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Physics analysis with Leibniz's differential operators d^n

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Abstract

We introduce a systematic approach to represent Leibniz's nth-order differential operator d^n as the ratio of an infinite product of infinitesimal difference operators to an infinitesimal parameter. Because every difference operator can be expressed as a difference of two shift operators that translate the argument of a function by finite amounts, Leibniz's differential operator d^n is eventually expressed as the infinite product of infinitesimal binomial operators consisting of the shift operators. We apply this strategy to demonstrate the derivation of the translation or time-evolution operators in quantum mechanics. This fills the logical gap in most textbooks on quantum mechanics that usually omit explicit derivations. Our approach could be employed in general physics or classical mechanics classes with which one can solve the equation of motion without prior knowledge of differential equations.

Keywords Differential equation · Differential operator · Difference operator · Translation · Time evolution

1 Introduction

Physics deals with quantitative analyses of the motion of a particle in space. Because of the postulates of the homogeneousness and isotropy of the three-dimensional Euclidean space and the homogeneousness of time, the space–time dependence of a physical quantity such as displacement is well described by an analytic (differentiable) function. Therefore, it is natural that classical mechanics gave birth to various analytic branches of mathematics including calculus and differential equation. Indeed, Sir Isaac Newton and Gottfried Wilhelm Leibniz invented calculus to solve physics problems independently. Leibniz's notation d for the differential operator is particularly useful in performing algebras of infinitesimal changes of mechanical variables. While the dots of Newton and the primes of Joseph–Louis Lagrange are compact notations for derivatives, Leibniz's notation is more convenient for the algebra of infinitesimal mechanical variables. For example, Leibniz's notation greatly simplifies the expressions in applying the change of variables and the chain rule.

Leibniz's differential operator d can be obtained from the corresponding finite *difference operator* Δ in the infinitesimal limit. When being applied to a mechanical variable $f(x)$ that depends on another variable x such as time or displacement, it is convenient to define the difference operator $\Delta_{a,\lambda}$ as

$$\Delta_{a,\lambda} f(x) = f\left[x + \left(\lambda + \tfrac{1}{2}\right)a\right] - f\left[x + \left(\lambda - \tfrac{1}{2}\right)a\right]. \quad (1)$$

Here, a is the parameter that determines the amount of the shift in the parameter x. The free parameter λ can be chosen as an arbitrary finite real number. Simple choices are $\lambda = 0$ for the central difference, $\lambda = \frac{1}{2}$ for the forward difference, and $\lambda = -\frac{1}{2}$ for the backward difference.

To define a higher order difference operator $\Delta^n_{a,\lambda}$ systematically, we introduce a *shift operator* $\mathscr{S}[c]$ that shifts the argument of a function by $+c$ as

$$\mathscr{S}[c] f(x) = f(x + c). \quad (2)$$

Apparently, the multiplication of two shift operators $\mathscr{S}[c]$ and $\mathscr{S}[c']$ is commutative

$$\mathscr{S}[c]\mathscr{S}[c'] f(x) = \mathscr{S}[c']\mathscr{S}[c] f(x) = f(x + c + c'). \quad (3)$$

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The difference operator $\Delta_{a,\lambda}$ can be expressed as a linear combination of the shift operators as

$$\Delta_{a,\lambda} \equiv \mathscr{S}\left[\left(\lambda+\frac{1}{2}\right)a\right] - \mathscr{S}\left[\left(\lambda-\frac{1}{2}\right)a\right]. \tag{4}$$

By making use of the expression in Eq. (4), we determine the second-order difference operator $\Delta^2_{a,\lambda}$ acting on $f(x)$ as

$$\begin{aligned}\Delta^2_{a,\lambda} f(x) =& \left\{\mathscr{S}\left[\left(\lambda+\frac{1}{2}\right)a\right] - \mathscr{S}\left[\left(\lambda-\frac{1}{2}\right)a\right]\right\}^2 f(x)\\ =& \left\{\mathscr{S}^2\left[\left(\lambda+\frac{1}{2}\right)a\right] - 2\mathscr{S}\left[\left(\lambda+\frac{1}{2}\right)a\right]\mathscr{S}\left[\left(\lambda-\frac{1}{2}\right)a\right] + \mathscr{S}^2\left[\left(\lambda-\frac{1}{2}\right)a\right]\right\} f(x)\\ =& f\left[x+2\left(\lambda+\frac{1}{2}\right)a\right] - 2f\left[x+\left(\lambda+\frac{1}{2}\right)a+\left(\lambda-\frac{1}{2}\right)a\right] + f\left[x+2\left(\lambda-\frac{1}{2}\right)a\right].\end{aligned} \tag{5}$$

