_1^{n+3}z_1^5z_3$, $d_3(U_1^n\times U_2y_8)=d_3(U_1^nU_3y_6)=U_1^nU_2^2U_3^2$, $d_3(U_1^n\Phi_3y_4)=U_1^n\Phi_3U_2^3$, $d_3(U_1^nU_2^2z_7)=U_1^{n+1}U_2^2U_3^2$, $d_3(U_1^nU_2^2y_7)=U_1^{n+2}U_2^3\Phi_3$, $d_3(U_1^nU_2^2z_3y_4)=U_1^nU_2^2(U_1y_4+U_2z_3)$, $d_3(U_1^nU_2^2U_3y_4)=d_3(U_1^nU_2^3y_6)=U_1^nU_2^5U_3$, $d_3(U_1^nU_2^4z_5)=U_1^{n+1}\times U_2^5U_3$, $d_3(U_1^nU_2^5y_4)=U_1^nU_2^8$, $d_3(U_1^nU_2^6z_3)=U_1^{n+2}U_2^7$, $d_3(U_1^nz_2y_7)=0$, $d_3(U_1^nz_2z_7)=0$, $d_3(U_1^nz_4z_5)=0$, $d_3(U_1^nz_2^3z_3)=0$, $d_3(U_1^nz_1z_2z_3^2)=0$, $d_3(U_1^nz_1^2z_2z_5)=0$, $d_3(U_1^nz_1^2z_3z_4)=0$, $d_3(U_1^nz_1z_2^2z_4)=0$, $d_3(U_1^nz_1z_8)=0$, $d_3(U_1^nz_1^3z_6)=0$, $d_3(U_1^nz_1^5z_4)=0$, $d_3(U_1^nz_1^3z_2^3)=0$, $d_3(U_1^nz_1^4z_2z_3)=0$, $d_3(U_1^nz_1^7z_2)=0$, $d_3(U_1^nU_2y_4^2)=0$, $d_3(U_1^nz_3z_6)=0$, $d_3(U_1^nU_2z_3z_5)=0$, $d_3(U_1^nU_3^3)=0$, $d_3(U_1^nU_2^3z_3^2)=0$, $d_3(U_1^nU_2U_3\Phi_3)=0$, $d_3(U_1^nU_2^2U_4)=0$, $d_3(U_1^nU_2^3U_3^2)=0$, $d_3(U_1^nU_2^4\Phi_3)=0$, $d_3(U_1^nU_2^6U_3)=0$, $d_3(U_1^nU_2^9)=0$, $d_3(U_1^n\Phi_5)=0$.

Table 17 continued

(10)	$d_3(U_1^n z_1z_2z_7) = d_3(U_1^n z_1z_4z_5) = d_3(U_1^n z_2z_4^2) = d_3(U_1^n y_8z_1^2) = U_1^{n+3}z_2z_7,$ $d_3(U_1^n y_6z_2^2) = d_3(U_1^n y_6z_3z_1) = d_3(U_1^n z_5z_3z_2) = d_3(U_1^n z_4z_2^2) = d_3(U_1^n y_4$ $\times z_5z_1) = d_3(U_1^n y_4z_4z_2) = U_1^{n+2}U_3z_3^2, d_3(U_1^n z_{10}) = U_1^{n+3}\Phi_5, d_3(U_1^n y_{10}^*)$ $= U_1^n(U_3^3 + U_2^2U_4 + U_2U_3\Phi_3), d_3(U_1^n y_{10}) = U_1^nU_3^3, d_3(U_1^n z_1^2z_2y_6) = U_1^{n+1}$ $\times U_2z_1^2(U_1y_6 + U_2z_5), d_3(U_1^n y_6z_1^4) = d_3(U_1^n z_1^2z_2^2z_4) = d_3(U_1^n z_1^3z_3z_4)$ $= d_3(U_1^n z_1^3z_2z_5) = U_1^{n+3}z_1z_2^2z_4, d_3(U_1^n z_4z_1^6) = U_1^{n+3}z_4z_1^5, d_3(U_1^n z_1^2z_2z_3^2)$ $= d_3(U_1^n z_2^5) = d_3(U_1^n z_1z_2^3z_3) = d_3(U_1^n z_1^3z_3y_4) = d_3(U_1^n z_1^2z_2^2y_4) = U_1^{n+2}U_2z_2^4,$ $d_3(U_1^n z_1^5z_2z_3) = d_3(U_1^n z_1^4z_2^3) = d_3(U_1^n z_1^6y_4) = U_1^{n+2}U_2z_3z_1^5, d_3(U_1^n z_1^8z_2)$ $= U_1^nU_2z_1^8, d_3(U_1^n z_1^4z_2y_4) = U_1^{n+1}U_2z_1^4(U_1y_4 + U_2z_3), d_3(U_1^n z_1z_2y_7)$ $= d_3(U_1^n z_1z_3z_6) = d_3(U_1^n z_2^2z_6) = U_1^{n+3}z_2y_7, d_3(U_1^n z_2y_4^2) = U_1^{n+2}U_2y_4^2,$ $d_3(U_1^n z_2y_8) = U_1^{n+1}U_2(U_1y_8 + U_2z_7), d_3(U_1^n y_4y_6) = U_1^nU_2^2(U_2y_6 + U_3y_4),$ $d_3(U_1^n z_1^4z_6) = U_1^{n+2}\Phi_3z_1^4, d_3(U_1^n z_1^2z_4y_4) = U_1^{n+1}U_3z_1^2(U_1y_4 + U_2z_3),$ $d_3(U_1^n z_3^2y_4) = U_1^nU_2^3z_3^2, d_3(U_1^n z_4y_6) = U_1^{n+1}U_3(U_1y_6{+}U_2z_5), d_3(U_1^n y_4z_6)$ $= U_1^{n+1}\Phi_3(U_1y_3 + U_2z_3), d_3(U_1^n z_2^3y_4) = d_3(U_1^n z_1z_2z_3y_4) = U_1^{n+1}U_2z_2^2$ $\times(U_1y_4 + U_2z_3), d_3(U_1^n z_1z_9) = 0, d_3(U_1^n z_5^2) = 0, d_3(U_1^n z_1z_2z_3z_4) = 0,$ $d_3(U_1^n z_2z_8) = 0, d_3(U_1^n z_3z_7) = 0, d_3(U_1^n z_7z_1^3) = 0, d_3(U_1^n z_1^3z_2^2z_3) = 0,$ $d_3(U_1^n z_4^2z_1^2) = 0, d_3(U_1^n z_1^2z_3z_5) = 0, d_3(U_1^n z_1z_2^2z_5) = 0, d_3(U_1^n z_1^4z_2z_4) = 0,$ $d_3(U_1^n y_7z_3) = 0, d_3(U_1^n z_1^5z_5) = 0, d_3(U_1^n z_2^3z_4) = 0, d_3(U_1^n z_1z_3^3) = 0,$ $d_3(U_1^n z_2^2z_3^2) = 0, d_3(U_1^n z_1^4z_3^2) = 0, d_3(U_1^n z_1^2z_2^4) = 0, d_3(U_1^n z_1^7z_3) = 0,$ $d_3(U_1^n z_1^6z_2^2) = 0, d_3(U_1^n y_9z_1) = 0, d_3(U_1^n z_6z_4) = 0, d_3(U_1^n y_4^2z_1^2) = 0,$ $d_3(U_1^n z_1^{10}) = 0, d_3(U_1^n y_7z_1^3) = 0, d_3(U_1^n z_6z_2z_1^2) = 0.$
(11)	$d_3(U_1^n z_1^3z_2^2y_4) = d_3(U_1^n z_1^4z_3y_4) = U_1^{n+2}z_1^2z_2^2(U_1y_4 + U_2z_3), d_3(U_1^n z_1^5y_6)$ $= U_1^{n+1}z_1^4(U_1y_6 + U_2z_5), d_3(U_1^n z_1^7y_4) = U_1^{n+1}z_1^6(U_1y_4 + U_2z_3), d_3(U_1^n z_1$ $\times z_3^2y_4) = d_3(U_1^n z_2^2z_3y_4) = U_1^{n+1}z_3^2(U_1y_4 + U_2z_3), d_3(U_1^n y_{10}z_1) = U_1^{n+2}$ $\times(U_1y_{10} + U_3z_7), d_3(U_1^n z_1^2z_3y_6) = d_3(U_1^n z_1z_2^2y_6) = U_1^{n+2}z_1z_3(U_1y_6$ $+U_2z_5), d_3(U_1^n z_1^2z_9) = d_3(U_1^n z_1z_2z_8) = d_3(U_1^n z_1^2y_9) = d_3(U_1^n z_1z_4z_6)$ $= U_1^{n+3}z_1z_9, d_3(U_1^n z_1^2y_4z_5) = U_1^{n+2}z_1z_5(U_1y_4 + U_2z_3), d_3(U_1^n z_7z_1^4)$ $= d_3(U_1^n z_4^2z_1^3) = U_1^{n+3}z_1^3z_7, d_3(U_1^n z_1^{11}) = U_1^{n+2}z_1^{10}, d_3(U_1^n z_1^2z_2z_3y_4)$ $= d_3(U_1^n z_1z_2^3y_4) = d_3(U_1^n z_1z_2^2z_3^2) = d_3(U_1^n z_1^2z_3^3) = d_3(U_1^n z_2^4z_3) = U_1^{n+3}$ $\times z_2^2z_3^2, d_3(U_1^n y_4^2z_1^3) = U_1^{n+3}z_1^2y_4^2, d_3(U_1^n y_8z_2z_1) = d_3(U_1^n y_6z_4z_1) = d_3(U_1^n$ $\times z_7z_3z_1) = d_3(U_1^n z_5^2z_1) = d_3(U_1^n z_5z_4z_2) = d_3(U_1^n z_7z_2^2) = d_3(U_1^n z_4^2z_3) =$ $= U_1^{n+3}z_3z_7, d_3(U_1^n z_1z_2^3z_4) = d_3(U_1^n z_1^3z_4y_4) = d_3(U_1^n z_1^2z_2z_3z_4) = d_3(U_1^n$ $\times z_1^2z_2^2z_5) = d_3(U_1^n z_1^3z_2y_6) = d_3(U_1^n z_1^3z_3z_5) = U_1^{n+3}z_2^3z_4, d_3(U_1^n z_1^5z_2z_4)$ $= d_3(U_1^n z_1^6z_5) = U_1^{n+3}z_1^5z_5, d_3(U_1^n y_{10}^*z_1) = U_1^{n+2}(U_1y_{10}^* + U_2(z_9 + y_9)$ $+U_3z_7), d_3(U_1^n z_1^5z_3^2) = d_3(U_1^n z_1^3z_2^4) = d_3(U_1^n z_1^4z_2^2z_3) = d_3(U_1^n z_1^5z_2y_4)$ $= U_1^{n+3}z_1^4z_3^2, d_3(U_1^n z_1^8z_3) = d_3(U_1^n z_1^7z_2^2) = U_1^{n+1}U_2^2z_1^6z_2^2, d_3(U_1^n z_6y_4z_1)$ $= d_3(U_1^n z_6z_3z_2) = d_3(U_1^n y_7z_2^2) = d_3(U_1^n y_7z_3z_1) = U_1^{n+3}z_3y_7, d_3(U_1^n y_7z_1^4)$ $= d_3(U_1^n z_6z_2z_1^3) = U_1^{n+3}z_6z_2z_1^2, d_3(U_1^n y_4y_6z_1) = U_1^{n+2}[U_1y_4y_6 + U_2(z_3y_6$