While we have ignored the λ dependence of the result, one should be careful with the dependence on λ if there is a discontinuity of the derivative at a point. Apparently, the denominator $\mathrm{d}x^n$ on the left-hand side does not mean $d(x^n) = nx^{n-1}\mathrm{d}x$ but means $(\mathrm{d}x)^n = a^n$ in the limit $a \to 0$. The explicit definition of the nth-order derivative of f in Eq. (7) is particularly useful in comparison with the usual recursive definition by applying single derivatives by n times implicitly. For example, we can make use of Eq. (7) in the derivation of the differential equation from a difference equation in the continuum limit at any order.

In this paper, we focus on the employment of Leibniz's notation presented in Eq. (7) based on the difference operator in Eq. (6) to derive the translation operator and time-evolution operator. Those operators are specific representations of the shift operator defined in Eq. (2). In fact, the derivation is equivalent to the proof of the Taylor-series expansions of a single variable function or a multivariable function. The computation of these operators allows us to find the solution to the equation of motion straightforwardly. In general physics, the operator approach is seldom employed mostly because students are not fully acquainted with analytic techniques of calculus and differential equation. However, systematic operator algebra can greatly simplify the analysis and such an operator approach is indeed very useful when students encounter classical mechanics or especially quantum mechanics later. We aim at freshman readers who are in the middle of taking calculus. Only the binomial theorem with integral power and the concept of limit is required to follow the main theme presented in the paper. Although we solve n linear equations, they can be solved step by step recursively. Hence, the reader does not need any prerequisite knowledge on linear algebra or differential equations.

The nth-order difference involves the spectrum of $f(x)$ evaluated in an interval $[x+n(\lambda+\frac{1}{2})a, x+n(\lambda-\frac{1}{2})a]$. This spectrum consists of $(n+1)$ terms $f[x+n(\lambda+\frac{1}{2})a]$, $f[x+(n-1)(\lambda+\frac{1}{2})a+(\lambda-\frac{1}{2})]$, $\cdots$, $f[x+n(\lambda-\frac{1}{2})a]$. As a result, the local behavior (nth-order differences) of a mechanical variable smears out of a point x to a finite vicinity of that fixed point. The coefficients of this linear combination of $(n+1)$ terms are straightforwardly identifiable as the binomial coefficient with the oscillating weighting factor that respects the subtraction mechanism of the difference operator demonstrated in Eqs. (1) and (5). This mechanism originates from the negative sign in definition (4). We apply the binomial expansion to the nth-order difference $\Delta^n_{a,\lambda}$ acting on $f(x)$ to find that

$$\begin{aligned}\Delta^n_{a,\lambda} f(x) =& \left\{\mathscr{S}\left[\left(\lambda+\frac{1}{2}\right)a\right] - \mathscr{S}\left[\left(\lambda-\frac{1}{2}\right)a\right]\right\}^n f(x)\\ =& \sum_{k=0}^{n}(-1)^{n-k}\frac{n!}{k!(n-k)!}\mathscr{S}^k\left[\left(\lambda+\frac{1}{2}\right)a\right]\mathscr{S}^{n-k}\left[\left(\lambda-\frac{1}{2}\right)a\right] f(x)\\ =& \sum_{k=0}^{n}(-1)^{n-k}\frac{n!}{k!(n-k)!} f\left[x+k\cdot\left(\lambda+\frac{1}{2}\right)a+(n-k)\cdot\left(\lambda-\frac{1}{2}\right)a\right].\end{aligned} \tag{6}$$

Note that Eq. (6) is an identity that does not carry any errors. The general expansion formula for the difference operator $\Delta^n_{a,\lambda}$ in Eq. (6) are consistent with those listed on p. 14 of Kelley [1].