Table 17 continued

$+z_5y_4)], d_3(U_1^n z_3^2 z_5) = d_3(U_1^n z_2 z_3 y_6) = d_3(U_1^n z_2 z_5 y_4) = d_3(U_1^n z_3 z_4 y_4)$
$= U_1^{n+1}U_2^2 z_3 z_5, d_3(U_1^n y_4^2 z_3) = U_1^{n+1}U_2^2 y_4^2, d_3(U_1^n y_{11}) = U_1^{n+1}U_2\Phi_5,$
$d_3(U_1^n z_7 y_4) = U_1^n U_3^2(U_1y_4+U_2z_3), d_3(U_1^n y_7 y_4) = U_1^n U_2\Phi_3(U_1y_4+U_2z_3),$
$d_3(U_1^n z_3 y_8) = U_1^n U_2^2(U_1y_8+U_2z_7), d_3(U_1^n y_6 z_5) = U_1^n U_2 U_3(U_1y_6+U_2z_5),$
$d_3(U_1^n z_{11}) = U_1^{n+1}U_3U_4, d_3(U_1^n\Phi_3 y_6) = U_1^nU_2^2U_3\Phi_3, d_3(U_1^nU_2y_6y_4) = U_1^n$
$= \times U_2^3(U_2y_6+U_3y_4), d_3(U_1^nU_3y_8) = d_3(U_1^nU_2y_{10}) = U_1^nU_2U_3^3, d_3(U_1^nU_2$
$\times y_{10}^*) = U_1^nU_2(U_2^2U_4+U_2U_3\Phi_3+U_3^3), d_3(U_1^nU_4y_4) = U_1^nU_2^3U_4, d_3(U_1^n\Phi_3$
$\times z_6) = U_1^{n+3}\Phi_3^2, d_3(U_1^nz_1^3y_8) = U_1^nz_1^2(U_1y_8+U_2z_7), d_3(U_1^nU_2z_3^2y_4) = U_1^n$
$\times U_2^4z_3^2, d_3(U_1^nU_2^2z_3y_6) = U_1^nU_2^4(U_1y_6+U_2z_5), d_3(U_1^nU_2^2z_9) = U_1^{n+1}U_2^3U_4,$
$d_3(U_1^nU_2^2z_3^3) = U_1^{n+1}U_2^4z_3^2, d_3(U_1^nU_2U_3z_7) = U_1^{n+1}U_2U_3^3, d_3(U_1^nU_2^2z_5y_4)$
$= U_1^nU_2^3U_3(U_1y_4+U_2z_3), d_3(U_1^nU_2^2y_9) = U_1^{n+1}U_2^2U_3\Phi_3, d_3(U_1^nU_2^2U_3y_6)$
$= d_3(U_1^nU_2^3y_8) = d_3(U_1^nU_2U_3^2y_4) = U_1^nU_2^4U_3^2, d_3(U_1^nU_2^2\Phi_3y_4) = U_1^nU_2^5\Phi_3,$
$d_3(U_1^nz_1^3z_2^2z_4) = 0, d_3(U_1^nU_2^4z_3y_4) = U_1^nU_2^6(U_1y_4+U_2z_3), d_3(U_1^nU_2^4z_7)$
$= U_1^{n+1}U_2^4U_3^2, d_3(U_1^nz_1z_2^2z_6) = 0, d_3(U_1^nz_1^3z_2z_3^2) = 0, d_3(U_1^nU_2^4y_7) = U_1^{n+1}$
$\times U_2^5\Phi_3, d_3(U_1^nU_2^5y_6) = d_3(U_1^nU_2^4U_3y_4) = U_1^nU_2^7U_3, d_3(U_1^nU_2^6z_5) = U_1^{n+1}$
$\times U_2^7U_3, d_3(U_1^nz_1^7z_4) = 0, d_3(U_1^nz_1^3z_8) = 0, d_3(U_1^nz_6z_1^5) = 0, d_3(U_1^nz_1^5z_2^3)$
$= d_3(U_1^nz_1^2z_2^3z_3) = 0, d_3(U_1^nz_1^4z_3z_4) = 0, d_3(U_1^nz_1^4z_2z_5) = 0, d_3(U_1^nz_1z_2^5)$
$= d_3(U_1^nz_1^2z_2y_7) = d_3(U_1^nz_1^2z_3z_6) = 0, d_3(U_1^nz_1^9z_2) = 0, d_3(U_1^nz_1^6z_2z_3) = 0,$
$d_3(U_1^nz_1z_2z_3z_5) = 0, d_3(U_1^nz_1z_3^2z_4) = 0, d_3(U_1^nz_2^3z_5) = 0, d_3(U_1^nz_2^2z_3z_4) = 0,$
$d_3(U_1^nz_1z_2y_4^2) = 0, d_3(U_1^nz_1z_2z_4^2) = 0, d_3(U_1^nz_4z_7) = 0, d_3(U_1^nz_2y_9) = 0,$
$d_3(U_1^nz_2z_3^3) = d_3(U_1^nz_2z_9) = d_3(U_1^nz_3z_8) = d_3(U_1^nz_4y_7) = d_3(U_1^nz_5z_6) = 0,$
$d_3(U_1^nz_1^2z_2z_7) = 0, d_3(U_1^nz_1^2z_4z_5) = 0, d_3(U_1^nz_{10}z_1) = 0, d_3(U_1^nU_3y_2^2) = 0,$
$d_3(U_1^nU_2^3y_4^2) = 0, d_3(U_1^nU_2z_5^2) = 0, d_3(U_1^n\Phi_3z_3^2) = 0, d_3(U_1^nU_2^3z_3z_5) = 0,$
$d_3(U_1^n\Phi_6) = 0, d_3(U_1^nU_2^2\Phi_5) = 0, d_3(U_1^nU_2\Phi_3^2) = 0, d_3(U_1^nU_2U_3U_4) = 0,$
$d_3(U_1^nU_3^2\Phi_3) = 0, d_3(U_1^nU_2^5z_3^2) = 0, d_3(U_1^nU_2^4U_4) = 0, d_3(U_1^nU_2^3U_3\Phi_3) = 0,$
$d_3(U_1^nU_2^2U_3^3) = 0.$

(12) $d_3(U_1^nz_1^4y_8) = d_3(U_1^nz_1^3z_2z_7) = d_3(U_1^nz_1^3z_4z_5) = d_3(U_1^nz_1^2z_2z_4^2) = U_1^{n+3}z_1$
$\times z_2z_4^2, d_3(U_1^nz_{10}z_1^2) = U_1^{n+3}z_{10}z_1, d_3(U_1^nz_1^2z_2y_8) = U_1^{n+2}z_1z_2(U_1y_8+U_2z_7)$
$d_3(U_1^nz_1^2z_4y_6) = U_1^{n+2}z_1z_4(U_1y_6+U_2z_5), d_3(U_1^nz_1^4z_4y_4) = U_1^{n+2}z_1^3z_4(U_1y_4$
$+U_2z_3), d_3(U_1^nz_1^4z_2y_6) = U_1^{n+2}z_1^3z_2(U_1y_6+U_2z_5), d_3(U_1^nz_1^4z_8) = U_1^{n+3}$
$\times z_1^4z_8, d_3(U_1^nz_1^3z_2z_3y_4) = d_3(U_1^nz_1^2z_2^3y_4) = U_1^{n+2}z_1z_2^3(U_1y_4+U_2z_3),$
$d_3(U_1^nz_1^6z_2y_4) = U_1^{n+2}z_1^5z_2(U_1y_4+U_2z_3), d_3(U_1^nz_1^6z_6) = U_1^{n+3}z_1^5z_6, d_3(U_1^n$
$\times z_1^2y_4z_6) = U_1^{n+2}z_1^2z_6(U_1y_4+U_2z_3), d_3(U_1^nz_1^2y_4y_6) = U_1^{n+2}z_2^2(U_2y_6+U_3y_4),$
$d_3(U_1^nz_1z_2z_3y_6) = d_3(U_1^nz_2^3y_6) = U_1^{n+2}z_2z_3(U_1y_6+U_2z_5), d_3(U_1^nz_{12})$
$= U_1^{n+2}\Phi_6, d_3(U_1^nz_1z_2z_5y_4) = d_3(U_1^nz_1z_3z_4y_4) = d_3(U_1^nz_2^2z_4y_4) = U_1^{n+2}$
$\times z_2z_5(U_1y_4+U_2z_3), d_3(U_1^nz_1^3z_2y_7) = d_3(U_1^nz_1^3z_3z_6) = d_3(U_1^nz_1^2z_2^2z_6)$
$= U_1^{n+3}z_1^2z_2y_7, d_3(U_1^nz_1^4z_2^2y_4) = d_3(U_1^nz_1^5z_3y_4) = U_1^{n+3}z_1z_2^5, d_3(U_1^nz_1^2z_2^5)$
$= d_3(U_1^nz_1^3z_2^3z_3) = d_3(U_1^nz_1^4z_2z_3^2) = U_1^{n+3}z_1^3z_2z_3^2, d_3(U_1^nz_1^4z_2^2z_4) = d_3(U_1^nz_1^5z_3$