By making use of the explicit expression for the nth-order difference operator $\Delta^n_{a,\lambda}$ in Eq. (6) and taking the limit $a \to 0$, we can express the nth-order derivative of f as

$$\begin{aligned}\frac{d^n f(x)}{\mathrm{d}x^n} \equiv& \lim_{a\to 0}\frac{\Delta^n_{a,\lambda} f(x)}{a^n}\\ =& \lim_{a\to 0}\frac{1}{a^n}\sum_{k=0}^{n}(-1)^{n-k}\frac{n!}{k!(n-k)!}\\ &\times f\left[x+k\cdot\left(\lambda+\frac{1}{2}\right)a+(n-k)\cdot\left(\lambda-\frac{1}{2}\right)a\right].\end{aligned} \tag{7}$$

The following is a heuristic example of applying the explicit form of the difference operator $\Delta^2_{a,\lambda}$ defined in Eq. (5). Consider a uniform string of total mass $M = \rho L$, density

ρ, and length L whose both ends are fixed at the same level. We neglect the gravitational force assuming that the uniform tension τ is strong enough. We slice the string into pieces of infinitesimal length $a \to 0^+$ in the region $[x-\frac{1}{2}a, x+\frac{1}{2}a]$ centered at x. To simplify the analysis, we restrict the string to vibrate on the vertical plane with the displacement $\psi(t,x)$ at time t and at the horizontal position x. We define θ_1 and θ_2 as the angles from the horizontal plane to the infinitesimal straight pieces in the domains $[x-\frac{1}{2}a, x]$ and $[x, x+\frac{1}{2}a]$, respectively. Let us consider the small angle limit in which $|\theta_1|, |\theta_2| \ll 1$ satisfying the following approximation:

$$\cos\theta_i \approx 1, \quad \sin\theta_i \approx \tan\theta_i \approx \theta_i, \tag{8}$$

where $i = 1, 2$. In that limit, the horizontal component of the tension is uniform as τ

$$T_1 \cos\theta_1 = T_2 \cos\theta_2 \;\to T_1 \approx T_2 = \tau, \tag{9}$$

where T_1 and T_2 are the tensions exerted on the infinitesimal straight pieces in the domains $[x-\frac{1}{2}a, x]$ and $[x, x+\frac{1}{2}a]$, respectively. The vertical component of the tension is linearly proportional to the vertical displacement relative to the left adjacent piece in the domain $[x-\frac{1}{2}a-a, x-\frac{1}{2}a]$ centered at $x-a$ and the right adjacent piece in the domain $[x+\frac{1}{2}a, x+\frac{1}{2}a+a]$ centered at $x+a$

$$T_1 \sin\theta_1 \approx \tau \tan\theta_1 = \tau\left[\frac{\psi(t,x-a)-\psi(t,x)}{a}\right], \tag{10a}$$

$$T_2 \sin\theta_2 \approx \tau \tan\theta_2 = \tau\left[\frac{\psi(t,x+a)-\psi(t,x)}{a}\right]. \tag{10b}$$

This approximation is valid as long as the deviation from the equilibrium point is small at any $x \in [0, L]$. Then, the mass $dm = \rho a$ of the infinitesimal element in the region $[x-\frac{1}{2}a, x+\frac{1}{2}a]$ centered at x satisfies the equation of motion

$$\begin{aligned}\rho a \frac{\partial^2}{\partial t^2}\psi(t,x)\\ = \tau\left[\frac{\psi(t,x+a)-\psi(t,x)}{a} - \frac{\psi(t,x)-\psi(t,x-a)}{a}\right].\end{aligned} \tag{11}$$

The $\Delta^2_{a,\lambda}$ operator defined in Eq. (5) can be immediately applied to find that equation (11) reduces to the wave equation

$$\begin{aligned}\frac{1}{v^2}\frac{\partial^2}{\partial t^2}\psi(t,x) =& \lim_{a\to 0}\frac{\Delta^2_{a,0}\psi(t,x)}{a^2}\\ =& \frac{\partial^2}{\partial x^2}\psi(t,x),\end{aligned} \tag{12}$$

where $v = \sqrt{\tau/\rho}$ is the speed of the wave. We remark that the derivation from Eqs. (11) to (12) in the continuum limit $a \to 0$ does not bring in any errors according to Eq. (7).