Table 17 continued

$\times z_4) = d_3(U_1^n z_1^5 z_2 z_5) = d_3(U_1^n z_1^6 y_6) = U_1^{n+3} z_1^3 z_2^2 z_4,\ d_3(U_1^n z_1^8 z_4) = U_1^{n+3} z_1^7 z_4,$
$d_3(U_1^n z_1 z_4 z_7) = d_3(U_1^n z_1^2 y_{10}) = d_3(U_1^n z_4^3) = U_1^{n+3} z_4 z_7,\ d_3(U_1^n z_1^{10} z_2) = U_1^{n+3}$
$\times z_1^9 z_2, d_3(U_1^n z_1^7 z_2 z_3) = d_3(U_1^n z_1^6 z_2^3) = U_1^{n+3} z_1^5 z_2^3, d_3(U_1^n z_2^4 z_4) = d_3(U_1^n z_1^2 z_2 z_4$
$\times y_4) = d_3(U_1^n z_1^2 z_2 z_3 z_5) = d_3(U_1^n z_1^3 z_5 y_4) = d_3(U_1^n z_1^3 z_3 y_6) = d_3(U_1^n z_1^2 z_2^2 y_6)$
$= d_3(U_1^n z_1^2 z_2^3 z_4) = d_3(U_1^n z_1 z_2^3 z_5) = d_3(U_1^n z_1 z_2^2 z_3 z_4) = U_1^{n+3} z_5 z_2^3, d_3(U_1^n z_1^2 z_2$
$\times y_4^2) = U_1^{n+3} z_1 z_2 y_4^2, d_3(U_1^n z_1 z_2 z_3^3) = d_3(U_1^n z_2^4 y_4) = d_3(U_1^n z_1^2 z_3^2 y_4) = d_3(U_1^n$
$\times z_2^3 z_3^2) = d_3(U_1^n z_1 z_2^2 z_3 y_4) = U_1^{n+3} z_2 z_3^3,\ d_3(U_1^n z_1 z_3 z_8) = d_3(U_1^n z_1 z_2 z_9)$
$= d_3(U_1^n z_2^2 z_8) = U_1^{n+3} z_3 z_8,\ d_3(U_1^n z_2 z_4 z_6) = d_3(U_1^n z_1 z_5 z_6) = d_3(U_1^n z_1 z_4 y_7)$
$= d_3(U_1^n z_1 z_2 y_9) = U_1^{n+3} z_5 z_6, d_3(U_1^n z_1^2 y_{10}^*) = U_1^{n+3}(z_4 z_7 + z_4 y_7 + z_1 z_9), d_3(U_1^n$
$\times y_4 z_8) = U_1^{n+1} U_4(U_1 y_4 + U_2 z_3), d_3(U_1^n z_2 y_4 y_6) = U_1^{n+1} U_3(U_1 y_4 y_6 + U_2 z_3 y_6$
$+U_2 z_5 y_4), d_3(U_1^n z_2 y_{10}) = U_1^{n+1} U_2(U_1 y_{10} + U_3 z_7), d_3(U_1^n z_4 y_8) = U_1^{n+1} U_3(U_1$
$\times y_8 + U_2 z_7), d_3(U_1^n z_2 z_3^2 y_4) = U_1^{n+1} U_2 z_3^2(U_1 y_4 + U_2 z_3), d_3(U_1^n z_6 y_6) = U_1^{n+1}$
$\times \Phi_3(U_1 y_6 + U_2 z_5),\ d_3(U_1^n z_2 y_{10}^*) = U_1^{n+1} U_2(U_1 y_{10}^* + U_2 z_9 + U_2 y_9 + U_3 z_7),$
$d_3(U_1^n z_2 z_3 y_7) = d_3(U_1^n z_3^2 z_6) = d_3(U_1^n z_1 y_4 y_7) = d_3(U_1^n z_2 y_4 z_6) = U_1^{n+2} U_2 z_3 y_7,$
$d_3(U_1^n z_4 y_4^2) = U_1^{n+2} U_4 y_4^2,\ d_3(U_1^n z_3 z_4 z_5) = d_3(U_1^n z_1 z_7 y_4) = d_3(U_1^n z_2 z_3 z_7)$
$= d_3(U_1^n z_4^2 y_4) = d_3(U_1^n z_2 z_5^2) = d_3(U_1^n z_1 z_5 y_6) = d_3(U_1^n z_2 z_4 y_6) = d_3(U_1^n z_2^2 y_8)$
$= d_3(U_1^n z_1 z_3 y_8) = U_1^{n+2} U_2 z_5^2, d_3(U_1^n y_4 y_8) = U_1^n U_2(U_2^2 y_8 + U_3^2 y_4), d_3(U_1^n z_3^2 y_6)$
$= d_3(U_1^n z_3 z_5 y_4) = U_1^n U_2^3 z_3 z_5,\ d_3(U_1^n y_4^3) = U_1^n U_2^3 y_4^2, d_3(U_1^n y_{12}) = U_1^n(U_3^2 \Phi_3$
$+U_2 U_3 U_4), d_3(U_1^n U_3 y_9) = U_1^{n+1} U_3^2 \Phi_3, d_3(U_1^n U_2 z_5 y_6) = U_1^n U_2^2 U_3(U_1 y_6 + U_2 z_5),$
$d_3(U_1^n U_2 y_4 z_7) = U_1^n U_2 U_3^2(U_1 y_4 + U_2 z_3), d_3(U_1^n U_2 z_3 y_8) = U_1^n U_2^3(U_1 y_8 + U_2 z_7),$
$d_3(U_1^n U_2 y_4 y_7) = U_1^n U_2 \Phi_3(U_1 y_4 + U_2 z_3), d_3(U_1^n U_2 z_{11}) = U_1^{n+1} U_2 U_3 U_4, d_3(U_1^n$
$\times U_2 y_{11}) = U_1^{n+1} U_2^2 \Phi_5, d_3(U_1^n \Phi_3 y_7) = U_1^{n+1} U_2 \Phi_3^2,\ d_3(U_1^n U_2^2 y_4 z_3^2) = U_1^n U_2^5 z_3^2,$
$d_3(U_1^n U_2^2 y_4 y_6) = U_1^n U_2^4(U_2 y_6 + U_3 y_4), d_3(U_1^n U_2^2 y_{10}^*) = U_1^n U_2^2(U_2^2 U_4 + U_2 U_3 \Phi_3$
$+U_3^3),\ d_3(U_1^n U_2 U_3 y_8) = d_3(U_1^n U_2^2 y_{10}) = d_3(U_1^n U_3^2 y_6) = U_1^n U_2^2 U_3^3,\ d_3(U_1^n U_2$
$\times \Phi_3 y_6) = d_3(U_1^n U_3 \Phi_3 y_4) = U_1^n U_2^3 U_3 \Phi_3, d_3(U_1^n U_2 U_4 y_4) = U_1^n U_2^4 U_4, d_3(U_1^n U_2^3$
$\times z_3^3) = U_1^{n+1} U_2^5 z_3^2, d_3(U_1^n U_2^2 U_4 z_3) = U_1^{n+1} U_2^4 U_4, d_3(U_1^n U_2 U_3 \Phi_3 z_3) = d_3(U_1^n$
$\times U_2^3 y_9) = U_1^{n+1} U_2^3 U_3 \Phi_3, d_3(U_1^n U_3^3 z_3) = d_3(U_1^n U_2^2 U_3 z_7) = U_1^{n+1} U_2^2 U_3^3, d_3(U_1^n$
$\times z_1^2 z_2 z_8) = 0,\ d_3(U_1^n z_1^2 z_3 z_7) = 0,\ d_3(U_1^n z_1 z_2 z_4 z_5) = 0,\ d_3(U_1^n z_1 z_2^2 z_7) = 0,$
$d_3(U_1^n z_1^4 z_3 z_5) = 0,\ d_3(U_1^n z_1^3 z_2^2 z_5) = 0,\ d_3(U_1^n z_1^3 z_2 z_3 z_4) = 0,\ d_3(U_1^n z_1 z_3 z_4^2) = 0,$
$d_3(U_1^n z_1^3 z_9) = 0, d_3(U_1^n z_1^2 z_5^2) = 0, d_3(U_1^n z_1^5 z_7) = 0, d_3(U_1^n z_1^4 z_4^2) = 0, d_3(U_1^n$
$\times z_1^7 z_5) = 0, d_3(U_1^n z_1^6 z_3^2) = 0, d_3(U_1^n z_2^2 z_4^2) = 0, d_3(U_1^n z_1^2 z_2^3 z_4) = 0,\ d_3(U_1^n$
$\times z_1^6 z_2 z_4) = 0,\ d_3(U_1^n z_1^5 z_2^2 z_3) = 0,\ d_3(U_1^n z_1^2 z_2^2 z_3^2) = 0,\ d_3(U_1^n z_1 z_2^4 z_3) = 0,$
$d_3(U_1^n z_1^4 z_2^4) = 0, d_3(U_1^n z_2^6) = 0, d_3(U_1^n z_1^3 z_3^3) = 0, d_3(U_1^n z_1^9 z_3) = 0, d_3(U_1^n z_1^2$
$\times z_4 z_6) = 0,\ d_3(U_1^n z_1^4 z_2 z_6) = 0,\ d_3(U_1^n z_1 z_2 z_3 z_6) = 0,\ d_3(U_1^n z_1 z_2^2 y_7) = 0,$
$d_3(U_1^n z_1^2 z_3 y_7) = 0, d_3(U_1^n z_1 z_3^2 z_5) = 0, d_3(U_1^n z_2^2 z_3 z_5) = 0, d_3(U_1^n z_2 z_3^2 z_4) = 0,$
$d_3(U_1^n z_1^8 z_2^2) = 0, d_3(U_1^n z_1^{12}) = 0, d_3(U_1^n z_1^3 y_9) = 0, d_3(U_1^n z_1^4 y_4^2) = 0, d_3(U_1^n$
$\times z_1 z_3 y_4^2) = 0, d_3(U_1^n z_1^5 y_7) = 0, d_3(U_1^n z_2^3 z_6) = 0, d_3(U_1^n z_2^2 y_4^2) = 0, d_3(U_1^n$
$\times z_3^4) = 0,\ d_3(U_1^n z_5 y_7) = 0,\ d_3(U_1^n z_6^2) = 0,\ d_3(U_1^n y_6^2) = 0,\ d_3(U_1^n z_5 z_7) = 0,$

Table 17 continued

	$d_3(U_1^n U_2^2 z_3 y_7) = 0, d_3(U_1^n z_1 y_{11}) = 0, d_3(U_1^n z_2 z_{10}) = 0, d_3(U_1^n U_2 U_3 y_4^2) = 0,$
	$d_3(U_1^n z_4 z_8) = 0, d_3(U_1^n z_3 z_9) = 0, d_3(U_1^n \Omega_1) = 0, d_3(U_1^n U_2 \Phi_6) = 0, d_3(U_1^n$
	$\times U_3 \Phi_5) = 0, d_3(U_1^n U_4 \Phi_3) = 0, d_3(U_1^n U_2^2 z_5^2) = 0, d_3(U_1^n z_3 y_9) = 0, d_3(U_1^n$
	$\times z_1 z_{11}) = 0.$
(13)	$d_3(U_1^n z_{13}) = U_1^{n+2}\Omega_1,\ d_3(U_1^n y_{13}) = U_1^{n+1}\Phi_3 U_4,\ d_3(U_1^n y_{13}^*) = U_1^{n+1} U_2 \Phi_6,$
	$d_3(U_1^n z_1 y_4 z_8) = U_1^{n+1} z_3 z_9, d_3(U_1^n z_1^5 y_8) = U_1^{n+2} z_1^4 (U_1 y_8 + U_2 z_7), d_3(U_1^n z_1^3$
	$\times z_2 y_8) = d_3(U_1^n z_1^3 z_4 y_6) = U_1^{n+3} z_2^2 z_4^2,\ d_3(U_1^n z_1^5 z_4 y_4) = d_3(U_1^n z_1^5 z_2 y_6)$
	$= U_1^{n+3} z_1^2 z_2^3 z_4, d_3(U_1^n z_1 z_2^6) = d_3(U_1^n z_1^4 z_3^3) = d_3(U_1^n z_1^3 z_2^3 y_4) = d_3(U_1^n z_1^3 z_2^2$
	$\times z_3^2) = d_3(U_1^n z_1^4 z_2 z_3 y_4) = d_3(U_1^n z_1^2 z_2^4 z_3) = U_1^{n+3} z_2^6, d_3(U_1^n z_1^2 z_2 z_3 y_6)$
	$= d_3(U_1^n z_1 z_2^3 y_6) = d_3(U_1^n z_1^2 z_2 z_5 y_4) = d_3(U_1^n z_1^2 z_3 z_4 y_4) = U_1^{n+3} z_2^2 z_3 z_5,$
	$d_3(U_1^n z_1^7 y_6) = U_1^{n+2} z_1^6 (U_1 y_6 + U_2 z_5), d_3(U_1^n z_1^3 y_4 z_6) = U_1^{n+3} z_2^3 z_6, d_3(U_1^n z_1^7$
	$\times z_2 y_4) = U_1^{n+3} z_1^4 z_2^4, d_3(U_1^n z_1^3 y_4 y_6) = U_1^{n+2} z_1^2 (U_1 y_6 y_4 + U_2 z_3 y_6 + U_2 z_5 y_4),$
	$d_3(U_1^n z_1 z_2^2 z_4 y_4) = U_1^{n+3} z_2 z_3^2 z_4, d_3(U_1^n z_1^5 z_2^2 y_4) = d_3(U_1^n z_1^6 z_3 y_4) = U_1^{n+2} z_1^4$
	$\times z_2^2 (U_1 y_4 + U_2 z_3), d_3(U_1^n z_1^4 z_3 y_6) = d_3(U_1^n z_1^3 z_2^2 y_6) = U_1^{n+2} z_1^3 z_3 (U_1 y_6 + U_2$
	$\times z_5), d_3(U_1^n z_1^4 z_5 y_4) = d_3(U_1^n z_1^3 z_2 z_4 y_4) = U_1^{n+2} z_1^3 z_5 (U_1 y_4 + U_2 z_3), d_3(U_1^n$
	$\times z_1^3 y_{10}) = U_1^{n+2} z_1^2 (U_1 y_{10} + U_3 z_7),\ d_3(U_1^n z_1^3 z_2^3 z_4) = U_1^{n+3} z_1^2 z_2^3 z_4,\ d_3(U_1^n$
	$\times z_1 z_2^4 y_4) = d_3(U_1^n z_1^2 z_2^2 z_3 y_4) = d_3(U_1^n z_1^3 z_3^2 y_4) = U_1^{n+2} z_2^4 (U_1 y_4 + U_2 z_3),$
	$d_3(U_1^n z_1^3 y_{10}^*) = U_1^{n+2} z_1^2 (U_1 y_{10}^* + U_2 z_9 + U_2 y_9 + U_3 z_7),\ d_3(U_1^n z_1 z_2 y_4 y_6)$
	$= U_1^{n+2} z_2 z_3 (U_2 y_6 + U_3 y_4), d_3(U_1^n z_1 z_2 z_3^2 y_4) = d_3(U_1^n z_2^3 z_3 y_4) = d_3(U_1^n z_2^2$
	$\times z_3^3) = U_1^{n+3} z_3^4, d_3(U_1^n z_1 z_6 y_6) = U_1^{n+3} z_5 y_7, d_3(U_1^n z_1^2 y_4 y_7) = d_3(U_1^n z_1 z_2$
	$\times y_4 z_6) = U_1^{n+2} z_1 y_7 (U_1 y_4 + U_2 z_3),\ d_3(U_1^n z_1 z_4^2 y_4) = d_3(U_1^n z_1^2 z_7 y_4)$
	$= U_1^{n+2} z_4^2 (U_1 y_4 + U_2 z_3),\ d_3(U_1^n z_1 y_4^3) = U_1^{n+2} y_4^2 (U_1 y_4 + U_2 z_3),\ d_3(U_1^n z_1$
	$\times z_2 y_{10}^*) = U_1^{n+3} (z_3 z_9 + z_5 z_7 + z_5 y_7),\ d_3(U_1^n z_1 z_2 y_{10}) = d_3(U_1^n z_1 z_4 y_8)$
	$= U_1^{n+3} z_5 z_7, d_3(U_1^n z_1^4 y_9) = U_1^{n+3} z_1^2 z_4 z_6, d_3(U_1^n z_1^5 y_4^2) = U_1^{n+3} z_1^4 y_4^2, d_3(U_1^n$
	$\times z_1^6 y_7) = d_3(U_1^n z_1^5 z_2 z_6) = U_1^{n+3} z_1^4 z_2 z_6, d_3(U_1^n z_1^2 z_5 y_6) = d_3(U_1^n z_1 z_2 z_4 y_6)$
	$= U_1^{n+2} z_1 z_5 (U_1 y_6 + U_2 z_5), d_3(U_1^n z_1^2 z_3 y_8) = d_3(U_1^n z_1 z_2^2 y_8) = U_1^{n+2} z_1 z_3 (U_1$
	$\times y_8 + U_2 z_7),\ d_3(U_1^n z_1 y_4 y_8) = U_1^{n+2} (U_1 y_8 y_4 + U_2 z_3 y_8 + U_2 z_7 y_4),\ d_3(U_1^n$
	$\times z_1 z_3 z_5 y_4) = d_3(U_1^n z_1 z_3^2 y_6) = U_1^{n+2} z_3^2 (U_1 y_6 + U_2 z_5),\ d_3(U_1^n z_1^3 z_2 z_8)$
	$= d_3(U_1^n z_1^4 z_9) = U_1^{n+3} z_1^3 z_9, d_3(U_1^n z_1^3 z_3 z_7) = d_3(U_1^n z_1^2 z_2 z_4 z_5) = d_3(U_1^n z_1^2$
	$\times z_2^2 z_7) = d_3(U_1^n z_1^3 z_5^2) = d_3(U_1^n z_1^2 z_3 z_4^2) = d_3(U_1^n z_1 z_2^2 z_4^2) = U_1^{n+3} z_1^2 z_3 z_7,$
	$d_3(U_1^n z_1^6 z_7) = d_3(U_1^n z_1^5 z_4^2) = U_1^{n+3} z_1^5 z_7, d_3(U_1^n z_1 y_{12}) = U_1^{n+2} (U_1 y_{12} + U_2 z_{11}$
	$+ U_3 y_9),\ d_3(U_1^n z_1^5 z_3 z_5) = d_3(U_1^n z_1^4 z_2^2 z_5) = d_3(U_1^n z_1^4 z_2 z_3 z_4) = d_3(U_1^n z_1^3 z_2^3 z_4)$
	$= U_1^{n+3} z_1^4 z_3 z_5, d_3(U_1^n z_1^8 z_5) = d_3(U_1^n z_1^7 z_2 z_4) = U_1^{n+3} z_1^7 z_5, d_3(U_1^n z_1^7 z_3^2) = d_3(U_1^n$
	$\times z_1^6 z_2^2 z_3) = d_3(U_1^n z_1^5 z_2^4) = U_1^{n+3} z_1^6 z_3^2, d_3(U_1^n z_1^{10} z_3) = d_3(U_1^n z_1^9 z_2^2) = U_1^{n+3} z_1^9 z_3,$
	$d_3(U_1^n z_1^{13}) = U_1^{n+3} z_1^{12}, d_3(U_1^n z_1^3 z_4 z_6) = U_1^{n+3} z_1^2 z_4 z_6, d_3(U_1^n z_2^4 z_5) = d_3(U_1^n z_2^3 z_3$
	$\times z_4) = U_1^{n+3} z_2^2 z_3 z_5,\ d_3(U_1^n z_2^2 z_3 y_6) = U_1^{n+2} z_3^2 (U_1 y_6 + U_2 z_5),\ d_3(U_1^n z_2 z_4 z_7)$
	$= U_1^{n+3} z_5 z_7,\ d_3(U_1^n z_2^2 y_9) = d_3(U_1^n z_4 y_7 z_2) = U_1^{n+3} z_3 y_9,\ d_3(U_1^n z_2 z_3 z_4 y_4)$
	$= d_3(U_1^n z_2^2 z_5 y_4) = U_1^{n+2} z_3 z_5 (U_1 y_6 + U_2 z_5),\ d_3(U_1^n z_2 z_5 y_6) = d_3(U_1^n z_2 z_7 y_4)$