This paper is organized as follows. In Sect. 2, we derive the translation operator $\mathscr{T}(a)$ that transforms a wave function $\psi(x)$ to $\psi(x-a)$ and generalize the method to compute the three-dimensional version $\mathscr{T}(\boldsymbol{a})$ and the time-evolution operator $\mathscr{U}(t'-t)$ that transforms a time-dependent field $\psi(t,\boldsymbol{x})$ to $\psi(t',\boldsymbol{x})$. The operator is employed in Sect. 3 to solve the equation of motion for a particle under a damping force. The application to solving the simple harmonic oscillator is given in Sect. 4 and we conclude in Sect. 5.

2 Translation and time-evolution operators

In quantum mechanics, the translation operator $\mathscr{T}(\boldsymbol{a}) \equiv \mathscr{S}(-\boldsymbol{a})$ is used to compute the wave function $\psi(\boldsymbol{x}-\boldsymbol{a})$ of a particle after a finite amount of *active* translation of the particle in space by a displacement of $+\boldsymbol{a}$ from the original wave function $\psi(\boldsymbol{x})$. Here, $\mathscr{S}(\boldsymbol{a})$ is the three-dimensional generalization of the shift operator defined in Eq. (2). The wave function $\psi(t,\boldsymbol{x})$ at a future time t' can be obtained by applying the time-evolution operator $\mathscr{U}(t'-t)$ to the original wave function $\psi(t,\boldsymbol{x})$ at $t' > t$. In constructing such an operator, the general expression in Eq. (7) for the nth-order derivative is particularly useful.

First, we consider the translation of wave function $\psi(x)$ by $\pm a\hat{\boldsymbol{e}}_1$ in one dimension, where $\hat{\boldsymbol{e}}_1$ is the unit basis vector of the corresponding Cartesian coordinate. If we set $\lambda = \frac{1}{2}$, then $\Delta_{a,\frac{1}{2}}$ represents the difference operator in the forward direction. We can express the value of $\psi(x+a)$ with a finite shift from x to $x+a$ as

$$\psi(x+a) = \psi(x) + [\psi(x+a)-\psi(x)] = \left(\mathbb{1} + \Delta_{a,\frac{1}{2}}\right)\psi(x), \tag{13}$$

where $\mathbb{1}$ is the identity operator. In a similar manner

$$\begin{aligned}\psi(x+a) =& \psi\left(x+\frac{n-1}{n}a\right) + \left[\psi(x+a) - \psi\left(x+\frac{n-1}{n}a\right)\right]\\ =& \left(\mathbb{1} + \Delta_{\frac{a}{n},\frac{1}{2}}\right)\psi\left(x+\frac{n-1}{n}a\right),\end{aligned} \tag{14}$$

where n is an arbitrary natural number. We can pull out the operator $\left(\mathbb{1} + \Delta_{\frac{a}{n},\frac{1}{2}}\right)$ from the wave function in Eq. (14) repeatedly to arrive at

$$\begin{aligned}\psi(x+a) =& \left(\mathbb{1} + \Delta_{\frac{a}{n},\frac{1}{2}}\right)^2\psi\left(x+\frac{n-2}{n}a\right)\\ =& \left(\mathbb{1} + \Delta_{\frac{a}{n},\frac{1}{2}}\right)^k\psi\left(x+\frac{n-k}{n}a\right)\\ =& \left(\mathbb{1} + \Delta_{\frac{a}{n},\frac{1}{2}}\right)^n\psi(x).\end{aligned} \tag{15}$$

According to Eq. (7), we can derive the following identities for any $k = 1$ through n:

$$\Delta^k_{\frac{a}{n},\frac{1}{2}}\psi(x) \rightarrow \left(\frac{a}{n}\frac{\mathrm{d}}{\mathrm{d}x}\right)^k \psi(x), \quad \text{as} \quad n \rightarrow \infty. \tag{16}$$

As a result, in the limit $n \rightarrow \infty$, the expression (15) reduces to

$$\begin{aligned}\psi(x+a) &= \lim_{n\rightarrow\infty}\left(\mathbb{1} + \Delta_{\frac{a}{n},\frac{1}{2}}\right)^n \psi(x)\\ &= \lim_{n\rightarrow\infty}\left(\mathbb{1} + \frac{a}{n}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \psi(x).\end{aligned} \tag{17}$$