Table 17 continued

$= d_3(U_1^n z_2 y_8 z_3) = d_3(U_1^n z_3 z_4 y_6) = U_1^{n+1} U_2^2 z_5^2, d_3(U_1^n z_2 y_7 y_4) = U_1^{n+1} U_2^2 z_3 y_7,$
$d_3(U_1^n z_2 z_5 z_6) = U_1^{n+3} z_5 y_7, d_3(U_1^n z_2^2 z_9) = d_3(U_1^n z_1 z_3 z_9) = U_1^{n+3} z_3 z_9, d_3(U_1^n z_3$
$\times y_{10}) = U_1^n U_2^2(U_1 y_{10} + U_3 z_7), d_3(U_1^n z_3 y_{10}^*) = U_1^n U_2^2(U_1 y_{10}^* + U_2 z_9 + U_2 y_9$
$+U_3 z_7), d_3(U_1^n z_3 y_4 y_6) = U_1^n U_2^2(U_1 y_6 y_4 + U_2 z_5 y_4 + U_2 y_6 z_3), d_3(U_1^n z_3^2 z_7)$
$= d_3(U_1^n z_5^2 z_3) = U_1^n U_2^2 z_3 z_7, d_3(U_1^n z_3^3 y_4) = U_1^n U_2^2 z_3^2(U_1 y_4 + U_2 z_3), d_3(U_1^n y_7 z_3^2)$
$= U_1^n U_2^2 y_7 z_3, d_3(U_1^n z_3 z_6 z_4) = U_1^n U_2^2 z_4 z_6, d_3(U_1^n y_4 z_9) = U_1^n U_2 U_4(U_1 y_4 + U_2 z_3),$
$d_3(U_1^n y_9 y_4) = U_1^n U_3 \Phi_3(U_1 y_4 + U_2 z_3), d_3(U_1^n z_4^2 z_5) = U_1^{n+2} U_3 z_4 z_5, d_3(U_1^n y_4^2 z_5)$
$= U_1^{n+1} U_2 U_3 y_4^2, d_3(U_1^n y_4 z_4 z_5) = U_1^{n+1} U_2^2 z_5^2, d_3(U_1^n z_5 y_8) = U_1^n U_2 U_3(U_1 y_8$
$+U_2 z_7), d_3(U_1^n y_6 y_7) = U_1^n U_2 \Phi_3(U_1 y_6 + U_2 z_5), d_3(U_1^n y_6 z_7) = U_1^n U_3^2(U_1 y_6$
$+U_2 z_5), d_3(U_1^n z_1 z_2^2 z_8) = 0, d_3(U_1^n z_1 z_2^4 z_4) = 0, d_3(U_1^n z_1^9 z_4) = 0, d_3(U_1^n z_1 z_4^3) = 0,$
$d_3(U_1^n z_1^4 z_2 z_7) = 0, d_3(U_1^n z_1^4 z_4 z_5) = 0, d_3(U_1^n z_1^3 z_2 z_4^2) = 0, d_3(U_1^n z_1^4 z_2 y_7) = 0,$
$d_3(U_1^n z_{10} z_1^3) = 0, d_3(U_1^n z_1^5 z_8) = 0, d_3(U_1^n z_1^7 z_6) = 0, d_3(U_1^n z_1^4 z_3 z_6) = 0, d_3(U_1^n$
$\times z_1^3 z_2^2 z_6) = 0, d_3(U_1^n z_2^5 z_3) = 0, d_3(U_1^n z_1^2 z_4 z_7) = 0, d_3(U_1^n z_1^2 z_2 z_9) = 0, d_3(U_1^n$
$\times z_1^5 z_2 z_3^2) = 0, d_3(U_1^n z_1 z_2 z_4 z_6) = 0, d_3(U_1^n z_1^3 z_2 z_3 z_5) = 0, d_3(U_1^n z_1^2 z_2^2 z_3 z_4) = 0,$
$d_3(U_1^n z_1^2 z_2 y_9) = 0, d_3(U_1^n z_1^3 z_2 y_4^2) = 0, d_3(U_1^n z_1^2 z_2 z_3^3) = 0, d_3(U_1^n z_1^2 z_3 z_8) = 0,$
$d_3(U_1^n z_1^4 z_2^3 z_3) = 0, d_3(U_1^n z_1^8 z_2 z_3) = 0, d_3(U_1^n z_2 y_4^2 z_3) = 0, d_3(U_1^n z_1^5 z_2^2 z_4) = 0,$
$d_3(U_1^n z_1 z_2 z_3 y_7) = 0, d_3(U_1^n z_1^6 z_3 z_4) = 0, d_3(U_1^n z_1^6 z_2 z_5) = 0, d_3(U_1^n z_1^2 z_2^3 z_5) = 0,$
$d_3(U_1^n z_1^9 z_4) = 0, d_3(U_1^n z_1^{11} z_2) = 0, d_3(U_1^n z_1^3 z_2^5) = 0, d_3(U_1^n z_1^7 z_2^3) = 0, d_3(U_1^n z_5$
$\times z_8) = 0, d_3(U_1^n z_1^2 z_5 z_6) = 0, d_3(U_1^n z_1^2 z_4 y_7) = 0, d_3(U_1^n z_5 z_4 z_2^2) = 0, d_3(U_1^n z_4^2$
$\times z_3 z_2) = 0, d_3(U_1^n z_1 z_3^2 z_6) = 0, d_3(U_1^n z_1 z_4 y_4^2) = 0, d_3(U_1^n z_6 z_7) = 0, d_3(U_1^n z_6$
$\times y_7) = 0, d_3(U_1^n z_1 z_2 z_3 z_7) = 0, d_3(U_1^n z_1 z_3 z_4 z_5) = 0, d_3(U_1^n z_1 z_2 z_5^2) = 0, d_3(U_1^n$
$\times z_3 z_{10}) = 0, d_3(U_1^n z_2 z_{11}) = 0, d_3(U_1^n z_7 z_2^3) = 0, d_3(U_1^n y_7 z_2^3) = 0, d_3(U_1^n z_2 y_{11})$
$= 0, d_3(U_1^n z_6 z_3 z_2^2) = 0, d_3(U_1^n z_3^2 z_5 z_2) = 0, d_3(U_1^n z_1 z_2^3 z_3^2) = 0, d_3(U_1^n z_4 z_9) = 0,$
$d_3(U_1^n y_9 z_4) = 0, d_3(U_1^n z_1 z_{12}) = 0.$

Table 18 The ring $\pi_*(MSp)$ in dimensions from 32 to 52

n	32	
$\pi_n(MSp)$	$22\mathbb{Z}$	
Generators	$z_1^8, 2z_2 z_1^6, z_1^4 z_2^2, 2z_1^4 z_4, z_1^4 y_4 + z_1^2 z_2^3, 2z_1^3 z_2 z_3, z_1^3 z_5, 2z_1^2 z_2 y_4, 2z_1^2 y_6,$	
n	32	
$\pi_n(MSp)$	$22\mathbb{Z}$	
Generators	$z_1^2 z_2 z_4, z_1^2 y_6 + z_1 z_3 z_4, 2z_4 y_4, 2z_2 y_6, z_2^2 y_4 + z_2 z_3^2, z_1 y_7, z_2 z_6, z_3 z_5,$	
n	32	33
$\pi_n(MSp)$	$22\mathbb{Z}$	$12\mathbb{Z}_2$
Generators	$z_4^2, y_4^2, z_1^2 z_3^2, 2z_8, 2y_8$	$\theta_1 z_1^4 z_2^2, \theta_1(z_1^4 y_4 + z_1^2 z_2^3), \theta_1(z_2^2 y_4 + z_2 z_3^2),$
n	33	
$\pi_n(MSp)$	$12\mathbb{Z}_2$	
Generators	$\theta_1(z_1^2 y_6 + z_1 z_3 z_4), \tau_4, \theta_1 z_1 y_7 = \theta_1 z_2 z_6 = \Phi_3 z_1 z_2, \theta_1 z_4^2 = \theta_1 z_1 z_7 =$	