The solution in Eq. (17) is crucial to derive the translation operator. By carrying out the binomial expansion, we can express the solution (17) as

$$\begin{aligned}\psi(x+a) &= \lim_{n\rightarrow\infty}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\left(\frac{a}{n}\right)^k\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k\psi(x)\\ &= \lim_{n\rightarrow\infty}\sum_{k=0}^{n}\frac{n!}{n^k(n-k)!}\frac{a^k}{k!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k\psi(x)\\ &= \sum_{k=0}^{\infty}\frac{a^k}{k!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k\psi(x)\\ &= e^{a\frac{\mathrm{d}}{\mathrm{d}x}}\psi(x),\end{aligned} \tag{18}$$

where we have used the identity

$$\lim_{n\rightarrow\infty}\frac{n!}{n^k(n-k)!} = 1. \tag{19}$$

It is worth mentioning that the derivation in Eq. (18) is analogous to the derivation of the infinite-series definition of Eulers' number e from Bernoulli's definition.

As a result, the shift operator defined in Eq. (2) is obtained as

$$\mathscr{S}[a] = e^{a\frac{\mathrm{d}}{\mathrm{d}x}}. \tag{20}$$

Substituting $-a$ for the a in Eq. (18), we find that

$$\psi(x-a) = \mathscr{T}(a)\psi(x), \qquad \mathscr{T}(a\hat{\boldsymbol{e}}_1) = e^{-a\frac{\mathrm{d}}{\mathrm{d}x}} = \mathscr{S}[-a]. \tag{21}$$

Elementary algebraic properties of the one-dimensional translation operator $\mathscr{T}(a)$ can be obtained by making use of Eq. (21) and the commutative property of the differential operator $\frac{\mathrm{d}}{\mathrm{d}x}$ with itself

$$\mathscr{T}(0) = \mathbb{1}, \tag{22a}$$

$$\begin{aligned}\mathscr{T}(a)\mathscr{T}(b) = \mathscr{T}(b)\mathscr{T}(a) &= \lim_{n\rightarrow\infty}\left(\mathbb{1} - \frac{a}{n}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \lim_{n'\rightarrow\infty}\left(\mathbb{1} - \frac{b}{n'}\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n'}\\ &= \lim_{n\rightarrow\infty}\left(\mathbb{1} - \frac{a+b}{n}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n\\ &= \mathscr{T}(a+b),\end{aligned} \tag{22b}$$

$$[\mathscr{T}(a)]^{-1} = \mathscr{T}(-a). \tag{22c}$$

Manifestly, we can employ the result in Eq. (21) to construct the translation operator in three-dimensional Euclidean space

$$\psi(\boldsymbol{x}-\boldsymbol{a}) = \mathscr{T}(\boldsymbol{a})\psi(\boldsymbol{x}), \qquad \boldsymbol{a} = a_1\hat{\boldsymbol{e}}_1 + a_2\hat{\boldsymbol{e}}_2 + a_1\hat{\boldsymbol{e}}_3. \tag{23}$$

By imposing one-dimensional translation operators to each direction, we obtain

$$\begin{aligned}\mathscr{T}(\boldsymbol{a})\psi(\boldsymbol{x}) &= \mathscr{T}(a_3\hat{\boldsymbol{e}}_3)\mathscr{T}(a_2\hat{\boldsymbol{e}}_2)\mathscr{T}(a_1\hat{\boldsymbol{e}}_1)\psi(\boldsymbol{x})\\ &= \mathscr{T}(a_3\hat{\boldsymbol{e}}_3)\mathscr{T}(a_2\hat{\boldsymbol{e}}_2)\psi(\boldsymbol{x}-a_1\hat{\boldsymbol{e}}_1)\\ &= \mathscr{T}(a_3\hat{\boldsymbol{e}}_3)\psi(\boldsymbol{x}-a_1\hat{\boldsymbol{e}}_1-a_2\hat{\boldsymbol{e}}_2)\\ &= \psi(\boldsymbol{x}-\boldsymbol{a}).\end{aligned} \tag{24}$$

Each of the one-dimensional translation operators is defined in terms of the exponentiated partial differential operator

$$\begin{aligned}\mathscr{T}(a_i\hat{\boldsymbol{e}}_i) &= \lim_{n\rightarrow\infty}\left(\mathbb{1} - \frac{a_i}{n}\frac{\partial}{\partial x_i}\right)^n\\ &= \sum_{k=0}^{\infty}\frac{(-a_i)^k}{k!}\frac{\partial^k\psi(x)}{\partial x_i^k}\\ &= e^{-a_i\frac{\partial}{\partial x_i}}\psi(x),\end{aligned} \tag{25}$$

where the repeated indices i are not summed over. At this stage, we must impose the commutation relations for the partial differential operators

$$\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] \equiv \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i} = 0, \qquad \forall i,\ j = 1,\ 2,\ 3. \tag{26}$$

The diagonal element $i = j$ has already been used in the derivation of Eq. (22). The commutation relation (26) can be applied to the product of the three one-dimensional translation operators to simplify the three-dimensional translation operator $\mathscr{T}(\boldsymbol{a})$ as

$$\begin{aligned}\mathscr{T}(\boldsymbol{a}) &= \mathscr{T}(a_3\hat{\boldsymbol{e}}_3)\mathscr{T}(a_2\hat{\boldsymbol{e}}_2)\mathscr{T}(a_1\hat{\boldsymbol{e}}_1)\\ &= \lim_{n_3\rightarrow\infty}\left[\mathbb{1} - \frac{a_3}{n_3}\frac{\partial}{\partial x_3}\right]^{n_3} \lim_{n_2\rightarrow\infty}\left[\mathbb{1} - \frac{a_2}{n_2}\frac{\partial}{\partial x_2}\right]^{n_2}\\ &\quad \lim_{n_1\rightarrow\infty}\left[\mathbb{1} - \frac{a_1}{n_1}\frac{\partial}{\partial x_1}\right]^{n_1}\\ &= \lim_{n\rightarrow\infty}\left[\mathbb{1} - \frac{1}{n}\left(a_1\frac{\partial}{\partial x_1} + a_2\frac{\partial}{\partial x_2} + a_3\frac{\partial}{\partial x_3}\right)\right]^n\\ &= \lim_{n\rightarrow\infty}\left[\mathbb{1} - \frac{1}{n}(\boldsymbol{a}\cdot\boldsymbol{\nabla})\right]^n\\ &= e^{-\boldsymbol{a}\cdot\boldsymbol{\nabla}},\end{aligned} \tag{27}$$

where the gradient operator $\boldsymbol{\nabla}$ is defined by

$$\nabla = \hat{\boldsymbol{e}}_1 \frac{\partial}{\partial x_1} + \hat{\boldsymbol{e}}_2 \frac{\partial}{\partial x_2} + \hat{\boldsymbol{e}}_3 \frac{\partial}{\partial x_3}. \tag{28}$$

The algebraic properties of the three-dimensional translation operator are identical to those for the one-dimensional counterpart listed in Eq. (22)

$$\mathscr{T}(\mathbf{0}) = \mathbb{1}, \tag{29a}$$

$$\begin{aligned}\mathscr{T}(\boldsymbol{a})\mathscr{T}(\boldsymbol{b}) &= \mathscr{T}(\boldsymbol{b})\mathscr{T}(\boldsymbol{a}) \\ &= \lim_{n\to\infty}\left[\mathbb{1} - \frac{1}{n}\boldsymbol{a}\cdot\nabla\right]^n \lim_{n'\to\infty}\left[\mathbb{1} - \frac{1}{n'}\boldsymbol{b}\cdot\nabla\right]^{n'} \\ &= \lim_{n\to\infty}\left[\mathbb{1} - \frac{1}{n}(\boldsymbol{a}+\boldsymbol{b})\cdot\nabla\right]^n \\ &= \mathscr{T}(\boldsymbol{a}+\boldsymbol{b}),\end{aligned} \tag{29b}$$

$$[\mathscr{T}(\boldsymbol{a})]^{-1} = \mathscr{T}(-\boldsymbol{a}). \tag{29c}$$

The one-dimensional translation operator can be generalized to obtain the time-evolution operator $\mathscr{U}(t'-t)$. The requirement of the operator $\mathscr{U}(t'-t)$ is

$$\begin{aligned}\psi(t',\boldsymbol{x}) &= \psi(t+\Delta t,\boldsymbol{x})\Big|_{\Delta t = t'-t} \\ &= \mathscr{U}(\Delta t)\psi(t,\boldsymbol{x})\Big|_{\Delta t=t'-t}.\end{aligned} \tag{30}$$