Table 18 continued

n	33		
$\pi_n(MSp)$	$12\mathbb{Z}_2$		
Generators	$= \Phi_2 z_1 z_4,\ \theta_1 z_3 z_5 = \Phi_2 z_2 z_3 = \Phi_1 z_3^2,\ \theta_1 z_1^3 z_5,\ \theta_1 z_1^2 z_3^2,\ \theta_1 y_4^2,\ \theta_1 z_1^8$		
n	34		
$\pi_n(MSp)$	$16\mathbb{Z}_2$		
Generators	$\theta_1^2 z_1^4 z_2^2, \theta_1 \tau_4, \Phi_1^2 z_3^2, \Phi_1 \Phi_4, \Phi_2 \Phi_3, \Phi_1 \tau_3, \theta_1^2 z_1 y_7, \theta_1^2 z_4^2, \theta_1^2 z_3 z_5, \theta_1^2 z_1^3 z_5,$		
n	34		
$\pi_n(MSp)$	$16\mathbb{Z}_2$		
Generators	$\theta_1^2 (z_1^4 y_4 + z_1^2 z_2^3),\ \theta_1^2 (z_2^2 y_4 + z_2 z_3^2),\ \theta_1^2 (z_1^2 y_6 + z_1 z_3 z_4),\ \theta_1^2 y_4^2,\ \theta_1^2 z_1^8,$		
n	34	35	36
$\pi_n(MSp)$	$16\mathbb{Z}_2$	0	$30\mathbb{Z}$
Generators	$\theta_1^2 z_1^2 z_3^2\ 2z_1^9, z_1^7 z_2, 2z_1^5 z_2^2, z_1^5 z_4, 2z_1^5 y_4, 2z_1 z_2^2 y_4, 2z_3^3, z_1^2 z_3 z_4,$		
n	36		
$\pi_n(MSp)$	$30\mathbb{Z}$		
Generators	$z_3^3 + z_2 z_3 y_4, 2z_9, z_2 z_7, 2z_1^2 y_7, 2z_1 z_4^2, z_3 z_6, 2y_4^2 z_1, 2y_6 z_3, z_5 z_4, z_2^3 z_3,$		
n	36		
$\pi_n(MSp)$	$30\mathbb{Z}$		
Generators	$z_2 z_3 z_4 + z_1 z_2 y_6, z_2 z_3 z_4 + z_1 z_4 y_4, 2z_5 z_1^4, 2y_8 z_1, y_7 z_2, z_1^3 z_3^2 + z_1^3 z_2 y_4,$		
n	36	37	
$\pi_n(MSp)$	$30\mathbb{Z}$	$17\mathbb{Z}_2$	
Generators	$2z_1^3 y_6, 2z_5 y_4, 2z_1^3 z_3^2, z_1 z_8, z_1^3 z_6, z_1^3 z_2^3\quad \theta_1 z_1^7 z_2, \theta_1 z_1^5 z_4, \theta_1 z_2^3 z_3, \tau_5, \Phi_5$		
n	37		
$\pi_n(MSp)$	$17\mathbb{Z}_2$		
Generators	$\theta_1 (z_3^3 + z_2 z_3 y_4),\ \theta_1 z_2 z_7 = \theta_1 z_5 z_4 = \Phi_2 z_2 z_4 = \Phi_2 z_5 z_1,\ \theta_1 z_1^2 z_3 z_4,$		
n	37		
$\pi_n(MSp)$	$17\mathbb{Z}_2$		
Generators	$\theta_1 (z_2 z_3 z_4 + z_1 z_2 y_6), \theta_1 (z_2 z_3 z_4 + z_1 z_4 y_4), \theta_1 (z_1^3 z_3^2 + z_1^3 z_2 y_4), \Phi_1 y_4^2,$		
n	37		
$\pi_n(MSp)$	$17\mathbb{Z}_2$		
Generators	$\theta_1 z_1 z_8 = \Phi_4 z_1^2, \theta_1 z_1^3 z_6, \theta_1 z_1^3 z_2^3, \theta_1 y_7 z_2 = \theta_1 z_6 z_3 = \Phi_3 z_1 z_3 = \Phi_3 z_2^2,$		
n	37	38	
$\pi_n(MSp)$	$17\mathbb{Z}_2$	$19\mathbb{Z}_2$	
Generators	$\Phi_1 z_3 z_5 = \Phi_2 z_3^2\ \theta_1^2 z_1^7 z_2, \theta_1^2 z_1^5 z_4, \theta_1^2 z_2^3 z_3, \theta_1^2 (z_3^3 + z_2 z_3 y_4), \Phi_3 \tau_1,$		
n	38		
$\pi_n(MSp)$	$19\mathbb{Z}_2$		
Generators	$\theta_1 \Phi_5, \theta_1^2 z_5 z_4, \theta_1^2 (z_1^3 z_3^2 + z_1^3 z_2 y_4), \theta_1^2 z_1 z_8,\ \theta_1^2 z_1^3 z_6, \theta_1^2 z_1^3 z_2^3, \theta_1 \Phi_1 z_3 z_5,$		
n	38		
$\pi_n(MSp)$	$19\mathbb{Z}_2$		
Generators	$\Phi_1 \tau_4,\ \theta_1^2 y_7 z_2,\ \theta_1^2 (z_2 z_3 z_4 + z_1 z_2 y_6),\ \theta_1^2 (z_2 z_3 z_4 + z_1 z_4 y_4), \theta_1^2 z_1^2 z_3 z_4,$		
n	38	39	40
$\pi_n(MSp)$	$19\mathbb{Z}_2$	$\mathbb{Z}_2$	$42\mathbb{Z}$

Table 18 continued

Generators	$\theta_1\Phi_1 y_4^2, \Phi_2\tau_2\Phi_1^2\Phi_4 = \Phi_1\Phi_2\Phi_3 \quad 2z_1^2z_2y_6, 2z_1^2z_4y_4, 2z_2^3y_4,$	
n	40	
$\pi_n(MSp)$	$42\mathbb{Z}$	
Generators	$z_1^2z_8 + z_1z_4z_5, 2z_2^5, z_1z_2^3z_3 + z_1^2z_2^2y_4, 2z_1^2z_8, z_6z_4, 2z_1^4z_2y_4, 2z_1^2z_2^2z_4,$	
n	40	
$\pi_n(MSp)$	$42\mathbb{Z}$	
Generators	$2y_{10}^*, 2z_2y_8, 2z_2^2y_6, z_1^3z_3z_4 + z_1^4y_6, 2z_1^6z_4, 2z_1^4z_6, 2z_1^4z_2^3, z_1^4z_2^3 + z_1^6y_4,$	
n	40	
$\pi_n(MSp)$	$42\mathbb{Z}$	
Generators	$z_5^2, 2z_1^8z_2, 2z_4y_6, 2y_{10}, z_1z_3y_6 + z_3^2z_4, z_1y_4z_5 + z_3^2z_4, 2z_2^2y_6, 2z_2y_4^2,$	
n	40	
$\pi_n(MSp)$	$42\mathbb{Z}$	
Generators	$z_1y_9, 2y_4y_6, 2y_4z_6, 2z_3^2y_4, z_1z_9, z_1^2z_4^2, z_2^3z_4, z_1^5z_5, z_2^2z_3^2, z_1^2z_2^4, z_1^6z_2^2,$	
n	40	41
$\pi_n(MSp)$	$42\mathbb{Z}$	$23\mathbb{Z}_2$
Generators	$z_1^{10}, z_3y_7, z_1^3y_7, z_1^2y_4^2, z_2z_8 \quad \theta_1(z_1^2z_8 + z_1z_4z_5), \theta_1z_1^2z_4^2, \theta_1z_1^{10},$	
n	41	
$\pi_n(MSp)$	$23\mathbb{Z}_2$	
Generators	$\theta_1(z_1z_2^3z_3 + z_1^2z_2^2y_4), \theta_1(z_1^3z_3z_4 + z_1^4y_6), \theta_1(z_1^4z_2^3 + z_1^6y_4), \theta_1z_1^6z_2^2,$	
n	41	
$\pi_n(MSp)$	$23\mathbb{Z}_2$	
Generators	$\theta_1(z_1y_4z_5 + z_3^2z_4), \theta_1(z_1z_3y_6 + z_3^2z_4), \theta_1z_3y_7 = \Phi_3z_2z_3, \Phi_2z_3z_4 =$	
n	41	
$\pi_n(MSp)$	$23\mathbb{Z}_2$	
Generators	$= \theta_1z_5^2 = \Phi_2z_2z_5, \theta_1z_2^2z_3^2, \theta_1z_1^2z_2^4, \theta_1z_1z_9 = \theta_1z_2z_8 = \Phi_4z_1z_2, \tau_6^*,$	
n	41	
$\pi_n(MSp)$	$23\mathbb{Z}_2$	
Generators	$\theta_1z_1^5z_5, \theta_1z_2^3z_4, \theta_1z_1y_9, \theta_1z_1^3y_7, \theta_1z_1^2y_4^2, \tau_6, \Phi_1(z_3^3 + z_2z_3y_4), \kappa_1$	
n	42	
$\pi_n(MSp)$	$29\mathbb{Z}_2$	
Generators	$\theta_1^2(z_1^3z_3z_4 + z_1^4y_6), \theta_1^2z_1^2z_4^2, \theta_1^2(z_1^2z_8 + z_1z_4z_5), \theta_1^2(z_1z_2^3z_3 + z_1^2z_2^2y_4),$	
n	42	
$\pi_n(MSp)$	$29\mathbb{Z}_2$	
Generators	$\theta_1^2(z_1y_4z_5 + z_3^2z_4), \theta_1^2(z_1z_3y_6 + z_3^2z_4), \theta_1^2z_1^{10}, \theta_1^2z_5^2, \theta_1^2(z_1^4z_2^3 + z_1^6y_4),$	
n	42	
$\pi_n(MSp)$	$29\mathbb{Z}_2$	
Generators	$\theta_1^2z_2^3z_4, \theta_1^2z_3y_7, \theta_1^2z_1^5z_5, \theta_1^2z_2^2z_3^2, \theta_1^2z_1^2z_2^4, \theta_1^2z_1z_9, \theta_1^2z_1^6z_2^2, \Phi_2\Phi_4, \Phi_1\Phi_5,$	
n	42	
$\pi_n(MSp)$	$29\mathbb{Z}_2$	