Equations (18) and (30) yield the explicit form of the time-evolution operator

$$\mathscr{U}(\Delta t) = e^{\Delta t\frac{\partial}{\partial t}}. \tag{31}$$

Note that the replacement $\Delta t = t'-t$ in Eq. (30) must be made after applying the operator $\mathscr{U}(\Delta t)$ to $\psi(t,\boldsymbol{x})$. The causal time-evolution operator, in which the reversed flow of time $t' < t$ is excluded, can be developed from the time-evolution operator (31) as

$$\mathscr{U}_+(\Delta t) = \theta(t'-t)e^{\Delta t\frac{\partial}{\partial t}}, \tag{32}$$

where $\theta(t'-t)$ is the Heaviside step function $\theta(t'-t)$

$$\theta(t'-t) = \begin{cases} 1, & t' \geq t, \\ 0, & \text{otherwise.}\end{cases} \tag{33}$$

3 Application to the linear damping force

The equation of motion for an object of mass m under a linear damping force is

$$m\ddot{x}(t) + b\dot{x}(t) = 0, \tag{34}$$

where b is the damping coefficient and $x(t)$ is the displacement. The time-evolution operator in Eq. (31) can be used to express the displacement $x(t')$. We modify Eq. (30) appropriate for the solution $x(t')$ as

$$\begin{aligned}x(t') &= \mathscr{U}(\Delta t)x(t)\Big|_{\Delta t=t'-t} \\ &= e^{\Delta t\frac{\partial}{\partial t}}x(t)\Big|_{\Delta t=t'-t} \\ &= \sum_{k=0}^{\infty}\frac{(t'-t)^k}{k!}x^{(k)}(t), \qquad x^{(k)}(t) = \left(\frac{\partial}{\partial t}\right)^k x(t).\end{aligned} \tag{35}$$

By taking the jth-order time derivative of Eq. (34), we find that the jth-order time derivative $x^{(j)}(t)$ also satisfies the same equation of motion

$$mx^{(j+2)}(t) + bx^{(j+1)}(t) = 0, \qquad j = 0,\ 1,\ 2,\ \cdots \tag{36}$$

Note that $x(t)$ does not contribute to the conditions (36). The recurrence relation (36) determines the time derivatives of $x(t)$ to all orders

$$\begin{aligned}x^{(2)}(t) &= \left(-\frac{b}{m}\right)\dot{x}(t), \\ x^{(3)}(t) &= \left(-\frac{b}{m}\right)x^{(2)}(t) = \left(-\frac{b}{m}\right)^2\dot{x}(t), \\ &\vdots \\ x^{(j+1)}(t) &= \left(-\frac{b}{m}\right)^j\dot{x}(t), \quad j = 0,\ 1,\ 2,\ \cdots\end{aligned} \tag{37}$$

The constraints in Eq. (37) can be imposed to the solution (35) obtained by the time-evolution operator. The result is

$$\begin{aligned}x(t') &= x(t) + \dot{x}(t)\sum_{k=1}^{\infty}\frac{1}{k!}\left(-\frac{b}{m}\right)^{k-1}(t'-t)^k \\ &= x(t) - \frac{m}{b}\dot{x}(t)\sum_{k=1}^{\infty}\frac{1}{k!}\left[-\frac{b}{m}(t'-t)\right]^k \\ &= x(t) - \frac{m}{b}\dot{x}(t)\left[e^{-\frac{b}{m}(t'-t)} - 1\right].\end{aligned} \tag{38}$$

The result in Eq. (38) reveals that $x(t')$ is completely determined once both $x(t)$ and $\dot{x}(t)$ are known.

4 Application to classical harmonic oscillator

The equation of motion for the classical simple harmonic oscillator consisting of a mass m and a spring with the spring constant k is

$$m\ddot{x}(t) + kx(t) = 0, \tag{39}$$

where $x(t)$ is the displacement of the mass from the equilibrium point at time t. The time-evolution operator in Eq. (31) can be used to express the displacement $x(t')$. We modify Eq. (30) appropriate for the solution $x(t')$ as

$$x(t') = \mathscr{U}(\Delta t)x(t)\Big|_{\Delta t=t'-t} = e^{\Delta t\frac{\partial}{\partial t}}x(t)\Big|_{\Delta t=t'-t} = \sum_{k=0}^{\infty}\frac{(t'-t)^k}{k!}x^{(k)}(t), \qquad x^{(k)}(t) = \left(\frac{\partial}{\partial t}\right)^k x(t). \tag{40}$$