Table 18 continued

Generators	$\Phi_1^2 y_4^2,\ \theta_1\tau_6^*,\ \theta_1\kappa_1,\ \theta_1\tau_6, \theta_1^2 z_1 y_9, \theta_1^2 z_1^3 y_7, \theta_1^2 z_1^2 y_4^2, \theta_1\Phi_1(z_3^3 + z_2 z_3 y_4),$		
n	42	43	44
$\pi_n(MSp)$	$29\mathbb{Z}_2$	0	$56\mathbb{Z}$
Generators	$\Phi_1^2 z_3 z_5, \Phi_1\tau_5, \Phi_2\tau_3\ 2z_1^3 y_8, 2z_1^3 z_2^2 y_4, 2z_1^5 y_6, 2z_1^7 y_4, 2z_1 y_{10},$		
n	44		
$\pi_n(MSp)$	$56\mathbb{Z}$		
Generators	$2z_2^2 z_3 y_4,\ z_1 z_{10},\ 2z_1 z_2^2 y_6,\ 2z_1^2 z_5 y_4, 2z_1^2 z_9, z_1 z_2 y_8 + z_1 z_4 y_6, 2z_1^3 z_3 z_5,$		
n	44		
$\pi_n(MSp)$	$56\mathbb{Z}$		
Generators	$z_1 z_4 y_6 + z_1 z_5^2, z_5 z_6, 2z_1^3 z_4^2,\ z_1^3 z_4 y_4 + z_1^3 z_2 y_6,\ 2z_1^6 z_5,\ 2z_1^3 z_2^4,\ 2z_1^7 z_2^2,$		
n	44		
$\pi_n(MSp)$	$56\mathbb{Z}$		
Generators	$2z_1^2 y_9,\ z_1^4 z_2^2 z_3 + z_1^5 z_2 y_4,\ z_1^3 z_2 y_6 + z_1^3 z_3 z_5,\ z_1 z_2^3 y_4 + z_1 z_2^2 z_3^2, 2z_2^4 z_3,$		
n	44		
$\pi_n(MSp)$	$56\mathbb{Z}$		
Generators	$2z_1^{11}, z_1 y_4 z_6 + z_2 z_3 z_6, z_4 y_7, z_3 z_4 y_4 + z_2 z_3 y_6, z_2 z_3 y_6 + z_3^2 z_5, 2z_3^2 z_5,$		
n	44		
$\pi_n(MSp)$	$56\mathbb{Z}$		
Generators	$2z_{11}, 2z_1^4 y_7, 2z_1^3 y_4^2, 2z_3 y_4^2, 2z_5 y_6, 2y_4 z_7, 2z_3 y_8, 2y_4 y_7, z_3 z_8, 2z_1 y_6 y_4,$		
n	44		
$\pi_n(MSp)$	$56\mathbb{Z}$		
Generators	$2z_1 y_{10}^*, z_1^2 z_2 y_7, z_1 z_2^5, z_1^3 z_8, z_1^3 z_2^2 z_4, z_1^7 z_4, z_1^5 z_6, z_1^5 z_2^3, z_2^2 z_3 z_4, z_1 z_2 y_4^2,$		
n	44	45	
$\pi_n(MSp)$	$56\mathbb{Z}$	$31\mathbb{Z}_2$	
Generators	$z_1^9 z_2, z_4 z_7, z_2 y_9, z_2 z_9, z_1^2 z_4 z_5, z_2 z_3^3\ \theta_1(z_1 z_2 y_8 + z_1 z_4 y_6), \theta_1 z_1 z_{10},$		
n	45		
$\pi_n(MSp)$	$31\mathbb{Z}_2$		
Generators	$\theta_1(z_1 z_4 y_6 + z_1 z_5^2), \theta_1(z_1^3 z_4 y_4 + z_1^3 z_2 y_6), \theta_1(z_1^3 z_2 y_6 + z_1^3 z_3 z_5), \tau_7, \tau_7^*$		
n	45		
$\pi_n(MSp)$	$31\mathbb{Z}_2$		
Generators	$\theta_1(z_1 z_2^3 y_4 + z_1 z_2^2 z_3^2), \theta_1 z_3 z_8 = \theta_1 z_2 z_9 = \Phi_4 z_1 z_3 = \Phi_4 z_2^2, \theta_1 z_1^2 z_4 z_5,$		
n	45		
$\pi_n(MSp)$	$31\mathbb{Z}_2$		
Generators	$\theta_1(z_3 z_4 y_4 + z_2 z_3 y_6),\ \theta_1(z_1^4 z_2^2 z_3 + z_1^5 z_2 y_4),\ \theta_1(z_1 y_4 z_6 + z_2 z_3 z_6),$		
n	45		
$\pi_n(MSp)$	$31\mathbb{Z}_2$		
Generators	$\theta_1(z_2 z_3 y_6 + z_3^2 z_5), \theta_1 z_1^7 z_4, \theta_1 z_1^5 z_6, \theta_1 z_1^3 z_2^2 z_4, \theta_1 z_1^5 z_2^3, \theta_1 z_1^9 z_2, \theta_1 z_1 z_2^5,$		
n	45		
$\pi_n(MSp)$	$31\mathbb{Z}_2$		

Table 18 continued

Generators $\theta_1 z_1^2 z_2 y_7, \theta_1 z_2^2 z_3 z_4, \theta_1 z_1 z_2 y_4^2, \theta_1 z_4 z_7 = \Phi_2 z_4^2 = \Phi_2 z_1 z_7, \theta_1 z_2 z_3^3, \Phi_6,$		
n	45	
$\pi_n(MSp)$	$31\mathbb{Z}_2$	
Generators $\theta_1 z_2 y_9 = \theta_1 z_4 y_7 = \theta_1 z_5 z_6 = \Phi_3 z_2 z_4 = \Phi_1 z_1 y_9 = \Phi_2 z_1 y_7 = \Phi_2 z_2 z_6,$		
n	45	46
$\pi_n(MSp)$	$31\mathbb{Z}_2$	$36\mathbb{Z}_2$
Generators $\theta_1 z_1^3 z_8, \Phi_2 y_4^2, \Phi_1 z_3 z_7 = \Phi_2 z_3 z_5 = \Phi_1 z_5^2, \Phi_2 z_3 y_7\, \theta_1^2 z_1 z_{10}, \Phi_1 \tau_6,$		
n	46	
$\pi_n(MSp)$	$36\mathbb{Z}_2$	
Generators $\theta_1^2(z_1^3 z_4 y_4 + z_1^3 z_2 y_6), \theta_1^2(z_1 z_2 y_8 + z_1 z_4 y_6), \theta_1^2(z_1 z_4 y_6 + z_1 z_5^2),$		
n	46	
$\pi_n(MSp)$	$36\mathbb{Z}_2$	
Generators $\theta_1^2(z_1^3 z_2 y_6 + z_1^3 z_3 z_5), \theta_1^2(z_1 z_2^3 y_4 + z_1 z_2^2 z_3^2), \theta_1^2 z_2^2 z_3 z_4, \theta_1^2 z_1 z_2 y_4^2,$		
n	46	
$\pi_n(MSp)$	$36\mathbb{Z}_2$	
Generators $\theta_1^2(z_1^4 z_2^2 z_3 + z_1^5 z_2 y_4),\ \theta_1^2(z_1 y_4 z_6 + z_2 z_3 z_6),\ \theta_1^2 z_3 z_8,\ \theta_1^2 z_1^7 z_4,\ \theta_1^2 z_1^5 z_6,$		
n	46	
$\pi_n(MSp)$	$36\mathbb{Z}_2$	
Generators $\theta_1^2 z_1^2 z_2 y_7, \theta_1^2 z_1 z_2^5, \theta_1^2(z_3 z_4 y_4 + z_2 z_3 y_6), \theta_1^2(z_2 z_3 y_6 + z_3^2 z_5), \theta_1^2 z_1^2 z_4 z_5$		
n	46	
$\pi_n(MSp)$	$36\mathbb{Z}_2$	
Generators $\theta_1^2 z_4 z_7,\ \theta_1 \Phi_2 z_3 y_7,\ \theta_1 \Phi_2 y_4^2,\ \theta_1 \Phi_6, \theta_1^2 z_1^3 z_8, \theta_1^2 z_1^3 z_2^2 z_4, \theta_1^2 z_1^5 z_2^3, \theta_1^2 z_1^9 z_2,$		
n	46	
$\pi_n(MSp)$	$36\mathbb{Z}_2$	
Generators $\theta_1 \Phi_1 z_3 z_7, \theta_1^2 z_2 z_3^3, \theta_1^2 z_2 y_9, \Phi_2 \tau_4, \Phi_1 \tau_6^*, \Phi_3 \tau_2, \Phi_4 \tau_1, \Phi_1 \kappa_1, \Phi_1 \tau_1 z_3^2$		
n	47	48
$\pi_n(MSp)$	$3\mathbb{Z}_2$	$77\mathbb{Z}$
Generators $\Phi_1^2 \Phi_5, \Phi_1 \Phi_2 \Phi_4 = \Phi_2^2 \Phi_3, \Phi_1^2 \tau_5 = \Phi_1 \Phi_2 \tau_3\ z_1^3 y_8 + z_1^2 z_2 y_7, 2z_1^2 z_2^5,$		
n	48	
$\pi_n(MSp)$	$77\mathbb{Z}$	
Generators $2z_1^2 z_2 z_4^2, 2z_1^2 z_{10}, 2z_1^2 z_2 y_8, 2z_1^2 z_4 y_6, 2z_1^4 z_4 y_4, 2z_1^4 z_2 y_6, 2z_2^3 y_6, 2z_2^2 z_4 y_4,$		
n	48	
$\pi_n(MSp)$	$77\mathbb{Z}$	
Generators $2z_1^6 z_2 y_4, 2z_1^2 y_4 y_6, 2z_1^3 z_2 y_7, z_1^4 z_2 z_3^2 + z_1^5 z_3 y_4, 2z_1^4 z_8, z_1^6 z_6 + z_1^4 z_2^2 z_4,$		
n	48	
$\pi_n(MSp)$	$77\mathbb{Z}$	
Generators $2z_1^4 z_2^2 z_4, 2z_1^8 z_4, 2z_1^6 z_6, z_1^8 y_4 + z_1^7 z_2 z_3, 2z_1^6 z_2^3, 2z_1^{10} z_2, 2z_1^2 y_4 z_6, 2z_1^2 z_2^5,$		
n	48	
$\pi_n(MSp)$	$77\mathbb{Z}$	

Table 18 continued

Generators	$z_1z_4z_7 + z_1^2y_{10}, z_1^3z_3y_6 + z_2^4z_4, 2z_2^4z_4, z_2^4z_4 + z_1^2z_2z_4y_4, 2z_4^3, 2z_2^3z_3^2,$	
n	48	
$\pi_n(MSp)$	$77\mathbb{Z}$	
Generators	$z_1^2y_{10} + z_1^2y_{10}^* + z_1z_2z_9 + z_1z_2y_9, 2z_2y_4y_6, 2z_2z_3^2y_4, 2z_1z_2y_9, z_2z_{10},$	
n	48	
$\pi_n(MSp)$	$77\mathbb{Z}$	
Generators	$2z_2y_{10}^*, 2z_2y_{10}, 2z_4y_8, 2z_6y_6, 2z_4^2y_4, 2z_4y_4^2, z_2^3z_3^2 + z_2^4y_4, 2z_1^2z_2y_4^2, z_2^6,$	
n	48	
$\pi_n(MSp)$	$77\mathbb{Z}$	
Generators	$2y_4z_8, z_1y_4z_7 + z_3z_4z_5, z_3z_4z_5 + z_1z_5y_6, z_2z_4y_6 + z_2^2y_8, 2z_3^2z_6, 2y_4^3,$	
n	48	
$\pi_n(MSp)$	$77\mathbb{Z}$	
Generators	$2y_4y_8, z_3z_5y_4 + z_3^2y_6, 2z_3^2y_6, z_2z_3y_7 + z_1y_4y_7, 2z_{12}, z_1^3z_9, z_2^2z_4^2, z_1^4z_4^2,$	
n	48	
$\pi_n(MSp)$	$77\mathbb{Z}$	
Generators	$z_1^8z_2^2, z_1^7z_5, z_1^4z_2^4, z_1^3y_9, z_2^3z_6, z_1^5y_7, z_1^{12}, z_1^4y_4^2, z_2z_3^2z_4, z_2^2y_4^2, z_1y_{11}, z_3^4,$	
n	48	49
$\pi_n(MSp)$	$77\mathbb{Z}$	$41\mathbb{Z}_2$
Generators	$z_1^2z_2^3z_4, z_1z_{11}, z_4z_8, z_3z_9, z_3y_9, y_6^2, z_5z_7\kappa_2, \theta_1(z_2z_4y_6 + z_2^2y_8),$	
n	49	
$\pi_n(MSp)$	$41\mathbb{Z}_2$	
Generators	$\theta_1z_1^4z_2^4, \theta_1z_1^8z_2^2, y_4^2\tau_1, \Omega_1, \theta_1y_6^2, \theta_1(z_1^6z_6 + z_1^4z_2^2z_4), \theta_1z_1^3z_9, \theta_1z_2^2z_4^2,$	
n	49	
$\pi_n(MSp)$	$41\mathbb{Z}_2$	
Generators	$\theta_1(z_1^4z_2z_3^2 + z_1^5z_3y_4), \theta_1(z_1^8y_4 + z_1^7z_2z_3), \theta_1z_2^2y_4^2, \theta_1z_1^4z_4^2, \theta_1z_1^2z_2^3z_4,$	
n	49	
$\pi_n(MSp)$	$41\mathbb{Z}_2$	
Generators	$\tau_8, \theta_1z_3^4, \theta_1(z_1^3z_3y_6 + z_2^4z_4), \theta_1(z_2^4z_4 + z_1^2z_2z_4y_4), \theta_1z_2z_3^2z_4, \theta_1z_1^5y_7,$	
n	49	
$\pi_n(MSp)$	$41\mathbb{Z}_2$	
Generators	$\theta_1(z_1z_4z_7 + z_1^2y_{10}), \theta_1(z_2^3z_3^2 + z_2^4y_4), \theta_1z_1^7z_5, \theta_1z_2^6, \theta_1z_1^4y_4^2, \theta_1z_1^3y_9,$	
n	49	
$\pi_n(MSp)$	$41\mathbb{Z}_2$	
Generators	$\theta_1(z_1^2y_{10} + z_1^2y_{10}^* + z_1z_2z_9 + z_1z_2y_9), \theta_1(z_1y_4z_7 + z_3z_4z_5), \theta_1z_2^3z_6,$	
n	49	
$\pi_n(MSp)$	$41\mathbb{Z}_2$	
Generators	$\theta_1(z_3z_4z_5 + z_1z_5y_6), \theta_1(z_3z_5y_4 + z_3^2y_6), \theta_1(z_2z_3y_7 + z_1y_4y_7), \theta_1z_1^{12}$	
n	49	
$\pi_n(MSp)$	$41\mathbb{Z}_2$	

Table 18 continued

Generators	$\theta_1 z_4 z_8 = \theta_1 z_1 z_{11} = \Phi_2 z_4^2 = \Phi_4 z_1 z_4, \Phi_1(z_2 z_3 y_6 + z_3^2 z_5), \theta_1 z_1 y_{11} =$	
n	49	
$\pi_n(MSp)$	$41\mathbb{Z}_2$	
Generators	$= \theta_1 z_2 z_{10} = \Phi_5 z_1 z_2, \theta_1 z_3 z_9 = \Phi_1 z_3 z_8 = \Phi_4 z_2 z_3, \theta_1 z_3 y_9 = \Phi_2 z_3 z_6 =$	
n	49	
$\pi_n(MSp)$	$41\mathbb{Z}_2$	
Generators	$= \Phi_1 z_4 y_7 = \Phi_3 z_3 z_4,\ \Phi_1(z_3 z_4 y_4 + z_3^2 z_5),\ \theta_1 z_5 z_7 = \Phi_1 z_4 z_7 =$	
n	49	50
$\pi_n(MSp)$	$41\mathbb{Z}_2$	$48\mathbb{Z}_2$
Generators	$= \Phi_2 z_2 z_7 = \Phi_2 z_4 z_5\, \theta_1 \kappa_2, \theta_1^2(z_2 z_4 y_6 + z_2^2 y_8), \theta_1^2 z_1^4 z_2^4, \theta_1^2 z_1^8 z_2^2,$	
n	50	
$\pi_n(MSp)$	$48\mathbb{Z}_2$	
Generators	$\theta_1 \tau_8, \theta_1 y_4^2 \tau_1, \theta_1 \Omega_1, \theta_1^2 y_6^2, \theta_1^2(z_1^4 z_2 z_3^2 + z_1^5 z_3 y_4), \theta_1^2(z_1^6 z_6 + z_1^4 z_2^2 z_4),$	
n	50	
$\pi_n(MSp)$	$48\mathbb{Z}_2$	
Generators	$\theta_1^2 z_1^3 z_9, \theta_1^2 z_2^2 z_4^2 \theta_1^2(z_1^8 y_4 + z_1^7 z_2 z_3), \theta_1^2 z_2^2 y_4^2, \theta_1^2 z_3^4, \theta_1^2 z_1^4 z_4^2, \theta_1^2 z_1^2 z_2^3 z_4,$	
n	50	
$\pi_n(MSp)$	$48\mathbb{Z}_2$	
Generators	$\theta_1^2(z_1^3 z_3 y_6 + z_2^4 z_4), \theta_1^2(z_2^4 z_4 + z_1^2 z_2 z_4 y_4), \theta_1^2 z_2 z_3^2 z_4, \theta_1^2 z_1^7 z_5, \theta_1^2 z_1^5 y_7,$	
n	50	
$\pi_n(MSp)$	$48\mathbb{Z}_2$	
Generators	$\theta_1^2(z_1 z_4 z_7 + z_1^2 y_{10}), \theta_1^2(z_2^3 z_3^2 + z_2^4 y_4), \theta_1^2(z_1 y_4 z_7 + z_3 z_4 z_5), \theta_1^2 z_1^4 y_4^2,$	
n	50	
$\pi_n(MSp)$	$48\mathbb{Z}_2$	
Generators	$\theta_1^2(z_1^2 y_{10} + z_1^2 y_{10}^* + z_1 z_2 z_9 + z_1 z_2 y_9), \theta_1^2(z_3 z_4 z_5 + z_1 z_5 y_6), \theta_1^2 z_1^3 y_9,$	
n	50	
$\pi_n(MSp)$	$48\mathbb{Z}_2$	
Generators	$\Phi_2 \tau_5, \theta_1^2 z_2^6, \theta_1^2(z_3 z_5 y_4 + z_3^2 y_6), \theta_1^2(z_2 z_3 y_7 + z_1 y_4 y_7), \theta_1^2 z_2^3 z_6, \theta_1^2 z_1^{12},$	
n	50	
$\pi_n(MSp)$	$48\mathbb{Z}_2$	
Generators	$\theta_1^2 z_1 z_{11}, \theta_1 \Phi_1(z_2 z_3 y_6 + z_3^2 z_5), \theta_1 \Phi_1(z_3 z_4 y_4 + z_3^2 z_5), \theta_1^2 z_3 y_9, \Phi_1 \tau_7,$	
n	50	51
$\pi_n(MSp)$	$48\mathbb{Z}_2$	0
Generators	$\theta_1^2 z_1 y_{11}, \theta_1^2 z_3 z_9, \theta_1^2 z_5 z_7, \Phi_1 \tau_7^*, \Phi_3 \tau_3, \Phi_1 \Phi_6, \Phi_2 \Phi_5, \Phi_1 \Phi_2 y_4^2$	

Remark In Table 18 we have used the following notations:

$$\tau_1 = U_1 y_4 + U_2 z_3, \tau_2 = U_1 y_6 + U_3 z_3, \tau_3 = U_2 y_6 + U_3 y_4, \tau_4 = U_1 y_8 + U_2 z_7, \tau_5 = U_2 y_8 + U_3 y_6, \tau_6 = U_1 y_{10} + U_3 z_7, \tau_6^* = U_1(y_{10}^* + y_{10}) + U_2(z_9 + y_9), \chi_1 = U_1 y_6 y_4 + U_2(y_6 z_3 + y_4 z_5), \tau_7 = U_2 y_{10} + U_3 y_8, \tau_8 = U_1 y_{12} + U_3(z_9 + y_9), \tau_7^* = U_2(y_{10}^* + y_{10}) + \Phi_3 y_6 + U_4 y_4, \chi_2 = U_1 y_8 y_4 + U_2(y_8 z_3 + y_4 z_7).$$

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