By taking the jth-order time derivative of Eq. (39), we find that the jth-order time derivative $x^{(j)}(t)$ also satisfies the same equation of motion

$$mx^{(j+2)}(t) + kx^{(j)}(t) = 0, \qquad j = 0,\ 1,\ 2,\ \cdots \tag{41}$$

As was presented in Eq. (37), the recurrence relation (41) determines the time derivatives of $x(t)$ to all orders

$$\begin{aligned} x^{(2j)}(t) &= \left(-\frac{k}{m}\right)^j x(t), \\ x^{(2j+1)}(t) &= \left(-\frac{k}{m}\right)^j \dot{x}(t), \\ j &= 0,\ 1,\ 2,\ \ldots \end{aligned} \tag{42}$$

The constraints in Eq. (42) can be imposed to the solution (40) obtained by the time-evolution operator. The result is

$$\begin{aligned} x(t') &= \sum_{j=0}^{\infty}\frac{(t'-t)^{2j}}{(2j)!}x^{(2j)}(t) + \sum_{j=0}^{\infty}\frac{(t'-t)^{2j+1}}{(2j+1)!}x^{(2j+1)}(t) \\ &= x(t)\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j)!}\left[\sqrt{\frac{k}{m}}(t'-t)\right]^{2j} \\ &\quad + \sqrt{\frac{m}{k}}\dot{x}(t)\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)!}\left[\sqrt{\frac{k}{m}}(t'-t)\right]^{2j+1} \\ &= x(t)\cos\left[\sqrt{\frac{k}{m}}(t'-t)\right] \\ &\quad + \sqrt{\frac{m}{k}}\dot{x}(t)\sin\left[\sqrt{\frac{k}{m}}(t'-t)\right], \end{aligned} \tag{43}$$

where we have made use of the identities

$$\cos x = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!}, \qquad \sin x = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}. \tag{44}$$

As shown in Eq. (38), the result in Eq. (43) reveals that $x(t')$ is completely determined once both $x(t)$ and $\dot{x}(t)$ are known.

5 Conclusion

In this paper, we have presented the explicit expansion form of Leibniz's differential operator d^n by employing elementary algebra only. The expansion formula for the application of this differential operator is systematically controlled by the binomial expansion with oscillating factor. We introduced difference operators $\Delta^n_{a,\lambda}$, the corresponding discrete version, and apply them to construction of the translation operator. The resultant operator is identical to the configuration-space representation of the quantum-mechanical translation operator. A generalization to three-dimensional Euclidean space is explicitly given. As applications, we apply the operator to solve elementary equations of motion. The motion of a particle under a linear damping force is demonstrated to be in the same mathematical structure as that of the exponential function. The same method is applied to solve the equation of motion for the simple harmonic oscillator.

Freshmen learn the Taylor-series expansion at the end of a 1-year calculus course. For example, Stewart [2] deals with the Taylor and Maclaurin series in section 11.10 (p. 759–) and partial derivatives in section 14.3 (p. 911–). Hence, it is too late to wait until the standard calculus course deals with the Taylor-series expansion. Furthermore, the strategy of arriving at the Taylor series expansion in usual calculus textbooks is not ready to implement in physics class. Under the assumption that the Taylor-series expansion of the solution of the equation of motion, Adler [3] introduced a teaching strategy applicable in introductory mechanics class. Although our strategy yields a result equivalent to the Frobenius method, the fundamental difference from the Frobenius method is that we have not assumed the existence and convergence of the Taylor-series expansion. One of the drawbacks of employing the Frobenius method is that it is difficult to convince students why the expansion formula represents the solution even if the practical procedure to reach the solution is quite straightforward; see, for example, Owens [4].

The application of the time-evolution operator in general physics looks non-standard as a teaching strategy. However, the approach we have demonstrated in this paper does not require advanced mathematical tricks. Complete knowledge that is required to understand the Taylor-series expansion is contained in this compact derivation. Our unique derivation relies on binomial expansion and the resultant power-series expansion of the exponential function. Unlike mathematics textbooks, we do not require any additional convergence test of the series, because it is automatically guaranteed by the collapse of the binomial expansion to an exponentiated operator. Our strategy of teaching the translation and time-evolution operators at the freshmen level greatly saves time in students' learning quantum mechanics when they encounter these operators for the first time. In addition, unnecessarily long calculus courses could be simplified to make physics and engineering major students understand how to deal with equations of motion without losing any theoretical rigorousness.

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